Energy-energy correlations in $\mathcal{N} = 4$ SYM

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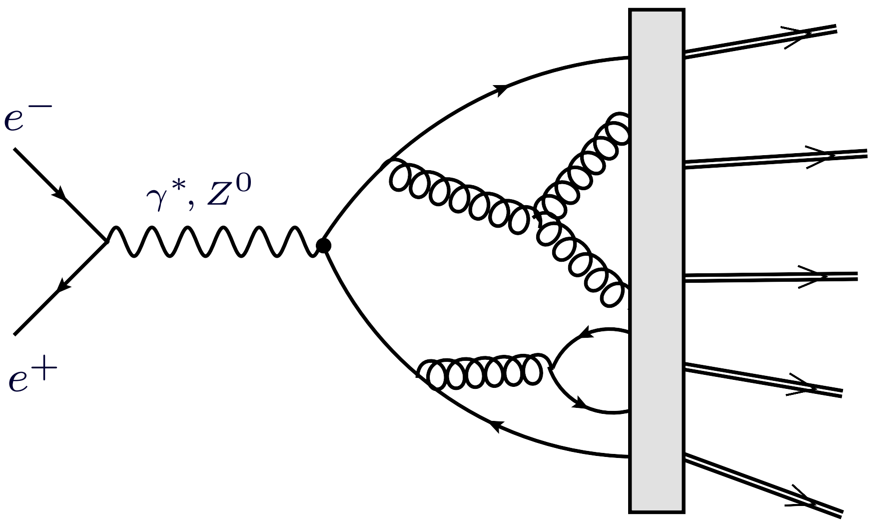
Can we define and compute

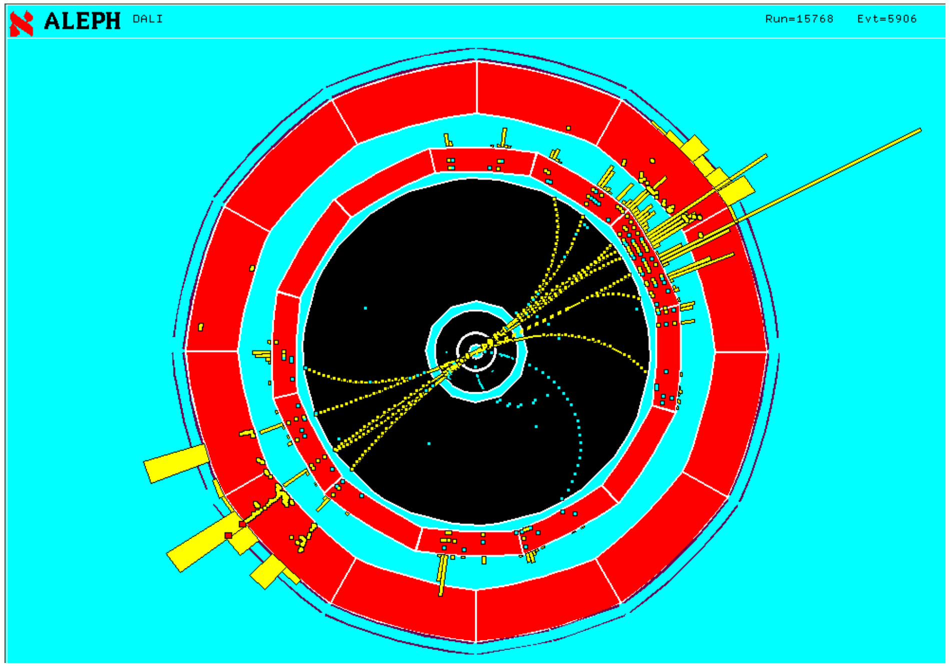
“real" physical observables

related to scattering amplitudes in $\mathcal{N} = 4$ SYM?

e^+e^- annihilation in QCD

✔ PETRA (1978-1986) and LEP (1989-2010)



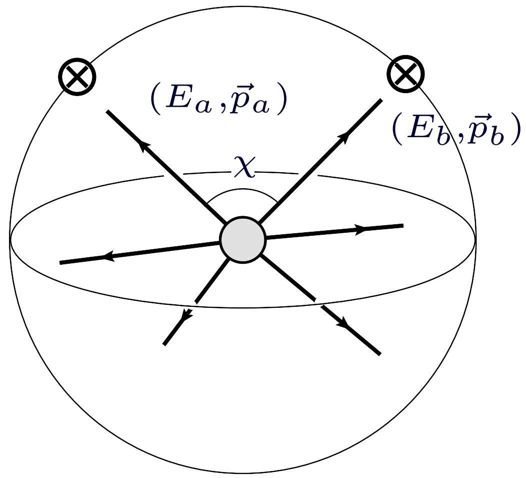


✔ Virtual photon or Z^0-boson decays into quarks and gluons that undergo hadronization

✔ Final states can be described using a class of *infrared finite* observables (event shapes): energy-energy correlations (EEC), thrust, heavy mass, . . .

