$\mathcal{N} = 4$ scattering amplitudes and the geometry of cluster coordinates

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Main questions

We want to obtain scattering amplitudes explicitly.
Two related questions:

- What functions?
- Of what arguments?

Two quick (incomplete) answers:

- Transcendental functions
- Cluster coordinates

Not all transcendental functions are good candidates. What is the right subclass of functions to consider?

Transcendental functions

We've learned that a large class of functions appearing in scattering amplitudes (or Wilson loops and correlation functions) are transcendental functions. These functions are iterated integrals of type

$$T_n(x) = \int^x d\log R_1(t_1) \int^{t_1} d\log R_2(t_2) \cdots \int^{t_{n-1}} d\log R_n(t_n),$$

where R_j are rational fractions. The integrals are taken around some contour in a higher dimensional space.
It is important to stress that, once we choose a way to parametrize the space (a way to put coordinates x on the manifold containing the integration path), the symbol describes the transcendental function modulo branch cuts completely and canonically.

Integration

Now the problem is to compute these integrals (or integrate the symbol). Given an integrable symbol

$$R_n \otimes \ldots \otimes R_1,$$

compute the integral

$$T_n(x) = \int^x d\log R_1(t_1) \int^{t_1} d\log R_2(t_2) \cdots \int^{t_{n-1}} d\log R_n(t_n).$$

It's best to split the problem in simpler problems. We will focus on the "indecomposable" part, which can not be written as products of lower transcendentality functions.

Transcendentality two

The symbol of a product of functions is the shuffle product of the symbols of the terms

$$\mathcal{S}(fg) = \mathcal{S}(f) \sqcup\!\sqcup \mathcal{S}(g),$$

where

$$(f_1 \otimes \cdots \otimes f_n) \sqcup\!\sqcup (g_1 \otimes \cdots \otimes g_m) = f_1 \otimes ((f_2 \otimes \cdots \otimes f_n) \sqcup\!\sqcup (g_1 \otimes \cdots \otimes g_m)) + g_1 \otimes ((f_1 \otimes \cdots \otimes f_n) \sqcup\!\sqcup (g_2 \otimes \cdots \otimes g_m)).$$

If we work modulo products, i.e. we project out the shuffles, the most general integrable symbol of transcendentality two is antisymmetric. It is a theorem that it can always be written as

$$\sum_{i<j} a_i \wedge a_j = \sum_i c_i (1 - x_i) \wedge x_i \to \sum_i c_i \{x_i\}_2.$$

The objects $\{x\}_2$ satisfy the dilogarithm identities. They are elements of a group called Bloch group B_2.

Transcendentality three

Here instead of antisymmetrizing as for transcendentality two, we apply the following operation

$$a \otimes b \otimes c \to a \wedge b \otimes c - a \wedge c \otimes b - c \wedge a \otimes b + c \wedge b \otimes a.$$

If the initial symbol was integrable, then, after this projection the answer is writable as

$$\sum_i c_i (1 - x_i) \wedge x_i \otimes x_i \to \sum_i c_i \{x_i\}_3.$$

The objects $\{x\}_3$ satisfy the trilogarithm identities. They are elements of a group called Bloch group B_3.
The function whose symbol we study is of type

$$-\sum_i c_i \, \mathrm{Li}_3(x_i) + \text{products}.$$

Transcendentality four

Here we *don't* completely understand the class of functions. But we can project and partially integrate to two kinds of objects

- Objects of type $\sum_{i<j}\{x_i\}_2 \wedge \{x_j\}_2 \in \Lambda^2 B_2$
- Objects of type $\sum_{ij}\{y_i\}_3 \otimes z_j \in B_3 \otimes \mathbb{C}^*$

If the $\Lambda^2 B_2$ part vanishes, then the answer contains at worst Li_4 (Goncharov).
The $\Lambda^2 B_2$ and $B_3 \otimes \mathbb{C}^*$ parts satisfy a consistency condition:

$$\sum_{ij}(\{x_i\}_2 \otimes (1-x_j) \wedge x_j - \{x_j\}_2 \otimes (1-x_i) \wedge x_i + \{y_i\}_2 \otimes y_i \wedge z_j) = 0.$$

Enhanced bootstrap?

