The tropical limit and the UV behavior of $\mathcal{N} = 4$ supergravity

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Based on: [PT, Vanhove 1202.3692, PT 1309.3551]

1. The tropical limit of string theory $= \alpha' \to 0$ limit for higher genus amplitudes

2. Connection with the UV behavior of half-maximal supergravity theories

Motivation

Conceptually we want to understand better the $\alpha' \to 0$ limit of string theory amplitudes; access to both string & field theory.
Technically, we want to be able to extract as much physics as possible from a closed string theory scattering amplitude in the limit $\alpha' \to 0$:

- Field theory amplitudes (= graph topologies or integrands),
- Ultraviolet divergences, ...

Technical advantages of string amplitudes

- Compact expressions due to conformal field theory techniques
- Superior organization of the amplitude
- Naturally supersymmetric

Drawbacks

- **Severe difficulties** to compute explicitly higher genus amplitudes

However there are on the market some $g = 2, 3$ amplitudes (see [Mafra's talk]) of which very little was known about the field theory limit.

The domain of application: $g \geq 2$

The mechanisms by which Feynman graphs are expected to be recovered from string theory amplitudes are not mysterious. In particular at one-loop, everything is well understood.

However for genus ≥ 2 a lot of technology is missing and we want to develop it using tropical geometry.

Applications ($d = 10$):

- Two and three loops in $\mathcal{N} = 8$ supergravity.
- Exact integrands at two-loop in Heterotic string models giving $\mathcal{N} = 4$ supergravity;

Tropical geometry also proposes a framework to think globally about the field theory limit of string theory amplitudes.

Digressive foreword: Target Space tropicalization

[Aharony, Hanany, Kol '97 *Webs of (p,q) Five-Branes*]: one of the first use of tropicalization techniques (prior to standard math. papers).

$\longrightarrow$ Brane configurations described by tropical curves in *decompactification* limit from $4d$ to $5d$: (p,q) Webs.

M-theory / complex 2-torus τ_M is dual to Type IIB / circle L_B

$$L_B \sim (\mathrm{Im}\, \tau_M)^{-1}$$

Target space tropicalization in this case = decouple the Kaluza-Klein modes of the M-theory extra dimension.

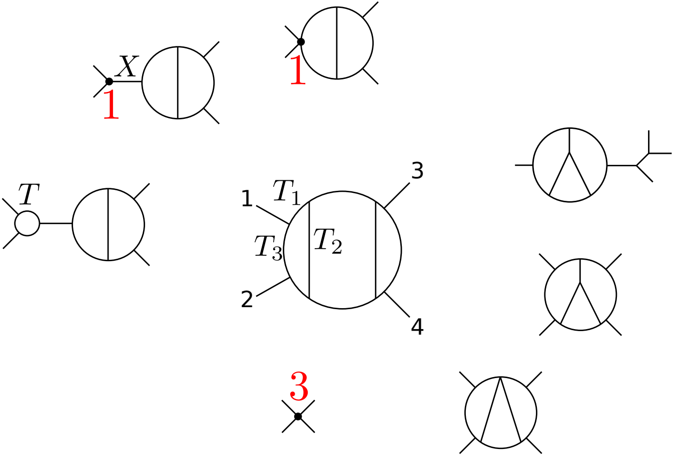
Point-like limit of string theory amplitudes

The outcome of the point-like limiting procedure can be written as:

$$\lim_{\alpha' \to 0} \int_{\mathcal{M}_{g,n}} \mathcal{F}^{(\alpha')}_{g,n} = \int_{M^{\text{trop}}_{g,n}} F^{\text{trop}}_{g,n}$$

Tropical Geometry

$M_{g,n}^{\text{trop}}$ is the moduli space of tropical graphs.
$\rightsquigarrow$ metric graphs with weighted vertices

