Form factors in N=4 SYM and their applications

Gang Yang

Humboldt Universität zu Berlin

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This talk is based on

1211.7028 and work in progress with Rutger Boels, Bernd Kniehl, Oleg Tarasov

and work in progress with Dhritiman Nandan and Matthias Wilhelm

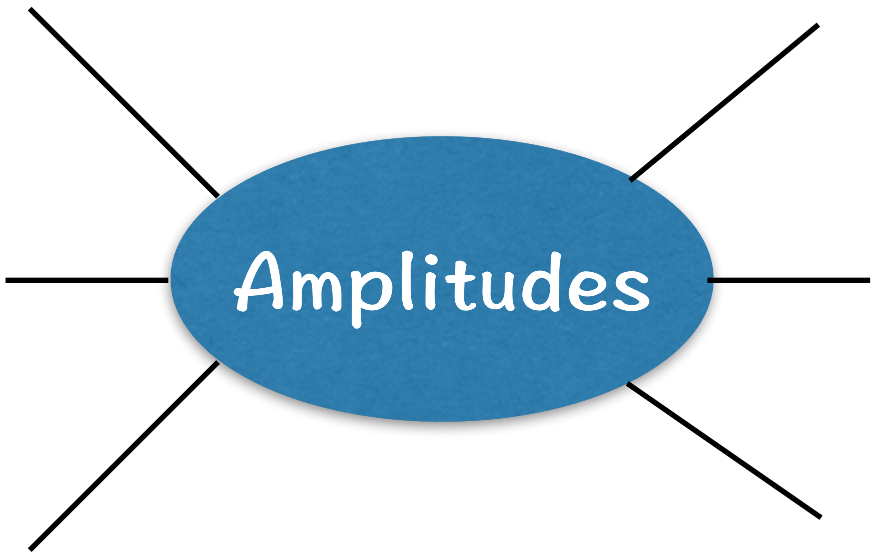
See also other work on form factors in N=4 SYM, including those of:

van Neerven, Alday, Maldacena, Zhiboedov, Brandhuber, Spense, Travaglini, GY, Bork, Kazakov, Vartanov, Gurdogan, Mooney, Gehrmann, Henn, Huber, Moch, Naculich; Engelund, Roiban, Young, Gao, Penante, Wen, …

Outline

- Motivations
- A brief review of form factors
- Sudakov form factor and cusp anomalous dimension
- Observables from form factors
- Outlook

Motivations



On-shell techniques are very efficient

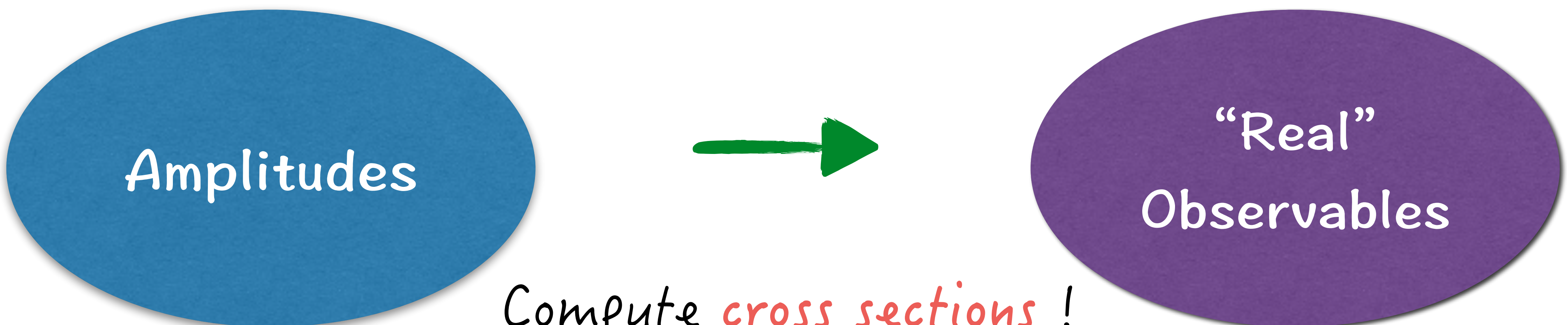
Amplitudes have surprising "hidden" symmetries and structures

(See many other talks in this conference)

How far can one apply them to compute other "off-shell" quantities, such as form factors and correlation functions?

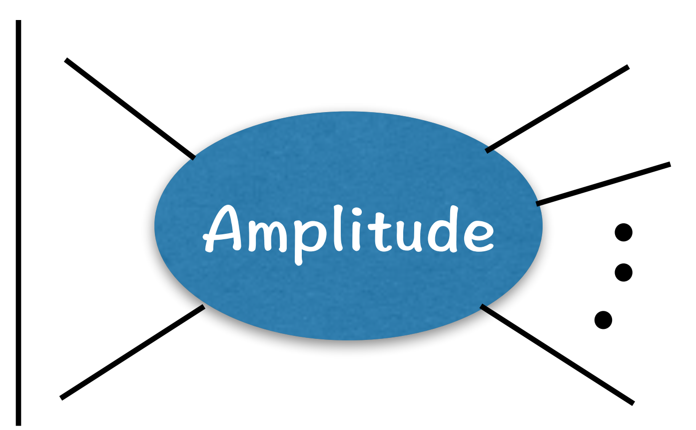
And how are the symmetries and structures generalized there?

Motivations



Compute cross sections !

$$\int d\mathrm{PS}_n \times \left| \quad \right|^2$$



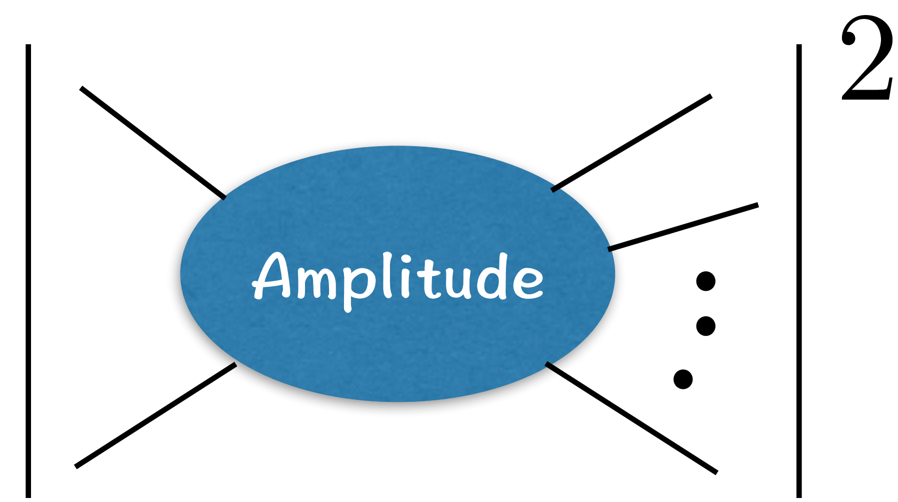
Much less explored area

(See Duhr's talk for some recent developments)

Motivations

Cross sections:

$$\int d\mathrm{PS}_n \times$$



Phase space integral
(in D-dimension!)

Regularization is also necessary

Phase space integrand
(external legs in D-dimension?)

Form factors can be a useful testing ground for such studies, while they also relate to interesting observables and quantities.

