Kinematic Algebras in Scattering Amplitudes

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Based on arXiv:1311.1151, with D. O'Connell
and work in progress with V. P. Nair and D. O'Connell

Outline

- Review of BCJ colour-kinematics duality
- Kinematic algebras from the scattering equations
- Kinematic algebra and geometry of Lie groups

Colour and Kinematics in Gauge Theory

How to decompose kinematics (k_i, ϵ_i) and colour (a_i)?
Tree-level:

$$\boxed{\mathcal{A}_n = \sum_{\text{non cyclic}} A(1, 2, \ldots, n)\, \mathrm{tr}(T^{a_1} T^{a_2} \cdots T^{a_n})}$$

- Lie algebra generators - T^{a_i}
- partial or colour-ordered amplitudes - $A(k_i, \epsilon_i)$

$$\boxed{\mathcal{A}_n = \sum_{\alpha \in \text{cubic}} \frac{n_\alpha c_\alpha}{D_\alpha}}$$

- colour factors - $c_\alpha = f^{\cdots} f^{\cdots} \cdots f^{\cdots} \qquad f^{abc} = \mathrm{tr}([T^a, T^b] T^c)$
- propagators - $D_\alpha(k_i)$
- kinematic numerators - $n_\alpha(k_i, \epsilon_i)$

Ambiguity: Jacobi identity $\quad c_\alpha \pm c_\beta \pm c_\gamma = 0$

BCJ Colour-Kinematics Duality

Bern, Carrasco, Johansson '08

Statement: it is possible to write gauge theory amplitudes as

$$\mathcal{A}_n = \sum_{\alpha \in \text{cubic}} \frac{n_\alpha c_\alpha}{D_\alpha}$$

such that numerators $n_\alpha(k_i, \epsilon_i)$ have symmetries of colour factors $c_\alpha(a_i)$

$$c_\alpha \pm c_\beta \pm c_\gamma = 0 \qquad \longleftrightarrow \qquad n_\alpha \pm n_\beta \pm n_\gamma = 0$$

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- $\Rightarrow$ **BCJ relations** for partial amplitudes A: $(n-3)!$ are independent.
  [Bjerrum-Bohr, Damgaard, Vanhove 09] [Stieberger 09] [Feng, Huang, Jia 10] [Cachazo 12]
- $\Leftarrow$ BCJ relations. BCJ numerators non-unique. [Bj-Bohr, Damgaard, Sondergaard, Vanhove 10] [Bjerrum-Bohr, Damgaard, RM, O'Connell 12] [Boels, Isermann 12] [Fu, Du, Feng 12]
- String theory insights: colour vs kinematics in heterotic [Tye, Zhang 10]
  pure spinor numerators [Mafra, Schlotterer, Stieberger 11]
- Loop level: conjecture for integrand. [BCJ 10]
- Kinematic algebra for numerators?

BCJ Double Copy to Gravity

Bern, Carrasco, Johansson '08

Statement: gravity amplitudes are obtained from gauge theory as

$$\mathcal{M}_n = \sum_{\alpha \in \text{cubic}} \frac{n_\alpha \tilde{n}_\alpha}{D_\alpha}$$

if $n_\alpha(k_i, \epsilon_i)$ are BCJ numerators. $(\epsilon_{\mu\nu} = \epsilon_\mu \tilde{\epsilon}_\nu)$

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Tree-level: proven, [Bern, Dennen, Huang, Kiermaier 10] [Cachazo, He, Yuan 13]

equivalent to **KLT relations** [Kawai, Lewellen, Tye 86] [Bern, Dixon,Perelstein,Rozowsky 98] [Bjerrum-Bohr, Damgaard, Feng, Sondergaard 10]

$$\mathcal{M}_n = \sum_{\substack{P,P' \in S_{n-3} \\ \text{3 fixed legs}}} A(P)\; S^{\text{KLT}}_{(n-3)}[P, P']\; \tilde{A}(P')$$

Loop level: conjecture for integrand. [BCJ 10]

studies of supergravity – see talks by Bern, Johansson, Roiban

BCJ Summary

Trace based

$$\mathcal{A}_n = \sum_{\text{non cyclic}} A(1,\ldots,n)\ \mathrm{tr}(T^{a_1}\cdots T^{a_n})$$

f^{abc} based

$$\mathcal{A}_n = \sum_{\alpha\in\text{cubic}} \frac{n_\alpha c_\alpha}{D_\alpha}$$

