Progress on Multiloop Scattering Amplitudes

new perspectives
on Feynman Integral Calculus

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Max-Planck-Institut für Physik
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Alexander von Humboldt
Stiftung / Foundation

Motivation

- Identify a unique Mathematical framework for any Multi-Loop Amplitude
- Simplify the calculations in High-Energy Physics
- Discover hidden properties of Feynman Amplitudes

Path

- Amplitudes Decomposition
- Multiloop *Integrand Reduction* and Multivariate Polynomial Division
- *Integrand Reduction* and the *minimal set* of Master Integrals
- Differential Equations for Feynman Integrals: Magnus Exponential
- Conclusions

Integr**and** Reduction **(Int′nd Red)**

- Very successful for **many-leg one-loop** amplitudes   Ossola, Papadopoulos, Pittau

$$
\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i \neq i_0}^{m-1} D_i \\
& + \tilde{P}(q) \prod_{i}^{m-1} D_i \,.
\end{aligned}
$$

Integr**and** Reduction **(Int'nd Red)**

- Very successful for **many-leg one-loop** amplitudes

Ossola, Papadopoulos, Pittau

Integ**ral** Identities **(IBP-id's, LI-id's,...)**

Chetyrkin, Tkachov; Laporta

Gehrmann, Remiddi

- Very successful for **many-loop** up to 4-legs amplitudes

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{\partial}{\partial k^\mu} v^\mu f(k, p_i) \quad = \quad 0. \qquad\qquad 2 p_i^\mu p_j^\nu \left(\sum_n p_n^{[\mu} \frac{\partial}{\partial p_n^{\nu]}} \right) I = 0$$

Integr**and** Reduction **(Int′nd Red)**

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Can we combine their advantages?

>>> *Badger's* talk

>>> *Larsen's* talk

>>> *Zhang's* talk

Integr**and** Reduction **(Int′nd Red)**

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Can we combine their advantages?

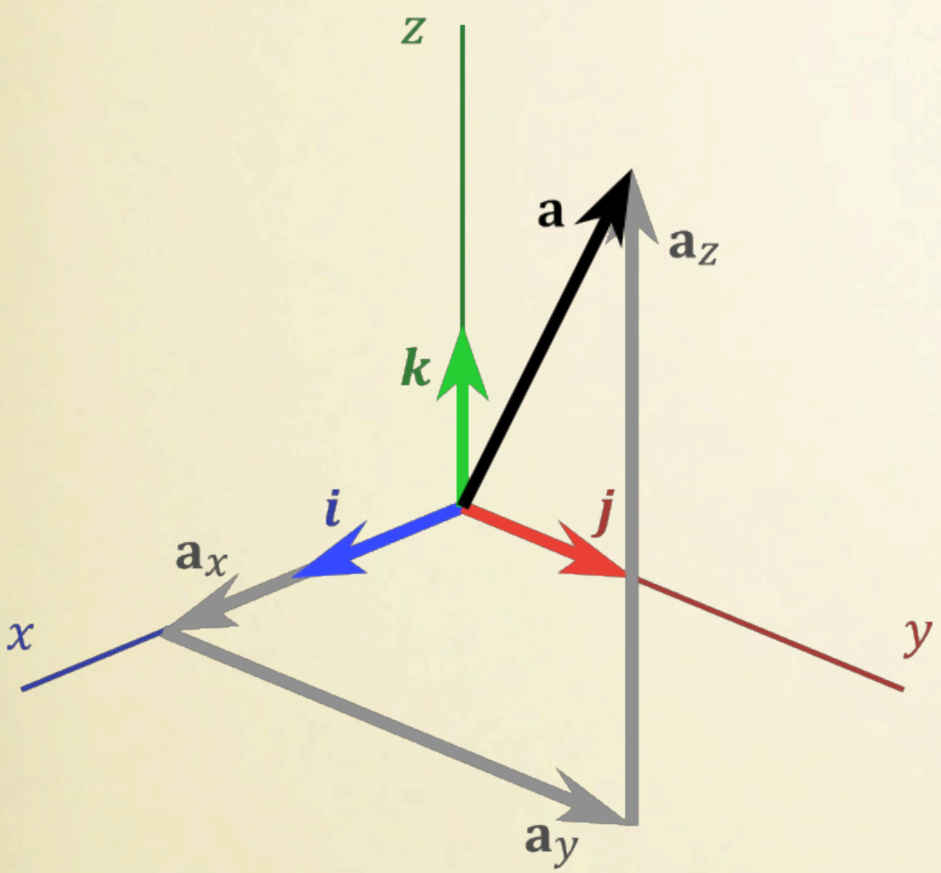
- New ideas to devise an all-order Int′nd Red′n Algorithm

Driving Principles Generic Properties of Feynman Amplitudes:

- *Unitarity & Factorization*
- *Loop-momentum-shift invariance*

Amplitudes Decomposition:

the algebraic way



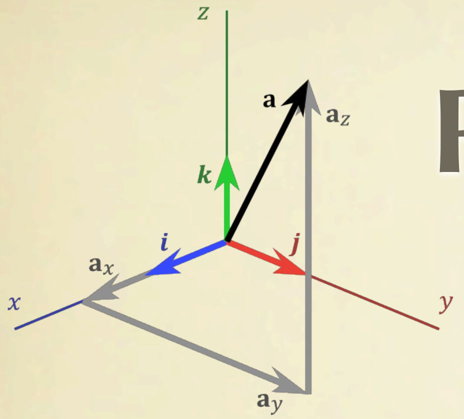
$\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$

- Basis: {i j k}
- Scalar product/Projection: to extract the components

$a_x = \mathbf{a}.\mathbf{i}$

$a_y = \mathbf{a}.\mathbf{j}$

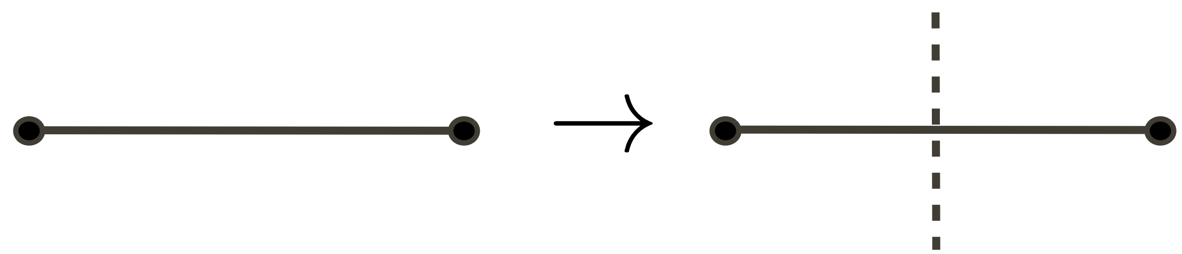
$a_z = \mathbf{a}.\mathbf{k}$



Projections :: On-Shell Cut-Conditions

vanishing denominators

$$\frac{1}{p^2 - m^2 - i0} \rightarrow \delta(p^2 - m^2)$$



Completeness Relations: cutting "1"