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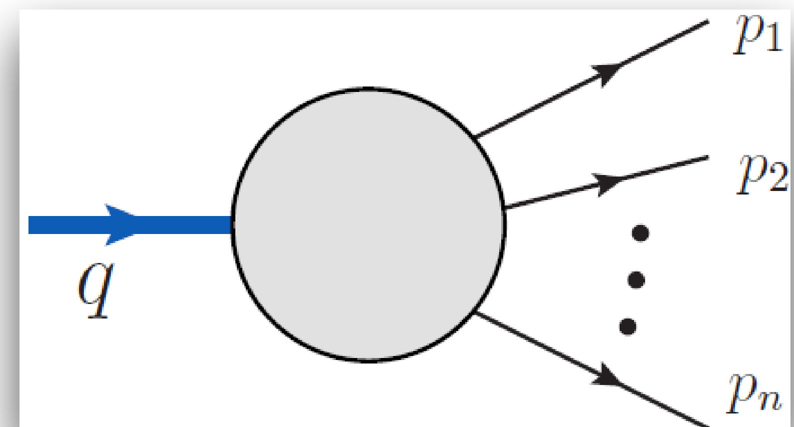
Form factors

Hybrids of on-shell states and off-shell operators:

$$F = \int d^4x\, e^{iq\cdot x}\langle 0|\mathcal{O}(x)|p_1\, p_2\cdots p_n\rangle$$
$$= \delta^4\Big(\sum_{i=1}^{n} p_i - q\Big)\langle 0|\mathcal{O}(q)|p_1 p_2\cdots p_n\rangle$$



$$q = \sum_i p_i, \qquad q^2 \neq 0$$

(work in momentum space)

$\langle 0|p_1 p_2\cdots p_n\rangle$

form factors

$\langle 0\,|\mathcal{O}(x_1)\mathcal{O}(x_2)...\mathcal{O}(x_n)|\,0\rangle$

Form factors in N=4 SYM

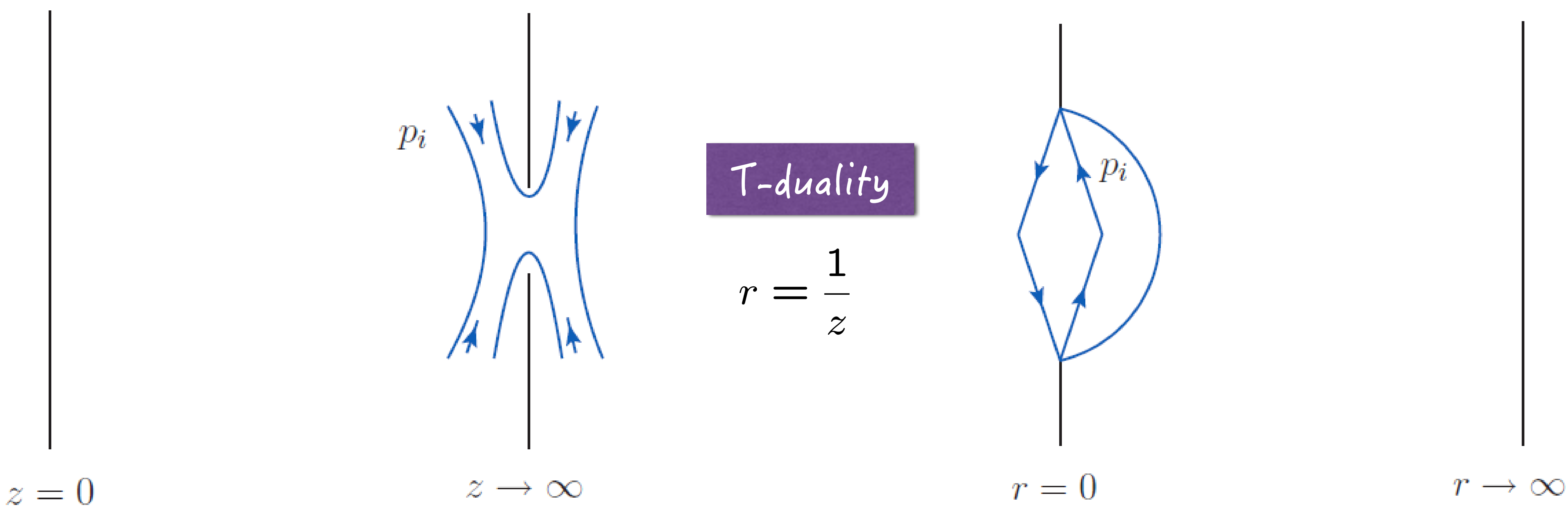
Van Neerven 1985

"Infrared Behaviour of On Shell Form Factors in an N = 4 Supersymmetric Yang-Mills Field Theory"

In 2007, via AdS/CFT a strong coupling picture was proposed by Alday and Maldacena.

Form factors at strong coupling

N=4 SYM ⟷ (AdS/CFT) Type IIB string theory in $AdS_5 \times S^5$

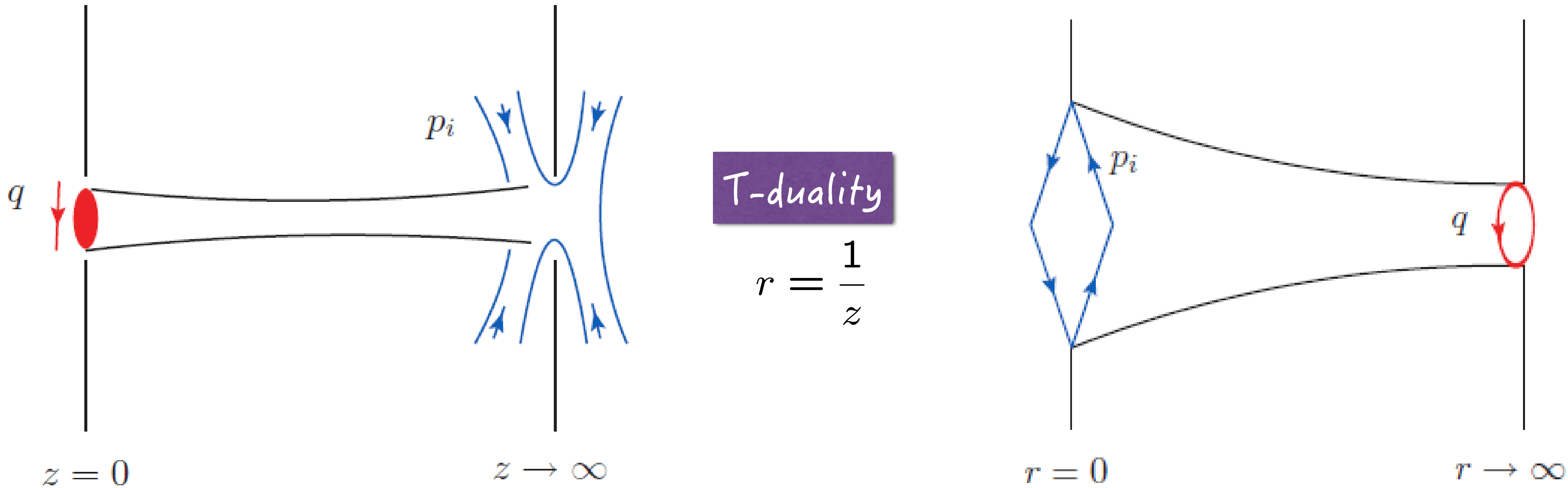


Amplitudes <-> minimal surface of Light-like Wilson loops

(Alday, Maldacena)

Form factors at strong coupling

N=4 SYM **AdS/CFT** ⟷ Type IIB string theory in $AdS_5 \times S^5$



Form factors as minimal surfaces in one period (Alday and Maldacena)

Y-system formulation (Maldacena and Zhiboedov in AdS3; Gao and GY in AdS5)

Indicate hidden structure

Perturbative Form factors

MHV structure of form factors: (Brandhuber, Spence, Travaglini, and GY)

$$F_n^{\mathrm{MHV}}(1^+, .., i_\phi, .., j_\phi, .., n^+; \mathrm{tr}(\phi^2)) = \delta^4(\sum_{i=1}^{n} p_i - q) \frac{\langle ij \rangle^2}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

$$q = \sum_i p_i, \quad p_i^2 = 0, \quad q^2 \neq 0$$

Parke-Taylor formula for amplitudes:

$$A_n^{\mathrm{MHV}}(1^+, .., i^-, .., j^-, .., n^+) = \delta^4(\sum_{i=1}^{n} p_i) \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

$$0 = \sum_i p_i, \quad p_i^2 = 0$$

Perturbative Form factors

Form factor/periodic Wilson line duality at one loop

(Brandhuber, Spence, Travaglini, GY)

BCFW, MHV, unitarity can be efficiently applied, and supersymmetric formulation for stress-tensor supermultiplet.