BCJ relations

$$0 = \sum_{P'\in S_{n-2}} S^{\mathrm{KLT}}_{(n-2)}[P,P']\ A(P')$$

Colour/kinematics duality

$\exists$ valid n_α such that

$$c_\alpha \pm c_\beta \pm c_\gamma = 0 \leftrightarrow n_\alpha \pm n_\beta \pm n_\gamma = 0$$

KLT relations

$$\mathcal{M}_n = \sum_{P,P'\in S_{n-3}} A(P)\ S^{\mathrm{KLT}}_{(n-3)}[P,P']\ \tilde{A}(P')$$

Double copy

$$\mathcal{M}_n = \sum_{\alpha\in\text{cubic}} \frac{n_\alpha \tilde{n}_\alpha}{D_\alpha}$$

Kinematic algebras from the scattering equations

- Review of scattering equations and CHY formulas (recall talk by Yuan)
- A self-dual-type vertex from the scattering equations
- Construction of BCJ numerators

Scattering Equations

Cachazo, He, Yuan '13

$$\sum_{b\neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0$$

- kinematic invariants $s_{ab} = 2k_a \cdot k_b \longrightarrow$ points $\sigma_a \in S^2$
- mom. conservation: $SL(2,\mathbb{C})$ invariance $\sigma_a \to \sigma'_a = \frac{A\,\sigma_a + B}{C\,\sigma_a + D}$, $AD - BC = 1$
- $(n-3)!$ distinct solutions $\sigma_a^{(I)}$

[Cachazo, Geyer 12] [CHY 13]

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- $(n-3)!$ distinct solutions $\sigma_a^{(I)}$ [Cachazo, Geyer 12] [CHY 13]

Crucial properties: consider Parke-Taylor factors $\sigma_{ab} = \sigma_a - \sigma_b$

$$A_{\rm PT}^{(I)}(1,2,3,\ldots,n) = \left[\frac{1}{\sigma_{12}\;\sigma_{23}\cdots\sigma_{n1}}\right]_{\sigma=\sigma^{(I)}}$$

- satisfy BCJ relations [Cachazo 12] $\Leftrightarrow$ have BCJ numerators [CHY 13] [Litsey, Stankowicz 13]
- satisfy KLT orthogonality: $\sum A_{\rm PT}^{(I)}\; S_{(n-3)}^{\rm KLT}\; A_{\rm PT}^{(J)} = 0 \quad$ if $I \neq J$ [Cachazo, Geyer 12] [CHY 13]

New Formulas for Amplitudes

Cachazo, He, Yuan '13

Gauge theory

$$A(1,2,\ldots,n) = \sum_{l=1}^{(n-3)!} \left[\frac{1}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}} \frac{\mathrm{Pf}'\Psi}{\det{}'\Phi} \right]_{\sigma=\sigma^{(l)}}$$

Gravity

$$\mathcal{M}_n = \sum_{l=1}^{(n-3)!} \left[\frac{(\mathrm{Pf}'\Psi)(\mathrm{Pf}'\tilde{\Psi})}{\det{}'\Phi} \right]_{\sigma=\sigma^{(l)}}$$

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On quantities $\det{}'\Phi(k_i,\sigma_i)$ and $\mathrm{Pf}'\Psi(k_i,\epsilon_i,\sigma_i)$,

- $\mathrm{Pf}'\Psi(k_i,\epsilon_i,\sigma_i)$ is gauge-invariant: $\epsilon_i^\mu \to \epsilon_i^\mu + c_i k_i^\mu$
- Both are permutation symmetric.

New Formulas for Amplitudes

Cachazo, He, Yuan '13

Definitions:

- $\det{}'\Phi = \frac{|\Phi|^{ijk}_{rst}}{(\sigma_{rs}\sigma_{st}\sigma_{ts})(\sigma_{ij}\sigma_{jk}\sigma_{ki})}$

$$\Phi_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}^2} & a \neq b \\ -\sum_{c \neq a} \frac{k_a \cdot k_c}{\sigma_{ac}^2} & a = b \end{cases}$$

[Cachazo, Geyer 12] [Hodges 12]

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- $\mathrm{Pf}{}'\Psi = 2\dfrac{(-1)^{i+j}}{\sigma_{ij}}\mathrm{Pf}\,(\Psi^{ij}_{ij})$

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \qquad 2n \times 2n$$

$$A_{ab} = \begin{cases} \dfrac{k_a \cdot k_b}{\sigma_{ab}} & a \neq b \\ 0 & a = b \end{cases} \qquad B_{ab} = \begin{cases} \dfrac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}} & a \neq b \\ 0 & a = b \end{cases} \qquad C_{ab} = \begin{cases} \dfrac{\epsilon_a \cdot k_b}{\sigma_{ab}} & a \neq b \\ -\displaystyle\sum_{c \neq a} \dfrac{\epsilon_a \cdot k_c}{\sigma_{ac}} & a = b \end{cases}$$

