Evaluating multiloop Feynman integrals by differential equations

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- General prescriptions and a simple one-loop example

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- Conclusion

[A.V. Kotikov’91, E. Remiddi’97, T. Gehrmann & E. Remiddi’00, J. Henn’13]

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Henn: use uniform transcendentality (UT)!

Reduction to master integrals

Evaluating a family of Feynman integrals associated with a given graph with general integer powers of the propagators (indices)

$$F_\Gamma(q_1,\ldots,q_n;d;a_1,\ldots,a_L)$$
$$= \int\ldots\int I(q_1,\ldots,q_n;k_1,\ldots,k_h;a_1,\ldots,a_L)\mathrm{d}^d k_1 \mathrm{d}^d k_2\ldots\mathrm{d}^d k_h$$

$$I(q_1,\ldots,q_n;k_1,\ldots,k_h;a_1,\ldots,a_L) = \frac{1}{(p_1^2-m_1^2)^{a_1}(p_2^2-m_2^2)^{a_2}\ldots}$$

The old straightforward analytical strategy:

to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

The standard modern strategy:

to derive, without calculation, and then apply IBP identities between the given family of Feynman integrals as recurrence relations.

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The whole problem of evaluation$\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

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- Solve DE

The crucial point:

choose all the master integrals as pure *functions of uniform* weight*, i.e uniform degree of transcendentality*

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Weight for numbers: n for $\zeta(n)$, $\mathrm{Li}_n(1/2)$ etc.

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A transition to a UT basis is a linear transformation in the space of master integrals and the corresponding matrix is rational with respect to dimension and kinematic invariants.

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A decisive criterion: if we arrive at canonical DE then we make a proper choice of UT master integrals!

An example: a one-loop massless propagator integral

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{a_1}[-(q-k)^2]^{a_2}} = \mathrm{i}\pi^{d/2} \frac{G(a_1,a_2)}{(-q^2)^{a_1+a_2+\epsilon-2}} \ ,$$

$$G(a_1,a_2) = \frac{\Gamma(a_1+a_2+\epsilon-2)\Gamma(2-\epsilon-a_1)\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4-a_1-a_2-2\epsilon)}$$

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$\Gamma(1+k\epsilon), \Gamma(k\epsilon)$ are UT, e.g.

$$\Gamma(1+\epsilon) = e^{-\gamma_{\mathrm{E}}\epsilon} \left(1 + \frac{\pi^2\epsilon^2}{12} - \frac{\epsilon^3\zeta(3)}{3} + \ldots \right)$$

$\Gamma(2-2\epsilon) \equiv (1-2\epsilon)\Gamma(1-2\epsilon)$ is not UT

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One can use Feynman parameters. For example,

$$\int \frac{\mathrm{d}^d k}{(-k^2+m^2)^{a_1}[-(q-k)^2]^{a_2}} \sim \int_0^1 \frac{\alpha^{a_2-1}(1-\alpha)^{1-a_2-\epsilon}}{[1+x\alpha]^{a_1+a_2+\epsilon-2}}$$

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A general rule: factors like $(1-\alpha)^{\pm\epsilon}$ or $\alpha^{\pm\epsilon}$ do not spoil UT

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$$\int \frac{\mathrm{d}^d k}{k^2 (k+p_1)^2 (k+p_1+p_2)^2 (k-p_3)^2} \to$$

$$\int \mathrm{d}^4 k \delta(k^2) \delta((k+p_1)^2) \delta((k+p_1+p_2)^2) \delta((k-p_3)^2) \sim \frac{1}{st}$$

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This gives the hint that after the multiplication by st we should obtain a UT Feynman integral.

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DE:

$$\partial_i f(\epsilon, x) = A_i(\epsilon, x) f(\epsilon, x) \,,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and each A_i is an $N \times N$ matrix.

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Conjecture: one can turn to a new basis by a linear transformation $f \to Bf$
(resulting in $A_m \to B^{-1} A_m B - B^{-1}(\partial_m B)$)
such that the DE will take the following *canonical* form

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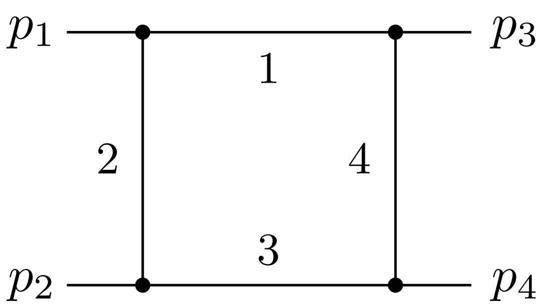
How to prove it? (A good mathematical problem.)

An example: the massless on-shell box diagram, i.e. with $p_i^2 = 0,\ i = 1, 2, 3, 4$



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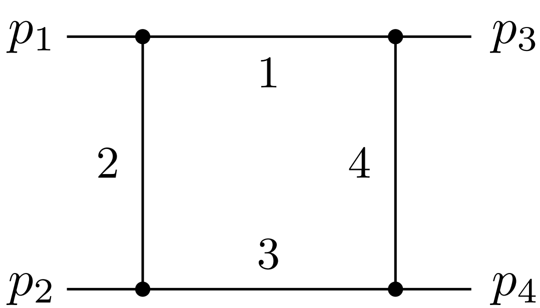

$$F_\Gamma(s, t; a_1, a_2, a_3, a_4, d)$$

$$= \int \frac{\mathrm{d}^d k}{(-k^2)^{a_1}[-(k+p_1)^2]^{a_2}[-(k+p_1+p_2)^2]^{a_3}[-(k-p_3)^2]^{a_4}} ,$$

where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$

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Choose them proportional to $G(2,1) = \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(\epsilon+1)}{\Gamma(1-2\epsilon)}$

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Choose them proportional to $G(2,1) = \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(\epsilon+1)}{\Gamma(1-2\epsilon)}$
Turn to a UT basis:

$$f = (-s)^{\epsilon}\{\epsilon t F(0,1,0,2), \epsilon s F(1,0,2,0), \epsilon^2 st F(1,1,1,1)\}$$

$\equiv \{f_1, f_2, f_3\}$

with $x = t/s$, $s = -1$.

DE in the new basis

$$f'(\epsilon, x) = \epsilon\, A(x)\, f(\epsilon, x)$$

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$$A(x) = \begin{pmatrix} -\frac{1}{x} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2}{x+1} - \frac{2}{x} & \frac{2}{x+1} & \frac{1}{x+1} - \frac{1}{x} \end{pmatrix}$$

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Solving DE in the ϵ-expansion, $f = \sum_{n=0} f^{(n)} \epsilon^n$

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{(n)}(x) = A(x) f^{(n-1)}(x)\,.$$

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$$f^{(n)}(x) = \int_0^x \mathrm{d}x' A(x') f^{(n-1)}(x') + g^{(n)}\,.$$

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In every order of the ϵ-expansion, one obtains a linear combination of integrals

$$\int_{0 \le x_1 \le \ldots x_k \le x} \frac{\mathrm{d}x_k}{x_k + a_k} \cdots \frac{\mathrm{d}x_1}{x_1 + a_1}$$

where $a_i = 0$ or 1.

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HPLs

$$H(a_1,a_2,\ldots,a_n;x)=\int_0^x f(a_1;t)H(a_2,\ldots,a_n;t)\mathrm{d}t\,,$$

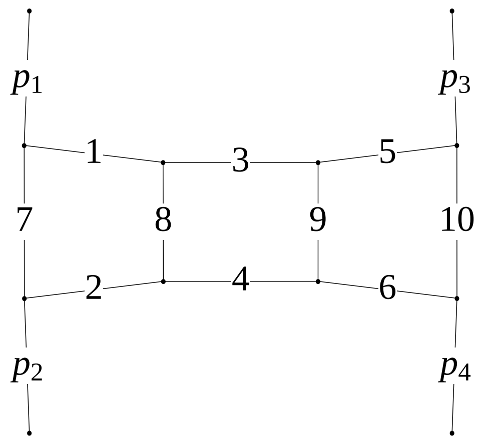
where $f(\pm 1;t)=1/(1\mp t),\;\; f(0;t)=1/t$

The result is $f_3 = \sum_{j=0} c_j(x,L)\epsilon^j$, with

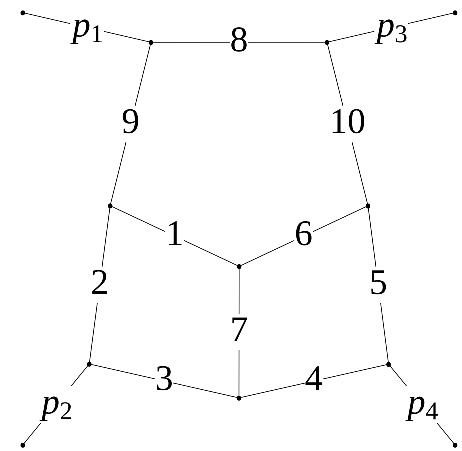
$$
\begin{aligned}
c_0 &= 4 \quad c_1 = 2L\,, \quad c_2 = -\frac{4}{3}\,\pi^2\,, \\
c_3 &= \pi^2 H_1(x) + 2H_{0,0,1}(x) - \frac{7}{6}\pi^2 L + 2H_{0,1}(x)L + H_1(x)L^2 - \frac{1}{3}L^3 - \frac{34}{3}\zeta_3\,, \\
c_4 &= -2H_{1,0,0,1}(x) - 2H_{0,0,1,1}(x) - 2H_{0,1,0,1}(x) - 2H_{0,0,0,1}(x) - 2H_{0,1,1}(x)L \\
&\quad - 2H_{1,0,1}(x)L + H_{0,1}(x)L^2 - H_{1,1}(x)L^2 + \frac{2}{3}H_1(x)L^3 - \frac{1}{6}L^4 \\
&\quad - \pi^2 H_{1,1}(x) + \pi^2 H_1(x)L - \frac{1}{2}\pi^2 L^2 + 2H_1(x)\zeta_3 - \frac{20}{3}L\zeta_3 - \frac{41}{360}\pi^4 + \cdots
\end{aligned}
$$

with $L = \log x$.