A recent line of research [Dixon, Drummond, Duhr, Henn, von Hippel, Pennington] was to start with a general ansatz for the symbol and impose various constraints on it: symmetry, parity, integrability, collinear limits, near collinear limits, Regge limits (see also [Goddard, Heslop, Khoze]). This turns out to be very constraining.
At transcendentality four one may instead start with the $B_3 \otimes \mathbb{C}^*$ and $\Lambda^2 B_2$ which satisfy the compatibility condition (see [Golden, Paulos, Spradlin, Volovich]).
At transcendentality six more interesting possibilities appear.
Of course, the product terms have to be dealt with separately.

Six-point kinematics

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle}, \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}.$$

In 2D language we have three cross-ratios:

$$u_1 = \frac{(23)(65)}{(25)(63)}, \qquad u_2 = \frac{(34)(16)}{(36)(14)}, \qquad u_3 = \frac{(45)(21)}{(41)(25)},$$

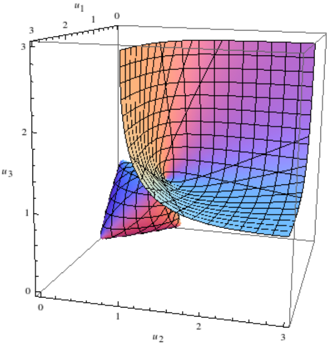
where $(ij) = z_i - z_j$.
The configuration space is either six points in $\mathbb{CP}^3$ or six points in $\mathbb{CP}^1$. The coordinates are of type $\langle ijkl\rangle$ or (ij) and are related by

$$\langle ijkl\rangle \to (mn),$$

where $ijklmn$ is an even permutation of 123456.

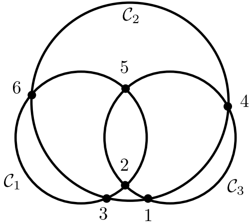
Euclidean region

This is a region where $x_{ij}^2 < 0$ (for signature $+---$) and which is free of branch cuts. Then $u_i > 0$ for $i = 1, 2, 3$.
We can instead think of the region $u_i > 0$ for $i = 1, 2, 3$ with no restriction on signature. If $(1 - u_1 - u_2 - u_3)^2 - 4u_1u_2u_3 > 0$ then have signature $++--$, if $(1 - u_1 - u_2 - u_3)^2 - 4u_1u_2u_3 < 0$ they have $+---$ signature. When $(1 - u_1 - u_2 - u_3)^2 - 4u_1u_2u_3 = 0$ the kinematics is conformally related to lower-dimensional kinematics.



Euclidean region in 2D representation

If four points have a real cross-ratio, then they belong on a circle. Using this we can show that in $++--$ signature all six points belong to the same circle. For Lorentzian signature they belong to three different circles



$$(1,2,3,4,5,6),(1,2,4,6,3,5),(1,2,4,6,5,3),(1,2,6,4,3,5),$$
$$(1,2,6,4,5,3),(1,3,2,4,6,5),(1,3,2,6,4,5),(1,3,4,2,6,5),\cdots$$

The arrangement of points on a circle has an interpretation in mathematics as a positive region in $\mathbb{CP}^1$, but the Lorentzian region with its three circles doesn't!

Collinear limits

In terms of cross-ratios the collinear limits are given by $u_3 = 0$ and $u_1 + u_2 = 1$ and cyclic permutations. In the 2D language the collinear limits are $z_2 \to z_3$ together with $z_5 \to z_6$ and cyclic permutations. After the limit we are left with four points on a circle so the collinear limit is parametrized by their (real) cross-ratio.

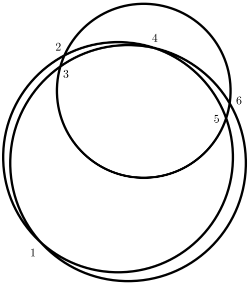


Figure: A way to approach the collinear limit from Lorentzian signature.

Cross-ratios . . .

If we have a number of (ordered) points in two dimensions we can present the cross-ratios they form in a graphical form. We form a convex polygon whose vertices are the initial points. Then to each diagonal in a triangulation we associate a cross-ratio.

