$\mathcal{N} = 4$ Scattering Amplitudes and the Deformed Graßmannian

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Based on

L. Ferro, T. Łukowski, C. Meneghelli, J. Plefka and M. Staudacher,
1212.0850 & 1308.3494

R. Frassek, N. Kanning, Y. Ko and M. Staudacher,
1312.1693

N. Kanning, T. Łukowski and M. Staudacher,
1403.3382

And to appear.

Related Work

D. Chicherin, S. Derkachov and R. Kirschner,
1306.0711 & 1309.5748

N. Beisert, J. Brödel and M. Rosso,
1401.7274

J. Brödel, M. de Leeuw and M. Rosso,
1403.3670

A Case for 3+1 Dimensions

Nature prefers Yang-Mills theory in exactly 1+3 dimensions:
Coordinates x^μ, momenta p^μ. So let us stay there!

Split index $\mu = 0, 1, 2, 3$ into spinorial indices $\alpha = 1, 2$ and $\dot\alpha = \dot 1, \dot 2$.

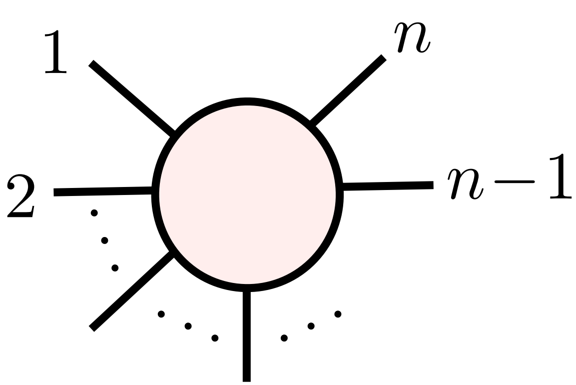
Interesting bijection $\mathbb{R}^{1,3} \to H(2\times 2), p^\mu \mapsto p_{\alpha\dot\alpha} = p_\mu \left(\sigma^\mu\right)_{\alpha\dot\alpha}$.
Reverse map $H(2\times 2) \to \mathbb{R}^{1,3}, p^\mu \mapsto \frac{1}{2}\,\mathrm{Tr}\, p_{\alpha\dot\alpha}\left(\bar\sigma^\mu\right)^{\dot\alpha\alpha}$.
Here $\sigma^\mu = (\mathbb{1}, \vec\sigma)$ and $\bar\sigma^\mu = (\mathbb{1}, -\vec\sigma)$ with Pauli matrices $\vec\sigma$. Explicitly:

$$p_{\alpha\dot\alpha} = \begin{pmatrix} p_0 + p_3 & p_1 - i\,p_2 \\ p_1 + i\,p_2 & p_0 - p_3 \end{pmatrix}$$

Gluons are labeled by momenta p^μ with $p^2 = p^\mu p_\mu = \det p_{\alpha\dot\alpha} = 0$ and helicity ± 1. Momentum factors: $p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$, shorthand for $= |\lambda_\alpha\rangle[\tilde\lambda_{\dot\alpha}|$.

Super-Spinor-Helicity and Amplitudes

There is a beautiful extension to maximally supersymmetric $\mathcal{N}=4$ theory:
One introduces for each leg j a Graßmann spinor η_j^A where $A=1,2,3,4$.
With $P_{\alpha\dot\alpha}=\sum_j \lambda_{j,\alpha}\tilde\lambda_{j,\dot\alpha}$ and $Q_\alpha^A=\sum_j \lambda_{j,\alpha}\eta_j^A$ the (color stripped) tree amplitudes for n particles are the known [Drummond, Henn '08] distributions



$$=\frac{\delta^4(P_{\alpha\dot\alpha})\delta^8(Q_\alpha^A)}{\langle 12\rangle\langle 23\rangle\ldots\langle n-1,n\rangle\langle n1\rangle}\mathcal{P}_n(\{\lambda_j,\tilde\lambda_j,\eta_j\}),$$

where $\langle \ell m\rangle=\epsilon^{\alpha\beta}\lambda_{\ell,\alpha}\lambda_{m,\beta}$ and $[\ell m]=\epsilon^{\dot\alpha\dot\beta}\tilde\lambda_{\ell,\dot\alpha}\tilde\lambda_{m,\dot\beta}$.
All external helicity configurations are generated by expansion in the η_j^A.
Super-helicity k corresponds to the terms of order η^{4k}.