Can be computed in perturbative QCD at high energy

Energy-energy correlation



- Function of the angle $0 \le \chi \le \pi$ between the detected particles [Basham,Brown,Ellis,Love]

- Conventional ('amplitude') approach

$$\text{EEC}(\chi) = \frac{1}{\sigma_{\text{tot}}} \sum_{a,b} \int d\sigma_{a+b+X} \frac{E_a E_b}{Q^2} \delta(\cos\theta_{ab} - \cos\chi)$$

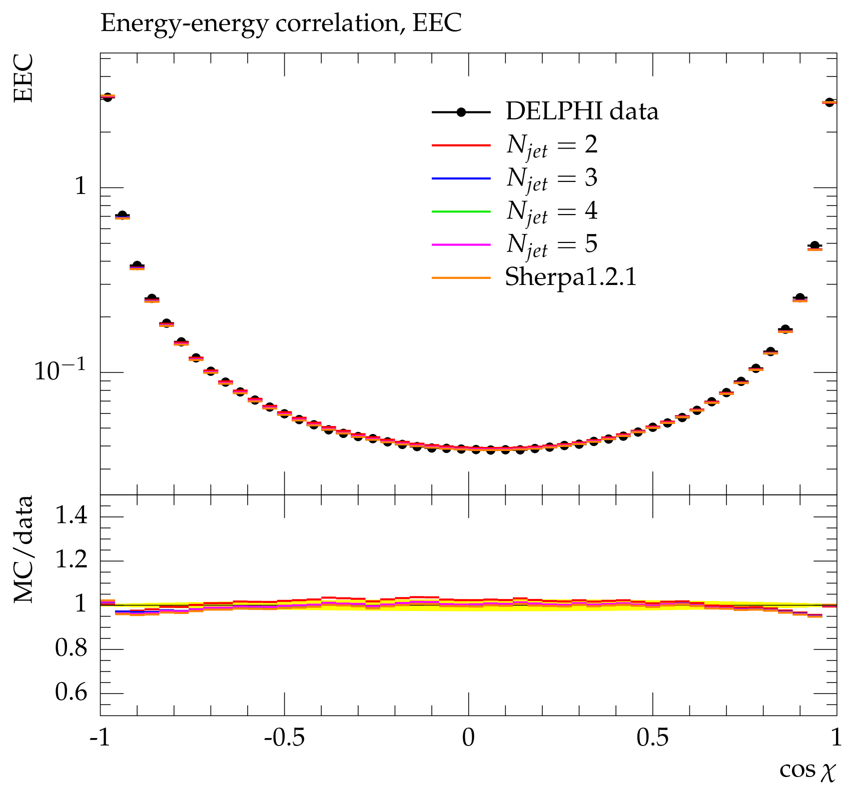
σ_{tot} total cross section $e^+e^- \to$ hadrons; total energy $\sum_a E_a = Q$

- Weak coupling expansion in QCD

$$\text{EEC}(\chi) = a\text{EEC}_{LO}(\chi) + a^2\text{EEC}_{NLO}(\chi) + O(a^3)$$

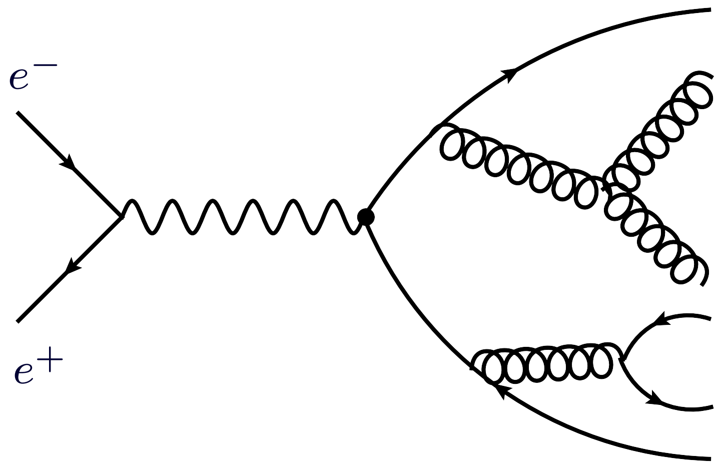
- Current status (1978 – today):
  - Very precise experimental data
  - Poor analytical control, EEC_{NLO} is evaluated numerically