(Brandhuber, Spence, Gurdogan, Mooney, Travaglini, GY)

(Bork, Kazakov, Vartanov)

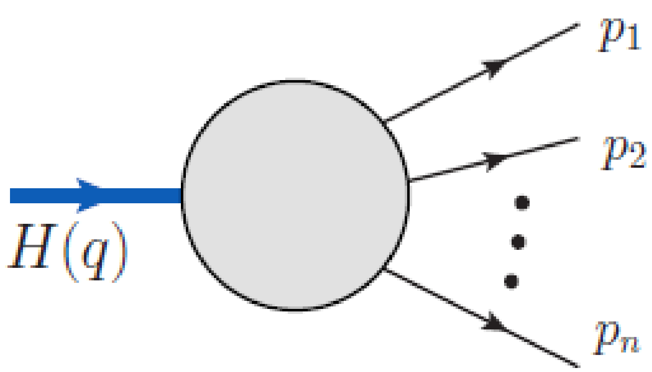
Form factors with more general operators. See Penante's talk.

(Engelund, Roiban)

(Brandhuber, Penante, Spence, Travaglini, Wen)

Form factors

Form factors are interesting and useful quantities themselves.



$$\mathcal{L}_{\text{eff.int.}} = -\frac{\lambda}{4} H \ \text{tr}\,(F_{\mu\nu}F^{\mu\nu})$$

effective vertex

(see Duhr's talk)

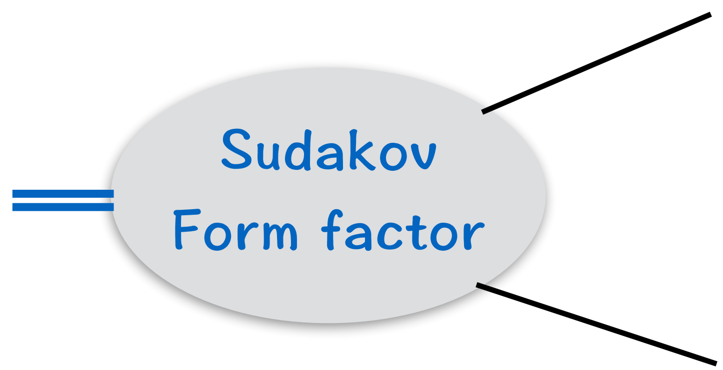
With Brandhuber and Travaglini, we found that the 2-loop 3-point form factor in N=4 SYM matches exactly the leading transcendental part of two-loop Higgs+3-gluon scattering in QCD.

(Brandhuber, Travaglini, GY) (Gehrmann, Jaquier, Glover, Koukoutsakis)

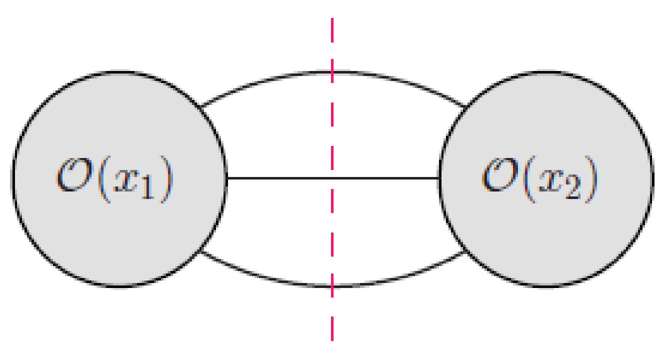
N=4 SYM ⟷ **QCD**

In this talk we will consider two other applications:

- Sudakov form factor -> Cusp anomalous dimension



- Cross section of form factors -> anomalous dimension of operators, and energy-energy correlation function.



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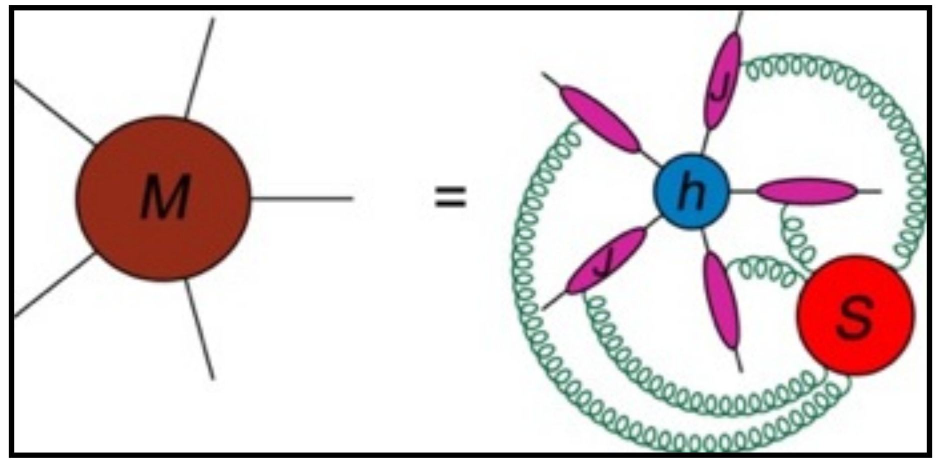
Cusp (soft) anomalous dimension

QCD factorization:



See Gardi's talk

[Mueller; Collins; Sen], [Korchemsky, Radyushkin], [Magnea, Sterman, Tejeda-Yeomans]

$$\mathcal{M}_n = J\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu), \epsilon\right) \times S\left(k_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu), \epsilon\right) \times h_n\left(k_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu), \epsilon\right)$$

Planar structure in N=4 SYM: [Bern, Dixon, Smirnov]

$$\mathcal{M}_n = \prod_{i=1}^{n}\left[\mathcal{M}^{[gg\to 1]}\left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon\right)\right]^{1/2} \times h_n\left(k_i, \mu, \alpha_s, \epsilon\right)$$

cusp anomalous dimension

Sudakov form factor $= \exp\left[-\frac{1}{4}\sum_{l=1}^{\infty} a^l \left(\frac{\mu^2}{-Q^2}\right)^{l\epsilon}\left(\frac{\hat{\gamma}_K^{(l)}}{(l\epsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\epsilon}\right)\right]$

N=4 SYM -> Leading transcendental part of QCD! (still a conjecture)

(Kotikov, Lipatov, Onishchenko, Velizhanin)

new relations, see Henn's talk

Cusp anomalous dimension

For planar N=4, it is in principle known to all loop orders via integrability:

(Beisert, Eden, Staudacher)

$$\Gamma_{\text{cusp}} = 2\left(\frac{\lambda}{8\pi^2}\right) - \frac{\pi^2}{3}\left(\frac{\lambda}{8\pi^2}\right)^2 + \frac{11\pi^4}{90}\left(\frac{\lambda}{8\pi^2}\right)^3 + \left(-2\zeta_3^2 - \frac{73\pi^6}{1260}\right)\left(\frac{\lambda}{8\pi^2}\right)^4 + \mathcal{O}(\lambda^5) \qquad (\lambda = g^2 N_c)$$

[Belitsky,Gorsky,Korchemsky'03], [Kotikov,Lipatov,Onishchenko,Velizhanin'04]
[Bern,Czakon,Dixon,Kosower,Smirnov'06],[Cachazo,Spradlin,Volovich'06]
[Gubser, Klebanov,Polyakov'02], [Frolov,Tseytlin'02], [Kruczenski'02], [Makeenko'02]

On the other hand, much less is known at non-planar order for cusp anomalous dimension, even the leading order result is not known!