Graßmannian Integrals and Amplitudes, I

A Graßmannian space $\mathrm{G}(k,n)$ is the set of k-planes intersecting the origin of an n-dimensional space. $k=1$ is ordinary projective space.

"Homogeneous" coordinates are packaged into a $k \times n$ matrix $C = (c_{ai})$.
C and $A \cdot C$ with $A \in \mathrm{GL}(k)$ correspond to the same "point" in $\mathrm{G}(k,n)$.

Build super-twistors $\mathcal{W}_j^{\mathcal{A}} = (\tilde{\mu}_j^{\alpha}, \tilde{\lambda}_j^{\dot{\alpha}}, \eta_j^{A})$ w. Fourier conjugates $\lambda_j^{\alpha} \to \tilde{\mu}_j^{\alpha}$.
Graßmannian integral formulation of tree-level $\mathrm{N}^{k-2}\mathrm{MHV}_n$ amplitudes:

$$\int \frac{d^{k \cdot n} C}{\mathrm{vol}(\mathrm{GL(k)})} \frac{\delta^{4k|4k}(C \cdot \mathcal{W})}{(1 \ldots k)(2 \ldots k+1) \ldots (n \ldots n+k-1)}$$

The $(i\, i+1 ... i+k-1)$ are the n cyclic $k \times k$ minors.
Integration is along "suitable contours". [Arkani-Hamed, Cachazo, Cheung, Kaplan '09]

Graßmannian Integrals and Amplitudes, II

For "most" points on $\mathrm{G}(k,n)$ we may use the $\mathrm{GL}(k)$ symmetry to write

$$C = \left(\begin{array}{c|cccc} & c_{1,k+1} & c_{1,k+2} & \cdots & c_{1,n} \\ \mathbb{I}_{k\times k} & \vdots & \vdots & \ddots & \vdots \\ & c_{k,k+1} & c_{k,k+2} & \cdots & c_{k,n} \end{array} \right)$$

The Graßmannian integral simplifies to

$$\int \frac{\prod_{a=1}^{k} \prod_{i=k+1}^{n} dc_{ai}}{(1\ldots k)(2\ldots k+1)\ldots(n\ldots n+k-1)} \prod_{a=1}^{k} \delta^{4|4}\big(\mathcal{W}_a^{\mathcal{A}} + \sum_{i=k+1}^{n} c_{ai}\mathcal{W}_i^{\mathcal{A}}\big)$$

Fourier-transforming back to spinor-helicity space, all tree-level N^{k-2}MHV$_n$ amplitudes may be obtained.

Symmetries

The amplitudes enjoy $\mathcal{N}=4$ superconformal symmetry ($\mathcal{A},\mathcal{B}=1\ldots8$):

$$J^{\mathcal{AB}}\cdot A_{n,k}=0\,,\quad \text{with}\quad J^{\mathcal{AB}}\in\mathfrak{psu}(2,2|4)$$

However, there is also a "non-local" dual superconformal symmetry:

$$\tilde{J}^{\mathcal{AB}}\cdot A_{n,k}=0\,,\quad \text{with}\quad \tilde{J}^{\mathcal{AB}}\in\mathfrak{psu}(2,2|4)^{\text{dual}}$$

Commuting J and $\tilde{J}$, one obtains Yangian symmetry. [Drummond, Henn, Plefka '09]
With twistor variables $\mathcal{W}_j^{\mathcal{A}}$ and the "local" generators $J_j^{\mathcal{AB}}=\mathcal{W}_j^{\mathcal{A}}\frac{\partial}{\partial\mathcal{W}_j^{\mathcal{B}}}$, we can succinctly express it as

$$J^{\mathcal{AB}}=\sum_{j=1}^{n}J_j^{\mathcal{AB}}\,,\quad \hat{J}^{\mathcal{AB}}=\sum_{i<j}J_i^{\mathcal{AC}}J_j^{\mathcal{CB}}-(i\leftrightarrow j)$$

This is how integrability first appeared in the planar scattering problem.