- *Our goal: develop more efficient method for computing EEC*



"e^+e^- annihilation" in $\mathcal{N}=4$ SYM

✔ Define EEC in $\mathcal{N}=4$ SYM and evaluate it at weak/strong coupling



✔ From QCD to $\mathcal{N}=4$ SYM: introduce an analog of the electromagnetic current

- ✗ (protected) half-BPS operator built from the six real scalars Φ^I (with $I=1,\dots,6$)

$$O^{IJ}_{\mathbf{20'}}(x) = \mathrm{tr}\left[\Phi^I\Phi^J - \tfrac{1}{6}\delta^{IJ}\Phi^K\Phi^K\right]$$

- ✗ To lowest order in the coupling, $O_{\mathbf{20'}}(x)$ produces a pair of scalars out of the vacuum
- ✗ For arbitrary coupling, the state $O_{\mathbf{20'}}(x)|0\rangle$ can be decomposed into an infinite sum over on-shell states with arbitrary numbers of scalars (s), gluinos (λ) and gluons (g)

$$\int d^4x\, \mathrm{e}^{iqx}\, O_{\mathbf{20'}}(x)|0\rangle = |ss\rangle + |ssg\rangle + |s\lambda\lambda\rangle + \dots$$

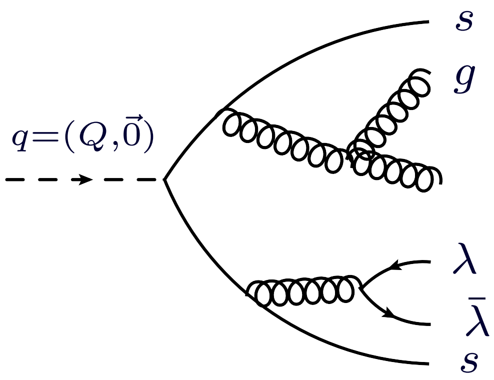
EEC in $\mathcal{N} = 4$ SYM

- ✔ Conventional approach

$$\mathrm{EEC}(\chi) = \frac{1}{\sigma_{\rm tot}} \sum_{a,b,X} \int \mathrm{dLIPS}\, \left|\mathcal{A}_{a+b+X}\right|^2 \frac{E_a E_b}{Q^2} \delta(\cos\chi - \cos\theta_{ab})$$

The amplitude of creation of the final state $|a, b, X = \text{everything}\rangle$

$$\mathcal{A}_{a+b+X} = \int d^4x\, \mathrm{e}^{iqx} \langle a, b, X | O_{\mathbf{20'}}(x) | 0\rangle$$

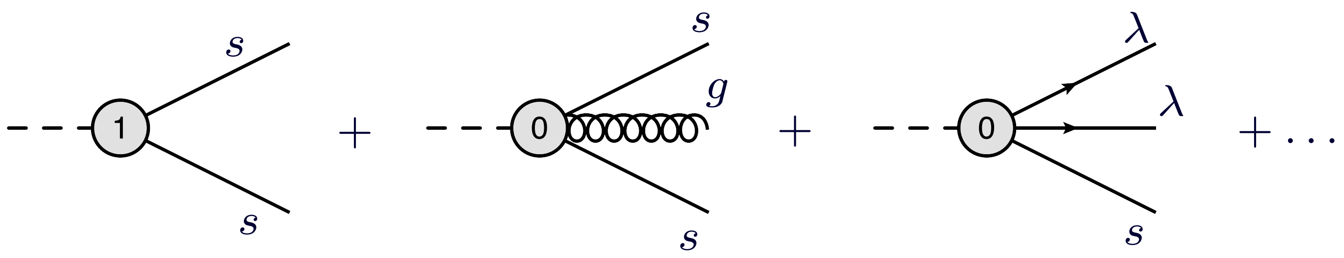


- ✔ Problems:
  - ✗ presence of infrared divergences in the transition amplitudes $\mathcal{A}_{a+b+X}$
  - ✗ integration over the Lorentz invariant phase space of the final states $d\mathrm{LIPS}$
  - ✗ need to sum over all final states $\sum_X$
  - ✗ no analytical results beyond one loop
- ✔ New approach: EEC can be computed from *correlation functions of energy flow operators*