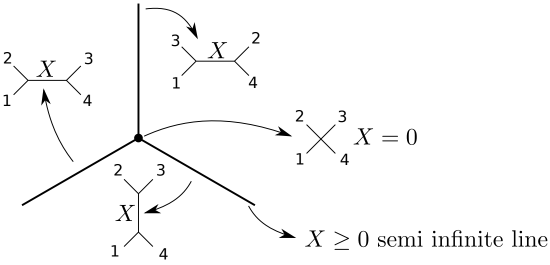


Genus of a graph = (# of loops) + ($\sum$ weights)

Examples of Tropical Moduli Spaces: $M_{0,n}^{\text{trop}}$

Tropical moduli space of genus zero graphs

- $M_{0,3}^{\text{trop}}$ is just a one-point set.
- $M_{0,4}^{\text{trop}}$ has more structure:



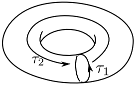
- $M_{0,5}^{\text{trop}}$ is even richer.

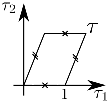
Tropical Moduli Space $M_{g,n}^{\text{trop}}$ [Caporaso *et al.*, Brannetti *et al.*]

Physically:

$M_{g,n}^{\text{trop}} = \{\text{Schwinger proper times graphs} + \text{counterterms vertices}\}.$

Point-like limit in string theory: $\alpha' \to 0$

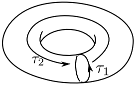


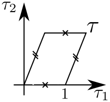


Genus one bosonic string partition function:

$$Z(\tau_1, \tau_2) = \mathrm{Tr}\, \left(q^{L_0 - 1/24} \bar{q}^{\bar{L}_0 - 1/24} \right), \quad q = \exp(2i\pi\tau), \quad \tau = \tau_1 + i\tau_2$$

Point-like limit in string theory: $\alpha' \to 0$





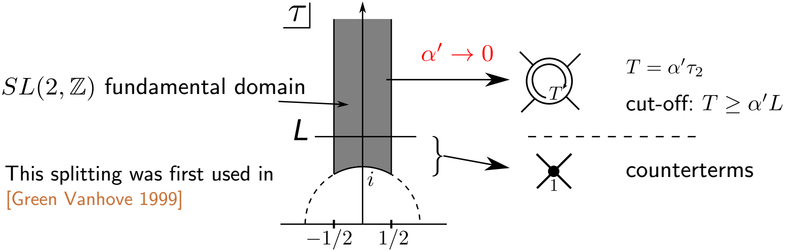
Genus one bosonic string partition function:

$$Z(\tau_1,\tau_2) = \mathrm{Tr}\left(e^{2i\pi\tau_1(L_0-\bar{L}_0)}e^{-\pi\alpha'\tau_2 m^2}\right), \quad m^2 = \frac{4n}{\alpha'},\ n = (-1,)\,0,1,2,\ldots$$

For strings absolute values are proper times while phases enforce the level-matching. Two different kind of regions when $\alpha' \to 0$:

$\tau_2 \to \infty$: massive states are projected out and the level matching condition is trivial $\Rightarrow$ edges of worldline graphs;

$\tau_2 \sim O(1)$: massive states do not decouple $\Rightarrow$ counterterms;



This splitting was first used in [Green Vanhove 1999]

"**Tropical geometry** *is a way to forget the phases of complex numbers*", [Itenberg, Mikhalkin '11]

The $\alpha' \to 0$ limit of string theory amplitudes [PT '13]

$$\int_{\mathcal{M}_{g,n}} \mathcal{F}_{g,n}{}^{(\alpha')}$$

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Point-like limit: $$\lim_{\alpha' \to 0} \int_{\mathcal{M}_{g,n}} \mathcal{F}_{g,n}{}^{(\alpha')}$$

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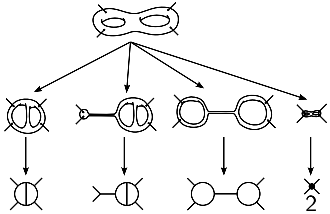
Point-like limit: $$\lim_{\alpha'\to 0} \int_{\mathcal{M}_{g,n}} \mathcal{F}_{g,n}{}^{(\alpha')}$$

Tropicalization

$\mathcal{M}_{g,n}$: Moduli space of Riemann surfaces of genus g with n punctures.