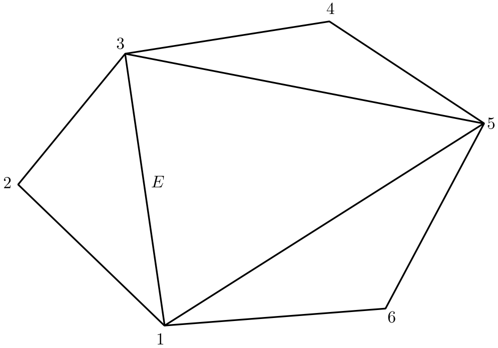


Figure: To the diagonal E we associate the cross-ratio $r(3,5,1,2) = r(1,2,3,5) = \frac{(12)(35)}{(23)(15)}$. Reading in opposite order to obtain the inverse.

...and mutations

For each triangulation we have three diagonals and therefore three cross-ratios. These cross-ratios are independent and can be used to describe the kinematics up to conformal transformations.

Flipping a diagonal in one of the quadrilaterals transforms one triangulation to another and also one set of cross-ratios to another. This is a *mutation*.

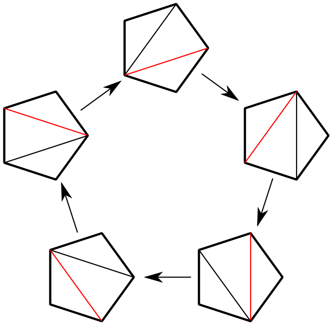


Figure: Mutations for five points. The red diagonal gets flipped.

Cluster algebras of geometric type

We start with a quiver (oriented graph). To each vertex i we associate cluster $\mathcal{A}$ coordinates x_i. We also define a skew-symmetric matrix

$$b_{ij} = (\#\text{arrows } i \to j) - (\#\text{arrows } j \to i).$$

Since only one of the terms above is nonvanishing, $b_{ij} = -b_{ji}$. A mutation at vertex k is obtained by applying the following operations on the initial quiver:

- for each path $i \to k \to j$ we add an arrow $i \to j$
- reverse all the arrows on the edges incident with k
- remove all the two-cycles that may have formed.

It is an involution; when applied twice in succession we obtain the initial cluster.

Mutation of cluster $\mathcal{A}$ coordinates

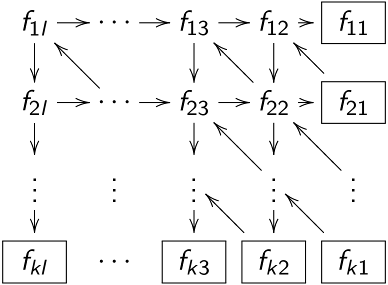
The mutation at k changes a_k to a'_k defined by

$$a_k a'_k = \prod_{i|b_{ik}>0} a_i^{b_{ik}} + \prod_{i|b_{ik}<0} a_i^{-b_{ik}},$$

and leaves the other cluster variables unchanged. (An empty product is set to one.)
Example: the A_2 cluster algebra can be expressed by a quiver $a_1 \to a_2$. Then, a mutation at a_1 replaces it by $a'_1 = \frac{1+a_2}{a_1} \equiv a_3$ and reverses the arrow. A mutation at a_2 replaces it by $a'_2 = \frac{1+a_1}{a_2} \equiv a_5$ and reverses the arrow.

Grassmannian cluster algebras

According to [Gekhtman, Shapiro, Vainshtein], the initial quiver for the $G_k(n)$ cluster algebra is given by[1]

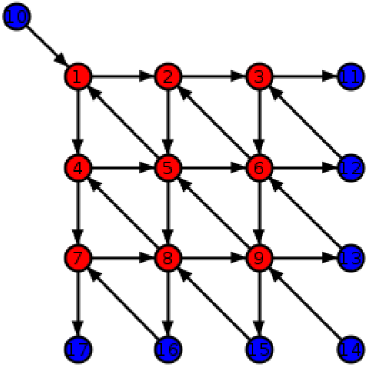


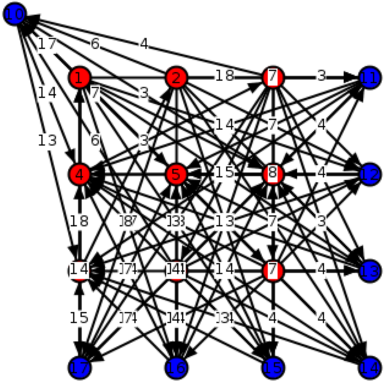
where

$$f_{ij} = \begin{cases} \frac{\langle i+1,\ldots,k,k+j,\ldots,i+j+k-1\rangle}{\langle 1,\ldots,k\rangle}, & i \le l-j+1, \\ \frac{\langle 1,\ldots,i+j-l-1,i+1,\ldots,k,k+j,\ldots,n\rangle}{\langle 1,\ldots,k\rangle}, & i > l-j+1 \end{cases}.$$