Massless three-loop four-point Feynman integrals on the light cone



(A)



(E)

$$F^{A}_{a_1,\ldots,a_{15}}(s,t;D) \;=\; \int\int\int \frac{\mathrm{d}^D k_1\,\mathrm{d}^D k_2\,\mathrm{d}^D k_3}{(-k_1^2)^{a_1}[-(p_1+p_2+k_1)^2]^{a_2}(-k_2^2)^{a_3}}$$

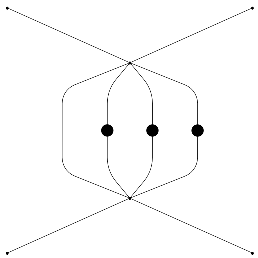
$$\times \frac{[-(k_1-p_3)^2]^{-a_{11}}[-(p_1+k_2)^2]^{-a_{12}}[-(k_2-p_3)^2]^{-a_{13}}}{[-(p_1+p_2+k_2)^2]^{a_4}(-k_3^2)^{a_5}[-(p_1+p_2+k_3)^2]^{a_6}[-(p_1+k_1)^2]^{a_7}}$$

$$\times \frac{[-(p_1+k_3)^2]^{-a_{14}}[-(k_1-k_3)^2]^{-a_{15}}}{[-(k_1-k_2)^2]^{a_8}[-(k_2-k_3)^2]^{a_9}[-(k_3-p_3)^2]^{a_{10}}}\;,$$

$$F^{E}_{a_1,\ldots,a_{15}}(s,t;D) \;=\; \int\int\int \frac{\mathrm{d}^D k_1\,\mathrm{d}^D k_2\,\mathrm{d}^D k_3}{[-(k_1-k_3)^2]^{a_1}[-(p_1+k_1)^2]^{a_2}}$$

$$\times \frac{[-(p_1+p_2+k_3)^2]^{-a_{11}}[-(p_1+k_2)^2]^{-a_{12}}[-(k_1-p_3)^2]^{-a_{13}}}{[-(p_1+p_2+k_1)^2]^{a_3}[-(p_1+p_2+k_2)^2]^{a_4}[-(k_2-p_3)^2]^{a_5}[-(k_2-k_3)^2]^{a}}$$

$$\times \frac{(-k_1^2)^{-a_{14}}(-k_2^2)^{-a_{15}}}{[-(k_1-k_2)^2]^{a_7}(-k_3^2)^{a_8}[-(p_1+k_3)^2]^{a_9}[-(k_3-p_3)^2]^{a_{10}}}\;.$$



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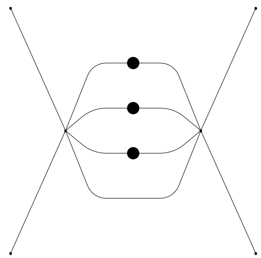

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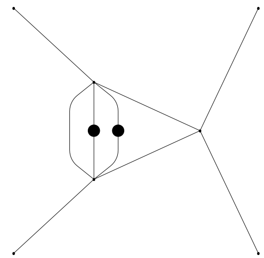

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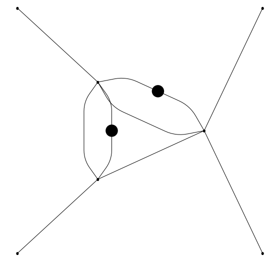

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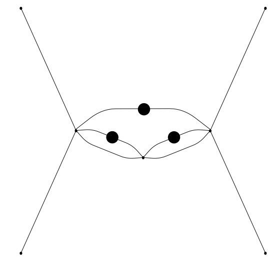

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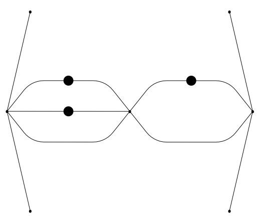

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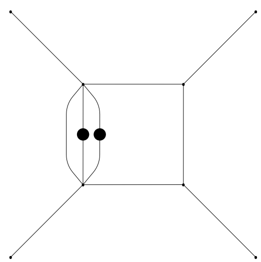

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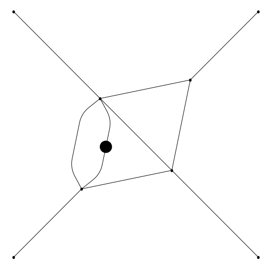

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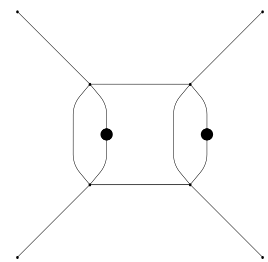
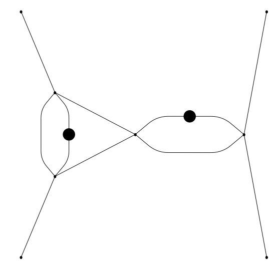


(9), (14)*



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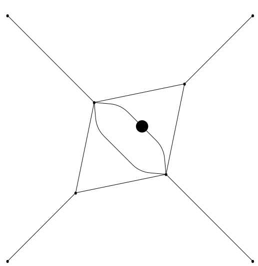

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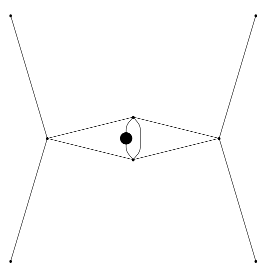

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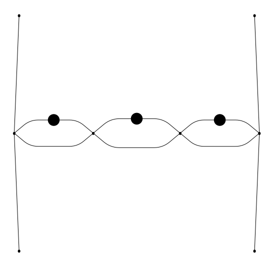

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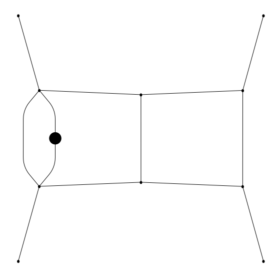

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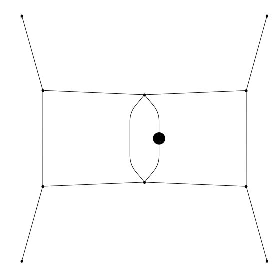

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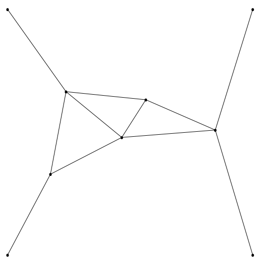

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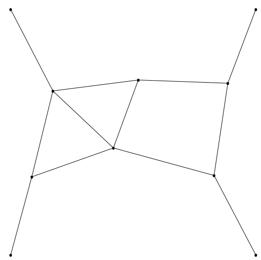

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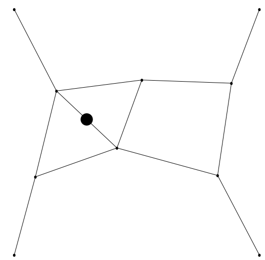

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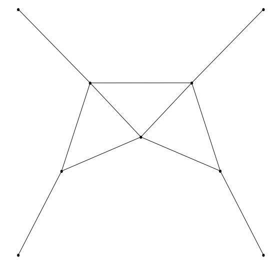

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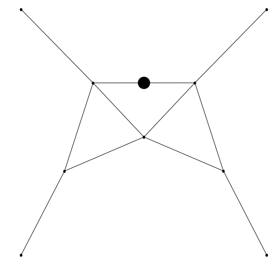

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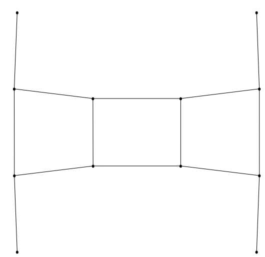
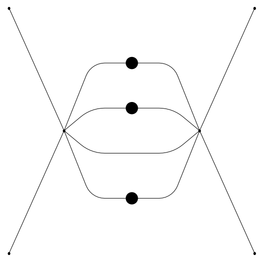
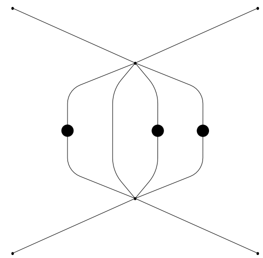


(24), (25)*,
(26)*



(1)



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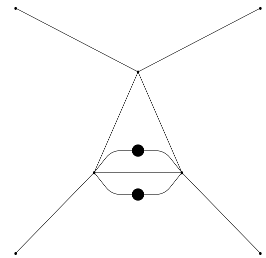

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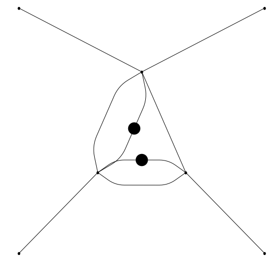

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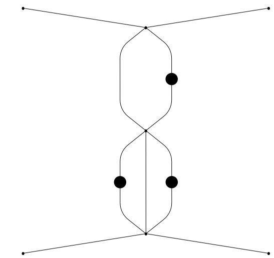

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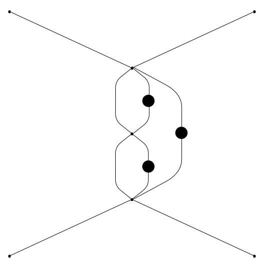

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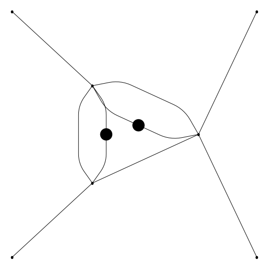

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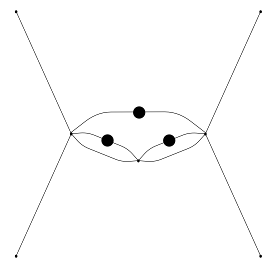

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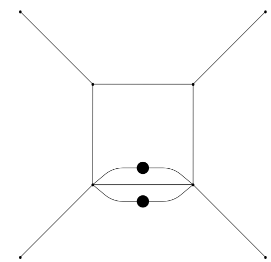

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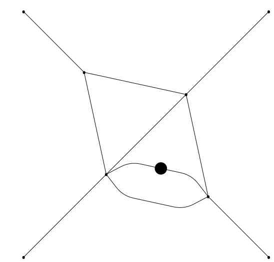

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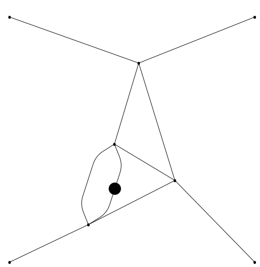

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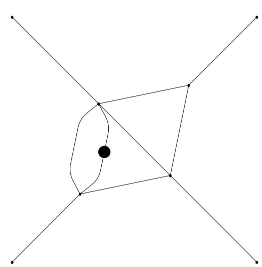

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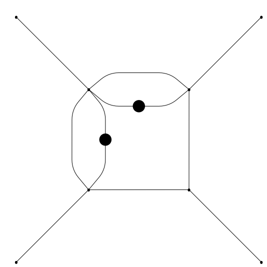

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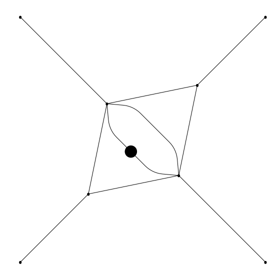

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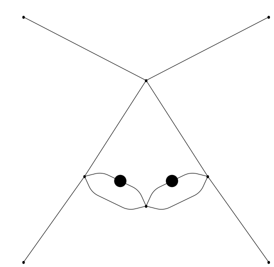