Dual Graßmannian Integrals and Amplitudes

In the dual description one can employ $4|4$ super momentum-twistors $\mathcal{Z}_j$.

With $\hat{k} = k - 2$, there is an equivalent "dual" description on $\mathrm{G}(\hat{k}, n)$:

[Mason, Skinner '09; Arkani-Hamed et.al. '09]

$$\frac{\delta^4(P_{\alpha\dot{\alpha}})\delta^8(Q^A_\alpha)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\int\frac{d^{\hat{k}\cdot n}\hat{C}}{\mathrm{vol}(\mathrm{GL}(\hat{\mathrm{k}}))}\frac{\delta^{4\hat{k}|4\hat{k}}(\hat{C}\cdot\mathcal{Z})}{(1\ldots\hat{k})\ldots(n\ldots\hat{k}-1)}$$

Note that the $k=2$ MHV part factors out.

The fact that the two formulations are related by a simple change of variables is due to dual conformal invariance, and thus Yangian invariance.

Deformed Symmetries

[Ferro, Łukowski, Meneghelli, Plefka, MS '12]

Of particular interest is the central charge generator of $\mathfrak{gl}(4|4)$:

$$C = \sum_{j=1}^{n} c_j \quad \text{with} \quad c_j = \lambda_j^\alpha \frac{\partial}{\partial \lambda_j^\alpha} - \tilde{\lambda}_j^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_j^{\dot{\alpha}}} - \eta_j^A \frac{\partial}{\partial \eta_j^A} + 2$$

For overall $\mathfrak{psu}(2,2|4)$ we have $C = 0$. So we can relax the "local" condition $c_j = 0$. This deforms the super helicities $h_j = 1 - \frac{1}{2} c_j$.

This yields something well-known: The Yangian in evaluation representation. Deforming the c_j switches on the parameters v_j. More below.

$$J^{\mathcal{AB}} = \sum_{j=1}^{n} J_j^{\mathcal{AB}}, \quad \hat{J}^{\mathcal{AB}} = \sum_{i<j} J_i^{\mathcal{AC}} J_j^{\mathcal{CB}} - (i \leftrightarrow j) + \sum_{j=1}^{n} v_j \, J_j^{\mathcal{AB}}$$

Deformed Graßmannian Integrals

[Ferro, Łukowski, MS, in preparation]

One could then ask how the Graßmannian contour formulas are deformed. The final answer is exceedingly simple, and very natural. With $v_j^- = v_j - \frac{c_j}{2}$

$$\int \frac{d^{k\cdot n}C}{\mathrm{vol}(\mathrm{GL(k)})} \frac{\delta^{4k|4k}(C\cdot\mathcal{W})}{(1\ldots k)^{1-v_1^-+v_n^-}\ldots(n\ldots k-1)^{1-v_n^-+v_{n-1}^-}}$$

A derivation will be sketched below. With $v_j^+ = v_j + \frac{c_j}{2}$ we can also write

$$\int \frac{d^{k\cdot n}C}{\mathrm{vol}(\mathrm{GL(k)})} \frac{\delta^{4k|4k}(C\cdot\mathcal{W})}{(1\ldots k)^{1+v_k^+-v_{k+1}^+}\ldots(n\ldots k-1)^{1+v_{k-1}^+-v_k^+}}$$

Note that it is not really the Graßmannian space $\mathrm{G}(k,n)$ as such that is deformed, but the integration measure on this space. $\mathrm{GL}(k)$ preserved!