EEC from amplitudes I

✔ Transition amplitude

$\mathcal{A}_{a+b+X} =$



✔ Matrix elements ($s_{ij} = (p_i + p_j)^2$ with $p_i^2 = 0$; 't Hooft coupling $a = g_{\rm YM}^2 N/(4\pi^2)$)

$$|\mathcal{A}_{ss}|^2 = \left|\langle s(p_1)s(p_2)|O_{\mathbf{20'}}|0\rangle\right|^2 = \frac{2}{s_{12}}\left[1 + aF_{\rm virt}(q^2)\right]$$

$$|\mathcal{A}_{ssg}|^2 = \left|\langle s(p_1)s(p_2)g(p_3)|O_{\mathbf{20'}}|0\rangle\right|^2 = a\,\frac{s_{12}}{s_{13}s_{23}}$$

$$|\mathcal{A}_{s\lambda\lambda}|^2 = \left|\langle \lambda(p_1)\lambda(p_2)s(p_3)|O_{\mathbf{20'}}|0\rangle\right|^2 = a\,\frac{2}{s_{12}}$$

✔ The total transition amplitude

$$\sigma_{\rm tot}(q) = \int \text{dLIPS}_2\, |\mathcal{A}_{ss}|^2 + \int \text{dLIPS}_3\, \left(|\mathcal{A}_{ssg}|^2 + |\mathcal{A}_{s\lambda\lambda}|^2\right) + O(a^2)$$

$$= \frac{1}{8\pi}\left[1 + aF_{\rm virt}(q^2)\right] + a\int \text{dLIPS}_3\, \frac{s_{12}^2 + 2s_{13}s_{23}}{s_{12}s_{13}s_{23}} + O(a^2) = \frac{1}{8\pi} + 0\cdot a + O(a^2)$$

Protected from perturbative corrections

EEC from amplitudes II

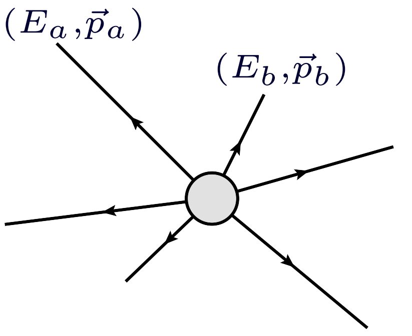
- Weighted cross-section

$$\mathrm{EEC} = \Big[\int \text{dLIPS}_2\, w(p_1,p_2)\, |\mathcal{A}_{ss}|^2 + \int \text{dLIPS}_3\, w(p_1,p_2,p_3) \big(|\mathcal{A}_{ssg}|^2 + |\mathcal{A}_{s\lambda\lambda}|^2\big) + O(a^2) \Big] / \sigma_{\rm tot}$$

- Weight factors for energy-energy correlations

$$w(p_1, p_2, \dots) = \sum_{a,b} \frac{E_a E_b}{q^2} \delta(\cos\theta_{ab} - \cos\chi)$$

$$\mathrm{dLIPS}_\ell = \prod_i \frac{d^4 p_i}{(2\pi)^3} \delta_+(p_i^2)\ \delta^{(4)}\Big(\sum p_i - q\Big)$$



- Conformal symmetry: EEC depends only on the scaling variable

$$z = \tfrac{1}{2}(1 - \cos\chi)\,, \qquad 0 < z < 1$$

- One-loop calculation (unprotected quantity) [Zhiboedov],[Engelund,Roiban]

$$\mathrm{EEC}_{\mathcal{N}=4} = \frac{a}{4z^2(1-z)} \ln\frac{1}{1-z} + O(a^2)$$

IR finite, positive definite function of $0 < \chi < \pi$

- Two-loop correction is hard to compute ($\sim 10^2$ diagrams)

EEC from correlation functions I

- ✔ Use the completeness condition $\sum_X |X\rangle\langle X| = 1$

$$\sigma_{\rm tot}(q) = \sum_X (2\pi)^4 \delta^{(4)}(q - p_X) |\mathcal{A}_{O_{\mathbf{20'}} \to X}|^2$$
$$= \int d^4x \, \mathrm{e}^{iqx} \sum_X \langle 0|O^\dagger(0)|X\rangle \, \mathrm{e}^{-ixp_X} \, \langle X|O(0)|0\rangle$$
$$= \int d^4x \, \mathrm{e}^{iqx} \, \langle 0|O^\dagger(x)O(0)|0\rangle = \int d^4x \, \mathrm{e}^{iqx} \, \frac{C_N}{(x^2 - i\epsilon x^0)^2} = \frac{1}{16\pi}(N^2-1)\theta(q^0)\theta(q^2)$$