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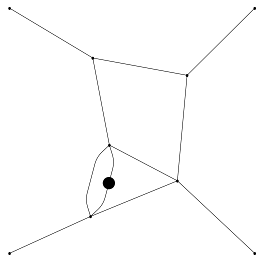
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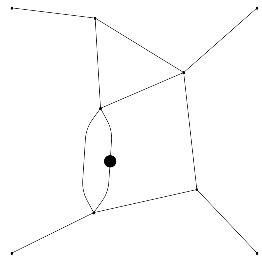
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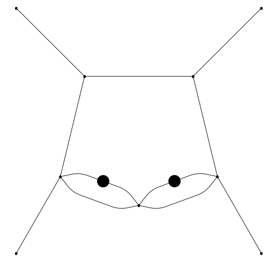
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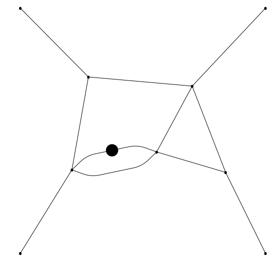
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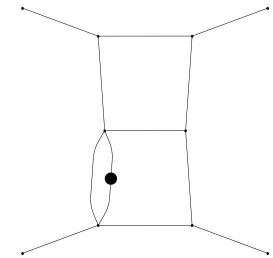
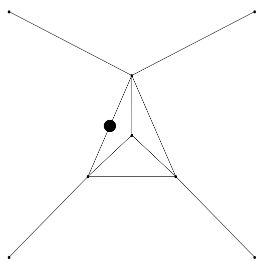

(29), (30)*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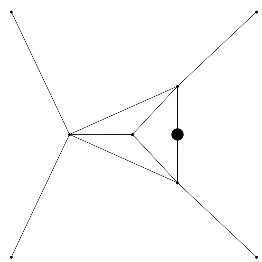
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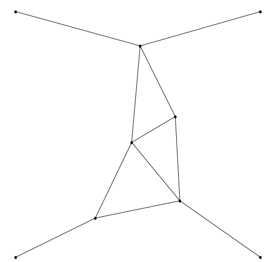
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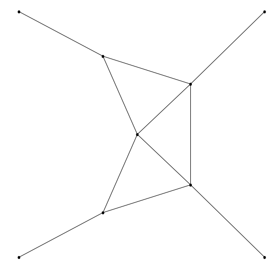
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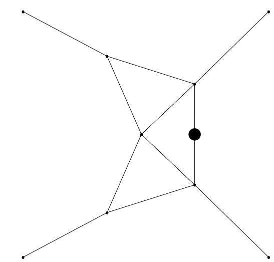
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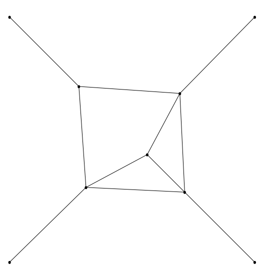
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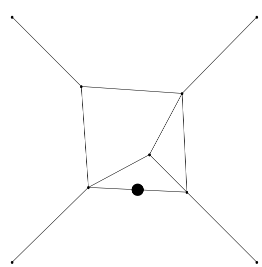
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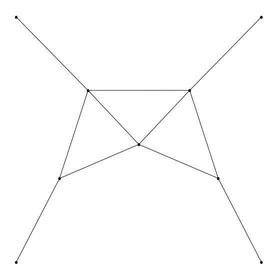
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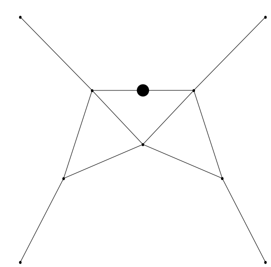
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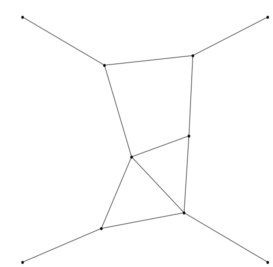
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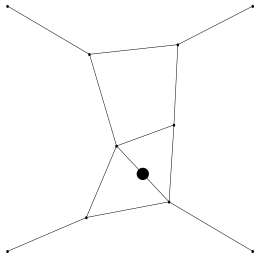

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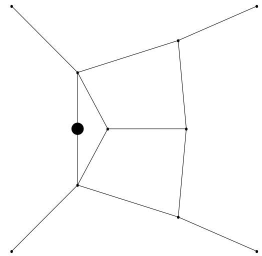

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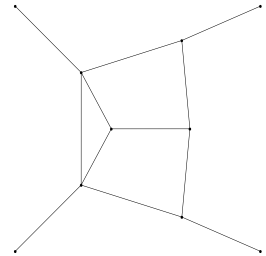

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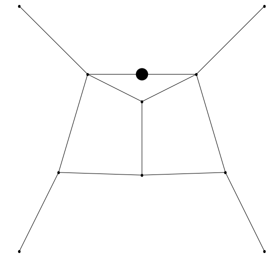

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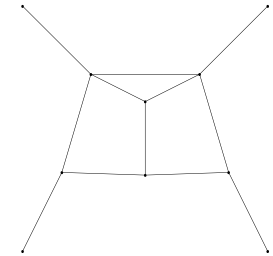

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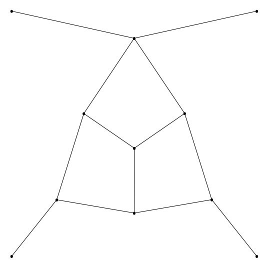

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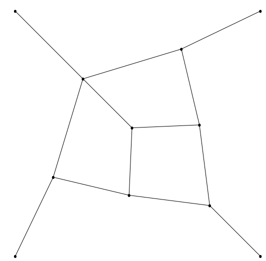

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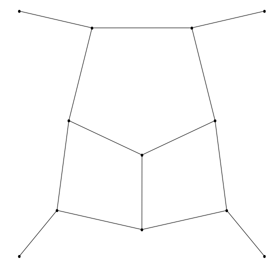

(39)*,

(40)*, (41)*

$$f_i^A = \epsilon^3 \, (-s)^{3\epsilon} \, \frac{e^{3\epsilon\gamma_{\rm E}}}{(i\pi^{D/2})^3} \; g_i^A \, .$$

The factor $(-s)^{3\epsilon}$ is to make the basis functions f_i^A dimensionless.
The factor ϵ^3 ensures that all basis functions admit a Taylor expansion around $\epsilon = 0$.

$$g_1^A = tF^A_{0,0,0,0,0,0,2,2,2,1,0,0,0,0,0,0}, \quad g_2^A = sF^A_{0,2,0,0,1,0,0,2,2,0,0,0,0,0,0},$$

$$g_3^A = \epsilon sF^A_{0,0,0,0,1,1,2,2,1,0,0,0,0,0,0}, \quad g_4^A = \epsilon sF^A_{0,0,0,1,2,0,2,1,1,0,0,0,0,0,0},$$

$$g_5^A = sF^A_{0,1,2,-1,0,1,0,2,2,0,0,0,0,0,0}, \quad g_6^A = s^2F^A_{0,2,2,0,2,1,0,1,0,0,0,0,0,0,0},$$

$$g_7^A = \epsilon stF^A_{0,0,0,0,1,1,2,2,1,1,0,0,0,0,0}, \quad g_8^A = \epsilon^2(s+t)F^A_{0,0,0,1,1,0,2,1,1,1,0,0,0,0,0},$$

$$g_9^A = \epsilon stF^A_{0,0,1,1,0,0,2,1,1,2,0,0,0,0,0}, \quad g_{10}^A = \epsilon s^2F^A_{0,0,1,1,2,1,2,1,0,0,0,0,0,0,0},$$

$$g_{11}^A = \epsilon^2(s+t)F^A_{0,1,0,0,1,0,1,1,2,1,0,0,0,0,0}, \quad g_{12}^A = -\epsilon(2\epsilon-1)sF^A_{1,1,0,0,1,1,0,2,1,0,0,0,0,0,0},$$

$$g_{13}^A = s^3F^A_{2,1,2,1,2,1,0,0,0,0,0,0,0,0,0}, \quad g_{14}^A = \epsilon sF^A_{0,0,1,1,0,0,2,1,1,2,0,0,-1,0,0},$$

$$g_{15}^A = \epsilon^3tF^A_{0,1,1,0,0,1,1,1,1,1,0,0,0,0,0}, \quad g_{16}^A = \epsilon^2s^2F^A_{0,1,2,0,0,1,1,1,1,1,0,0,0,0,0},$$

$$g_{17}^A = \epsilon^3sF^A_{0,1,1,0,1,1,1,1,1,0,0,0,0,0,0}, \quad g_{18}^A = \epsilon^2s^2F^A_{0,0,1,1,1,1,2,1,1,1,0,0,-1,0,0},$$

$$g_{19}^A = \epsilon^2s^2tF^A_{0,0,1,1,1,1,2,1,1,1,0,0,0,0,0}, \quad g_{20}^A = \epsilon^3s(s+t)F^A_{0,1,1,0,1,1,1,1,1,1,0,0,0,0,0},$$

$$g_{21}^A = \epsilon^2s^2tF^A_{0,1,1,0,1,1,1,2,1,1,0,0,0,0,0}, \quad g_{22}^A = \epsilon^2s^2tF^A_{1,1,0,0,1,1,1,2,1,1,0,0,0,0,0},$$

$$g_{23}^A = \epsilon^2s^2F^A_{1,1,0,0,1,1,1,2,1,1,-1,0,0,0,0}, \quad g_{24}^A = \epsilon^3s^3tF^A_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,0},$$

$$g_{25}^A = \epsilon^3s^3F^A_{1,1,1,1,1,1,1,1,1,1,-1,0,0,0,0}, \quad g_{26}^A = \epsilon^3s^3F^A_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,0}$$

With the variable $x = t/s$, the differential equations take the following form,

$$\partial_x f(x,\epsilon) = \epsilon \left(\frac{a}{x} + \frac{b}{1+x} \right) f(x,\epsilon)\,.$$

where a and b are $N \times N$ matrices with constant indices, with $N = 26$ and $N = 41$, respectively for cases A and E.

With the variable $x = t/s$, the differential equations take the following form,

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where a and b are $N \times N$ matrices with constant indices, with $N = 26$ and $N = 41$, respectively for cases A and E. A particular case of the Knizhnik-Zamolodchikov equation.

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where a and b are $N \times N$ matrices with constant indices, with $N = 26$ and $N = 41$, respectively for cases A and E.
A particular case of the Knizhnik-Zamolodchikov equation.