Deformed Dual Graßmannian Integrals

[Ferro, Łukowski, MS, in preparation]

It is equally natural to ask how the dual Graßmannian integrals deform. Using the parameters v_j^-, we found

$$\frac{\delta^4(P_{\alpha\dot{\alpha}})\delta^8(Q_\alpha^A)}{\langle 12\rangle^{1-v_{1-k}^-+v_n^-}\dots\langle n1\rangle^{1-v_{n-k}^-+v_{n-1}^-}}\times$$
$$\times\int\frac{d^{\hat{k}\cdot n}\hat{C}}{\mathrm{vol}(\mathrm{GL}(\hat{\mathrm{k}}))}\frac{\delta^{4\hat{k}|4\hat{k}}(\hat{C}\cdot\mathcal{Z})}{(1\dots\hat{k})^{1-v_n^-+v_{n-1}^-}\dots(n\dots\hat{k}-1)^{1-v_{n-1}^-+v_{n-2}^-}}$$

There is a similar formula in terms of v_j^+.

Note that both the MHV-prefactor and the contour integral are deformed.

Why?

Why should we consider this deformation? Here are some of the reasons:

- It is fun!
- As we shall see, it is very natural from the point of view of integrability.
- In fact, constructing amplitudes by integrability (arguably) requires it.
- Amplitudes are related to the spectral problem, where it is indispensable.
- Most importantly: It promises to provide a natural infrared regulator!

The last point was our original motivation. Interestingly, we recently learned that this deformation had been already studied as an infrared regulator in twistor theory in the early seventies by Penrose and Hodges.

Meromorphicity Lost and Gained

Let us take another look at the deformed Graßmannian contour integral:

$$\int \frac{d^{k\cdot n}C}{\mathrm{vol}(\mathrm{GL(k)})} \frac{\delta^{4k|4k}(C\cdot\mathcal{W})}{(1\dots k)^{1-v_1^-+v_n^-}\dots(n\dots k-1)^{1-v_n^-+v_{n-1}^-}}$$

Choosing the parameters v_j^- to be non-integer, we see that the poles in the variables c_{ai} turn into branch points.

Important point: We can no longer use the BCFW recursion relations, as they are based on the residue theorem, which does not apply anymore.

Sounds bad?

What we can hope to gain is complete meromorphicity in suitable combinations of the deformation parameters v_j^-. This should fix the contours.

A Toy Meromorphicity Experiment

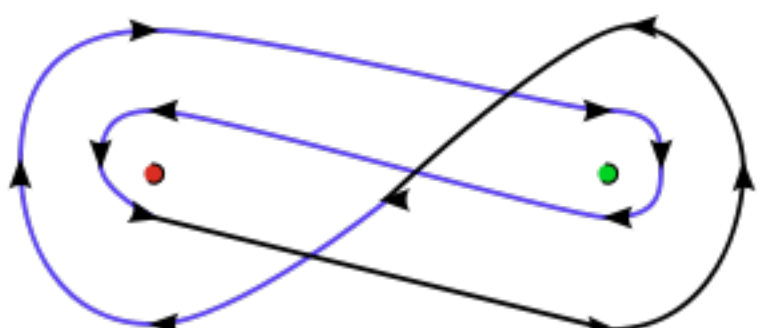
Consider Euler's first integral, the beta function $B(v_1, v_2)$.

$$\int_0^1 dc \, \frac{1}{c^{1-v_1}(1-c)^{1-v_2}}$$

For $v_1, v_2 \in \mathbb{N}$ Euler derived $\frac{(v_1-1)!(v_2-1)!}{(v_1+v_2-1)!}$. The analytic continuation for arbitrary $v_1, v_2 \in \mathbb{C}$ is $\frac{\Gamma(v_1)\Gamma(v_2)}{\Gamma(v_1+v_2)}$. Meromorphic in both v_1 and v_2.
This is not obvious from the integral. This problem was fixed by [Pochhammer '90]:

$$\frac{1}{(1-e^{2\pi i v_1})(1-e^{2\pi i v_2})} \int_{\mathcal{C}} dc \, \frac{1}{c^{1-v_1}(1-c)^{1-v_2}}$$

where the contour $\mathcal{C}$ goes at least two times through the cut:



[Wikipedia, the free encyclopedia]

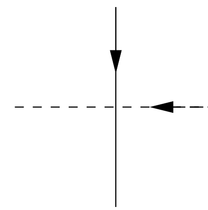
Yangian Invariants as Spin Chain States, I

[Frassek, Kanning, Ko, MS '13; Chicherin, Derkachov, Kirschner '13]

How to construct, generally and systematically, Yangian invariants?
It was recently proposed to identify them as special spin-chain states $|\Psi\rangle$.