New Formulas for Amplitudes

Cachazo, He, Yuan ’13

Remarks:

- Formulas can be cast as integrals over n points on S^2, localised on scattering equations.
- Generalise twistor-string formulas to any D. [Witten 03] [Roiban, Spradlin, Volovich 04] [Cachazo, Geyer 12] [Cachazo, Skinner 12]
- Obtained from string theories in ambitwistor space – see talk by Skinner [Mason, Skinner 13] [Berkovitz 13] [Adamo, Casali, Skinner 13] [Geyer, Lipstein, Mason 14]
- Reminescent of Gross-Mende limit – see talk by Bjerrum-Bohr [Bjerrum-Bohr, Damgaard, Tourkine, Vanhove 14]
- One loop extension proposed. [Adamo, Casali, Skinner 13]

Search for Kinematic Algebra

- BCJ numerators for each solution of scattering equations exist.

  [CHY 13] [Litsey, Stankowicz 13]

- Colour-kinematics duality: colour factors – Lie algebra of $SU(N)$

  BCJ numerators – **?**

- Kinematic algebra associated to the scattering equations?

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Strategy:

- CHY formulas rely on Parke-Taylor factors (MHV-like).
- Kinematic algebra know for MHV case. Close to self-dual theory, whose kinematic vertex is the structure cte of Lie algebra. [RM, O'Connell 11]
- Can be generalised through the scattering equations.

A Vertex from the Scattering Eqns

For a solution $\{\sigma_a\}$ of the scattering equations, define

$$\boxed{X_{a,b} = \frac{s_{ab}}{\sigma_a - \sigma_b}} \qquad \text{so that} \qquad \sum_{b \neq a} X_{a,b} = 0$$

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- Define "off-shell" version: A, B are collections of external particles

$$X_{A,B} = \sum_{a \in \{A\}} \sum_{b \in \{B\}} X_{a,b} \qquad \text{e.g.} \;\; X_{1+2,3} = X_{1,3} + X_{2,3}$$

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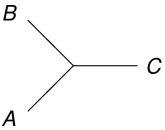
- Interpret as kinematic "vertex": partition $\{A, B, C\}$



$$= X_{A,B}\, f^{a_A a_B a_C}$$

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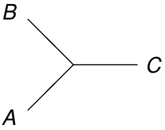

$$= X_{A,B}\, f^{a_A a_B a_C}$$

  - Consistency requires **1**) $X_{A,B} = -X_{B,A}$ **2**) $X_{A,B} = X_{B,C} = X_{C,A}$
  - Follows from scattering equations!

A Kinematic Algebra from the Scattering Eqns

Can $X_{A,B}$ be a structure constant?

Suppose

$$[V_A^+, V_B^+] = i\, X_{A,B} V_{A+B}^+$$

Consistency: need Jacobi identity

$$[[V_A^+, V_B^+], V_C^+] + [[V_B^+, V_C^+], V_A^+] + [[V_C^+, V_A^+], V_B^+] = 0$$

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follows from

$$X_{A,B}X_{A+B,C} + X_{B,C}X_{B+C,A} + X_{C,A}X_{C+A,B} =$$
$$= X_{A,B}(X_{A,C} + X_{B,C}) + X_{B,C}(X_{B,A} + X_{C,A}) + X_{C,A}(X_{C,B} + X_{A,B}) = 0$$

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Can write BCJ numerators for ***X*-amplitudes**.

- e.g. at 4 pts

$$\mathcal{A}_4^X = \frac{X_{1,2}X_{3,4}\, f^{a_1 a_2 b} f^{b a_3 a_4}}{s_{12}} + \frac{X_{2,3}X_{1,4}\, f^{a_2 a_3 b} f^{b a_1 a_4}}{s_{14}} + \frac{X_{3,1}X_{2,4}\, f^{a_3 a_1 b} f^{b a_2 a_4}}{s_{13}}$$

Relation to Self-Dual Vertex

$X_{A,B}$ is a generalisation of the self-dual vertex

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- In $D = 4$, there are **2 special solutions** of scattering equations.

  light cone coords $(u, v, z, \bar{z})$:

$$\sigma_a = \frac{k_{a\bar{z}}}{k_{au}} \longrightarrow X_{a,b} = k_{az}k_{bu} - k_{au}k_{bz} \qquad \text{SD vertex}$$

$$\bar{\sigma}_a = \frac{k_{az}}{k_{au}} \longrightarrow \bar{X}_{a,b} = k_{a\bar{z}}k_{bu} - k_{au}k_{b\bar{z}} \qquad \text{ASD vertex}$$