The matrices a and b for case A are on the next slide.

$$
\begin{pmatrix}
-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4}{3} & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{6} & 0 & 0 & -1 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & -8 & -\frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & -\frac{2}{3} & -\frac{40}{9} & 0 & 0 & -1 & -24 & -2 & \frac{4}{3} & 0 & 0 & 0 & -\frac{8}{3} & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 8 & 0 & 0 & \frac{3}{2} & 24 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{23}{27} & \frac{17}{54} & -\frac{1}{6} & -\frac{56}{9} & -\frac{14}{9} & \frac{1}{6} & -1 & -\frac{20}{3} & 0 & 0 & -2 & 0 & 0 & 0 & \frac{8}{3} & -2 & -\frac{2}{3} & 0 & 0 & 1 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
-\frac{4}{3} & -\frac{4}{3} & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\
-\frac{4}{3} & -\frac{4}{3} & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\
\frac{20}{9} & \frac{19}{9} & -2 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & -20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & -4 & 2 & 0 & -3 & 0 & 0 \\
-\frac{8}{3} & \frac{41}{18} & -\frac{7}{2} & \frac{68}{3} & \frac{14}{9} & \frac{7}{2} & 0 & 48 & 4 & 3 & -12 & 3 & 1 & 0 & 0 & 0 & -2 & 0 & -6 & 0 & 6 & -2 & 0 & 2 & 1 & 0 \\
-\frac{28}{9} & -\frac{7}{6} & \frac{9}{2} & \frac{20}{3} & \frac{22}{9} & \frac{3}{2} & 3 & 16 & 0 & 3 & 12 & 0 & 1 & 0 & -16 & 4 & 6 & 0 & -2 & -12 & 2 & -3 & -3 & 1 & 1 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{12} & 0 & 0 & -2 & -\frac{2}{3} & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & \frac{16}{3} & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{23}{27} & -\frac{17}{54} & \frac{1}{6} & \frac{56}{9} & \frac{14}{9} & -\frac{1}{6} & 1 & \frac{20}{3} & 0 & 0 & 2 & 0 & 0 & 0 & -\frac{8}{3} & 2 & \frac{2}{3} & 0 & 0 & -2 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
\frac{28}{9} & -\frac{1}{9} & -7 & -\frac{40}{3} & -4 & -2 & -3 & -16 & 0 & 0 & -12 & 0 & 0 & 0 & 16 & -4 & -4 & 0 & 0 & 12 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{3} & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{17}{9} & -7 & \frac{40}{3} & \frac{28}{9} & 7 & 0 & 0 & 0 & 6 & 0 & 6 & 2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\
-\frac{28}{9} & -\frac{16}{9} & 6 & \frac{32}{3} & \frac{8}{9} & -5 & 3 & 16 & 0 & -2 & 12 & -6 & -2 & 0 & -16 & 4 & 8 & -6 & 0 & -12 & -2 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & -6 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Three singularities, at $x = 0$, $x = -1$, and $x = \infty$ corresponding to the limits $s = 0$, $u = 0$, and $t = 0$, respectively.

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$$f_i^A(x, \epsilon) = \sum_{j=0}^{6} \epsilon^j f_i^{A,j}(x) + \mathcal{O}(\epsilon^7) \, .$$

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The iterative solution in ϵ for all functions f_i can be expressed in terms of harmonic polylogarithms of argument x and with indices drawn from $0,-1$, up to boundary constants.

For planar graphs we expect the limit $u \to 0$, i.e. $x \to -1$ to be finite.
The solution should be real for $x > 0$, i.e. when s and t have the same sign.

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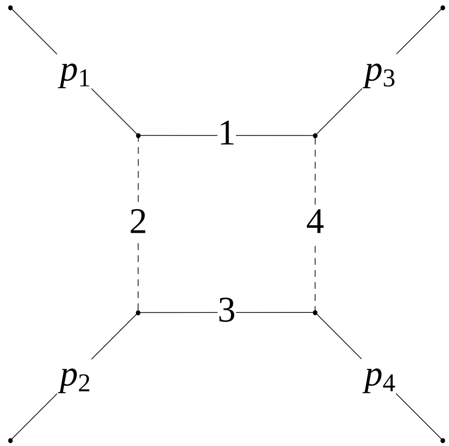
The solution should be real for $x > 0$, i.e. when s and t have the same sign.
These conditions fix almost everything:
the only additional information needed can easily be obtained from f_1:

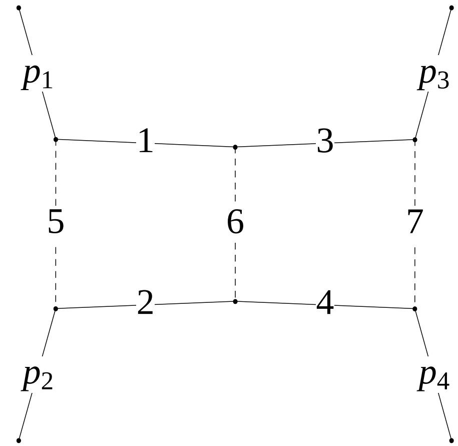
$$
\begin{aligned}
f_1^A &= e^{3\epsilon\gamma_{\rm E}}\Gamma^4(1-\epsilon)\Gamma(1+3\epsilon)/\Gamma(1-4\epsilon) \\
&= 1 - \epsilon^2\,\frac{\pi^2}{4} - 29\epsilon^3\,\zeta_3 - \epsilon^4\,\frac{71}{160}\pi^4 + \epsilon^5\,\left(\frac{29}{4}\pi^2\zeta_3 - \frac{1263}{5}\zeta_5\right) \\
&\quad + \epsilon^6\,\left(-\frac{11539}{24192}\pi^6 + \frac{841}{2}\zeta_3^2\right) + \mathcal{O}(\epsilon^7)\,.
\end{aligned}
$$

$$
\begin{aligned}
f_{26}^{A}(x,\epsilon) = & -\frac{4}{9} + \frac{13\pi^2\epsilon^2}{36} + \frac{1}{2}\epsilon H_{\{0\}}(x) \\
& + \epsilon^3\left(\frac{9}{4}\pi^2 H_{\{-1\}}(x) - \frac{15}{8}\pi^2 H_{\{0\}}(x) + \frac{9}{2}H_{\{-1,0,0\}}(x)\right. \\
& \left. - \frac{9}{2}H_{\{0,0,0\}}(x) - \frac{71\zeta_3}{18}\right) \\
& + \epsilon^4\left(\frac{61\pi^4}{720} + \frac{21}{4}\pi^2 H_{\{-1,-1\}}(x) - \frac{25}{4}\pi^2 H_{\{-1,0\}}(x)\right. \\
& - \frac{21}{4}\pi^2 H_{\{0,-1\}}(x) + \frac{25}{4}\pi^2 H_{\{0,0\}}(x) \\
& + \frac{21}{2}H_{\{-1,-1,0,0\}}(x) - 27H_{\{-1,0,0,0\}}(x) \\
& - \frac{21}{2}H_{\{0,-1,0,0\}}(x) + 27H_{\{0,0,0,0\}}(x) + \frac{21}{2}H_{\{-1\}}(x)\zeta_3 \\
& \left. - 2H_{\{0\}}(x)\zeta_3\right) + \ldots
\end{aligned}
$$

Two-loop four-point Feynman integrals for Bhabha scattering

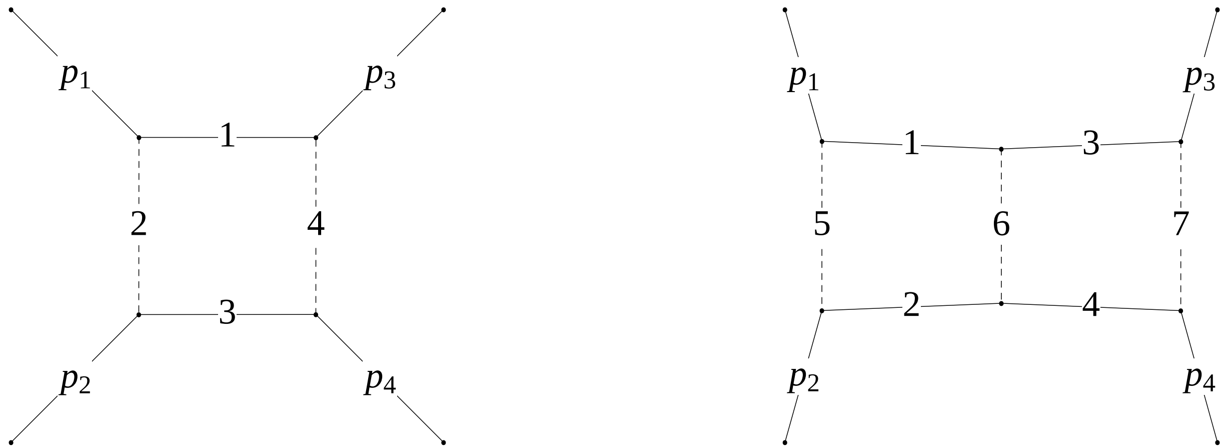


(1)



(2a)

Two-loop four-point Feynman integrals for Bhabha scattering



(1) (2a)

$$G_{a_1,\ldots,a_4}(s,t,m^2;D)$$

$$= \int \frac{\mathrm{d}^D k}{[-k^2+m^2]^{a_1}[-(k+p_1)^2]^{a_2}[-(k+p_1+p_2)^2+m^2]^{a_3}[-(k-p_3)^2]^{a_4}}\,,$$

$$G_{a_1,a_2,\ldots,a_9}(s,t,m^2;D) \quad = \quad \int\int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2}{(-k_1^2+m^2)^{a_1}[-(k_1+p_1+p_2)^2+m^2]^{a_2}}$$

$$\times \frac{[-(k_2+p_1)^2]^{-a_8}[-(k_1-p_3)^2]^{-a_9}}{[-k_2^2+m^2]^{a_3}[-(k_2+p_1+p_2)^2+m^2]^{a_4}[-(k_1+p_1)^2]^{a_5}[-(k_1-k_2)^2]^{a_6}[-(k_2-p_3)}$$

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Results for some of the master integrals for 2a

[VS’02, G. Heinrich & VS’04, M. Czakon, J. Gluza & T. Riemann’04–06]

$$G_{a_1,a_2,\ldots,a_9}(s,t,m^2;D) = \int\int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2}{(-k_1^2+m^2)^{a_1}[-(k_1+p_1+p_2)^2+m^2]^{a_2}}$$

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$$\frac{-s}{m^2} = \frac{(1-x)^2}{x}\,, \qquad \frac{-t}{m^2} = \frac{(1-y)^2}{y}\,.$$

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$$\frac{-s}{m^2} = \frac{(1-x)^2}{x}\,, \qquad \frac{-t}{m^2} = \frac{(1-y)^2}{y}\,.$$

Due to invariance under inversions of x and y, it is sufficient to consider $|x| < 1, |y| < 1$.