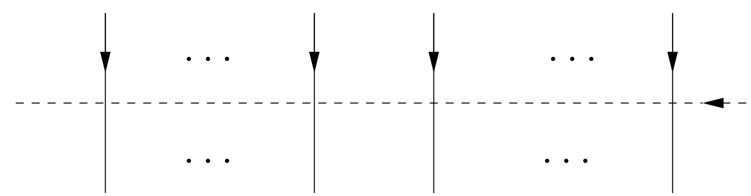
How does the Yangian appear for spin chains with $\mathfrak{gl}(m|n)$ symmetry?
Package the "local" generators $J_j^{\mathcal{AB}}$ into a Lax operator $L_j(u, v'_j)$:

$$L_j(u, v'_j) = 1 + \frac{1}{u - v'_j}\, e_{\mathcal{AB}}\, J_j^{\mathcal{AB}} =$$



Then build up a monodromy matrix $M^{\mathcal{AB}}(u, \{v'_j\})$:

$$M(u) = L_1(u, v'_1) \ldots L_n(u, v'_n) =$$



Here multiplication is both a tensor product and a matrix product.

Yangian Invariants as Spin Chain States, II

[Frassek, Kanning, Ko, MS '13; Chicherin, Derkachov, Kirschner '13]

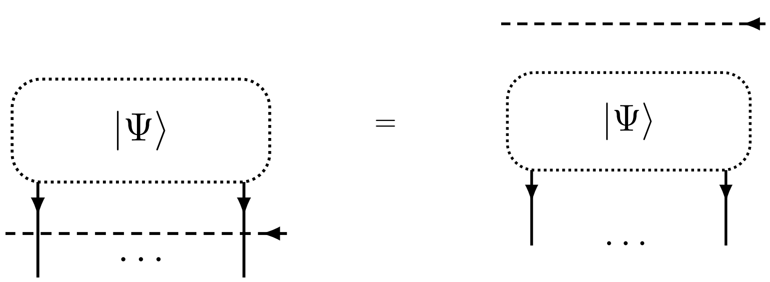
The Yangian generators, see above, appear by expanding at $u = \infty$:

$$M^{\mathcal{A}\mathcal{B}}(u) = \delta^{\mathcal{A}\mathcal{B}} + \frac{1}{u}\, J^{\mathcal{A}\mathcal{B}} + \frac{1}{u^2}\, \hat{J}^{\mathcal{A}\mathcal{B}} + \ldots$$

Note that the deformation of the $\hat{J}^{\mathcal{A}\mathcal{B}}$ indeed appears naturally.

Yangian invariance is now elegantly encoded as

$$M^{\mathcal{A}\mathcal{B}}(u) \cdot |\Psi\rangle = \delta^{\mathcal{A}\mathcal{B}} |\Psi\rangle \quad \text{or even} \quad M(u) \cdot |\Psi\rangle = |\Psi\rangle$$



In usual spin chains we take the trace, and study $\mathrm{Tr}\, M(u) \cdot |\Psi\rangle = t(u)|\Psi\rangle$.

Yangian Invariants and Bethe Ansatz, I

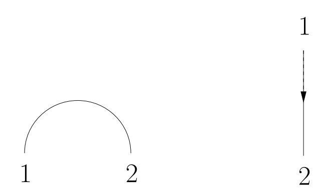
[Frassek, Kanning, Ko, MS '13]

Therefore, the machinery of the algebraic Bethe ansatz may be applied. Already in the simpler case of $\mathfrak{gl}(n)$ compact reps much of the structure of the [Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka '12] on-shell diagramatics is found.

Let us use "twistor variables" $\mathcal{W}_j$ in the fundamental rep of $\mathfrak{gl}(n)$.