Cusp (soft) anomalous dimension

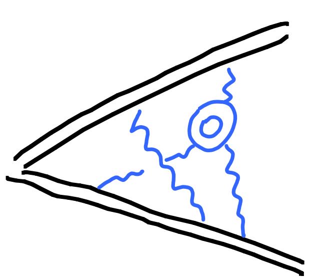
Leading order non-planar starts at four loop, due to the new group factor (quartic Casimir) that appears at this order. (See e.g Henn and Huber)

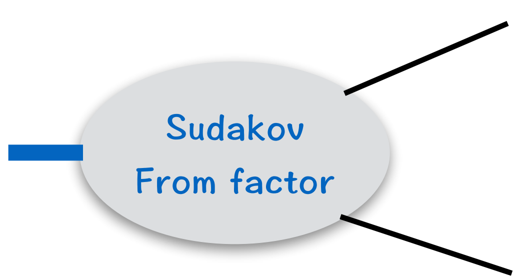
$$\log\langle W\rangle = g^2 C_F w_1 + g^4 C_F C_A w_2 + g^6 C_F C_A^2 w_3 + g^8 \left[C_F C_A^3 w_{4a} + \frac{d_F^{abcd} d_A^{abcd}}{N_F} w_{4b} \right]$$

For SU(N) case, this is:

$$\log\langle W\rangle = \frac{N^2-1}{2N} \left\{ g^2 w_1 + g^4 N w_2 + g^6 N^2 w_3 + g^8 \left[N^3 \left(w_{4a} + \frac{1}{24} w_{4b} \right) + N \frac{1}{4} w_{4b} \right] \right\}$$

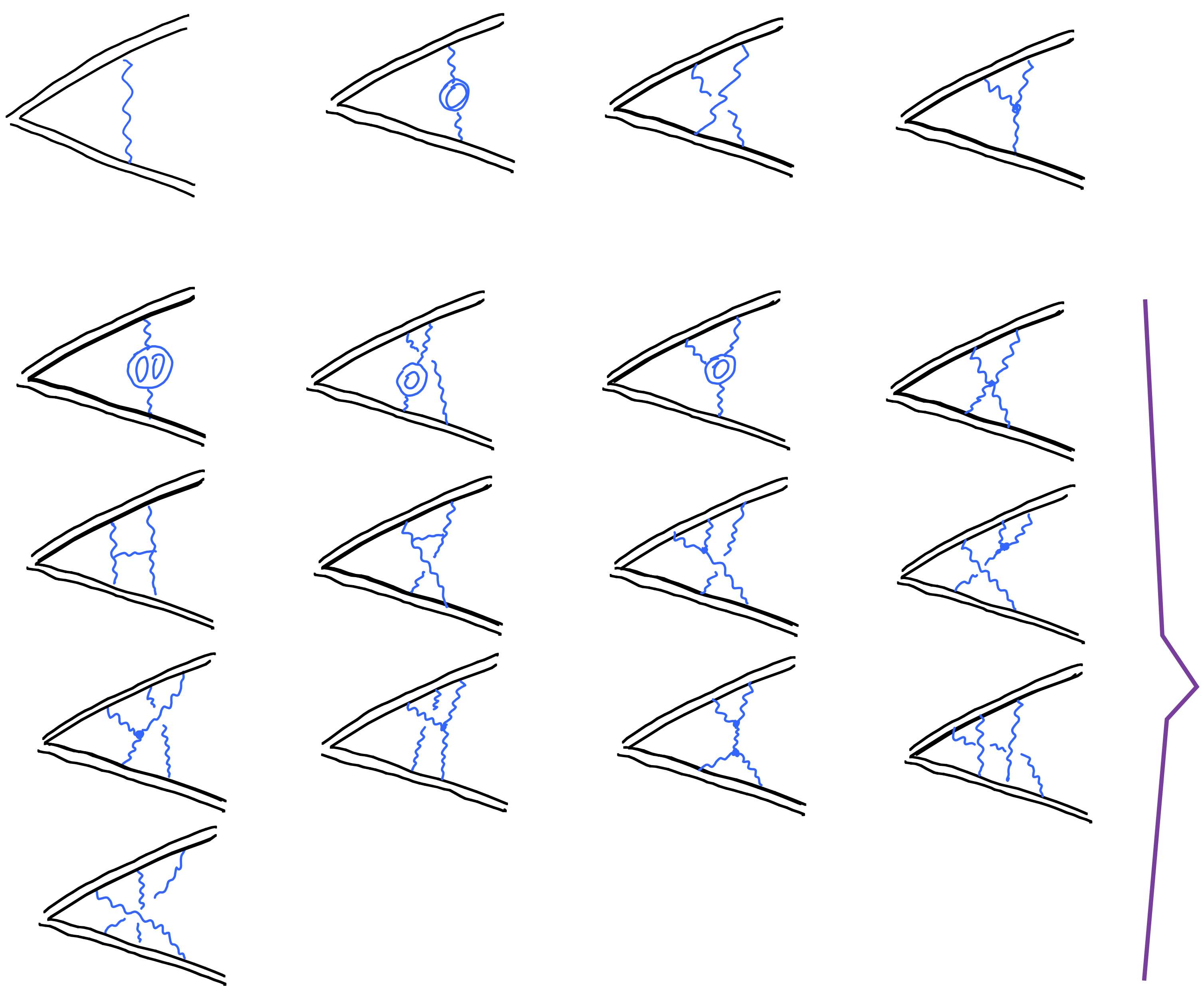
An honest calculation would be to compute cusp Wilson line or Sudakov form factors.





Wilson line computation

Web graphs via non-abelian exponential theorem: [Gatheral; Frenkel and Taylor]



three-loop

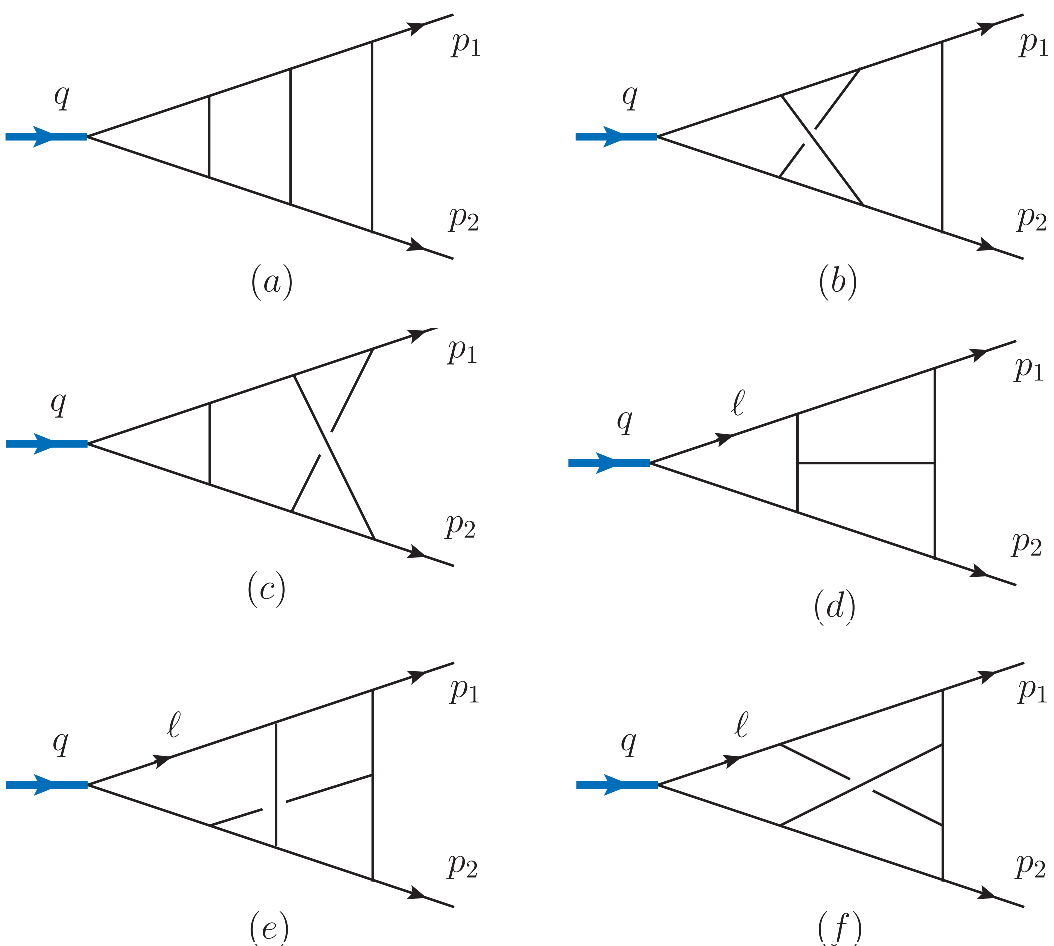
See the talks of Gardi and Henn for discussion and many recent developments

Three-loop form factor in N=4 SYM

Compact expression via color-kinematic duality:

(Boels, Kniehl, Tarasov, GY)

Results via unitarity first obtained by: (Gehrmann, Henn, Huber)