(i)

(ii)



$$\mathcal{M}_{g,n} = \bigcup_G \mathcal{D}_G$$

(i) prop. of massive/massles states

(ii) $\int$ phases = level-matching

In each $\mathcal{D}_G$, the integral descends to the tropical graph (=Schwinger proper-time form) of the Feynmann graph G

(i) get a particular **Fourier-Jacobi** q-expansion to perform,

(ii) extract the **residues**.

In type II amplitudes with maximal supersymmetry, no need to extract residues. Use simple tools; tropical one-forms and tropical period matrices.

Final result :

$$\lim_{\alpha'\to 0} \int_{\mathcal{M}_{g,n}} \mathcal{F}_{g,n} = \sum_G \int_{M^{\mathrm{trop}}(G)} F^{\mathrm{trop}}_{g,n}(G)$$

$$= \int_{M^{\mathrm{trop}}_{g,n}} F^{\mathrm{trop}}_{g,n}$$

$\longrightarrow$ field theory amplitude renormalized by string theory

Results

Successfully applied to [PT '13 and unpublished];

- ✓ Derived the form of the field theory limit for the four-graviton type II amplitude from [D'Hoker & Phong]'s written by [Green, J.Russo, Vanhove ('08)] from the original computation of [Bern *et al.* '98]
- ✓ Showed that the genus three amplitude of [Gomez and Mafra '13] reproduces the set of 12 graph topologies found by Bern *et al.*
  $\rightarrow$ Need now to get the actual form of the integrand, no more technical tools needed there.

Next we discuss a more general example of field theory limit, at 2 loops in heterotic string. Step (i) alone gives already interesting results.
For further developments (step (ii))

- $\rightarrow$ Compute the Fourier-Jacobi expansion of the $\langle X(z)X(0)\rangle$ propagator
- $\rightarrow$ understand better in particular **tropical theta functions** with characteristics. Define a "tropical prime form" ?

[*work in progress*]

Restrictions

Schwinger proper time rep's exist anyway, the question is rather to know if one should expect some more technical/conceptual understanding to take this limit.

Genus $g \leq 4$ facilitates an explicit parametrization of the decomposition $\mathcal{M}_{g,n} = \bigcup_G \mathcal{D}_G$ which is done in terms of **period matrices** (for $g \geq 5$ one faces Schottky problem)

One should start with actual string amplitudes; but there are either unsolved issues or frightening ones;

- In RNS Witten's supermoduli space non-projectedness for $g \geq 5$
- In Pure Spinors the λ-ghost regulator "everybody is afraid of [Mafra's talk]" for $g \geq 5$

Connection with an analysis on the UV behavior of N=4 supergravity

Half-maximal supergravity

Half-maximal supergravities ($\mathcal{N} = 4$ in $D = 4$) have a richer structure than maximal supergravity and allows couplings to n_v ($\mathcal{N} = 4$) vector multiplets matter fields.

	2	**3/2**	**1**	**1/2**	**0**
$\mathcal{N} = 4$ supergravity	1	4	6	4	2
$\mathcal{N} = 4$ SYM matter			1	4	6

$\longrightarrow$ the *pure* theory ($n_v = 0$) was expected to diverge at **3 loops** due to an R^4 counterterm.

$\mathcal{N} = 4$ supergravity amplitudes from CHL models [PT, Vanhove '12]

CHL models are $\mathbb{Z}_N$ orbifolds of the bosonic sector of heterotic string, produce $(4, 0)$ models with **tunable** $n_v = 22, 14, 10, 6, 4$ (possibly $n_v = 2, 0$). [Chaudhuri Hockney Lykken '95]

Conventions: bosonic sector = holomorphic.

Warm-up: one-loop point-like limit

Systematics for one-loop n-point amplitudes: “Bern-Kosower rules” [90’s].