- the richness of factorization

$$1 = i\ (-i)$$

$$\Box = (i\gamma^\mu \partial_\mu)(-i\gamma^\nu \partial_\nu)$$

$$1_{n\times n} = \sum_n |\psi_n\rangle\langle\psi_n|$$

On-shellness for Tree-Level Amplitudes

Cauchy's Residue Theorem

$$\oint \frac{dz}{z(z-z_1)(z-z_2)\cdots(z-z_n)} = 0 \quad \Longleftrightarrow \quad \frac{(-1)}{z_1 z_2 \cdots z_n} = \frac{1}{z_1(z_1-z_2)\cdots(z_1-z_n)} + \frac{1}{(z_2-z_1)z_2\cdots(z_2-z_n)} + \cdots\cdots + \frac{1}{(z_n-z_1)(z_n-z_2)\cdots(z_n-z_{n-1})z_n}$$

On-shellness for Tree-Level Amplitudes

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$\Longleftrightarrow$

Partial Fractioning

$$\frac{(-1)}{z_1 z_2 \cdots z_n} = \frac{1}{z_1(z_1-z_2)\cdots(z_1-z_n)} + \frac{1}{(z_2-z_1)z_2\cdots(z_2-z_n)} + \cdots\cdots + \frac{1}{(z_n-z_1)(z_n-z_2)\cdots(z_n-z_{n-1})z_n}$$

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On-shell condition (cuts)

$$(q_i - z_i\eta)^2 - m_i^2 = 0 \ , \qquad z_i = \frac{q_i^2 - m_i^2}{2\eta . q_i} \ ,$$

Vaman, Yao (2005)

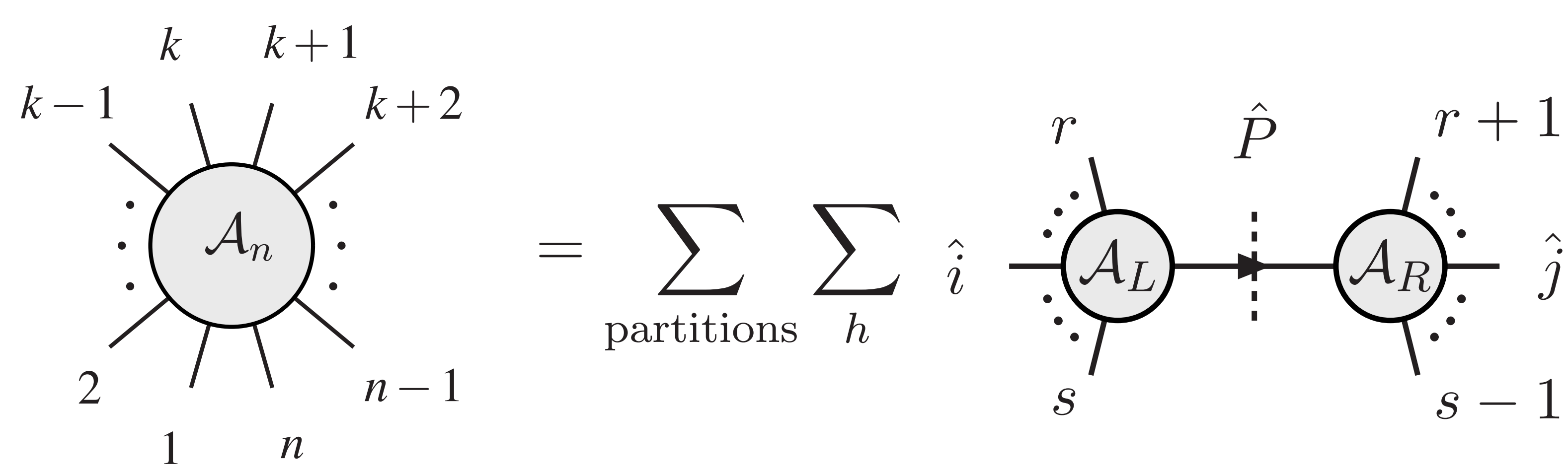
Denominator decomposition

$$\begin{aligned}(-1)\frac{1}{q_1^2-m_1^2}\frac{1}{q_2^2-m_2^2}\cdots\frac{1}{q_n^2-m_n^2} = & \frac{1}{q_1^2-m_1^2}\frac{1}{(q_2-z_1\eta)^2-m_2^2}\cdots\frac{1}{(q_n-z_1\eta)^2-m_n^2} \\ & + \frac{1}{(q_1-z_2\eta)^2-m_1^2}\frac{1}{q_2^2-m_2^2}\cdots\frac{1}{(q_n-z_2\eta)^2-m_n^2} \\ & + \cdots\cdots \\ & + \frac{1}{(q_1-z_n\eta)^2-m_1^2}\frac{1}{(q_2-z_n\eta)^2-m_2^2}\cdots\frac{1}{q_n^2-m_n^2}\end{aligned}$$

Tree-Level Amplitudes

- Cauchy's Residue Theorem $$\frac{1}{2\pi i}\oint \frac{\mathcal{A}_n(z)}{z} = \mathcal{A}_n(\infty) = \mathcal{A}_n(0) + \sum_{\text{poles}} \text{Res}\mathcal{A}_n(z)$$

If $\mathcal{A}_n(\infty) = 0$, then one obtaines the relation $$\mathcal{A}_n(0) = -\sum_{\text{poles}} \text{Res}\mathcal{A}_n(z)$$



$$\mathcal{A}_n(p_1^{h_1}, \ldots, p_n^{h_n}) = \sum_{\text{partition}} \sum_{h} \mathcal{A}_L(p_r, \ldots, \hat{p}_i, \ldots, p_s, -\hat{P}^h_{r:s}) \frac{1}{P^2} \mathcal{A}_R(\hat{P}^h_{r:s}, p_{s+1}, \ldots, \hat{p}_j, \ldots, p_{r-1})$$

BCFW Recurrence Relation

Britto, Cachazo, Feng, Witten

Multi-pole expansion of Tree-level Amplitudes!

Tree-level decomposition

by

partial fractioning:

is this an accident?

Cut-Conditions at Loop-level

- Loop momentum decomposition $\ell_\mu = x_1\, p_\mu + x_2\, q_\mu + x_3\, \varepsilon^+_\mu + x_4\, \varepsilon^-_\mu$

$$\begin{cases} p^\mu = \frac{\langle p|\gamma_\mu|p]}{2} & q^\mu = \frac{\langle q|\gamma_\mu|q]}{2} \\ \epsilon^\mu_+ = \frac{\langle p|\gamma_\mu|q]}{2} & \epsilon^\mu_- = \frac{\langle q|\gamma_\mu|p]}{2} \\ p^2 = q^2 = \epsilon_i^2 = 0 \end{cases}$$

$$\epsilon_+ \cdot \epsilon_- = -p \cdot q$$

- On-shell condition $\delta\left(\ell_i^2 - m_i^2\right)$

- under Multiple On-shellness Conditions :

- the loop-momentum becomes complex ;
- some of its components (if not all) are frozen;
- the left over free components are *integration*-variable

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- some of its components (if not all) are frozen;
- the left over free components are *integration*-variable