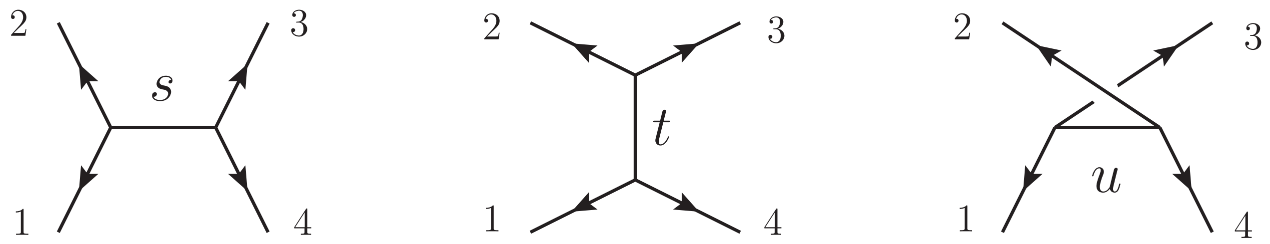


Basis	Numerator factor	Color factor	Symmetry factor
(a)	s_{12}^2	$8\,N_c^3\,\delta^{a_1 a_2}$	2
(b)	s_{12}^2	$4\,N_c^3\,\delta^{a_1 a_2}$	4
(c)	s_{12}^2	$4\,N_c^3\,\delta^{a_1 a_2}$	4
(d)	$(p_2 - p_1)\cdot\ell - p_1\cdot p_2$	$2\,N_c^3\,\delta^{a_1 a_2}$	2
(e)	$-(p_2 - p_1)\cdot\ell + p_1\cdot p_2$	$2\,N_c^3\,\delta^{a_1 a_2}$	1
(f)	$(p_2 - p_1)\cdot\ell - p_1\cdot p_2$	0	2

Color kinematic duality

(Bern, Carrasco and Johansson)

Tree level:



$$\mathcal{A}_4^{(0)}(1,2,3,4) \;=\; \frac{c_s\, n_s}{s} + \frac{c_t\, n_t}{t} + \frac{c_u\, n_u}{u}$$

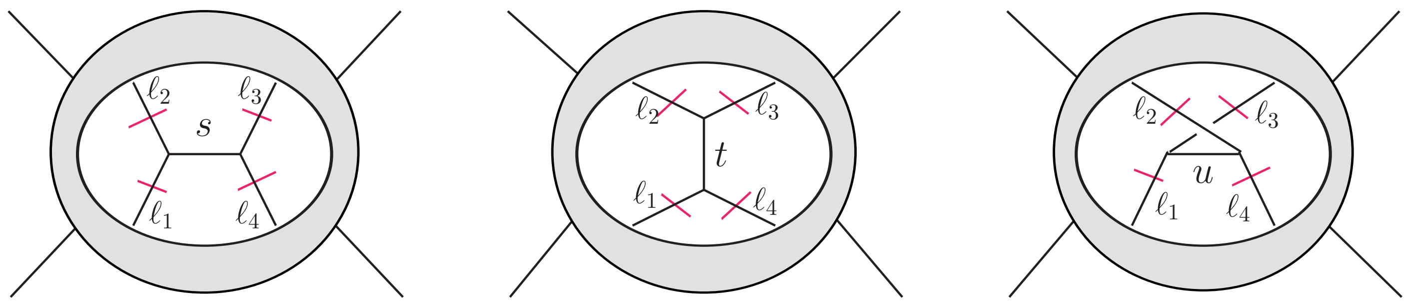
$$c_s = \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}, \qquad c_t = \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}, \qquad c_u = \tilde{f}^{a_1 a_3 b}\tilde{f}^{b a_2 a_4}$$

$$c_s = c_t + c_u \qquad \Rightarrow \qquad n_s = n_t + n_u$$

Color kinematic duality

(Bern, Carrasco and Johansson)

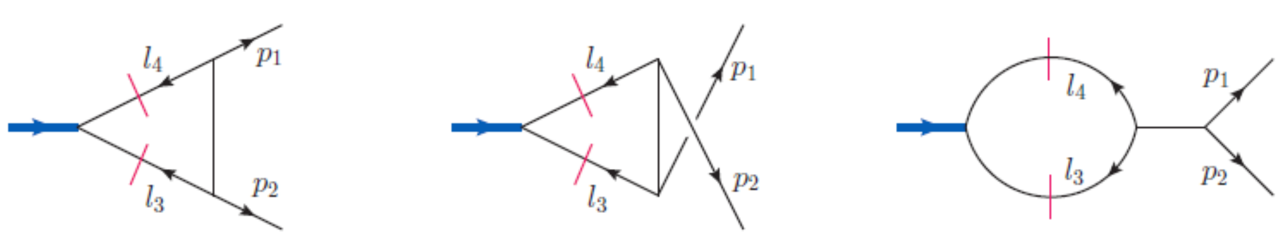
Loop level:



(more details in talks by Monteiro and Johansson)

$$\mathcal{A}_n^{(l)} = \sum_{\Gamma_i} \int \prod_j^l d^D \ell_j \frac{1}{S_i} \frac{C_i N_i}{\prod_a D_a} \qquad C_i = C_j + C_k \quad \Rightarrow \quad N_i = N_j + N_k$$

Generalization to form factors: (Boels, Kniehl, Tarasov, GY)



(a) (b) (c)

$$n_{(a)} = s_{12}\,, \qquad n_{(b)} = s_{12}\,, \qquad n_{(c)} = 0$$

General strategy

(Bern, Carrasco, Dixon, Johansson, Roiban)

(Boels, Kniehl, Tarasov, GY)

1. Generate all topologies (no-triangle property)
2. Ansatz for master integrals (power-counting requirement), and all other numerators are fixed via color-kinematic duality

$$N_d^{\text{ansatz}}(p_1, p_2, \ell) = \alpha_1 \ell \cdot p_1 + \alpha_2 \ell \cdot p_2 + \alpha_3 p_1 \cdot p_2$$

3. Symmetry of the numerator

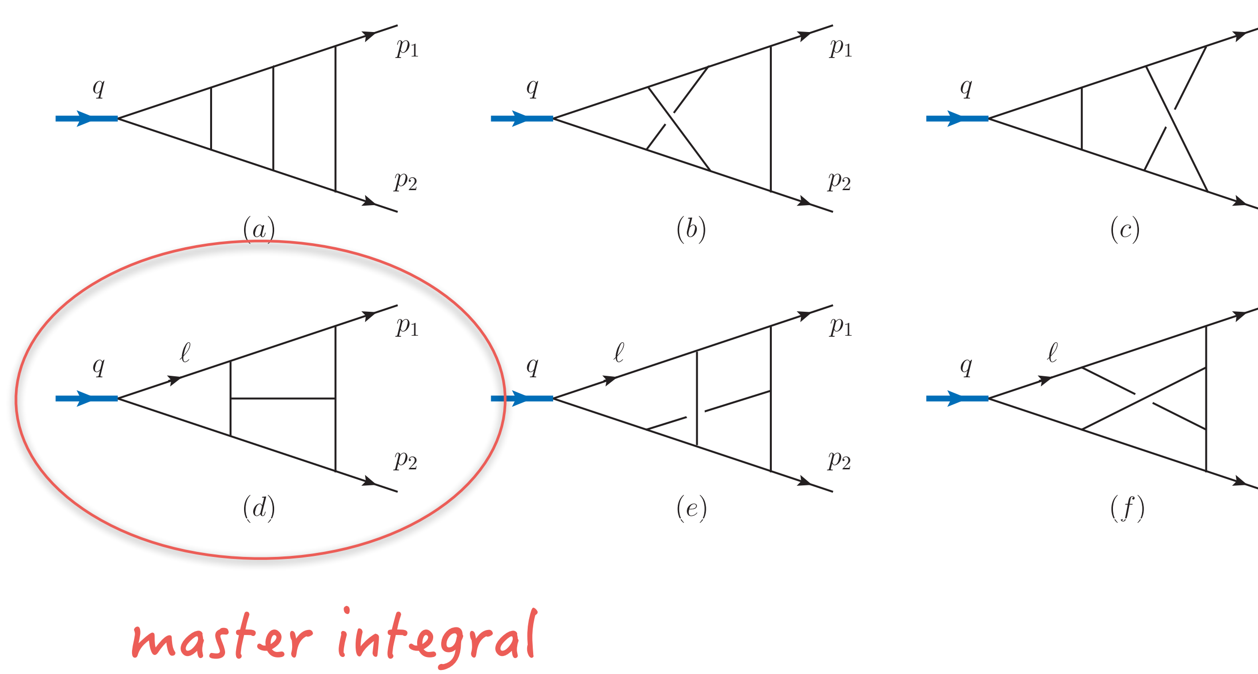
$$\{p_1, p_2, \ell\} \quad \Longleftrightarrow \quad \{p_2, p_1, q-\ell\} \quad \Rightarrow \quad \alpha_2 = -\alpha_1$$