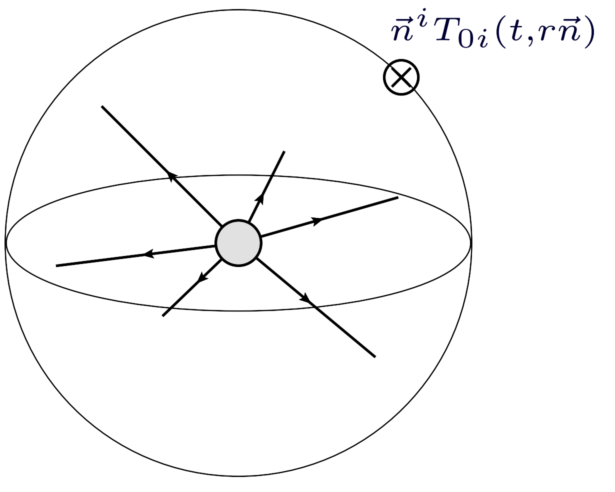
Wightman correlation function (no time ordering!), protected for 1/2-BPS operators

- ✔ Generalization to EEC

$$\mathrm{EEC} \sim \sum_X \langle 0|O^\dagger(x)|X\rangle w(X) \langle X|O(0)|0\rangle = \langle 0|O^\dagger(x)\mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)O(0)|0\rangle$$

Energy flow operator

$$\mathcal{E}(\vec{n})|X\rangle = \sum_a E_a \, \delta^{(2)}(\Omega_{\vec{p}_a} - \Omega_{\vec{n}})|X\rangle$$



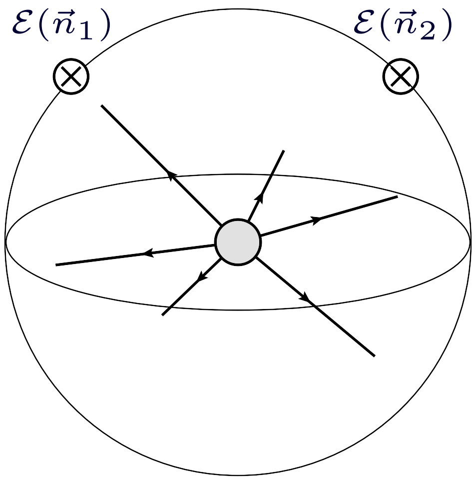
- ✔ Relation to the energy-momentum tensor in $\mathcal{N}=4$ SYM

[Sveshnikov,Tkachov],[Korchemsky,Oderda,Sterman]

$$\mathcal{E}(\vec{n}) = \int_0^\infty dt \lim_{r\to\infty} r^2 \, \vec{n}^i T_{0i}(t, r\vec{n})$$

EEC from correlation functions II

- Energy flow correlations [Korchemsky,Sterman],[Belitsky,Korchemsky,Sterman],[Hofman,Maldacena]



$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\rangle_q = \sigma_{\rm tot}^{-1} \int d^4x \, \mathrm{e}^{iqx} \langle 0|O^\dagger(x)\, \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\, O(0)|0\rangle$$

Energy flow in the direction of $\vec{n}_1$ and $\vec{n}_2$

- Average over the orientations $\vec{n}_1$ and $\vec{n}_2$ with the relative angle χ kept fixed

$$\mathrm{EEC} = \int d\Omega_1 d\Omega_2 \, \delta(\vec{n}_1 \cdot \vec{n}_2 - \cos\chi) \langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\rangle_q / q^2$$

- Multi-fold integral of the 4-point *Wightman* function

$$\mathrm{EEC} \sim \underbrace{\int d^4x \, \mathrm{e}^{iqx}}_{\text{Fourier}} \underbrace{\int_0^\infty dt_1 dt_2 \lim_{r_i\to\infty} r_1^2 r_2^2}_{\text{Detector limit}} \underbrace{\langle 0|O^\dagger(x)\, T_{0\vec{n}_1}(x_1) T_{0\vec{n}_2}(x_2)\, O(0)|0\rangle_W}_{\text{Wightman corr. function}} \Big|_{x_i = (t, r\vec{n}_i)}$$

$\mathcal{N} = 4$ SUSY and analytic continuation

- ✔ How to obtain $\langle O T_{\mu\nu} T_{\rho\sigma} O \rangle$?
  - ✗ Starting point: 4pt function of scalar half-BPS operators

$$\langle O(x_1) O(x_2) O(x_3) O(x_4) \rangle_{\mathrm{E}} = \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} \Phi(u,v;a)\,, \qquad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}$$

  - ✗ O and $T_{\mu\nu}$ are members of the $\mathcal{N} = 4$ energy-momentum tensor multiplet

$$\mathcal{T}(x,\theta,\bar\theta) = O(x) + \theta\sigma^\mu\bar\theta\; R_\mu(x) + \ldots + \theta\sigma^\mu\bar\theta\;\theta\sigma^\nu\bar\theta\; T_{\mu\nu}(x) + \ldots + (\theta)^4 \mathcal{L}_{\mathcal{N}=4\ SYM}(x)$$

  - ✗ $\langle O T_{\mu\nu} T_{\rho\sigma} O \rangle$ is obtained from $\langle OOOO \rangle$ by $\mathcal{N} = 4$ conformal SUSY

$$\stackrel{\mathcal{N}=4\ SUSY}{\Longrightarrow} \quad \langle O(x_1) T_{\mu\nu}(x_2) T_{\rho\sigma}(x_3) O(x_4) \rangle_{\mathrm{E}} = D_{\mu\nu\rho\sigma}(\partial_2,\partial_3) \langle O(x_1) O(x_2) O(x_3) O(x_4) \rangle_{\mathrm{E}}$$

    with $D_{\mu\nu\rho\sigma}$ a 4th-order differential operator.