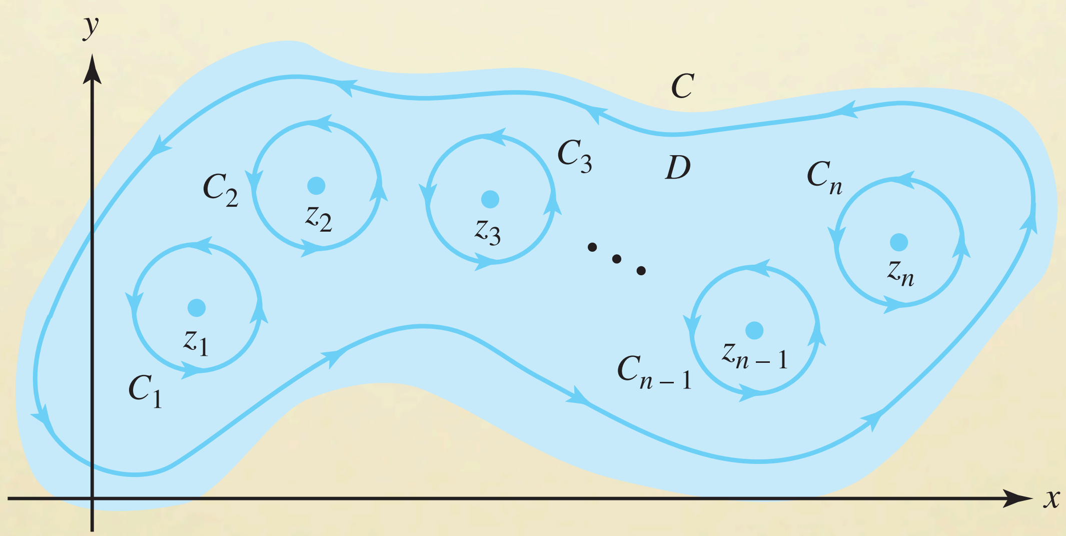
To *integrate* or not to *integrate*: that is the question

To *integrate...*

Cut-Integration
by
Cauchy's residue theorem
(and its generalization)

Loop=1

Britto, Cachazo, Feng
Forde; P.M.
Britto, Buchbinder, Cachazo, Feng
Britto, Feng, P.M.
P.M.
Britto, Mirabella
Badger
MIrabella, Peraro, P.M.



Loop > 1

Buchbinder, Cachazo
Larsen, Kosower
Larsen, Caron-Huot, Kosower, Johansson
Zhang, Huang, Sogaard

...or not to *integrate*

Cut-Integration replaced
by
partial fractioning
(and its generalization)

Multi-Loop Integrand-Reduction
by
Polynomial Division

Ossola & P.M. (2011)

Badger, Frellesvig, Zhang (2011)

Zhang (2012)

Mirabella, Ossola, Peraro, & P.M (2012)

Problem: what is the form of the residues?

"find the right variables encoding the cut-structure"

variables

- ISP's = Irreducible Scalar Products:
  - q-components which can variate under cut-conditions
  - spurious: vanishing upon integration
  - non-spurious: non-vanishing upon integration $\Rightarrow$ MI's

Ossola & P.M. (2011)

A simple idea

Remainder Theorem

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \;, \qquad deg(r) < deg(g)$$

$$g(x) = (x - x_0) : \quad \Rightarrow \quad \frac{f(x)}{(x - x_0)} = q(x) + \frac{r_0}{(x - x_0)} \;, \quad r_0 = f(x_0)$$

Multivariate Polynomial Division

Zhang (2012);
Mirabella, Ossola, Peraro, & P.M. (2012)

- **Ideal** $$\mathcal{J}_{i_1\cdots i_n} = \langle D_{i_1}, \cdots, D_{i_n} \rangle \equiv \left\{ \sum_{\kappa=1}^{n} h_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) : h_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

- **Groebner Basis** $$\mathcal{G}_{i_1\cdots i_n} = \{g_1(\mathbf{z}), \ldots, g_m(\mathbf{z})\}$$

$$\mathcal{J}_{i_1\ldots i_n} = \langle g_1, \ldots, g_m \rangle \equiv \left\{ \sum_{\kappa=1}^{m} \tilde{h}_\kappa(\mathbf{z}) g_\kappa(\mathbf{z}) : \tilde{h}_\kappa(\mathbf{z}) \in P(\mathbf{z}) \right\}$$

n-ple cut-conditions $$D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$$

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Groebner Basis $\mathcal{G}_{i_1 \cdots i_n} = \{ g_1(\mathbf{z}), \ldots, g_m(\mathbf{z}) \}$

$$\mathcal{J}_{i_1 \ldots i_n} = \langle g_1, \ldots, g_m \rangle \equiv \left\{ \sum_{\kappa=1}^{m} \tilde{h}_\kappa(\mathbf{z}) g_\kappa(\mathbf{z}) : \tilde{h}_\kappa(\mathbf{z}) \in P(\mathbf{z}) \right\}$$

n-ple cut-conditions $\quad D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$

Polynomial Division $\mathcal{N}_{i_1 \cdots i_n}(\mathbf{z}) = \Gamma_{i_1 \cdots i_n} + \Delta_{i_1 \cdots i_n}(\mathbf{z})$,

Remainder ~ Residue $\Delta_{i_1 \cdots i_n}(\mathbf{z})$

Quotients $\Gamma_{i_1 \cdots i_n} = \sum_{i=1}^{m} \mathcal{Q}_i(\mathbf{z}) g_i(\mathbf{z})$ belongs to the ideal $\mathcal{J}_{i_1 \cdots i_n}$,

$$= \sum_{\kappa=1}^{n} \mathcal{N}_{i_1 \cdots i_{\kappa-1} i_{\kappa+1} \cdots i_n}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \ .$$

Multi-Loop Integrand Recurrence

Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

$$\frac{\mathcal{N}_{i_1 \ldots i_n}}{D_{i_1} \cdots D_{i_n}} = \sum_{\kappa=1}^{n} \frac{\mathcal{N}_{i_1 \ldots i_{\kappa-1} i_{\kappa+1} \ldots i_n} \; D_{i_\kappa}}{D_{i_1} \cdots D_{i_{\kappa-1}} D_{i_\kappa} D_{i_{\kappa+1}} \cdots D_{i_n}} + \frac{\Delta_{i_1 \ldots i_n}}{D_{i_1} \cdots D_{i_n}}$$

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remainder = residue

$$\mathcal{I}_{i_1 \cdots i_n} = \sum_{\kappa=1}^{k} \mathcal{I}_{i_1 \cdots i_{\kappa-1} i_{\kappa+1} i_n} + \frac{\Delta_{i_1 \cdots i_n}}{D_{i_1} \cdots D_{i_n}} \; .$$

n-denominator integrand

(n-1)-denominator integrand

Multi-Loop Integrand Recurrence

Mirabella, Ossola, Peraro, & ***P.M.*** (2013)