The simplest is the $n = 2$, $k = 1$ two-site invariant, with $C = \begin{pmatrix} 1 & c_{12} \end{pmatrix}$,

$$|\Psi_{2,1}\rangle \simeq \oint \frac{dc_{12}}{c_{12}^{1+s_2}} \delta^n(\mathcal{W}_1 + c_{12}\mathcal{W}_2)$$



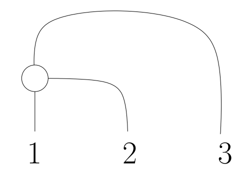
Here the contour is circular around zero, and $s_2 \in \mathbb{N}$ is a Dynkin label.

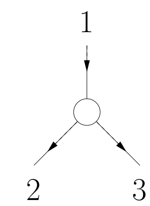
Yangian Invariants and Bethe Ansatz, II

[Frassek, Kanning, Ko, MS '13]

The next simplest cases are the three-site invariants with $n = 3$.

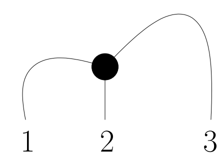
For $k = 1$ one gets, with $C = \begin{pmatrix} 1 & c_{12} & c_{13} \end{pmatrix}$,

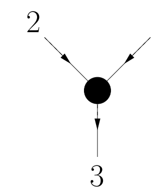




$$|\Psi_{3,1}\rangle \simeq \oint \frac{dc_{12}}{c_{12}^{1+s_2}} \frac{dc_{13}}{c_{13}^{1+s_3}} \, \delta^n(\mathcal{W}_1 + c_{12}\mathcal{W}_2 + c_{13}\mathcal{W}_3)$$

while for $k = 2$ one gets, with $C = \begin{pmatrix} 1 & 0 & c_{13} \\ 0 & 1 & c_{23} \end{pmatrix}$,





$$|\Psi_{3,2}\rangle \simeq \oint \frac{dc_{13}}{c_{13}^{1+s_1}} \frac{dc_{23}}{c_{23}^{1+s_2}} \, \delta^n(\mathcal{W}_1 + c_{13}\mathcal{W}_3) \, \delta^n(\mathcal{W}_2 + c_{23}\mathcal{W}_3)$$

All contours are closed and encircle zero.

Bethe Ansatz, Permutations, and Yangian Invariants

Since we solve $M(u) \cdot |\Psi\rangle = |\Psi\rangle$ and not $\mathrm{Tr}\, M(u) \cdot |\Psi\rangle = t(u)|\Psi\rangle$ the Bethe ansatz is more constraining. Apart from the Bethe roots, we find

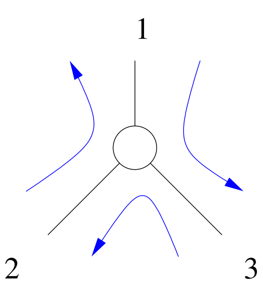
$$\prod_{j=1}^{n}(u - v_j^+) = \prod_{j=1}^{n}(u - v_j^-)$$

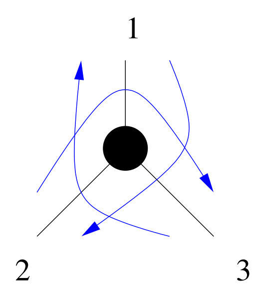
Thus, Yangian invariance requires the existence of a permutation σ with

$$v_{\sigma(j)}^+ = v_j^-$$

Exactly the condition of [Beisert, Broedel, Rosso '14] for deformed on-shell diagrams.
Showed relation to diagramatics in [Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka '12].

$$\begin{pmatrix} v_1^+ & v_2^+ & v_3^+ \\ \downarrow & \downarrow & \downarrow \\ v_3^- & v_1^- & v_2^- \end{pmatrix} \leftrightarrow$$





$$\leftrightarrow \begin{pmatrix} v_1^+ & v_2^+ & v_3^+ \\ \downarrow & \downarrow & \downarrow \\ v_2^- & v_3^- & v_1^- \end{pmatrix}$$

Direct Construction of Yangian Invariants

[Chicherin, Derkachov, Kirschner '13]