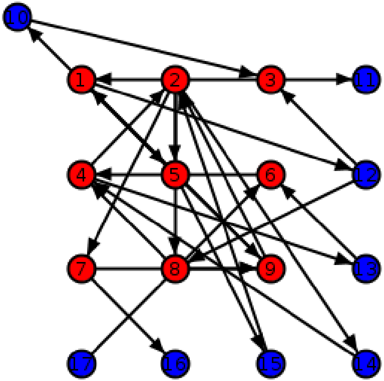
[1]Here we are presented a flipped version of the quiver and with the arrows reversed with respect to the quivers of that ref.

Examples of mutations





$\mathcal{X}$ coordinates



$$
\begin{aligned}
x_2 \to \; & \frac{1}{x_1 x_2 x_4 x_5 x_6 x_7 x_9} \times \\
& (((((x_1 x_3 x_4 x_5 x_6 + x_1^2 x_4 x_6^2) x_8 + x_1 x_2 x_4^2 x_6 x_9) x_{10}) x_{14} + \\
& (x_1 x_2 x_4 x_5 x_9 x_{10} x_{12} + ((x_1 x_3 x_4 x_5^2 + x_1^2 x_4 x_5 x_6) x_{10}) x_{13}) x_{15}) x_{16} + \\
& ((x_2 x_3 x_4 x_5 x_6 x_7 x_8 + (((x_3 x_5^2 x_6 + x_1 x_5 x_6^2) x_7) x_8 + \\
& (x_1 x_3 x_5 x_6 + x_1^2 x_6^2) x_8^2 + x_1 x_2 x_4 x_6 x_8 x_9) x_{10}) x_{14} + \\
& ((x_2^2 x_4 x_5 x_7 x_9 + ((x_2 x_5^2 x_7 + x_1 x_2 x_5 x_8) x_9) x_{10}) x_{12} + \\
& (x_2 x_3 x_4 x_5^2 x_7 + ((x_3 x_5^3 + x_1 x_5^2 x_6) x_7 + \\
& (x_1 x_3 x_5^2 + x_1^2 x_5 x_6) x_8) x_{10}) x_{13}) x_{15}) x_{17})
\end{aligned}
$$

Poisson brackets

The cluster algebra has a Poisson bracket which can be quite useful.
If two cluster $\mathcal{X}$ coordinates x_i and x_j are in the same cluster and are linked by an arrow $i \to j$, then their Poisson bracket is $\{x_i, x_j\} = x_i x_j$. If they are not connected, then $\{x_i, x_j\} = 0$. One can show that the Poisson bracket is compatible with mutations.
In general the Poisson bracket of two cross-ratios is complicated; only when they belong to the same cluster we can compute it easily. But given two cross-ratios, it can be hard to find a cluster which contains both of them (especially if the cluster algebra is of infinite type).

Sklyanin brackets

If we arrange the n momentum twistors in a $4 \times n$ matrix and if the first four of them are linearly independent, then we can go to frame where this matrix reads

$$\begin{pmatrix} \mathbf{1}_4 & y_{ij} \end{pmatrix},$$

where the y matrix is $4 \times (n-4)$-dimensional. All the four-brackets can be written in terms of y_{ij}.
We define

$$\{y_{ij}, y_{ab}\} = (\operatorname{sgn}(a-i) - \operatorname{sgn}(b-j)) y_{ib} y_{aj},$$
$$\{f(y), g(y)\} = \frac{\partial f}{\partial y_{ij}} \{y_{ij}, y_{ab}\} \frac{\partial g}{\partial y_{ab}}.$$

A Li_3 identity . . .