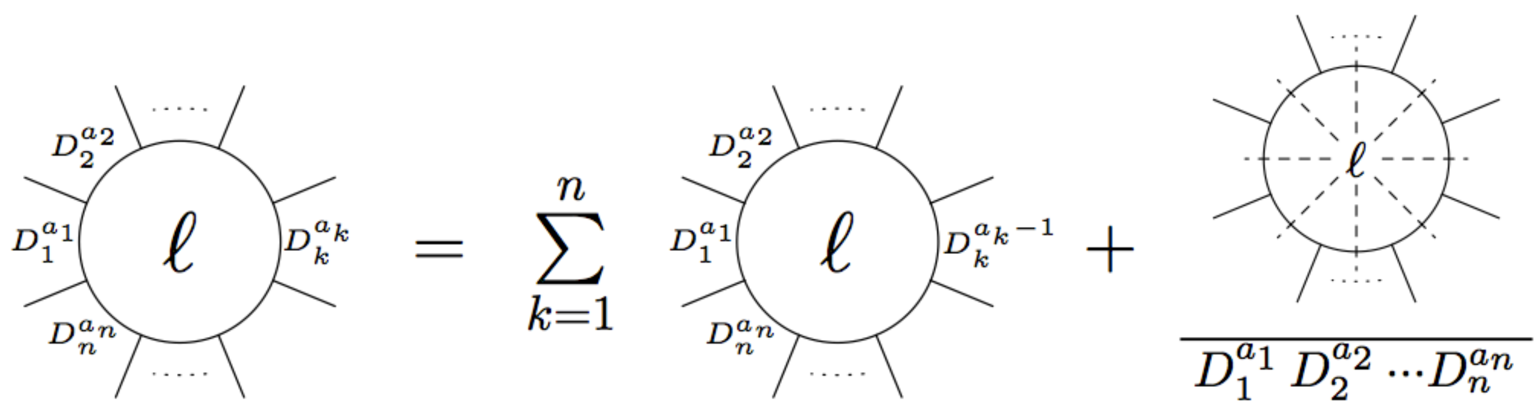
- ☑ D-reg
- ☑ Higher powers of denominators
- ☑ Arbitary kinematics

remainder = residue

$$\mathcal{I}_{\underbrace{i_1\cdots i_1}_{a_1}\cdots\underbrace{i_n\cdots i_n}_{a_n}} = \sum_{k=1}^{n} \mathcal{I}_{\underbrace{i_1\cdots i_1}_{a_1}\cdots\underbrace{i_k\cdots i_k}_{a_k-1}\cdots\underbrace{i_n\cdots i_n}_{a_n}} + \frac{\Delta_{i_1\cdots i_1\cdots i_n\cdots i_n}}{D_{i_1}^{a_1}\cdots D_{i_n}^{a_k}},$$

n-denominator integrand

(n-1)-denominator integrand



Multi-Loop Integrand Decomposition

☑ Multi-(particle)-pole decomposition

$$\mathcal{I}_{i_1\cdots i_n} = \frac{\mathcal{N}_{i_1\cdots i_n}}{D_{i_1} D_{i_2} \cdots D_{i_n}}$$

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-1}}$$

$$+ \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

Multi-Loop Integrand Decomposition

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Tree-level
decomposition
by
partial fractioning:
is this an accident?

Apparently no!

Parametric ***form of the residues*** is ***process independent***

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max-1}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max-1}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max-1}}}$$

$$+ \sum_{1=i_1<<i_{\max-2}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max-2}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max-2}}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

The actual ***values of the coefficients*** in the residues are are ***process dependent***

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2}\cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2}\cdots D_{i_{\max}-1}}$$

$$+ \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2}\cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

☑ Parametric form of the residues is process independent.

Fit-on-cuts...

Knowing the parametric form of residues is ***mandatory***!!!

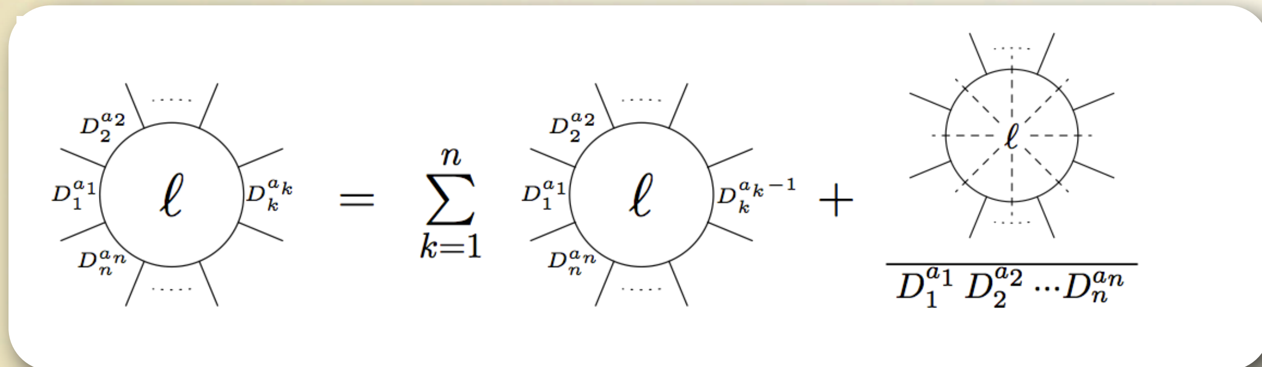
$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-1}}$$
$$+ \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

Use your favorite generator (how about ***GoSam****?), and* ***sample*** *I(q's) as many time as the number of unknown coefficients*

- ☑ Parametric form of the residues is process independent.
- ☑ Actual values of the coefficients is process dependent.

...Divide and Conquer

Mirabella, Ossola, Peraro, & ***P.M.*** (2013)



remainder = residue

$$\mathcal{I}_{\underbrace{i_1 \cdots i_1}_{a_1} \cdots \underbrace{i_n \cdots i_n}_{a_n}} = \sum_{k=1}^{n} \mathcal{I}_{\underbrace{i_1 \cdots i_1}_{a_1} \cdots \underbrace{i_k \cdots i_k}_{a_k - 1} \cdots \underbrace{i_n \cdots i_n}_{a_n}} + \frac{\Delta_{i_1 \cdots i_1 \cdots i_n \cdots i_n}}{D_{i_1}^{a_1} \cdots D_{i_n}^{a_k}} ,$$

n-denominator integrand

(n-1)-denominator integrand

just apply the ***polynomial division***
to the integrand you want to reduce:
analytic/algebraic reduction

☑ No need for the explicit cut-solutions

The Maximum-Cut Theorem

Mirabella, Ossola, Peraro, & P.M. (2012)

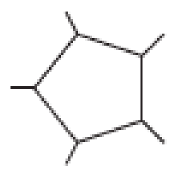
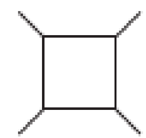
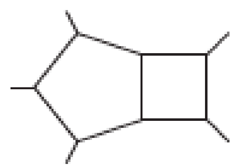
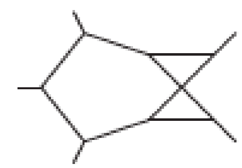
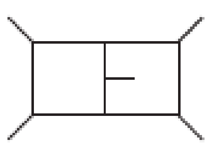
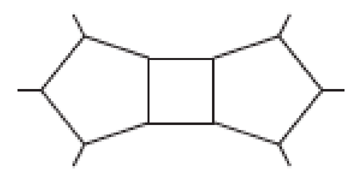
At any loop ℓ, loops we define *maximum cut* as the set of vanishing denominators

$$D_0 = D_1 = \ldots = 0$$

which constrains completely the components of the loop momenta.
We assume that, in non-exceptional phase-space points, a maximum-cut has a finite number n_s of solutions, each with multiplicity one.
Then,