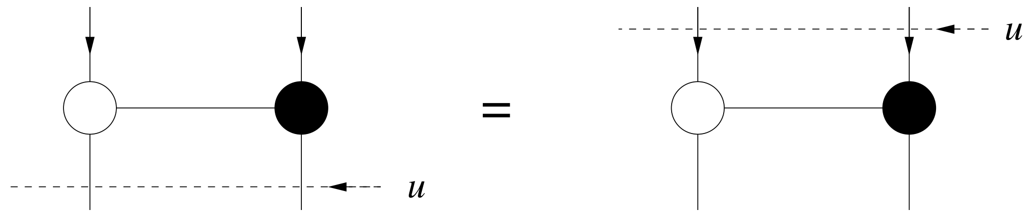
The Bethe ansatz is interesting, but constructing the states is hard.
A more direct method uses an intertwiner, which in twistor variables reads

$$\mathcal{B}_{jk}(u) = \left(-\mathcal{W}^k \cdot \frac{\partial}{\partial \mathcal{W}^j}\right)^u$$

Note $u \in \mathbb{C}$. Representation changing. Satisfies Yang-Baxter. Intertwines:

$$L_j(u, u_j) L_k(u, u_k) \mathcal{B}_{jk}(u_j - u_k) = \mathcal{B}_{jk}(u_j - u_k) L_j(u, u_k) L_k(u, u_j)$$

Graphical Depiction:



Use to make a Bethe-like ansatz to construct the invariants $|\Psi\rangle$.
Use intertwining relation to show $M(u) \cdot |\Psi\rangle = |\Psi\rangle$ iff for "correct" $\bar{u}_k$.

General Construction

[Broedel, De Leeuw Rosso; Kanning, Łukowski, MS '13]

Every on-shell diagram corresponds to some permutation σ. [Arkani-Hamed et.al. '12].

Resolve into "adjacent" transpositions: $\sigma = \tau_1 \dots \tau_P = (j_1 k_1) \dots (j_P k_P)$

Bethe-like ansatz

$$|\Psi\rangle = \mathcal{B}_{j_1 k_1}(\bar{u}_1) \dots \mathcal{B}_{j_P k_P}(\bar{u}_P)|\mathbf{0}\rangle$$

Bethe-like equations yield $\bar{u}_p = v_{\tau_p(k_p)} - v_{\tau_p(j_p)}$ with $\tau_p = (j_1 k_1) \dots (j_p k_p)$.

Example

Let us quickly look at $n = 4$, $k = 2$:

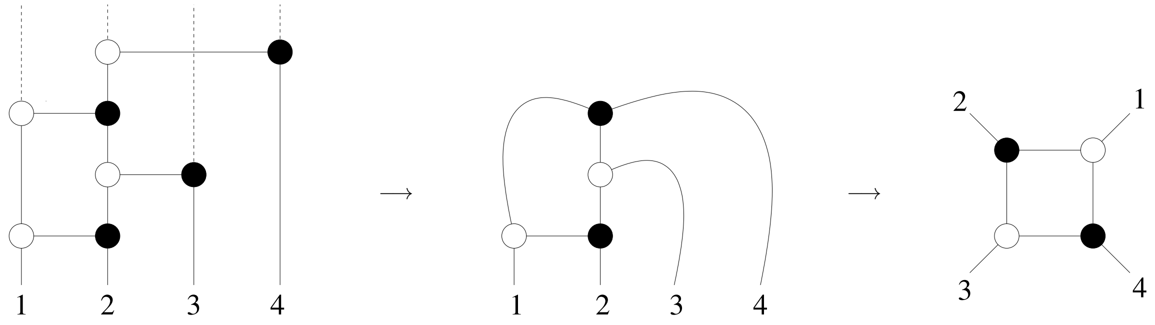
Permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (12)(23)(12)(24)$$

Yangian invariant:

$$|\Psi_{4,2}\rangle = \mathcal{B}_{12}(v_1 - v_2)\mathcal{B}_{23}(v_1 - v_3)\mathcal{B}_{12}(v_2 - v_3)\mathcal{B}_{24}(v_2 - v_4)|\mathbf{0}\rangle$$