- ✔ How to analytically continue $\Phi(u,v;a)$ from Euclid to Minkowski+Wightman? [Lüscher,Mack]
  - ✗ Most easily done using the Mellin transform

$$\Phi(u,v;a) = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M(j_1,j_2;a)\, u^{j_1} v^{j_2}$$

    by the simple replacement $x_{ij}^2 \to x_{ij}^2 - i\epsilon x_{ij}^0$ for $i<j$ in u, v.

All-loop prediction for EEC

Master formula

$$\mathrm{EEC}(\chi) = \frac{1}{4z^2(1-z)} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} \underbrace{M(j_1,j_2;a)}_{\text{corr. function}} \underbrace{K(j_1,j_2)}_{\text{detector}} \underbrace{\left(\frac{1-z}{z}\right)^{j_1+j_2}}_{\text{angular dependence}}$$

The dependence on the angle χ enters through

$$z = (1-\cos\chi)/2$$

Detector function depending on the type of observable (energy, charge, . . .) but not on the coupling

$$K_{EE}(j_1,j_2) = \frac{2\,\Gamma(1-j_1-j_2)}{\Gamma(j_1+j_2)[\Gamma(1-j_1)\Gamma(1-j_2)]^2}$$

The dependence on the coupling constant comes from the Mellin amplitude

$$\Phi(u,v;a) = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M(j_1,j_2;a)\, u^{j_1} v^{j_2}$$

$$M(j_1,j_2;a) = \underbrace{aM^{(1)}(j_1,j_2) + a^2 M^{(2)}(j_1,j_2)}_{\text{are known}} + \dots$$

Mellin amplitude also known at strong coupling via AdS/CFT [Arutyunov,Frolov]]

EEC at two loops

- ✔ Four-point correlator at weak coupling [Eden,Schubert,ES],[Bianchi et al]

$$\Phi(u,v;a) = a\,\Phi^{(1)}(u,v) + a^2\Big\{\frac{1}{2}(1+u+v)\left[\Phi^{(1)}(u,v)\right]^2 + 2\left[\Phi^{(2)}(u,v) + \frac{1}{u}\Phi^{(2)}(v/u,1/u) + \frac{1}{v}\Phi^{(2)}(1/v,u/v)\right]\Big\} + O(a^3)$$

- ✔ Euclidean ladder (or 'scalar box') integrals $\Phi^{(1)}$ and $\Phi^{(2)}$ [Usyukina,Davydychev]

- ✔ Mellin amplitude to two loops:

$$M(j_1,j_2) = aM^{(1)}(j_1,j_2) + a^2\left[\tfrac{1}{2}\widetilde{M}^{(2)}(j_1,j_2) + \widetilde{M}^{(2)}(j_1,j_2-1) + 2M^{(2)}(j_1,j_2) + 4M^{(2)}(j_1,-1-j_1-j_2)\right]$$

$$M^{(1)}(j_1,j_2) = -\frac{1}{4}\left[\Gamma(-j_1)\Gamma(-j_2)\Gamma(1+j_1+j_2)\right]^2$$

$$M^{(2)}(j_1,j_2) = -\frac{1}{4}\Gamma(-j_1)\Gamma(-j_2)\Gamma(1+j_1+j_2) \times \int \frac{dj_1'\,dj_2'}{(2\pi i)^2} M^{(1)}(j_1',j_2')\frac{\Gamma(j_1'-j_1)\Gamma(j_2'-j_2)\Gamma(1+j_1+j_2-j_1'-j_2')}{\Gamma(1-j_1')\Gamma(1-j_2')\Gamma(1+j_1'+j_2')}$$

$$\widetilde{M}^{(2)}(j_1,j_2) = \int \frac{dj_1'\,dj_2'}{(2\pi i)^2} M^{(1)}(j_1-j_1',j_2-j_2')M^{(1)}(j_1',j_2')$$