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- Relation: $$s_{ab} = \frac{X_{a,b}\,\bar{X}_{a,b}}{k_{au}\,k_{bu}} = [ab]\langle ab\rangle$$

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- X and $\bar{X}$ are structure constants of a diffeomorphism Lie algebra:

  $$V_A^{\pm} = -2e^{-iK_A\cdot x}\epsilon_A^{\pm}\cdot\partial \qquad [V_A^+, V_B^+] = i\,X_{A,B}V_{A+B}^+ \quad [V_A^-, V_B^-] = i\,\bar{X}_{A,B}V_{A+B}^-$$

  BCJ manifest in (A)SD gauge theory and gravity. [RM, O'Connell 11]

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- X-amplitudes correspond to $(- + + + \cdots +)$: vanish for general X

Beyond the Self-Dual Sector

2 special solutions in $D = 4$:

MHV amplitudes require both vertices: $X_{A,B}$ and $\bar{X}_{A,B}$.
Notice that

$$X_{a,b} = \frac{s_{ab}}{\sigma_a - \sigma_b} = (\bar{\sigma}_a - \bar{\sigma}_b)\, k_{a u}\, k_{b u} \qquad \bar{X}_{a,b} = \frac{s_{ab}}{\bar{\sigma}_a - \bar{\sigma}_b} = (\sigma_a - \sigma_b)\, k_{a u}\, k_{b u}$$

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Generic solution in any D:

Define

$$\boxed{X_{a,b} = \frac{s_{ab}}{\sigma_a - \sigma_b} \qquad \bar{X}_{a,b} = (\sigma_a - \sigma_b)\, h_a\, h_b} \quad \longrightarrow \quad s_{ab} = \frac{X_{a,b}\, \bar{X}_{a,b}}{h_a\, h_b}$$

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Symmetry between X and $\bar{X}$ is broken.

Vertex consistency requires
$$\boxed{\sum_a h_a = 0 \qquad \sum_a \sigma_a\, h_a = 0}$$

so that $\displaystyle\sum_{b\neq a} X_{a,b} = 0\,, \quad \sum_{b\neq a} \bar{X}_{a,b} = 0\,.$ No other constraint on the h_a.

BCJ Numerators

Can use algebras X and $\bar{X}$ to write BCJ numerators for Parke-Taylor factor.

$$\mathcal{A}_{\mathrm{PT}} = \sum_{\text{non cyclic}} \frac{\mathrm{tr}(T^{a_1} T^{a_2} \cdots T^{a_n})}{\sigma_{12}\sigma_{23}\cdots\sigma_{n1}}$$

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Guidance from 4D MHV case: in this limit, cubic graph has vertices X, except for single vertex $\bar{X}$ attached to reference particle. [RM, O'Connell 11]

$$X_{n,B} \to 0 \qquad\qquad \bar{X}_{n,B} \;\text{ finite}$$

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Then
$$\sum_a \sigma_a h_a = 0 \;\Rightarrow\; h_n \to 0 \quad \text{with} \quad \sigma_n h_n \;\; \text{finite}$$

Guidance from $4D$ MHV case: in this limit, cubic graph has vertices X, except for single vertex $\bar{X}$ attached to reference particle. [RM, O'Connell 11]

$$X_{n,B} \to 0 \qquad\qquad \bar{X}_{n,B} \;\; \text{finite}$$

BCJ numerators given by

$$\boxed{n_{\text{PT}} = \frac{1}{\sigma_n^2(\sigma_n h_n)^2}\, \bar{X}_{n,\cdot}\, X_{\cdot,\cdot} X_{\cdot,\cdot} \cdots X_{\cdot,\cdot}}$$

Complete BCJ Numerators

Obtained BCJ numerators for the Parke-Taylor factor:

$$\mathcal{A}_{\mathrm{PT}} = \sum_{\text{non cyclic}} \frac{\mathrm{tr}(T^{a_1} T^{a_2} \cdots T^{a_n})}{\sigma_{12}\sigma_{23} \cdots \sigma_{n1}} = \sum_{\alpha \in \text{cubic}} \frac{n_{\alpha}^{\mathrm{PT}} c_{\alpha}}{D_{\alpha}}$$

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such that

$$c_\alpha \pm c_\beta \pm c_\gamma = 0 \quad \longleftrightarrow \quad n_\alpha \pm n_\beta \pm n_\gamma = \sum_{l=1}^{(n-3)!} \left[(n_\alpha^{\mathrm{PT}} \pm n_\beta^{\mathrm{PT}} \pm n_\gamma^{\mathrm{PT}}) \frac{\mathrm{Pf}'\Psi}{\det{}'\Phi}\right]_{\sigma=\sigma^{(l)}} = 0$$

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- More naive approach to BCJ?