Theorem 4.1 (Maximum cut)**.** *The residue at the maximum-cut is a polynomial paramatrised by* n_s *coefficients, which admits a univariate representation of degree* $(n_s - 1)$*.*

Examples of Maximum-Cuts

diagram	Δ	n_s	diagram	Δ	n_s
	c_0	1		$c_0 + c_1 z$	2
	$\sum_{i=0}^{3} c_i z^i$	4		$\sum_{i=0}^{3} c_i z^{\,i}$	4
	$\sum_{i=0}^{7} c_i z^{\,i}$	8		$\sum_{i=0}^{7} c_i z^{\,i}$	8

One-Loop Integrand-Reduction

One-Loop Integrand Decomposition

- Choice of 4-dimensional basis for an m-point residue

$$e_1^2 = e_2^2 = 0\,, \qquad e_1 \cdot e_2 = 1\,, \qquad e_3^2 = e_4^2 = \delta_{m4}\,, \qquad e_3 \cdot e_4 = -(1-\delta_{m4})$$

- Coordinates: $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5) \equiv (x_1, x_2, x_3, x_4, \mu^2)$

$$q^\mu_{4\text{-dim}} = -p^\mu_{i_1} + x_1\, e_1^\mu + x_2\, e_2^\mu + x_3\, e_3^\mu + x_4\, e_4^\mu\,, \qquad q^2 = q^2_{4\text{-dim}} - \mu^2$$

- Generic numerator

$$\mathcal{N}_{i_1 \cdots i_m} = \sum_{j_1,\ldots,j_5} \alpha_{\vec{j}}\, z_1^{j_1}\, z_2^{j_2}\, z_3^{j_3}\, z_4^{j_4}\, z_5^{j_5}\,, \qquad (j_1 \ldots j_5) \quad \text{such that} \quad \text{rank}(\mathcal{N}_{i_1 \cdots i_m}) \leq m$$

- Residues

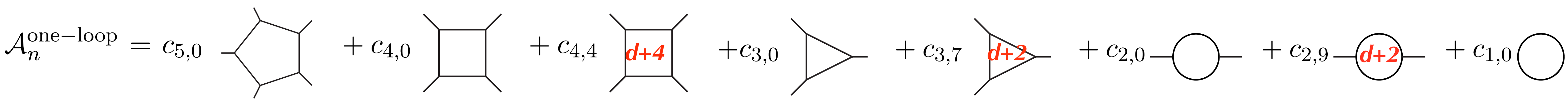
$$\begin{aligned}
\Delta_{i_1 i_2 i_3 i_4 i_5} &= c_0 \\
\Delta_{i_1 i_2 i_3 i_4} &= c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4) \\
\Delta_{i_1 i_2 i_3} &= c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4) \\
\Delta_{i_1 i_2} &= c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \\
\Delta_{i_1} &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}$$

Ossola, Papadopoulos, Pittau
Ellis, Giele, Kunszt, Melnikov

One-Loop Integrand Decomposition

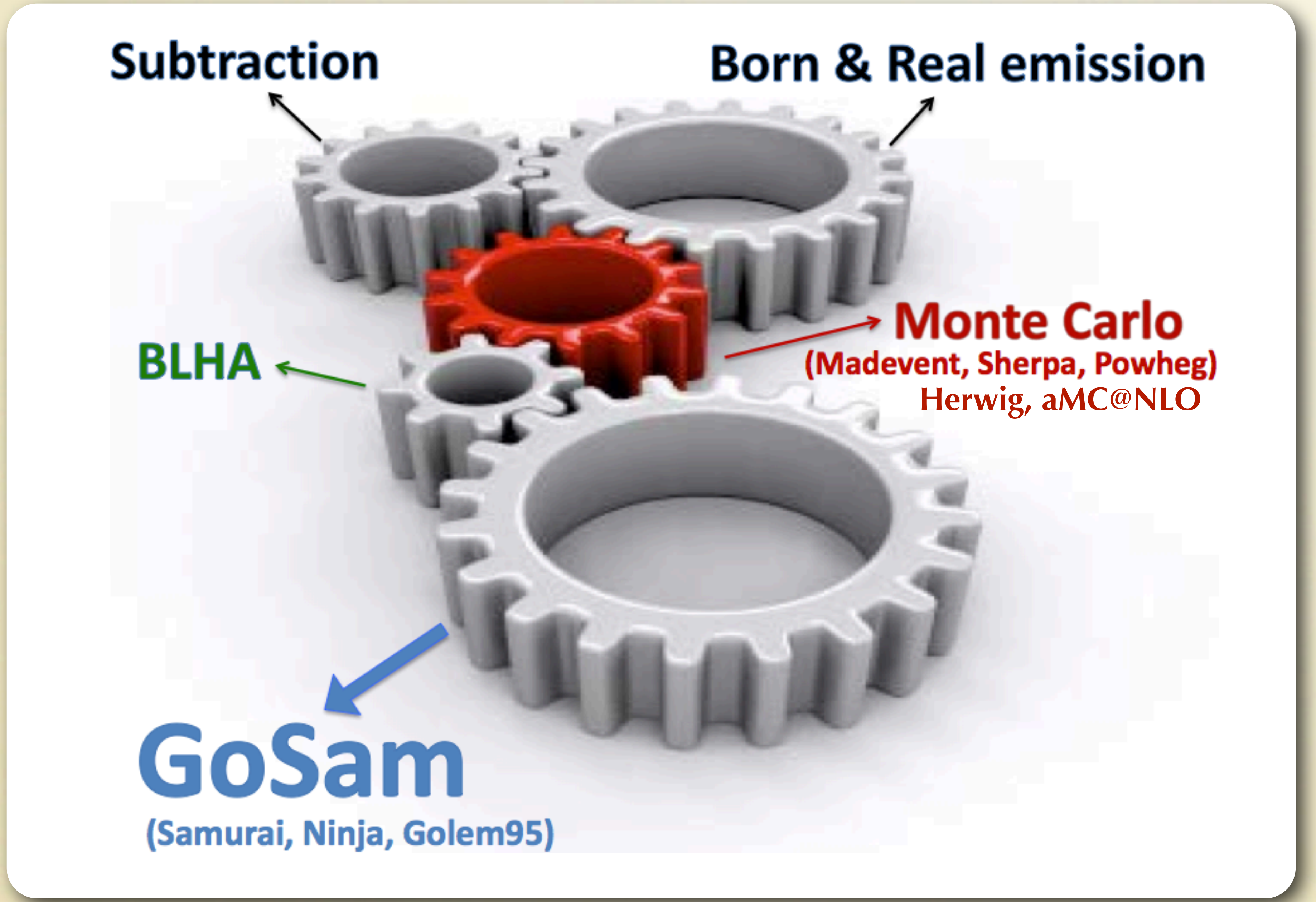
$$\mathcal{A}_n^{\text{one-loop}} = \int d^{-2\epsilon}\mu \int d^4 q \; A_n(q, \mu^2)\,, \qquad A_n(q,\mu^2) \equiv \frac{\mathcal{N}_n(q,\mu^2)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{n-1}} \qquad \bar{D}_i = (\bar{q} + p_i)^2 - m_i^2 = (q+p_i)^2 - m_i^2 - \mu^2$$