Warm-up exercise: One loop

- ✔ Master formula at one loop

$$\text{EEC}^{(1-\text{loop})} = \frac{a}{4z^2(1-z)} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M^{(1)}(j_1, j_2; a) K(j_1, j_2) \left(\frac{1-z}{z}\right)^{j_1+j_2}$$

Mellin amplitude

$$M^{(1)}(j_1, j_2) = -\frac{1}{4}\left[\Gamma(-j_1)\Gamma(-j_2)\Gamma(1+j_1+j_2)\right]^2$$

$$K(j_1, j_2) = \frac{2\,\Gamma(1-j_1-j_2)}{\Gamma(j_1+j_2)[\Gamma(1-j_1)\Gamma(1-j_2)]^2}$$

- ✔ Change integration variable $j_1 + j_2 \to j_1$

$$\begin{aligned}
\text{EEC}^{(1-\text{loop})} &= -\frac{a}{4z^2(1-z)} \int \frac{dj_1 dj_2}{(2\pi i)^2} \frac{j_1^2}{2(j_1-j_2)^2 j_2{}^2} \frac{\pi}{\sin(\pi j_1)} \left(\frac{1-z}{z}\right)^{j_1} \\
&= \frac{a}{4z^2(1-z)} \int \frac{dj_1}{2\pi i} \frac{\pi}{j_1 \sin(\pi j_1)} \left(\frac{1-z}{z}\right)^{j_1} \\
&= \frac{a}{4z^2(1-z)} \sum_{k=-1}^{-\infty} \frac{(-1)^k}{k} \left(\frac{1-z}{z}\right)^{k} \\
&= \frac{a}{4z^2(1-z)} \ln\frac{1}{1-z}
\end{aligned}$$

EEC at two loops I

Final result for EEC

$$\text{EEC}_{\mathcal{N}=4} = \frac{1}{4z^2(1-z)}\left\{aF_1(z) + a^2\left[(1-z)F_2(z) + \tfrac{1}{4}F_3(z)\right]\right\}, \qquad z = \tfrac{1}{2}(1-\cos\chi)$$

$F_w(z)$ are linear combinations of functions of homogenous weight $w = 1, 2, 3$

$$F_1(z) = -\ln(1-z)$$

$$F_2(z) = 4\sqrt{z}\left[\text{Li}_2\left(-\sqrt{z}\right) - \text{Li}_2\left(\sqrt{z}\right) + \frac{1}{2}\ln z \ln\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]$$
$$+ (1+z)\left[2\text{Li}_2(z) + \ln^2(1-z)\right] + 2\ln(1-z)\ln\left(\frac{z}{1-z}\right) + z\frac{\pi^2}{3},$$

$$F_3(z) = (1-z)(1+2z)\left[\ln^2\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\ln\left(\frac{1-z}{z}\right) - 8\text{Li}_3\left(\frac{\sqrt{z}}{\sqrt{z}-1}\right) - 8\text{Li}_3\left(\frac{\sqrt{z}}{\sqrt{z}+1}\right)\right]$$
$$- 4(z-4)\text{Li}_3(z) + 6(3+3z-4z^2)\text{Li}_3\left(\frac{z}{z-1}\right) - 2z(1+4z)\zeta_3 + 2\big[(3-4z)z\ln z$$
$$+ 2(2z^2-z-2)\ln(1-z)\big]\text{Li}_2(z) + \frac{1}{3}\ln^2(1-z)\left[4(3z^2-2z-1)\ln(1-z)\right.$$
$$\left.+ 3(3-4z)z\ln z\right] + \frac{\pi^2}{3}\left[2z^2\ln z - (2z^2+z-2)\ln(1-z)\right]$$

EEC at two loops II

- ✔ Sum of basis functions with nontrivial arguments and rational prefactors R_2 and R_3

$$\mathrm{EEC}^{(2\mathrm{loops})} = \sum R_2(z,\sqrt{z})\, W_2(z,\sqrt{z}) + \sum R_3(z,\sqrt{z})\, W_3(z,\sqrt{z})$$

  Weight two $W_2 = \{\mathrm{Li}_2, \ln\ln, \pi^2\}$

  Weight three $W_3 = \{\mathrm{Li}_3, \mathrm{Li}_2 \ln, \ln\ln\ln, \pi^2 \ln, \zeta_3\}$

- ✔ W_2 and W_3 depend on $\sqrt{z} = \left|\sin(\chi/2)\right|$, but EEC is manifestly invariant under $\sqrt{z} \to -\sqrt{z}$

- ✔ Scattering amplitudes have homogenous weight in planar $\mathcal{N}=4$ SYM at weak coupling

$$A_{a+b+X} \sim \exp\left(-\text{Div}(1/\epsilon) + \sum_\ell a^\ell W_{2\ell} + O(\epsilon)\right)$$