Singular points

$$x = 0 \leftrightarrow s = \infty \,, \qquad x = 1 \leftrightarrow s = 0 \qquad x = -1 \leftrightarrow s = 4m^2$$

A branch cut in the s-channel starting at $s = 4m^2$ and
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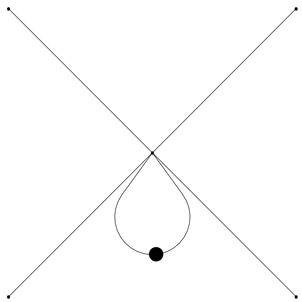
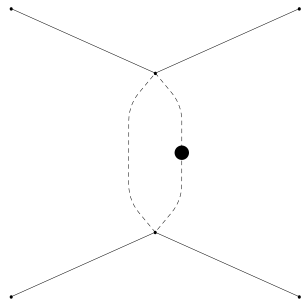
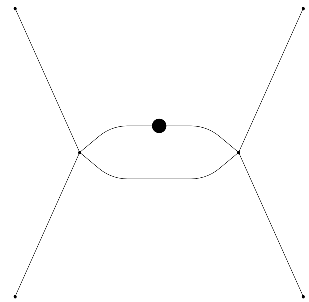
No branch cuts at $u = 0$, where $s + t + u = 4m^2$, and hence $x = -y$.
No singularity at $s = 0$
The analytic result should be real-valued in the $s < 0, t < 0$, i.e. $0 < x < 1, 0 < y < 1$.



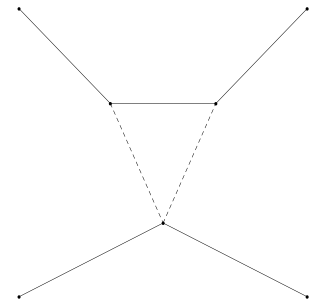
(b) (1)



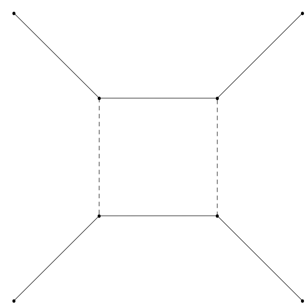
(b) (2)



(b) (3)



(b) (4)



(b) (5)

$$f_i = (m^2)^\epsilon \, e^{2\epsilon\gamma_{\rm E}} \, g_i$$

with

$$\begin{aligned}
g_1 =& \epsilon \, G_{2,0,0,0} \, , \\
g_2 =& \epsilon t \, G_{0,2,0,1} \, , \\
g_3 =& \epsilon \, \sqrt{(-s)(4m^2-s)} \, G_{2,0,1,0} \, , \\
g_4 =& -2\epsilon^2 \, (4m^2-t)(-t) \, G_{1,1,0,1} \, , \\
g_5 =& -2\epsilon^2 \, \sqrt{(-s)(4m^2-s)} t \, G_{1,1,1,1} \, .
\end{aligned}$$

The normalization is such that

$$f_i = \sum_{k \geq 0} \epsilon^k \, f_i^{(k)} \; .$$

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We obtain

$$\mathsf{d} \, f = \epsilon \, \mathsf{d} \tilde{A} \, f$$

with

$$\tilde{A} = \Bigg[\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{pmatrix} \log x + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & -2 \end{pmatrix} \log(1+x) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \end{pmatrix} \log y$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \log(1+y) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \log(1-y) +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 1 \end{pmatrix} \log(x+y) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2 & 1 \end{pmatrix} \log(1+xy) \Bigg] .$$

A solution in terms of Chen iterated integrals

$$f(x,y,\epsilon) = \mathbb{P}\, e^{\epsilon \int_C d\tilde{A}}\, g(\epsilon)\,,$$

which can be evaluated in terms of multiple polylogarithms. For example,

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which can be evaluated in terms of multiple polylogarithms. For example,

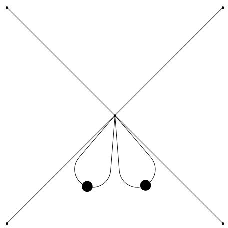
$$\begin{aligned}
f_5 =& \epsilon\Big[4H_0(x)\Big] + \epsilon^2\Big[4G_0(y)H_0(x) - 8G_1(y)H_0(x)\Big] \\
&+ \epsilon^3\Big[-8G_0(y)H_{-1,0}(x) + 4G_0(y)H_{0,0}(x) - 8H_0(x)G_{1,0}(y) + 16H_0(x)G_{1,1}(y) \\
&+ 4H_0(x)G_{-\frac{1}{x},0}(y) - 8H_0(x)G_{-\frac{1}{x},1}(y) + 4H_0(x)G_{-x,0}(y) - 8H_0(x)G_{-x,1}(y) \\
&+ 8H_{-1,0}(x)G_{-\frac{1}{x}}(y) + 8H_{-1,0}(x)G_{-x}(y) - 4H_{0,0}(x)G_{-\frac{1}{x}}(y) - 4H_{0,0}(x)G_{-x}(y) \\
&+ 4G_{-\frac{1}{x},0,0}(y) - 8G_{-\frac{1}{x},0,1}(y) - 4G_{-x,0,0}(y) + 8G_{-x,0,1}(y) + 8H_{-2,0}(x) \\
&- 16H_{-1,-1,0}(x) + 8H_{-1,0,0}(x) - 4H_{0,0,0}(x) + \frac{10}{3}\pi^2 G_{-\frac{1}{x}}(y) - 2\pi^2 G_{-x}(y) \\
&- \frac{2}{3}\pi^2 G_0(y) - \frac{4}{3}\pi^2 H_{-1}(x) - \frac{7}{3}\pi^2 H_0(x) + 8\zeta_3\Big] + \mathcal{O}(\epsilon^4)\,.
\end{aligned}$$

$$G(a_1, \ldots a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)\,,$$

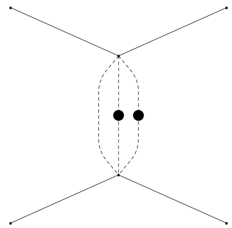
with

$$G(a_1; z) = \int_0^z \frac{dt}{t - a_1}\,, \qquad a_1 \neq 0\,.$$

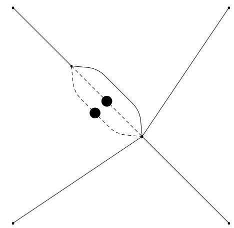
For $a_1 = 0$, we have $G(\vec{0}_n; x) = \frac{1}{n!} \log^n(x)$.



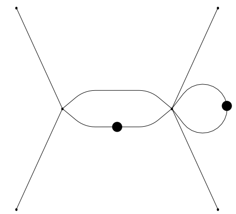
(1)



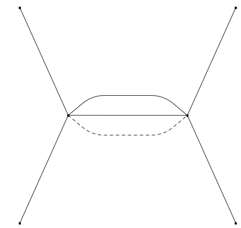
(2)



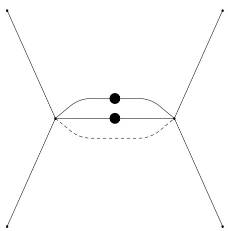
(3)



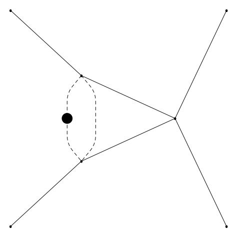
(4)



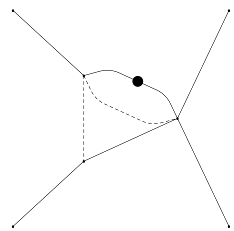
(5)†



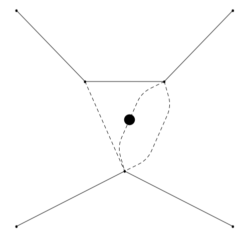
(6)



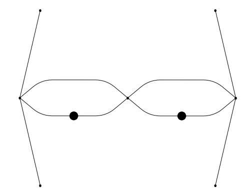
(7)



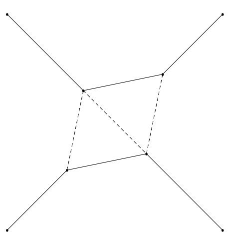
(8)



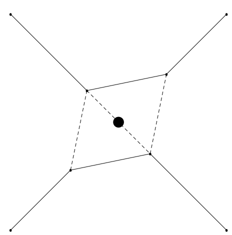
(9)



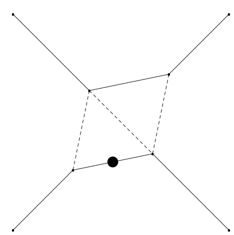
(10)



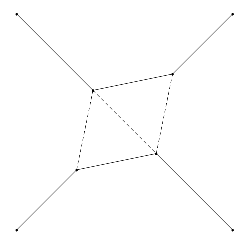
(11)



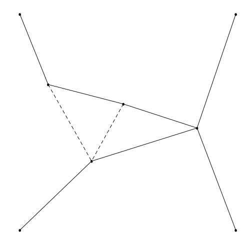
(12)



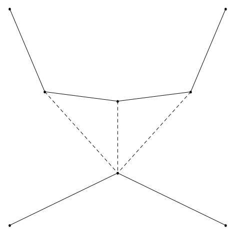
(13)



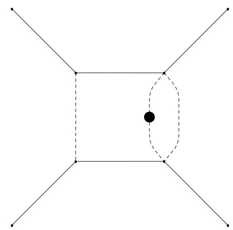
(14)†



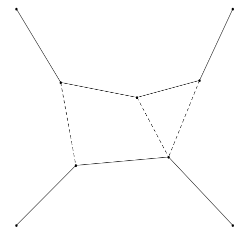
(15), (16)†



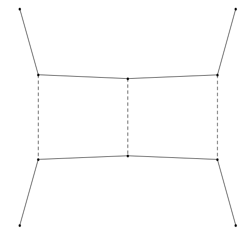
(17),(18)†



(19)



(20),(21)†



(22),(23)*

$$\mathrm{d}\, f = \epsilon\, \mathrm{d}\tilde{A}\, f$$

with

$$\mathsf{d}\, f = \epsilon\, \mathsf{d}\tilde{A}\, f$$

with

$$\begin{aligned}\tilde{A} =& B_1\ \log(x) + B_2\ \log(1+x) + B_3\ \log(1-x) + B_4\ \log(y) + B_5\ \log(1+y) \\ &+ B_6 \log(1-y)\ + B_7 \log(x+y) + B_8 \log(1+xy) \\ &+ B_9 \log(x+y-4xy+x^2y+xy^2) + B_{10} \log\left(\frac{1+Q}{1-Q}\right) \\ &+ B_{11} \log\left(\frac{(1+x)+(1-x)Q}{(1+x)-(1-x)Q}\right) + B_{12} \log\left(\frac{(1+y)+(1-y)Q}{(1+y)-(1-y)Q}\right)\end{aligned}$$

$$\mathrm{d}\, f = \epsilon\, \mathrm{d}\tilde{A}\, f$$

with

$$\begin{aligned}\tilde{A} =& B_1 \log(x) + B_2 \log(1+x) + B_3 \log(1-x) + B_4 \log(y) + B_5 \log(1+y) \\ &+ B_6 \log(1-y) + B_7 \log(x+y) + B_8 \log(1+xy) \\ &+ B_9 \log(x+y-4xy+x^2y+xy^2) + B_{10} \log\left(\frac{1+Q}{1-Q}\right) \\ &+ B_{11} \log\left(\frac{(1+x)+(1-x)Q}{(1+x)-(1-x)Q}\right) + B_{12} \log\left(\frac{(1+y)+(1-y)Q}{(1+y)-(1-y)Q}\right)\end{aligned}$$

$$Q = \sqrt{\frac{(x+y)(1+xy)}{x+y-4xy+x^2y+xy^2}}\,,$$

- At order ϵ, ϵ^2 and ϵ^3, the arguments of the logarithms in $\tilde{A}$ are the same as at one loop.
- At order ϵ^4 all basis functions except f_{11} have arguments as at one loop.