We use a bar to denote objects living in $d = 4 - 2\epsilon$ dimensions $\qquad \bar{\not{q}} = \not{q} + \not{\mu}\,, \quad$ with $\quad \bar{q}^2 = q^2 - \mu^2\,.$



The GoSam Project 2.0

Cullen van Deurzen Greiner Heinrich Luisoni
Mirabella Ossola Peraro Reichel Schlenk
von Soden-Fraunhofen Tramontano ***P.M.***



MC Interfaces

Beyond SM

EW Physics

Top Physics

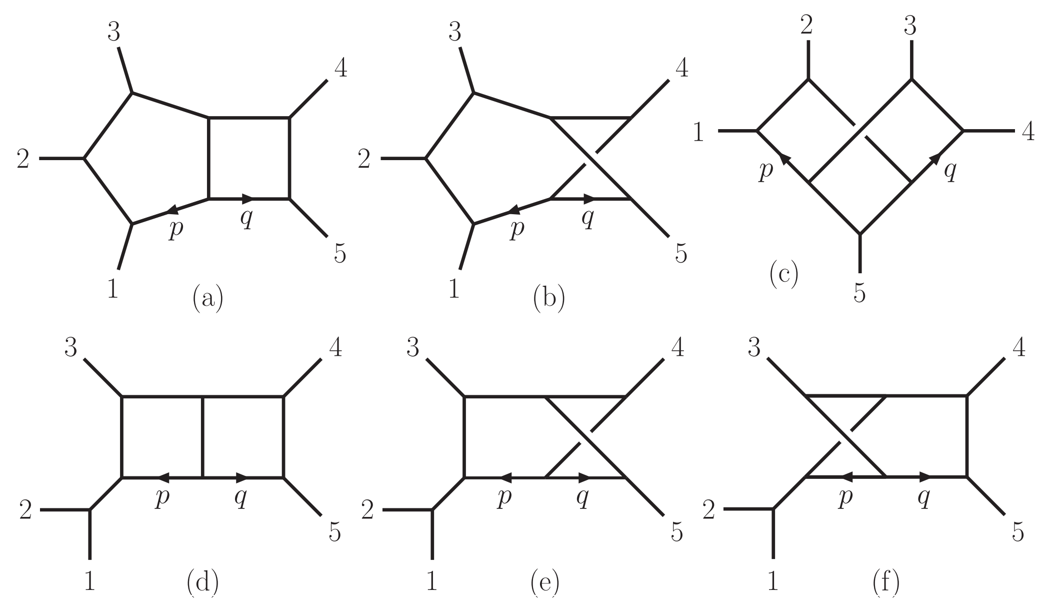
Diphoton and jets

Higgs & Jets

Two-Loop Integrand-Reduction

2-loop 5-point amplitudes in N=4 SYM & N=8 Sugra

Mirabella, Ossola, Peraro, & ***P.M***. (2012)



2-loop 5-point amplitudes in N=4 SYM & N=8 Sugra

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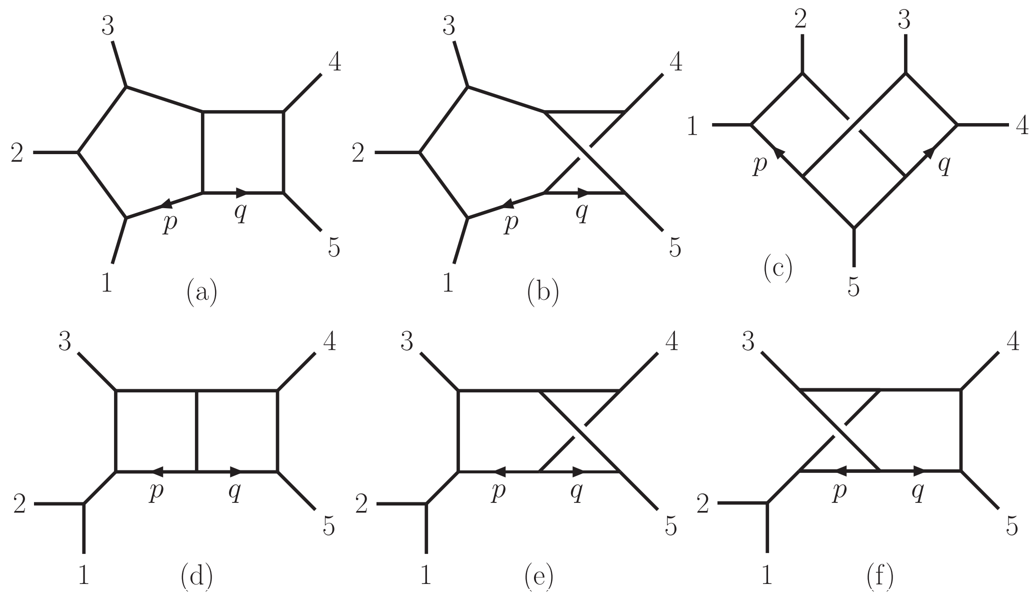
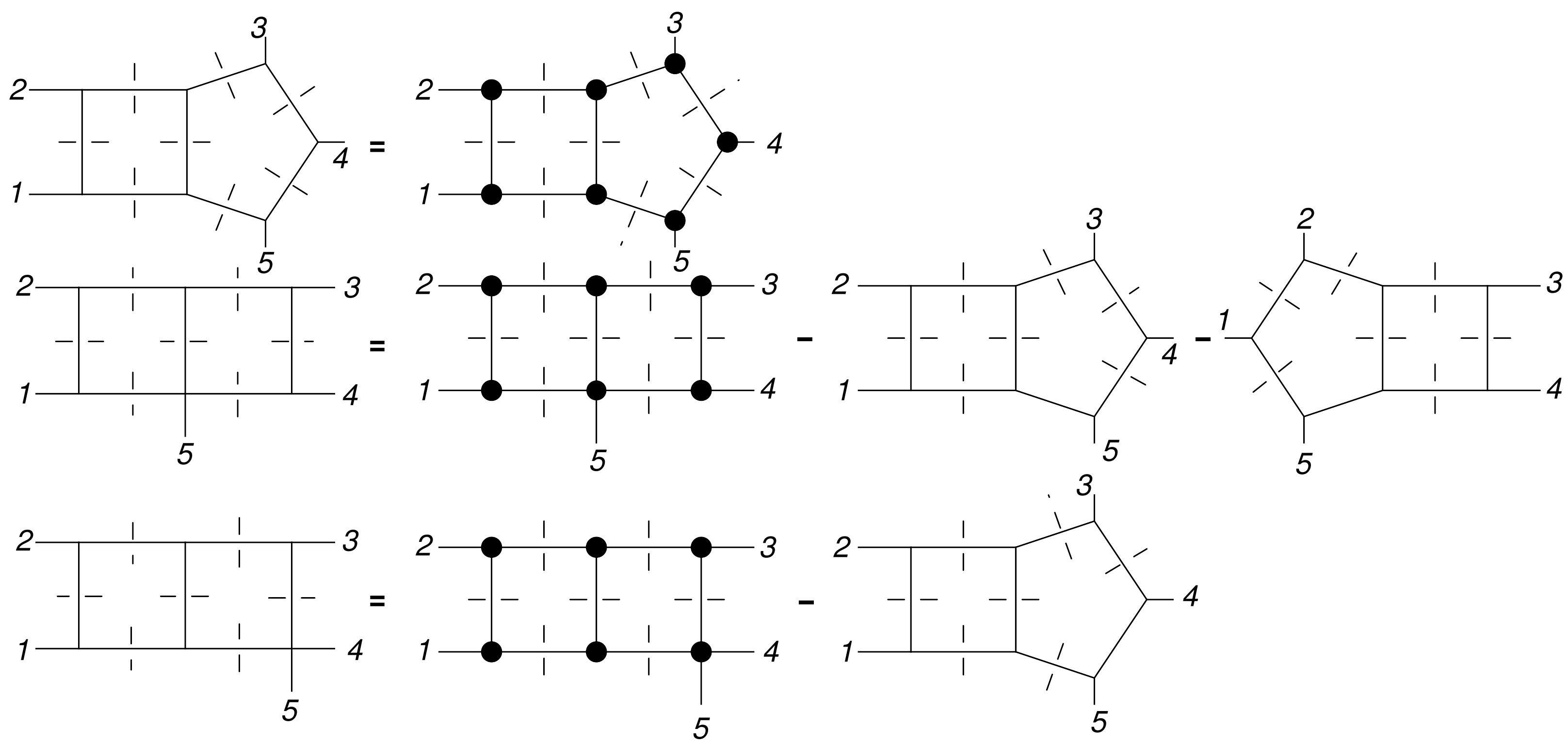
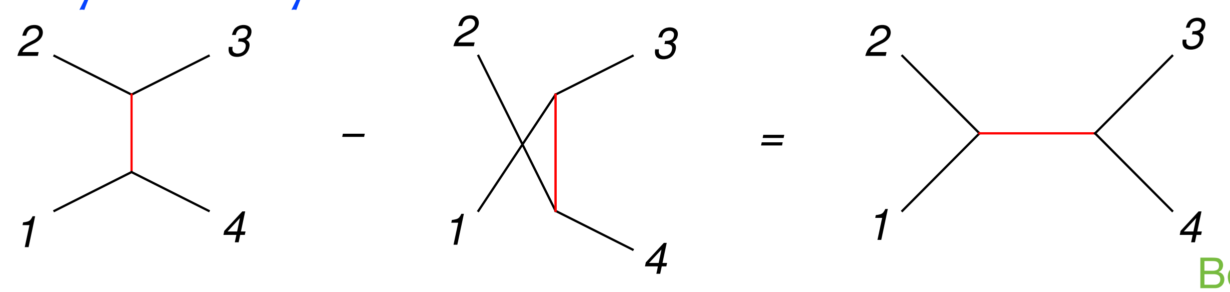