Kinematic Algebra and the Geometry of Lie Groups

- Review of geometry of Lie groups
- Try kinematic algebra
- Simplest examples

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Work in progress!

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Admit Cartan-Killing metric ds^2, vielbein E^a_α, $\quad ds^2 = E^a_\alpha \, E^a_\beta \, d\theta^\alpha \otimes d\theta^\beta$

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$$\Rightarrow\ 4(c_{ab,cd} + c_{bc,ad} + c_{ca,bd}) = t_a \cdot \left([t_b,[t_c,t_d]] + [t_d,[t_b,t_c]] + [t_c,[t_d,t_b]]\right) + \ldots$$
$$= 0$$

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$$V_{(3)e}^{ab}\, V_{(3)}^{ecd} = V_{(3)\ \ \nu}^{\mu_a \mu_b}(k_a, k_b, -k_a - k_b)\, V_{(3)}^{\nu \mu_c \mu_d}(-k_c - k_d, k_c, k_d) \\ \times (2\pi)^D \delta^D(k_a + k_b + k_c + k_d)$$

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Differences wrt Lie algebra:

- 3-pt vertex does not satisfy Jacobi identity

$$V_{(3)e}^{ab}\, V_{(3)}^{ecd} + V_{(3)e}^{bc}\, V_{(3)}^{ead} + V_{(3)e}^{ca}\, V_{(3)}^{ebd} \neq 0$$

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- there are higher-point vertices

Idea: take $f^{abc} \to V_{(3)}^{abc}$ and deform generators with higher point vertices.

4-pt Amplitude

Deform generator $t_a = (E^{-1})_a^{\alpha} \dfrac{\partial}{\partial\theta^{\alpha}}$ with 4-pt vertex:

$$(E^{-1})_a^{\alpha} = \delta_a^{\alpha} - \frac{1}{2} V_{a\lambda}^{(3)\alpha}\, \theta^{\lambda} + \left(\frac{1}{12} V_{a\lambda}^{(3)b} V_{b\sigma}^{(3)\alpha} + x\, V_{a\lambda,\sigma}^{(4)\ \ \alpha} \right) \theta^{\lambda}\theta^{\sigma} + \ldots$$

where Feynman "stripped" numerator is $N_{ab,cd} = V_{ab}^{(3)e}\, V_{ecd}^{(3)} + V_{ab,cd}^{(4)}$.

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Our "stripped" numerator is evaluated at $\theta = 0$:

$$n_{ab,cd} = \frac{1}{4} \left(t_a \cdot [t_b, [t_c, t_d]] + t_b \cdot [[t_c, t_d], t_a] + t_c \cdot [t_d, [t_a, t_b]] + t_d \cdot [[t_a, t_b], t_c] \right)\Big|_{\theta=0}$$

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Gives valid numerator iff $x = 1/3$:

$$\boxed{n_{ab,cd} \;=\; N_{ab,cd} - \frac{1}{3}(N_{ab,cd} + N_{bc,ad} + N_{ca,bd}) \;\simeq\; N_{ab,cd}}$$

5-pt Amplitude

Consider the generator at the next order:

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Our numerator gives:

$$\boxed{n_{ab,c,de}} = \frac{1}{5}\big(t_a \cdot [t_b,[t_c,[t_d,t_e]]] + t_b \cdot [[t_c,[t_d,t_e]],t_a] + t_c \cdot [[t_d,t_e],[t_a,t_b]] + t_d \cdot [t_e,[[t_a,t_b],t_c]] + t_e \cdot [[[t_a,t_b],t_c],t_d]\big)\Big|_{\theta=0} \simeq \boxed{N_{ab,c,de}}$$

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Valid BCJ numerator! Holds at higher points?

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- Total "kinematic algebra" by deformation of Lie algebra:

$$(E^{-1})_a^\alpha = \delta_a^\alpha - \frac{1}{2} V^{(3)\alpha}_{a\lambda}\, \theta^\lambda + \left(\frac{1}{12} V^{(3)b}_{a\lambda} V^{(3)\alpha}_{b\sigma} + \frac{1}{3} V^{(4)\ \ \alpha}_{a\lambda,\sigma} \right) \theta^\lambda \theta^\sigma - \frac{1}{4} V^{(5)\ \ \ \ \alpha}_{a\lambda,\sigma,\rho}\, \theta^\lambda \theta^\sigma \theta^\rho + \ldots$$

  Higher points? Loop level? Why?