4. A simple unitarity cut

$$\left[N_d(p_1, p_2, \ell) - (\ell - p_1)^2\right]\Big|_{\text{maximal cut}} = 0$$

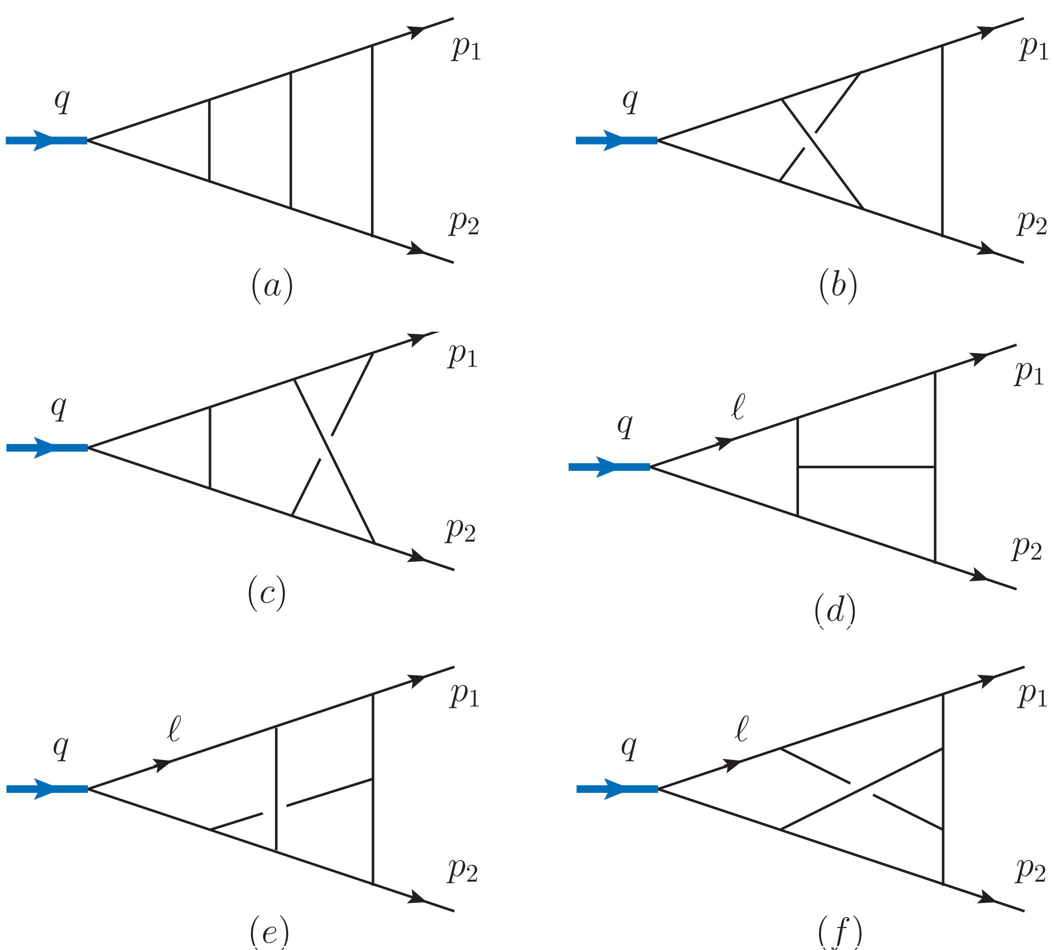
$$\Rightarrow \quad \alpha_1 = -1, \qquad \alpha_3 = -1$$

5. Finally, check with all unitarity cuts.



Three-loop form factor in N=4 SYM

(Boels, Kniehl, Tarasov, GY)



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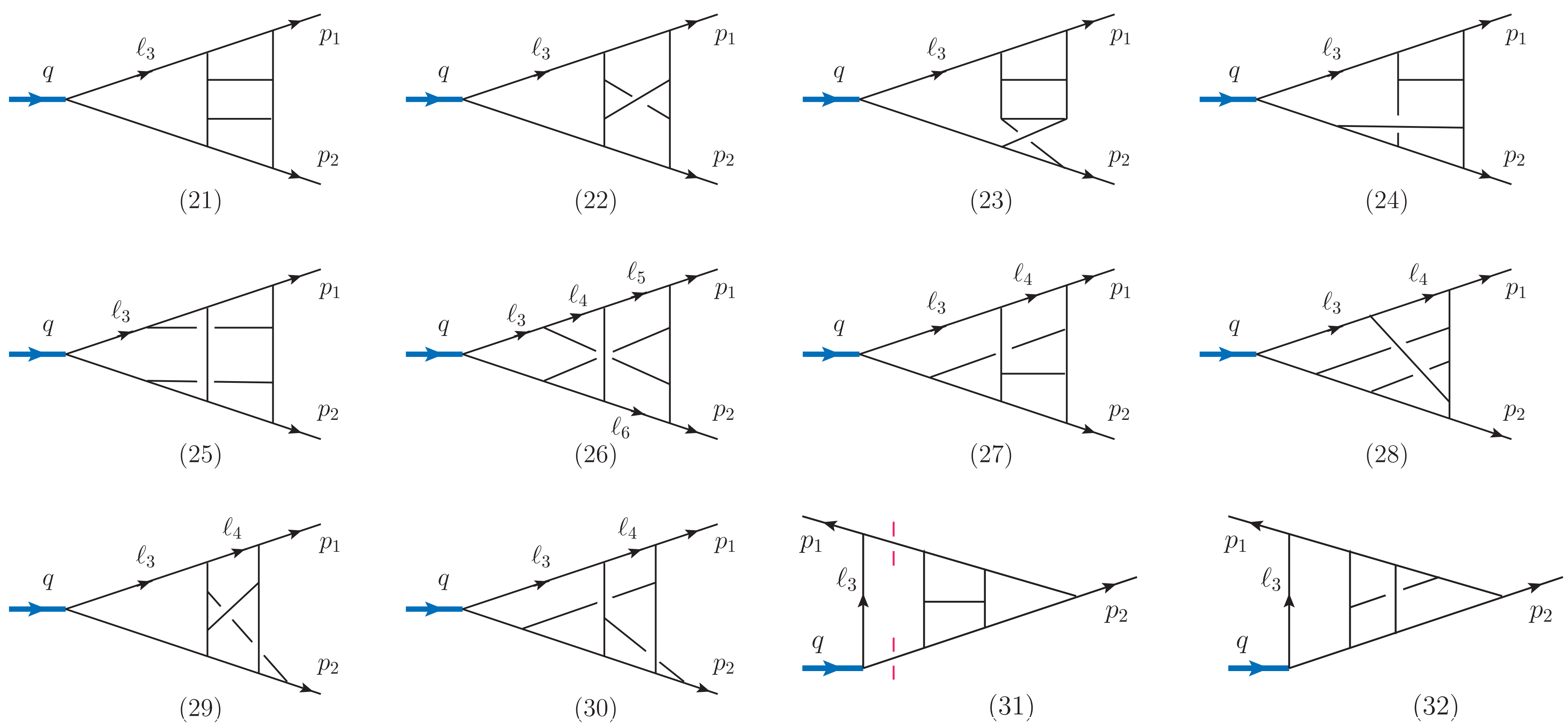
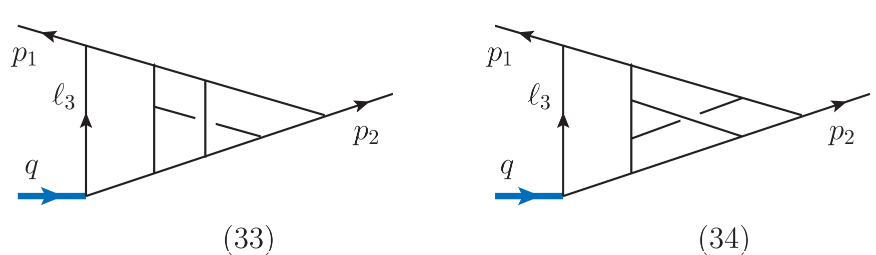
(e)	$-(p_2 - p_1) \cdot \ell + p_1 \cdot p_2$	$2\, N_c^3\, \delta^{a_1 a_2}$	1
(f)	$(p_2 - p_1) \cdot \ell - p_1 \cdot p_2$	0	2