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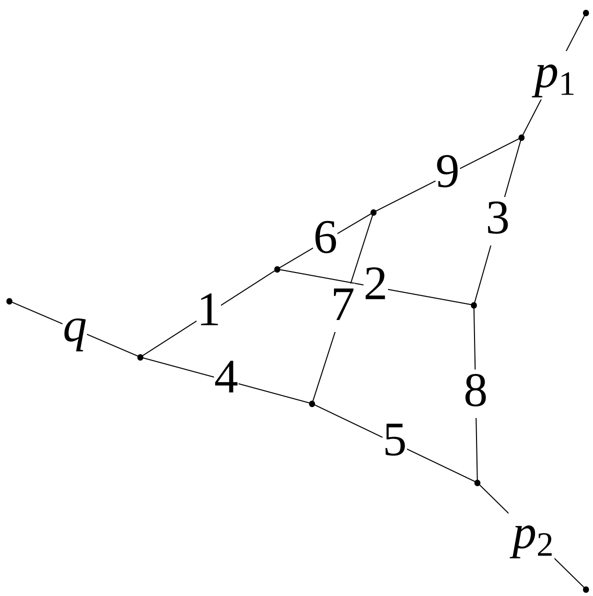
- At order ϵ^4 all basis functions except f_{11} have arguments as at one loop.

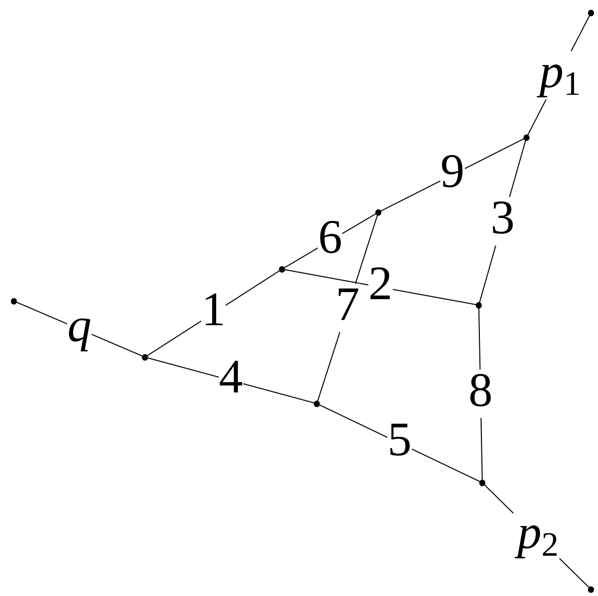
For example,

$$
\begin{aligned}
f_{23} =& \epsilon^2\Big[-12H_{0,0}(x)\Big] + \epsilon^3\Big[-16G_0(y)H_{0,0}(x) + 32G_1(y)H_{0,0}(x) + 8H_{2,0}(x) \\
&+ 16H_{-1,0,0}(x) - 4H_{0,0,0}(x) + \frac{4}{3}\pi^2 H_0(x) + 4\zeta_3\Big] + \epsilon^4\Big[32G_0(y)H_{-2,0}(x) \\
&- 32H_{-2,0}(x)G_{-\frac{1}{x}}(y) - 32H_{-2,0}(x)G_{-x}(y) + 64G_{1,0}(y)H_{0,0}(x) - 128G_{1,1}(y)H_{0,0}(x) \\
&- 32H_{0,0}(x)G_{-\frac{1}{x},0}(y) + 64H_{0,0}(x)G_{-\frac{1}{x},1}(y) - 32H_{0,0}(x)G_{-x,0}(y) \\
&+ 64H_{0,0}(x)G_{-x,1}(y) - 16H_0(x)G_{-\frac{1}{x},0,0}(y) + 32H_0(x)G_{-\frac{1}{x},0,1}(y) \\
&+ 16H_0(x)G_{-x,0,0}(y) - 32H_0(x)G_{-x,0,1}(y) + 64G_0(y)H_{-1,0,0}(x) \\
&- 64H_{-1,0,0}(x)G_{-\frac{1}{x}}(y) - 64H_{-1,0,0}(x)G_{-x}(y) \\
&- 48G_0(y)H_{0,0,0}(x) + 48H_{0,0,0}(x)G_{-\frac{1}{x}}(y) + 48H_{0,0,0}(x)G_{-x}(y) - 120H_{-3,0}(x) \\
&+ \frac{52}{3}\pi^2 H_{0,0}(x) + 48H_{3,0}(x) + 128H_{-2,-1,0}(x) - 120H_{-2,0,0}(x) - 48H_{-2,1,0}(x) \\
&+ 64H_{-1,-2,0}(x) - 32H_{-1,2,0}(x) - 48H_{2,-1,0}(x) + 32H_{2,0,0}(x) + 16H_{2,1,0}(x) \\
&+ 64H_{-1,-1,0,0}(x) - 80H_{-1,0,0,0}(x) + 76H_{0,0,0,0}(x) + \frac{8}{3}\pi^2 G_0(y)H_0(x) \\
&- \frac{40}{3}\pi^2 H_0(x)G_{-\frac{1}{x}}(y) + 8\pi^2 H_0(x)G_{-x}(y) - 16\zeta_3 H_{-1}(x) - 28\zeta_3 H_0(x) \\
&+ \frac{8}{3}\pi^2 H_{-2}(x) - \frac{4}{3}\pi^2 H_2(x) - \frac{4\pi^4}{15}\Big] + \mathcal{O}(\epsilon^5)
\end{aligned}
$$

Evaluating single-scale diagrams by DE



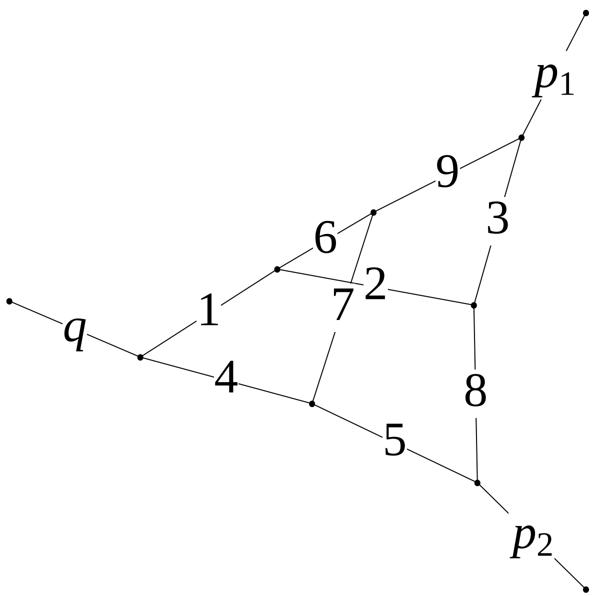
Evaluating single-scale diagrams by DE



A three-loop form-factor integral called A_{92}.

[J.M. Henn, A. Smirnov & V.S.'13]

Evaluating single-scale diagrams by DE

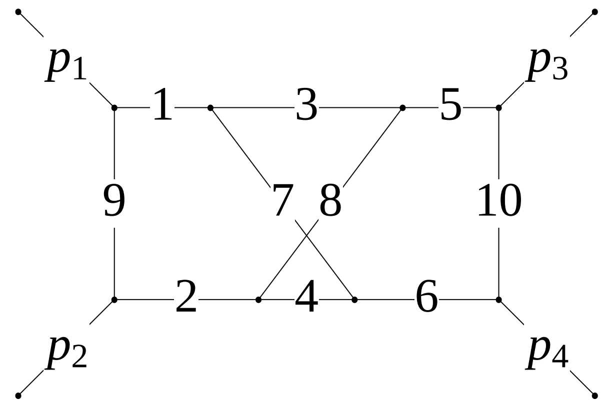


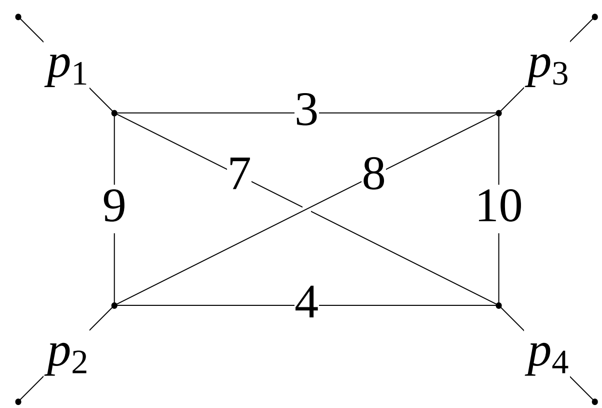
A three-loop form-factor integral called A_{92}.

[J.M. Henn, A. Smirnov & V.S.'13]

$p_2^2 = 0 \to p_2^2 \neq 0$, with $x = p_2^2/q^2$.