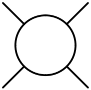
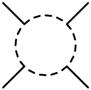
$$\mathcal{A}^{1-loop}_{4-graviton} = R^4 \int \mathrm{d}\mu^{g=1} \mathcal{Z}_{CHL} \mathcal{W}_{\text{bos}} e^{\mathcal{Q}}$$

- $e^{\mathcal{Q}}$: Koba-Nielsen factor $\leftrightarrow$ denominator of the graph
- $\boxed{\mathcal{Z}_{CHL} = \frac{1}{q} + (n_v + 2) + O(q)}$ is the part. fct, $\boxed{q = e^{2i\pi\tau} \to 0}$
- $\mathcal{W}_{\text{bos}} = \frac{\langle (:\partial X e^{ikx}:)^4 \rangle}{\langle (:e^{ikx}:)^4 \rangle} \sim \sum \partial G^4$ with $G = \langle XX \rangle$

  ↳ kinematical information $\leftrightarrow$ numerator of Feynman graph

Dictionary: $(\partial G)^n \leftrightarrow \ell^n$

residue extraction: $\mathcal{Z}_{CHL}\mathcal{W}_{bos} \longrightarrow \underbrace{(\mathcal{W}_{bos})\,|_q}_{\ell^0,\ell^2} + (n_v + 2) \underbrace{(\mathcal{W}_{bos})\,|_{q^0}}_{\ell^4} + O(q)$

Worldline picture:  $+\, n_v$  (—— : grav., - - - : matt.)

Non-renormalization theorem at two loops [PT, Vanhove '12]

The four-graviton Heterotic string amplitude of [D'Hoker, Phong] is

$$\mathcal{A}_{4-graviton}^{2-loop} = R^4 \int \mathrm{d}\mu^{g=2} \overline{\mathcal{Y}}_{susy} \mathcal{Z}_{CHL} \mathcal{W}_{bos} e^{\mathcal{Q}}$$

- $\mathcal{W}_{bos} \sim \sum (\partial G)^4 \sim \ell^4$
- $\overline{\mathcal{Y}}_{susy} = (k_1 - k_2) \cdot (k_3 - k_4) \ \Delta(z_1, z_2) \Delta(z_3, z_4) + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)$

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no compensating $1/k_i.k_j$ pole because of right-moving SUSY

$\longrightarrow$ R^4 is not renormalized beyond one-loop in CHL models and $\mathcal{D}^2 R^4$ appears instead [PT, Vanhove '12]
$\longrightarrow$ Explanation from string theory of the observed vanishing of the 3-loop divergence of [Bern Davies Dennen Huang '12].

$\mathcal{D}^2 R^4$ is a valid counterterm for $\mathcal{N} = 4$ [Bossard Howe Stelle Vanhove '11], this naturally suggests a 4-loop divergence with $n_v = 0$.
This divergence and related ones have been found now in [BDDH '12], [BDD '13], [BDD, Smirnov, Smirnov '13] and [Bern's talk]

$\longrightarrow$ [Bossard Howe Stelle '12,'13] described the structure of the counterterm, relation to the anomaly and an argument on the validity of the theorem at $2-3-4$ loops and breakdown for ≥ 5 loops.

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$$\mathcal{A}^{2-loop}_{4-graviton} = R^4 \int \mathrm{d}\mu^{g=2} \overline{\mathcal{Y}}_{susy} \mathcal{Z}_{CHL} \mathcal{W}_{bos} e^{\mathcal{Q}}$$

- $\mathcal{W}_{bos} \sim \sum (\partial G)^4 \sim \ell^4$
- $\overline{\mathcal{Y}}_{susy} \sim k^2$

so the field theory naively looks like

$$k^2 R^4 \int \frac{\mathrm{d}\ell}{\ell} \ell^4$$

but the power counting is:

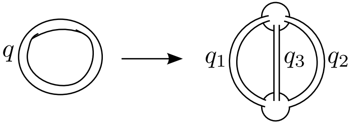
$$\mathcal{A}^{2-loop}_{4-gravitons} \sim R^4 \int \frac{\mathrm{d}\ell}{\ell} \ell^4 \sim k^2 R^4 \int \frac{\mathrm{d}\ell}{\ell} \ell^2$$