Non-planar four-loop form factors

(Boels, Kniehl, Tarasov, GY)

In a similar way, we obtain a compact integrand with at most tensor-2 numerators:





$$
\begin{aligned}
N_{21} = \; & -(\ell_3\cdot p_1)^2 - (\ell_3\cdot p_2)^2 - 6(\ell_3\cdot p_1)(\ell_3\cdot p_2) \\
& + (p_1\cdot p_2)\big[2(\ell_3\cdot \ell_3) + 4(\ell_3\cdot p_1) + p_1\cdot p_2\big] \\
& + (\alpha_1+1)\big[(\ell_3\cdot p_{12} - p_1\cdot p_2)^2 \\
& - \tfrac{2}{7}(\ell_3\cdot(\ell_3 - p_{12}) + p_1\cdot p_2)(p_1\cdot p_2)\big]
\end{aligned}
$$

Integration is very challenging. (work in progress)

see the talks of Henn and Smirnov for new ideas

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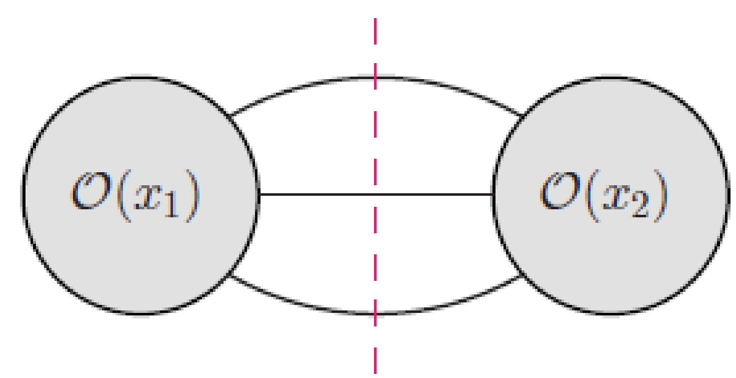
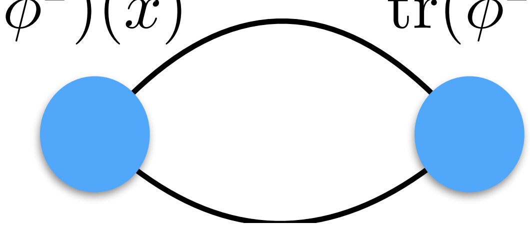


Structure of 2pt correlator

Position space:



$$\langle 0|\mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle = \frac{1}{(x^2)^{\Delta(g)}}$$

$$= \frac{1}{(x^2)^{\Delta_0}}\Big\{1 + g^2\,\gamma_1\,\log(x^{-2}) + g^4\big[\gamma_2\,\log(x^{-2}) + \frac{\gamma_1^2}{2}\log(x^{-4})\big] + \mathcal{O}(g^5)\Big\}$$

with the dimension of the operator:

$$\Delta(g) = \Delta_0 + \gamma_1\,g^2 + \gamma_2\,g^4 + \mathcal{O}(g^5) \qquad\qquad \Delta_0 = n(1-\epsilon)\,, \quad n \geq 2$$

Structure of 2pt correlator

Momentum space: $\Delta(g) = \Delta_0 + \gamma_1 g^2 + \gamma_2 g^4 + \mathcal{O}(g^5)$ $\Delta_0 = n(1-\epsilon)\,, \quad n \geq 2$

$$\Pi(q^2) \quad := \quad \int d^D x \, e^{iqx} \langle 0 | \mathcal{T} \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

$$\longrightarrow \quad \int d^D x \, e^{iq\cdot x} \frac{1}{(x^2)^\Delta} = 4^{D/2-\Delta} \pi^{D/2} \frac{\Gamma(1-\Delta)\Gamma(\frac{D}{2}-\Delta)}{\Gamma(2-\frac{D}{2})\Gamma(\frac{D}{2}-1)} \frac{1}{(q^2)^{D/2-\Delta}}$$

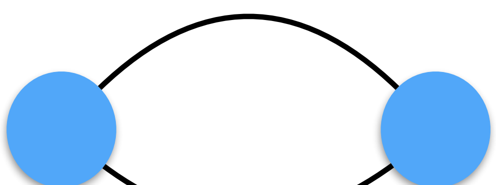
UV Singularity in epsilon expansion and for small g. (Penati, Santambrogio)

$$\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) + \sum_l g^{2l} \left(\frac{\gamma_l}{\epsilon^2} + \text{singular terms depend on } \gamma_{i<l} + \mathcal{O}(\epsilon^0) \right)$$

(Renormalization is necessary)

Tree level example

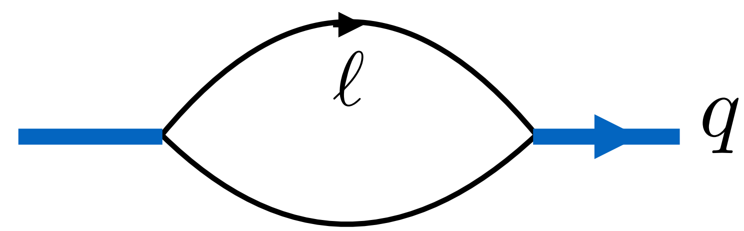
Position space:



$\mathrm{tr}(\phi^2)(x)$ $\mathrm{tr}(\phi^2)(0)$

$$\frac{1}{(x^2)^{2-2\epsilon}}$$

Momentum space:



$$\int d^D\ell \, \frac{1}{\ell^2(\ell-q)^2} \sim \frac{(-q^2)^{-\epsilon}}{\epsilon}$$

Divergence!

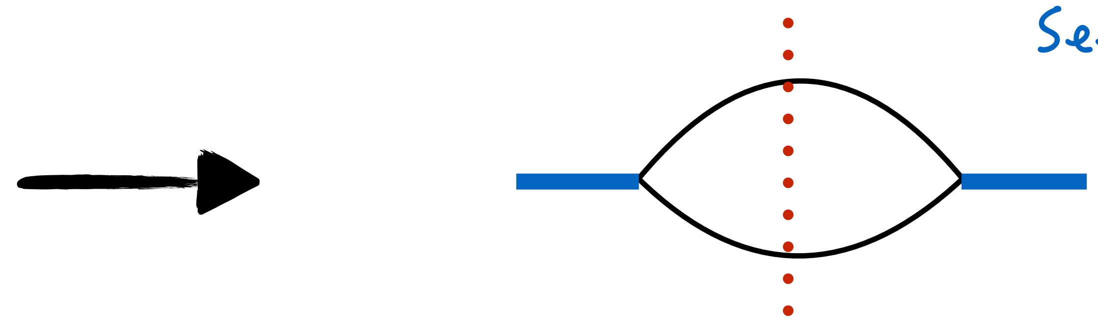
Structure of 2pt correlator

By optical theorem, the cut of two-point correlation function is given as

$$\mathrm{Im}[\Pi(q^2)] = \sigma_{\mathrm{tot}}(q) = \sum_X \delta^D(q - p_X)\,|\langle 0|\mathcal{O}(0)|X\rangle|^2$$

$$\int d^D\ell\,\frac{1}{\ell^2(\ell-q)^2} \sim \frac{(-q^2)^{-\epsilon}}{\epsilon} = \frac{1}{\epsilon} - \epsilon\log(-q^2) + \frac{\epsilon}{2}\log^2(-q^2) + \mathcal{O}(\epsilon^2)$$