☑ integrand-reduction



Integrand Red′n & Color-Kinematic Duality

Jacoby identity for trees



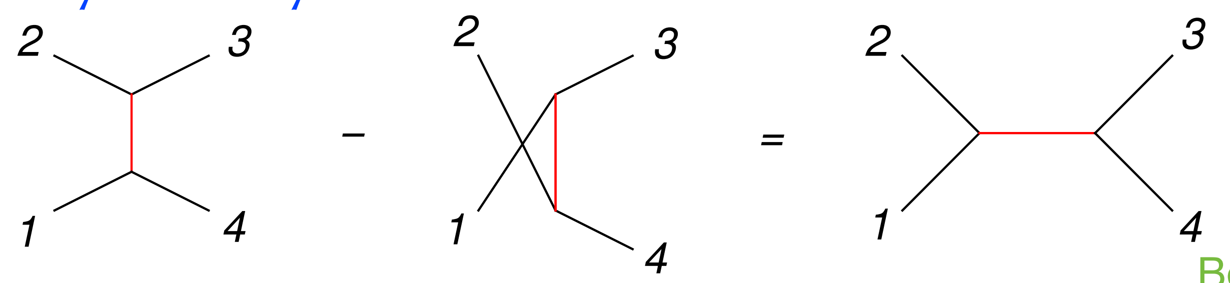
Bern Carrasco Johansson

kinematic term of scattering amplitudes fulfills the same algebra as the **color** term

Integrand Red’n & Color-Kinematic Duality

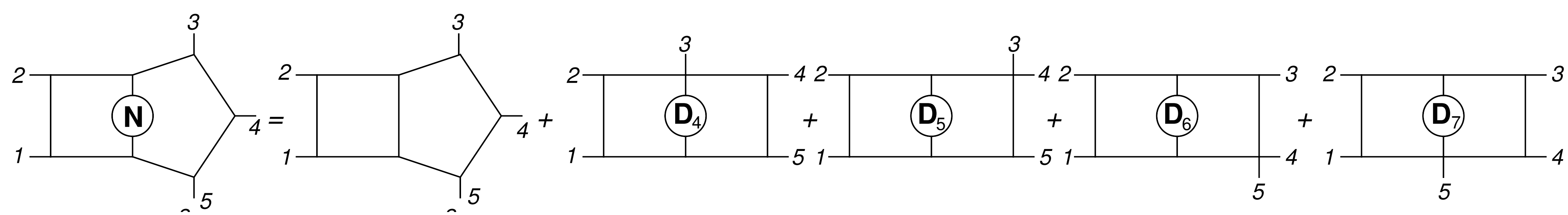
Schubert & ***P.M.*** (2013)