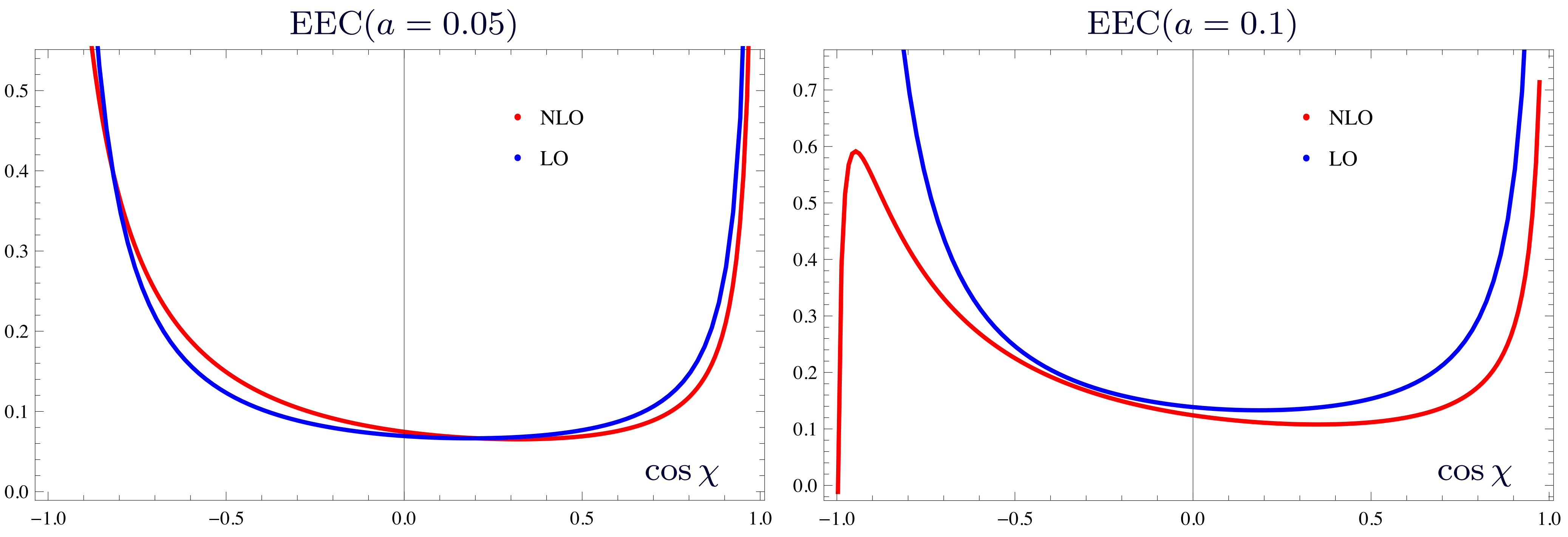
  This property is *'minimally'* violated for EEC after the phase space integration

$$\mathrm{EEC}(\chi) = \frac{1}{\sigma_{\mathrm{tot}}} \sum_{a,b} \int \mathrm{dLIPS}\, |A_{a+b+X}|^2\, \frac{E_a E_b}{Q^2} \delta(\cos\chi - \cos\theta_{ab})$$

  But it is restored in the back-to-back kinematics $\chi \to \pi$, or $z \to 1$

- ✔ $\mathrm{EEC}^{(2-\mathrm{loop})}$ involves some of the transcendental functions that also appear in the two-loop result for the quarks (= n_f dependent) contribution to EEC in QCD [Dixon,Schreiber]

From weak to strong coupling



- ✔ At weak coupling $\mathrm{EEC}_{\mathcal{N}=4}$ is remarkably similar to EEC_{QCD}
- ✔ Going from one to two loops, EEC flattens
- ✔ This agrees with the strong coupling prediction for EEC in planar $\mathcal{N} = 4$ SYM [Hofman,Maldacena]

$$\mathrm{EEC}_{\mathcal{N}=4} \overset{a\to\infty}{\sim} \frac{1}{2}\left[1 + a^{-1}\left(1 - 6z(1-z)\right) + O(a^{-3/2})\right]$$

No jets at strong coupling

Conclusions and open questions

- ✔ Energy flow correlations are good/nontrivial physical observables in $\mathcal{N}=4$ SYM
- ✔ Relation to energy flow correlations in QCD (most complicated part)?
- ✔ All symmetries of $\mathcal{N}=4$ SYM are preserved, what is the manifestation of integrability?
- ✔ Interpolation between weak and strong coupling?
- ✔ Bootstrap?
- ✔ Other proposals for ‘good’ observables?