Two-loop Fourier-Jacobi expansion (step (i) of $\alpha' \to 0$)

$$\mathcal{Z}_2^{CHL} = \frac{1}{q_1 q_2 q_3} + 2 \sum_{1 \le i < j \le 3} \frac{1}{q_i q_j} + (n_v + 2) \sum_{i=1}^{3} \frac{1}{q_i} + 0 + O(q_i)\,.$$

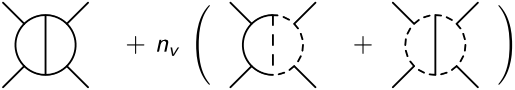
$\to$ explicitly checked for $N = 1$ and $N = 2$ (from [Dabholkar, Gaiotto '06])

$q \; = e^{2i\pi\tau}$ $\qquad$ $q_i = e^{2i\pi\tau_i}$



$\ln q_i \sim$ edges lengths

Vanishing of the constant term in $\mathcal{Z}_{CHL} \longrightarrow$ at least two powers of ∂G are soaked up in $\mathcal{W}_{bos}$. **This explains how inner loop momentum is transmuted to factorized derivatives.** Worldsheet point of view on the $D = 5$ cancellation described in [Bern's talk]

Worldline: 

$+$ non-planar

(——— : gravity, - - - : matter)

Conclusion

The $\alpha' \to 0$ limit of string theory amplitude defines a renormalized field theory amplitude written as an integral over the tropical moduli space.

The tropical procedure tells unambiguously where you should look for the various pieces of your amplitude in the string theory side.

Technology to handle higher genus limit of maximally supersymmetric amplitudes

Outlook

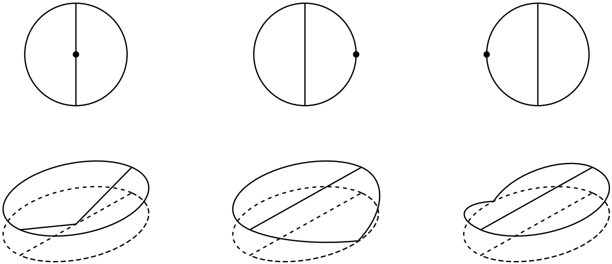
- Extract the complete tropical limit of [Gomez Mafra]

$\rightsquigarrow$ Complete step (ii) by obtaining corrections to the tropical limit of

$$\langle X(z)X(w)\rangle = G^{\text{trop}}(Z, W) + O(q_i)$$

Extra slides

Tropical theta characteristics

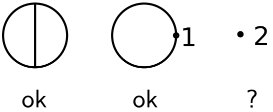


Theta characteristics:

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \qquad \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

Example of application at genus two

Possible to define a splitting with a hard Schwinger proper time cutoff as at one loop, which should give an integral over $\mathcal{M}_2^{\text{trop}}$ as:

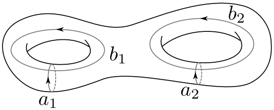


$\longrightarrow$ integral over bounded genus two $SP(4, \mathbb{Z})$ fundamental domain, difficult to access [D'Hoker, Green, Pioline, R. Russo '14], even numerically. [*work in progress*]

The measure $\mathrm{d}\mu_{bos}$

There are g independent holomorphic differentials $\omega_1, ..., \omega_g$, normalized along the a_I cycles and defining the period matrix along the b_J cycles:

$$\int_{a_I} \omega_J = \delta_{IJ} \quad \int_{b_I} \omega_J = \Omega_{IJ}$$



Ω is a $g \times g$ symmetric matrix with positive-definite imaginary part; it has $g(g+1)/2$ independent coefficients.

g	$3g-3$	$g(g+1)/2$	
1	1	1	✓
2	3	3	✓
3	6	6	✓
4	9	10	✓ Schottky problem, but solved.
5	12	15	? Schottky problem.