On-Shell diagramatics:



Contours

As pointed out by [Chicherin, Derkachov, Kirschner '13] $\mathcal{B}_{jk}(u)$ acts like a BCFW shift:

$$\mathcal{B}_{jk}(u) = \left(-\mathcal{W}^k \cdot \frac{\partial}{\partial \mathcal{W}^j} \right)^u \simeq \int_{\mathcal{C}} \frac{d\alpha}{\alpha^{1+u}} \, e^{\alpha \mathcal{W}^k \cdot \partial_{\mathcal{W}^j}}$$

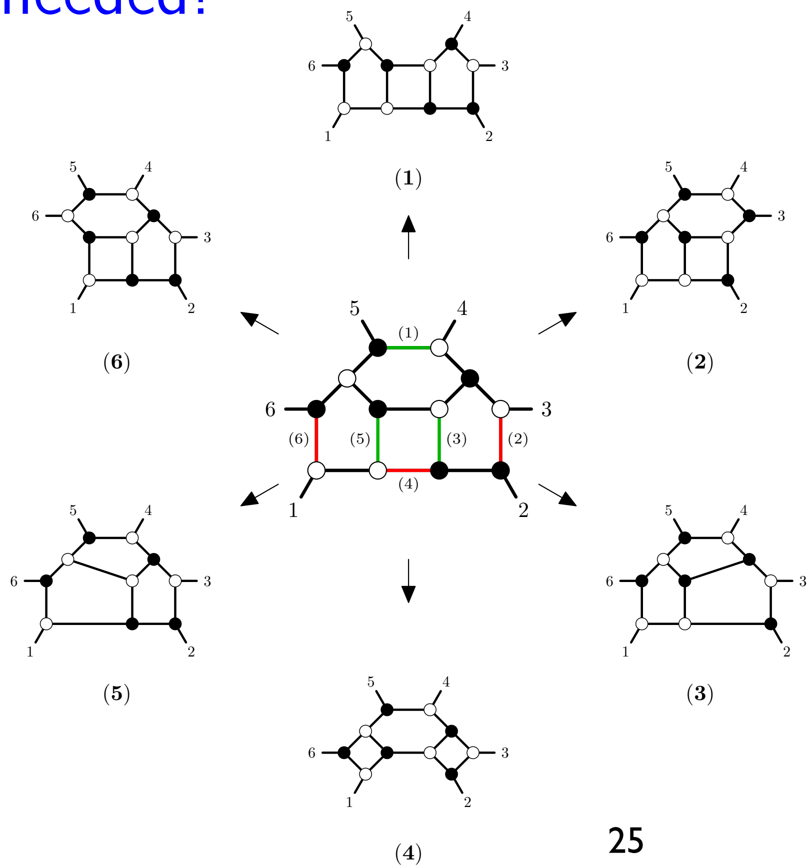
Recall super-twistors $\mathcal{Z}_j^{\mathcal{A}} = (\tilde{\mu}_j^{\alpha}, \tilde{\lambda}_j^{\dot{\alpha}}, \eta_j^A)$ w. Fourier conjugates $\lambda_j^{\alpha} \to \tilde{\mu}_j^{\alpha}$. This is however merely formal, unless the contour $\mathcal{C}$ is rigorously specified. Note that

- a Hankel contour does not work, in general
- for $u \neq 0$ BCFW recursion, based on residue theorem, no longer works

The Top-Cell

For the top-cell of the Graßmannian with general n, k the permutation σ is just a cyclic shift by k. This allows to derive the general deformed Graßmannian integral stated initially. [Ferro, Łukowski, MS, in preparation]

Important: The top-cell is the deformed tree-level amplitude. BCFW-decomposition breaks down when deforming, as shown in [Beisert, Broedel, Rosso '14]. But it is not needed!



[Figure from arXiv: 1401.7274: Beisert, Broedel, Rosso '14]

Outlook

Establish that the deformed Graßmannian is useful for loop calculations.