We have found the first 40-term trilogarithm identity of cluster type:

$$\left\{-\frac{\langle 125\rangle\langle 134\rangle}{\langle 123\rangle\langle 145\rangle}\right\}_3+\left\{-\frac{\langle 126\rangle\langle 145\rangle}{\langle 124\rangle\langle 156\rangle}\right\}_3+\left\{-\frac{\langle 126\rangle\langle 145\rangle\langle 234\rangle}{\langle 123\rangle\langle 146\rangle\langle 245\rangle}\right\}_3+ \frac{1}{3}\left\{-\frac{\langle 136\rangle\langle 145\rangle\langle 235\rangle}{\langle 123\rangle\langle 156\rangle\langle 345\rangle}\right\}_3+\text{cyclic}-\text{anticyclic}=0.$$

It is possible to associate $\{x\}_3 \to \text{function}(x)$ such that the identity is satisfied. Mathematicians use

$$\mathrm{L}_3(z) := \Re\Big(\mathrm{Li}_3(z) - \mathrm{Li}_2(z)\log|z| - \frac{1}{3}\log^2|z|\log(1-z)\Big), \quad z \in \mathbb{C},$$

which satisfy “clean” functional equations. However, these functions are only real analytic, not complex analytic. We can find functions which are complex analytic instead.

. . . and its Poisson portrait

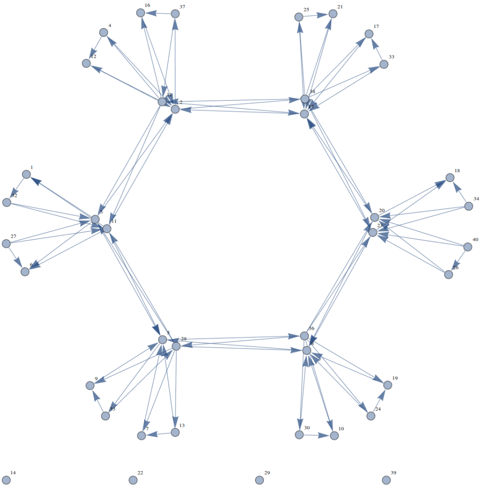


Figure: The oriented graph encoding the Poisson brackets of the 40 arguments of the Li_3 identity. There is an arrow between vertices i and j if the $\{\log X_i, \log X_j\} = 1$.

Some explicit results

We will use the notation

$$\langle ij|klmn\rangle \equiv \frac{\langle ijkl\rangle\langle ijmn\rangle}{\langle ijlm\rangle\langle ijnk\rangle}.$$

At six-point NMHV [Dixon, Drummond, Henn] found a way to express the two-loop answer in terms of two functions Ω_2 and $\tilde{\Omega}_2$.

$$\begin{aligned}\tilde{\Omega}_2|_{\Lambda^2 B_2} = & -\{\langle 36|1254\rangle\}_2\wedge\{\langle 34|1652\rangle\}_2-\{\langle 36|1254\rangle\}_2\wedge\{\langle 16|2543\rangle\}_2\\ & -\{\langle 14|2365\rangle\}_2\wedge\{\langle 34|1652\rangle\}_2-\{\langle 14|2365\rangle\}_2\wedge\{\langle 16|2543\rangle\}_2\\ & +\{\langle 25|1634\rangle\}_2\wedge\{\langle 56|1432\rangle\}_2+\{\langle 25|1634\rangle\}_2\wedge\{\langle 23|1456\rangle\}_2\\ & +\{\langle 25|1634\rangle\}_2\wedge\{\langle 12|3654\rangle\}_2+\{\langle 25|1634\rangle\}_2\wedge\{\langle 45|1236\rangle\}_2.\end{aligned}$$

$$\begin{aligned}\tilde{\Omega}_2|_{\Lambda^2 B_2} = & \{\langle 36|1254\rangle\}_2\wedge\{\langle 34|1652\rangle\}_2+\{\langle 14|2365\rangle\}_2\wedge\{\langle 16|2543\rangle\}_2\\ & -\{\langle 25|1634\rangle\}_2\wedge\{\langle 45|1236\rangle\}_2-\{\langle 25|1634\rangle\}_2\wedge\{\langle 56|1432\rangle\}_2.\end{aligned}$$