K_4





A straightforward strategy to evaluate Feynman integrals: integrate consecutively over Feynman parameters

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Results obtained within this scenario

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Computer codes to perform such parametric integration

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An attempt to evaluate K_4 with all the external momenta on the light cone

[C. Bogner & M. Lüders'13]

$$
F^C_{a_1,\ldots,a_{15}}(s,t;D) \quad = \quad \frac{1}{(i\pi^{D/2})^3} \int\int\int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3}{(-k_1^2)^{a_1}[-(p_1+p_2+k_1)^2]^{a_2}[-(k_1+k_3)^2]^{a_3}}
$$

$$
\times \frac{[-(k_1+k_2)^2]^{-a_{11}}[-(p_1+k_3)^2]^{-a_{12}}[-(p_1+k_2)^2]^{-a_{13}}}{[-(p_1+p_2+k_1+k_2)^2]^{a_4}[-(k_1+k_2+k_3)^2]^{a_5}[-(p_1+p_2+k_1+k_2+k_3)^2]^{a_6}}
$$

$$
\times \frac{[-(p_3+k_1)^2]^{-a_{14}}[-(p_3+k_3)^2]^{-a_{15}}}{(-k_3^2)^{a_7}(-k_2^2)^{a_8}[-(p_1+k_1)^2]^{a_9}[-(k_1+k_2+k_3-p_3)^2]^{a_{10}}} \ .
$$

$$F^C_{a_1,\ldots,a_{15}}(s,t;D) \quad = \quad \frac{1}{(i\pi^{D/2})^3} \int\int\int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3}{(-k_1^2)^{a_1}[-(p_1+p_2+k_1)^2]^{a_2}[-(k_1+k_3)^2]^{a_3}}$$

$$\times \frac{[-(k_1+k_2)^2]^{-a_{11}}[-(p_1+k_3)^2]^{-a_{12}}[-(p_1+k_2)^2]^{-a_{13}}}{[-(p_1+p_2+k_1+k_2)^2]^{a_4}[-(k_1+k_2+k_3)^2]^{a_5}[-(p_1+p_2+k_1+k_2+k_3)^2]^{a_6}}$$

$$\times \frac{[-(p_3+k_1)^2]^{-a_{14}}[-(p_3+k_3)^2]^{-a_{15}}}{(-k_3^2)^{a_7}(-k_2^2)^{a_8}[-(p_1+k_1)^2]^{a_9}[-(k_1+k_2+k_3-p_3)^2]^{a_{10}}} \; .$$

$$K_{a_1,a_2,\ldots,a_6} = F^C_{0,0,a_1,a_2,0,0,a_3,a_4,a_5,a_6,0,\ldots,0} \; .$$

$$\hat{K}_{a_1,a_2,\ldots,a_6,a'} = F^C_{0,a',a_1,a_2,0,0,a_3,a_4,a_5,a_6,0,\ldots,0} \; ,$$

where $a' \leq 0$.

We choose a UT basis

$$f = e^{3\epsilon\gamma_E}(-s)^{-3\epsilon} g \; = (f_1, f_2, \ldots, f_{10})$$

$$g_1 = \epsilon^3 t K_{0,0,1,2,2,2}, \qquad g_2 = \epsilon^3 (s+t) K_{1,2,0,0,2,2},$$

$$g_3 = \epsilon^3 s K_{1,2,2,2,0,0}, \qquad g_4 = 2\epsilon^4 (s+t) \hat{K}_{1,2,1,1,2,1,-1} + 2\epsilon^5 s K_{2,1,1,1,1,1},$$

$$g_5 = 4\epsilon^5 t K_{2,1,1,1,1,1}, \qquad g_6 = 4\epsilon^5 (s+t) K_{1,1,2,1,1,1},$$

$$g_7 = 4\epsilon^5 s K_{1,1,1,1,2,1}, \qquad g_8 = -2\epsilon^4 s (s+t) K_{2,2,1,1,1,1},$$

$$g_9 = -\, 2\epsilon^4 s t K_{1,1,2,2,1,1}, \qquad g_{10} = -2\epsilon^4 (s+t) t K_{1,1,1,1,2,2},$$

We choose a UT basis

$$f = e^{3\epsilon\gamma_E}(-s)^{-3\epsilon} g \; = (f_1, f_2, \ldots, f_{10})$$

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$$g_3 = \epsilon^3 s K_{1,2,2,2,0,0}, \qquad g_4 = 2\epsilon^4 (s+t) \hat{K}_{1,2,1,1,2,1,-1} + 2\epsilon^5 s K_{2,1,1,1,1,1},$$

$$g_5 = 4\epsilon^5 t K_{2,1,1,1,1,1}, \qquad g_6 = 4\epsilon^5 (s+t) K_{1,1,2,1,1,1},$$

$$g_7 = 4\epsilon^5 s K_{1,1,1,1,2,1}, \qquad g_8 = -2\epsilon^4 s(s+t) K_{2,2,1,1,1,1},$$

$$g_9 = -2\epsilon^4 st K_{1,1,2,2,1,1}, \qquad g_{10} = -2\epsilon^4 (s+t) t K_{1,1,1,1,2,2},$$

DE

$$\partial_x f(x,\epsilon) = \epsilon \left[\frac{A}{x} + \frac{B}{1+x} \right] f(x,\epsilon) \, .$$

$$
A = \begin{pmatrix}
-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{3} & \frac{2}{3} & -\frac{1}{6} & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{7}{6} & \frac{1}{12} & -\frac{1}{12} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & 4 & 5 & -3 & -3 & -\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{3} & \frac{5}{3} & \frac{1}{3} & 4 & \frac{7}{3} & -\frac{7}{3} & -\frac{11}{3} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\
-\frac{4}{3} & \frac{10}{3} & -\frac{10}{3} & 0 & \frac{20}{3} & \frac{10}{3} & -\frac{10}{3} & \frac{5}{3} & -\frac{2}{3} & \frac{2}{3} \\
-\frac{14}{3} & \frac{8}{3} & \frac{4}{3} & 8 & \frac{22}{3} & -\frac{16}{3} & -\frac{20}{3} & -\frac{2}{3} & -\frac{7}{3} & \frac{4}{3} \\
\frac{10}{3} & \frac{8}{3} & \frac{4}{3} & 8 & \frac{22}{3} & -\frac{16}{3} & -\frac{20}{3} & -\frac{2}{3} & \frac{2}{3} & -\frac{5}{3}
\end{pmatrix},
$$

$$
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & -5 & -\frac{7}{3} & \frac{5}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{12} & -\frac{1}{12} \\
-\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 4 & \frac{5}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{1}{3} & -\frac{5}{3} & -\frac{1}{3} & -4 & -\frac{7}{3} & \frac{7}{3} & -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \\
0 & -2 & 0 & 8 & 4 & -2 & 0 & -3 & 0 & 0 \\
\frac{10}{3} & -\frac{4}{3} & \frac{10}{3} & 0 & \frac{10}{3} & \frac{20}{3} & \frac{10}{3} & -\frac{2}{3} & \frac{5}{3} & -\frac{2}{3} \\
0 & -6 & 0 & -8 & -4 & 2 & 0 & 0 & 0 & -3
\end{pmatrix}.
$$

The first three integrals can be expressed in terms of gamma functions.

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Using the limit $x \to 0$ to fix boundary conditions.

An interplay with expansion by regions which gives contributions with $x^{0\epsilon}$ (h-h-h) and $x^{-3\epsilon}$ (c-c-c).

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Using the limit $x \to 0$ to fix boundary conditions.

An interplay with expansion by regions which gives contributions with $x^{0\epsilon}$ (h-h-h) and $x^{-3\epsilon}$ (c-c-c).

Terms with $x^{k\epsilon}$ at $k > 0$ are absent!

It was possible to evaluate the LO (c-c-c) terms analytically. For example, for $K_{2,2,1,1,1,1}$:

$$x^{-3\epsilon}\left[-\frac{421}{5}\zeta_5\log(x)+\frac{29}{12}\pi^2\zeta_3\log(x)-\frac{421i\pi\zeta_5}{10}+\frac{5597\zeta(3)^2}{36}\right.$$
$$\left.+\frac{29}{24}i\pi^3\zeta_3+\frac{31601\pi^6}{2177280}+O(x)\right] \; .$$

An example of result

$$K^{(0)}(x,\epsilon) = e^{3\epsilon\gamma_E}(-s)^{-3\epsilon}(1-4\epsilon)(1-5\epsilon)\epsilon^4 K_{1,1,1,1,1,1}(x,\epsilon)\,,$$

An example of result

$$K^{(0)}(x,\epsilon) = e^{3\epsilon\gamma_E}(-s)^{-3\epsilon}(1-4\epsilon)(1-5\epsilon)\epsilon^4 K_{1,1,1,1,1,1}(x,\epsilon)\,,$$

$$\begin{aligned}
K^{(0)}(x,\epsilon) = \quad & 2\zeta_3\epsilon^3 \\
& +\epsilon^4\left[3i\pi\zeta_3 + \frac{3\pi^4}{20} + 2i\pi H_{-3}(x) + \frac{1}{2}\pi^2 H_{-2}(x) - \frac{1}{2}i\pi^3 H_{-1}(x) - 3H_{-1}(x)\zeta_3\right. \\
& -2H_{-3,-1}(x) + H_{-2,-2}(x) - i\pi H_{-2,0}(x) + H_{-1,-3}(x) - \pi^2 H_{-1,-1}(x) \\
& \left.+\frac{1}{2}\pi^2 H_{-1,0}(x) + H_{-2,-1,0}(x) + H_{-1,-2,0}(x) - i\pi H_{-1,0,0}(x) - 2H_{-1,-1,0,0}(x)\right] \\
& +\mathcal{O}(\epsilon^5)\,.
\end{aligned}$$

Massless four-point integrals with two off-shell legs

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NNLO QCD corrections to the production of two off-shell vector bosons in hadron collisions.