See e.g Britto's talk



$\sim 2\pi i$

The order of divergence is decreased by one.

$$2\pi i\left\{1 + \mathcal{O}(\epsilon^1) + \sum_l g^{2l}\left(\frac{\gamma_l}{\epsilon} + \text{singular terms depend on } \gamma_{i<l} + \mathcal{O}(\epsilon^0)\right)\right\}$$

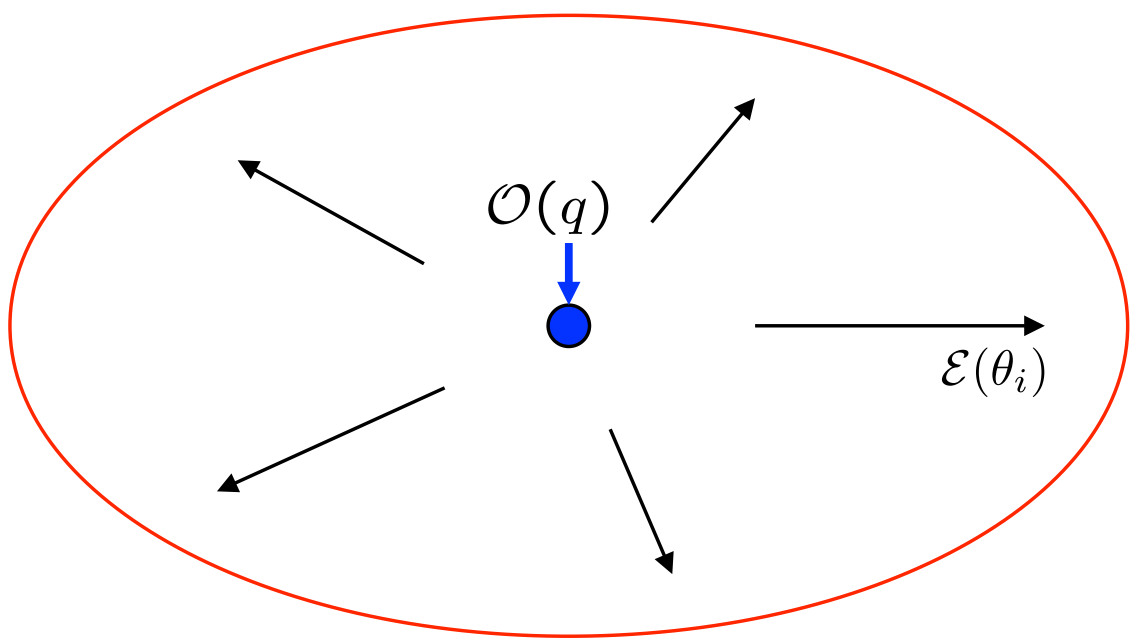
This allows us to extract anomalous dimension via cross section type of computation.

For BPS operators, there is no divergence.

Energy-energy correlators

[Basham, Brown, Ellis, Love]

Observables in N=4 SYM:



[Hofman, Maldacena]; [Roiban, Engelund]; [Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov]

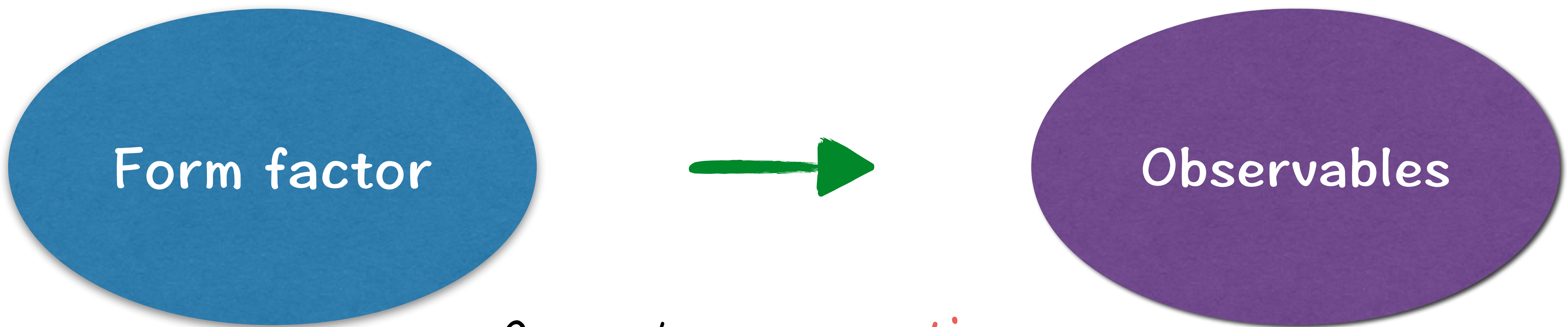
(See Sokatchev's talk)

$$\frac{\langle 0|\mathcal{O}^\dagger \mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)\mathcal{O}|0\rangle}{\langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle} \longleftrightarrow \sigma_w(q) = \sum_X (2\pi)^4\delta^{(4)}(q-k_X)w(X)|\langle X|O(0)|0\rangle|^2$$

$$\sum_X |X\rangle\langle X| = 1, \quad \mathcal{E}|X\rangle = w_{\mathcal{E}}(X)|X\rangle$$

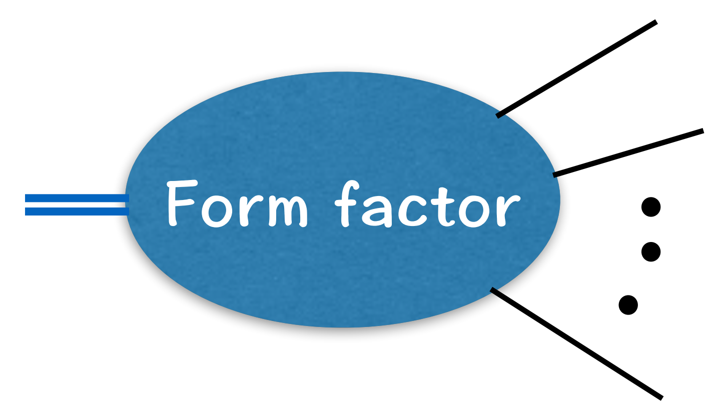
Energy-energy correlation function is given as the "weighted" cross section with form factors as the building blocks.

Cross sections with form factors



Compute cross sections

$$\int d\mathrm{PS}_n \times \text{(weight factor)} \left| \quad \right|^2$$



Form factors serve a useful testing ground for such studies, and for also computing interesting observables and quantities.

Summary and outlook

- Four-loop form factor -> Non-planar cusp anomalous dimension
  work in progress with Boels, Kniehl, Tarasov
- On-shell techniques for "real" observables and application for anomalous dimensions   work in progress with Nandan and Wilhelm
- Techniques for cross sections -> phase space integration   (see Duhr's talk)
- and phase space integrand: |Amplitude|^2 or |FF|^2 (hidden structure?)
- Cross section picture at strong coupling?

Thank you for your attention!

Phase space Integrands

Original Parke-Taylor paper considered |A|^2:

$$|\mathcal{M}_n(+++++\ldots)|^2 = c_n(g,N)\,[\,0 + \mathcal{O}(g^4)\,] \quad (1)$$

$$|\mathcal{M}_n(-++++\ldots)|^2 = c_n(g,N)\,[\,0 + \mathcal{O}(g^4)\,] \quad (2)$$

$$|\mathcal{M}_n(--+++\ldots)|^2 = c_n(g,N)\,\Big[\,(1\cdot 2)^4 \sum_P \frac{1}{(1\cdot 2)(2\cdot 3)(3\cdot 4)\ldots(n\cdot 1)} + \mathcal{O}(N^{-2}) + \mathcal{O}(g^2)\,\Big] \quad (3)$$

Similar result holds for MHV form factor:

$$|F_n^{\mathrm{MHV}}|^2 = q^4 \sum_P \frac{1}{s_{12}s_{23}\ldots s_{n1}}$$

Do other non-MHV cases such as |NMVH|^2 have simple structure?