More on Ω and $\tilde{\Omega}$

These functions are the *first* examples of functions with both $\Lambda^2 B_2$ and $B_3 \otimes \mathbb{C}^*$ expressible in terms of simple cross-ratios.
There is a construction [Goncharov] of a $B_3 \otimes \mathbb{C}^*$ from a given $\Lambda^2 B_2$ but it doesn't lead to simple cross-ratios in B_3. Therefore, this example is significant mathematically.
We also have

$$\Omega_2+\tilde{\Omega}_2+\star\tilde{\Omega}_2 = 4\,\mathrm{Li}_4(\langle 12|3456\rangle)-\mathrm{Li}_4(\langle 14|2356\rangle)-2\,\mathrm{Li}_4(\langle 14|2536\rangle)+ 2\,\mathrm{Li}_4(\langle 14|2563\rangle)+4\,\mathrm{Li}_4(\langle 16|2345\rangle)-4\,\mathrm{Li}_4(\langle 23|1456\rangle)- 2\,\mathrm{Li}_4(\langle 25|1346\rangle)-2\,\mathrm{Li}_4(\langle 25|1364\rangle)+2\,\mathrm{Li}_4(\langle 25|1436\rangle)+ 4\,\mathrm{Li}_4(\langle 34|1256\rangle)-\mathrm{Li}_4(\langle 36|1245\rangle)-2\,\mathrm{Li}_4(\langle 36|1254\rangle)- 2\,\mathrm{Li}_4(\langle 36|1425\rangle)-4\,\mathrm{Li}_4(\langle 45|1236\rangle)+4\,\mathrm{Li}_4(\langle 56|1234\rangle)+\text{products},$$

where $\star$ is the parity conjugation.

Some three-loop answers

At three loops (transcendentality six) MHV we have several partial integrations we can compute: $B_2 \wedge (B_2 \wedge B_2)$, $B_3 \wedge B_3$, $(B_3 \otimes \mathbb{C}^*) \otimes B_2$. We use the symbol found by [Dixon, Drummond, Henn].

- $B_2 \wedge (B_2 \wedge B_2) = B_2 \otimes (B_2 \wedge B_2) - (B_2 \wedge B_2) \otimes B_2$. Surprisingly, the $(B_2 \wedge B_2) \otimes B_2$ part vanishes. The full answer is very simple

$$\left\{\frac{(23)(56)}{(25)(36)}\right\}_2 \wedge \left(\left\{\frac{(23)(56)}{(25)(36)}\right\}_2 \wedge \left\{-\frac{(12)(36)}{(16)(23)}\right\}_2\right) + \text{dihedral permutation}$$

Some three-loop answers

- The $B_3 \wedge B_3$ part is

$$-\frac{2}{3}(4\alpha_2-1)\{r(1432)\}_3\wedge\{r(1452)\}_3+\frac{1}{24}(5-32\alpha_2)\{r(1432)\}_3\wedge\{r(1$$
$$\frac{1}{48}(7-32\alpha_2)\{r(1432)\}_3\wedge\{r(2635)\}_3+\frac{1}{16}(7-32\alpha_2)\{r(1432)\}_3\wedge\{r(26$$
$$\frac{1}{48}(-24\alpha_1-64\alpha_2+9)\{r(1452)\}_3\wedge\{r(1524)\}_3+\frac{1}{24}(32\alpha_2-7)\{r(1524)$$
$$\frac{1}{96}(-24\alpha_1-64\alpha_2+9)\{r(1634)\}_3\wedge\{r(1643)\}_3+\frac{1}{6}\{r(1423)\}_3\wedge\{r(1452)$$
$$\frac{1}{6}\{r(1423)\}_3\wedge\{r(1524)\}_3+\frac{1}{12}\{r(1423)\}_3\wedge\{r(1634)\}_3+\frac{1}{12}\{r(1432)\}_3\wedge\{$$
$$\frac{1}{4}\{r(1452)\}_3\wedge\{r(1532)\}_3-\frac{1}{12}\{r(1623)\}_3\wedge\{r(1634)\}_3+$$
$$\frac{1}{24}\{r(1623)\}_3\wedge\{r(1643)\}_3+\text{dihedral permutations},$$

where α_1 and α_2 are the constants which have been fixed later [Caron-Huot, He] $\alpha_1=-\frac{3}{8}$, $\alpha_2=\frac{7}{32}$.

Conclusions

- The notion of symbol of a transcendental function is useful in understanding and simplifying scattering amplitudes.
- Cluster coordinates seem to play an important role, but the interplay with supersymmetry is not completely understood.
- Transcendentality four functions are poorly understood mathematically, but explicit answers arising in physics can help to build and guide mathematical intuition.
- Beyond MHV, not all the cross-ratios are $\mathcal{X}$ coordinates of the same cluster algebra. SUSY generalization of the cluster algebras?

Thank you!