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Planar diagrams

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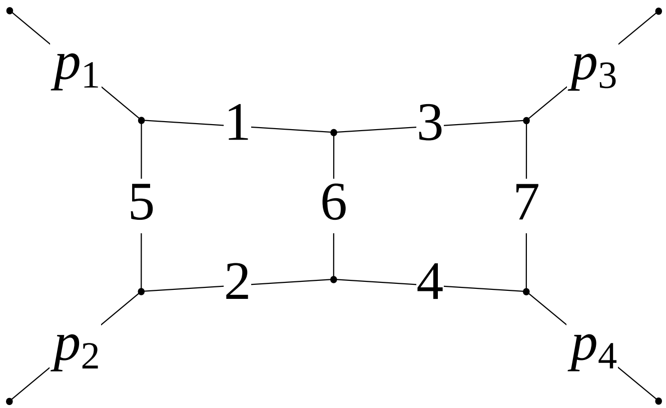
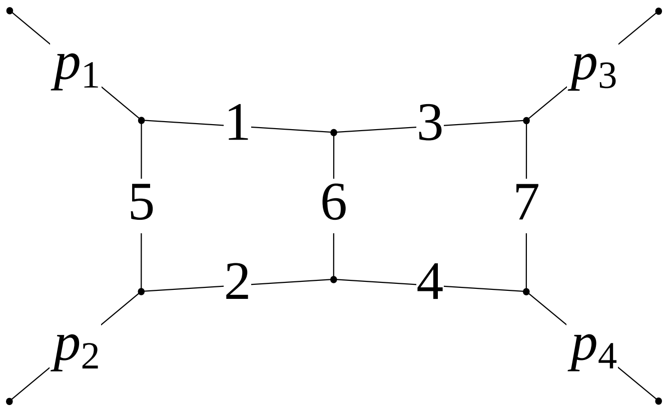


Massless four-point integrals with two off-shell legs

NNLO QCD corrections to the production of two off-shell vector bosons in hadron collisions.
Planar diagrams

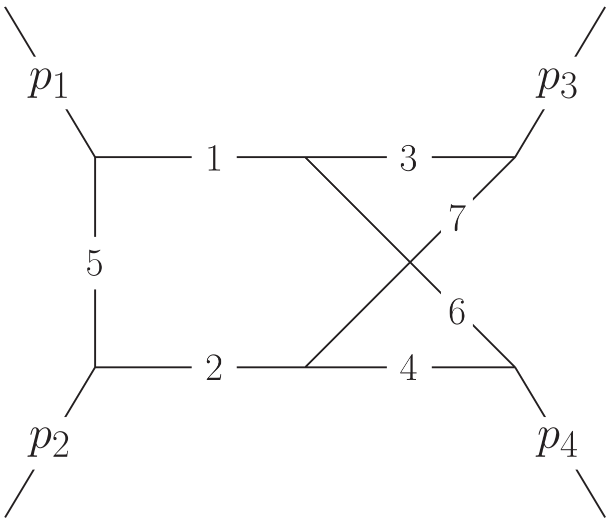


- P_{12}: $p_1 = -q_3,\ p_2 = -q_4,\ p_3 = q_1,\ p_4 = q_2$;
- P_{13}: $p_1 = -q_3,\ p_2 = q_1,\ p_3 = -q_4,\ p_4 = q_2$;
- P_{23}: $p_1 = q_2,\ p_2 = -q_4,\ p_3 = -q_3,\ p_4 = q_1$.

where $q_1^2 = 0, q_2^2 = 0$ and $q_3^2 = M_3^2$, $q_4^2 = M_4^2$

Nonplanar diagrams

Nonplanar diagrams



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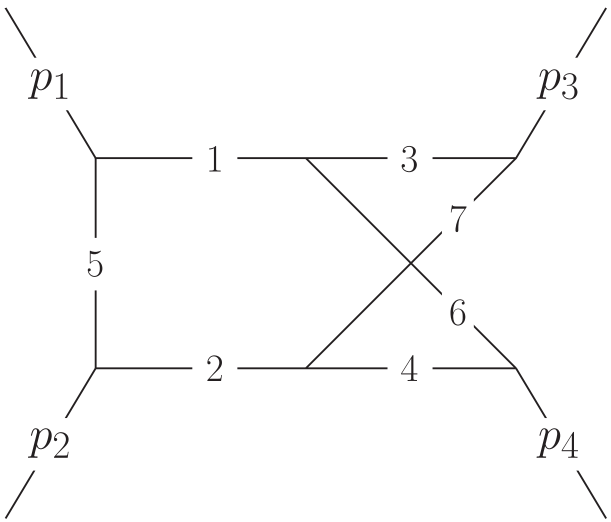

- N_{12}: $p_1 = -q_4,\ p_2 = -q_3,\ p_3 = q_2,\ p_4 = q_1$;
- N_{13}: $p_1 = -q_4,\ p_2 = q_2,\ p_3 = -q_3,\ p_4 = q_1$;
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[T. Gehrmann, L. Tancredi & E. Weihs’13]

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Get rid of a square root:

$$\frac{S}{M_3^2} = (1+x)(1+xy), \quad \frac{T}{M_3^2} = -xz, \quad \frac{M_4^2}{M_3^2} = x^2 y.$$

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In terms of x, y, z, the physical region is

$$x > 0, \quad y > 0, \quad y < z < 1.$$

$$
G_{a_1,\ldots,a_9} = \int\int \frac{\mathrm{d}^D k_1 \, \mathrm{d}^D k_2}{[-k_1^2]^{a_1}[-(k_1+p_1+p_2)^2]^{a_2}[-k_2^2]^{a_3}[-(k_2+p_1+p_2)^2]^{a_4}}
$$

$$
\times \frac{[-(k_2+p_1)^2]^{-a_8}[-(k_1-p_3)^2]^{-a_9}}{[-(k_1+p_1)^2]^{a_5}[-(k_1-k_2)^2]^{a_6}[-(k_2-p_3)^2]^{a_7}}
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DE

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where $\xi = x, y$ or z.

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$$\tilde{A} = \sum_{i=1}^{15} \tilde{A}_{\alpha_i}\, \log(\alpha_i)\,,$$

where the $\tilde{A}_{\alpha_i}$ are *constant* matrices.

In the *planar* case,
the arguments of the logarithms α_i (*letters*) are

$$\alpha = \{x, y, z, 1+x, 1-y, 1-z, 1+xy, z-y,\\ 1+y(1+x)-z, xy+z, 1+x(1+y-z), 1+xz, 1+y-z,\\ z+x(z-y)+xyz, z-y+yz+xyz\}$$

with only a linear dependence on x, y, z.

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Solving DE iteratively order-by-order in ϵ

$$f = \sum_{n=0}^{4} f^{(n)} \epsilon^n + \mathcal{O}(\epsilon^5).$$

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Integrate first in x, then in y, then in z.

We have decided to obtain results directly in the physical region $x > 0, y > 0, y < z < 1$
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For P_{23} (and for all non-planar families) the limit $y, z \to 1$ is singular, with a typical behaviour

$$f \sim f_a x^{-n_1 \epsilon} + f_b x^{-n_2 \epsilon} \left[(z - y)(1 - z)\right]^{-n_3 \epsilon}$$

To evaluate the LO asymptotics in the limit $x \to 0$, $z \to 1$, $y \to 1$ we applied expansion by regions

[M. Beneke & V.S.'98]

implemented in the open computer code `asy.m`

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Obtaining the boundary conditions also from the consistency of DE.

Because of a linear dependence of the letters on x, y, z, results can be expressed in terms of multiple (Goncharov) polylogarithms

$$G(a_1, \dots a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t) \,,$$

with

$$G(a_1; z) = \int_0^z \frac{dt}{t - a_1} \,, \qquad a_1 \neq 0 \,.$$

For $a_1 = 0$, we have $G(\vec{0}_n; x) = \frac{1}{n!} \log^n(x)$.

$$g_{28}^{\mathrm{P23}} = \epsilon^2\Big(2\epsilon p_2^2(p_3^2-s)G_{1,0,0,1,1,2,1,0,0} - 4\epsilon p_2^2(p_3^2-s)G_{1,1,0,0,1,1,2,0,0}$$
$$+ \quad 4\epsilon^2 s(-p_3^2+s)G_{1,1,1,1,1,1,1,-1,0}\Big) =$$

```
-(ep^2*(11*Pi^2 - (18*I)*Pi*G[0, z] +
       G[0, y]*((6*I)*Pi + 6*G[0, z]) - (6*I)*Pi*G[1, z] -
       6*G[0, y]*G[-((1 + y - z)/y), x] + 6*G[1, z]*G[-((1 + y - z)/y), x] +
       G[0, x]*((-12*I)*Pi + 6*G[0, y] - 6*G[0, z] - 6*G[1, z] -
         6*G[-1 + z, y]) + ((6*I)*Pi + 6*G[1, z])*G[-1 + z, y] +
       G[-((1 + y - z)/y), x]*((6*I)*Pi + 6*G[-1 + z, y]) +
       (6*I)*Pi*G[-z^(-1), x] + 6*G[0, z]*G[-z^(-1), x] -
       (12*I)*Pi*G[z, y] - 6*G[0, z]*G[z, y] - 6*G[1, z]*G[z, y] -
       12*G[0, 0, z] - 6*G[0, 1, z] - 6*G[1, 0, z] -
       6*G[-((1 + y - z)/y), 0, x] - 6*G[-1 + z, 0, y] +
       6*G[-1 + z, -1 + z, y] + 6*G[-z^(-1), 0, x] + 6*G[z, 0, y] -
       6*G[z, -1 + z, y])) - ep^3*(((16*I)/3)*Pi^3 + (Pi^2*G[-1, x])/3 +
      (Pi^2*G[0, x])/3 - (19*Pi^2*G[0, z])/3 - (31*Pi^2*G[1, z])/3 +
      (Pi^2*G[-y^(-1), x])/3 - (Pi^2*G[z, y])/3 - (24*I)*Pi*G[-1, 0, x] +
      (12*I)*Pi*G[-1, -(1 + y - z)^(-1), x] +
      (12*I)*Pi*G[-1, -((1 + y - z)/y), x] + (12*I)*Pi*G[-1, -z^(-1), x] +
      (12*I)*Pi*G[-1, -(z/y), x] + (80*I)*Pi*G[0, 0, x] -
      (6*I)*Pi*G[0, 0, y] + G[-(z/y), x]*(8*Pi^2 - 16*G[0, 0, z]) +
      (34*I)*Pi*G[0, 0, z] + (24*I)*Pi*G[0, 1, z] -
      (24*I)*Pi*G[0, -(1 + y - z)^(-1), x]  +...
```

Checks:

- Comparison with the equal mass case, $M_3^2 = M_4^2 = M^2$

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- Using `FIESTA3` [A. Smirnov'13]
- Using analytic continuation over contour in the complex plane starting from a point in an unphysical region where the boundary conditions are simple.

For the *nonplanar* families N_{12} and N_{13} we choose the same parametrization as in the planar case

$$S = M^2(1+x)(1+xy), \quad T = -M^2 xz, \quad M_3^2 = M^2, \quad M_4^2 = M^2 x^2 y$$

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The physical region is

$$x < 1/y, \quad 0 < y < 1, \quad 0 < z < 1.$$

For N_{12} and N_{13}, we have the following letters

$$\{x, 1+x, 1-y, y, 1+xy, 1+x(1+y-z), 1-z, y-z, 1+y-z, \\ 1+y+xy-z, z, -y+z, xy+z, 1+x+xy-xz, 1+xz, \\ 1+y+2xy-z+x^2yz, z-y(1-z-xz)\}$$

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$$\{x, 1+x, 1-y, y, 1+y, 1-xy, 1+xy, 1-y(1-2z), 1+y-2yz,$$
$$1-xy^2-y(1-x-2z+2xz), 1-xy(1-2z), 1+x(y-2yz),$$
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$$(1+y)(1+xy)-2zy(1+x), 1-y+2yz,$$
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In the physical region, all the letters are sign-definite. All iterated integrals needed for calculating the vector of the master integrals can be written in a manifestly real form, so that imaginary parts appear only through explicit factors of i coming from the boundary conditions.

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