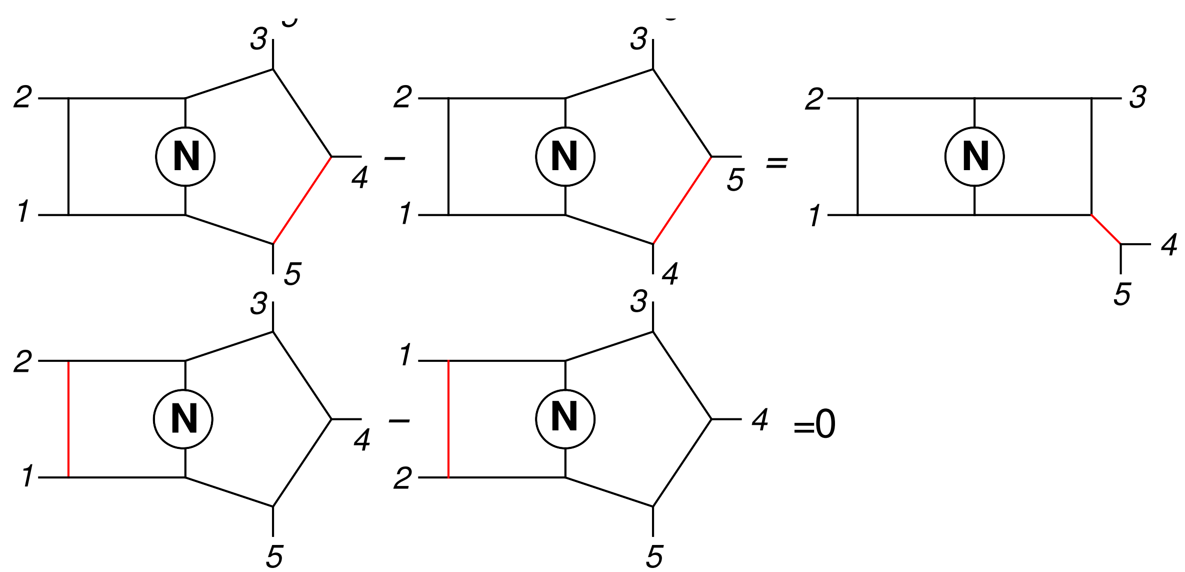
Jacoby identity for trees



Bern Carrasco Johansson

☑ integrand-reduction



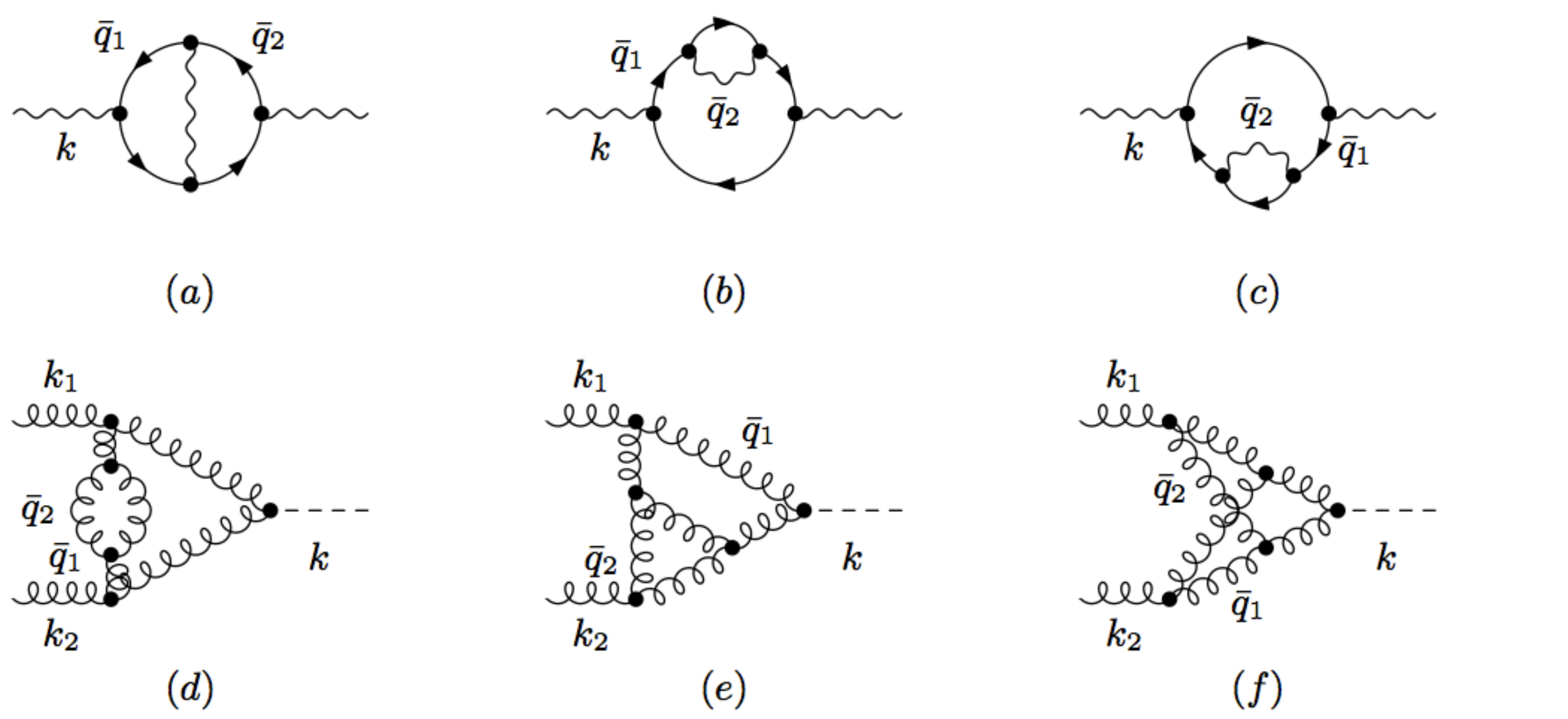


Schubert & ***P.M.***

confirming the result of Carrasco & Johansson

Integrand Red′n & Dim-reg Amplitudes

Mirabella, Ossola, Peraro, & ***P.M***. (2013)



Int'nd Red @ Higher-Loop: **it works!**

Badger, Frellesvig, Zhang

Mirabella, Ossola, Peraro, & ***P.M.***

issue:

independent monomials

are **not** a **minimal** set

Int'nd Red @ Higher-Loop: it works!

Badger, Frellesvig, Zhang

Mirabella, Ossola, Peraro, & ***P.M.***

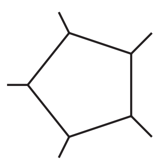
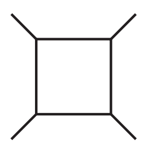
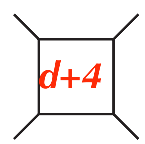
issue:
independent monomials
are **not** a **minimal** set

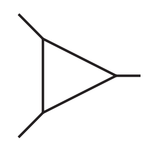
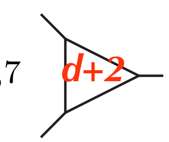
...but this is also the case at 1-loop

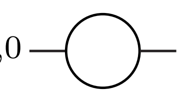
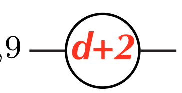
One-Loop Integrand Decomposition

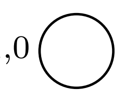
$$\mathcal{A}_n^{\mathrm{one-loop}} = \int d^{-2\epsilon}\mu \int d^4 q \; A_n(q,\mu^2) \;, \qquad A_n(q,\mu^2) \equiv \frac{\mathcal{N}_n(q,\mu^2)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{n-1}} \qquad \bar{D}_i = (\bar{q}+p_i)^2 - m_i^2 = (q+p_i)^2 - m_i^2 - \mu^2$$

We use a bar to denote objects living in $d = 4-2\epsilon$ dimensions $\qquad \bar{\not{q}} = \not{q} + \not{\mu} \,, \quad$ with $\quad \bar{q}^2 = q^2 - \mu^2 \;.$

$\mathcal{A}_n^{\mathrm{one-loop}} = c_{5,0}$  $+ c_{4,0}$  $+ c_{4,4}$ 

$+ c_{3,0}$  $+ c_{3,7}$ 

$+ c_{2,0}$  $+ c_{2,9}$ 

$+ c_{1,0}$ 

One-Loop Integrand Decomposition

$$\mathcal{A}_n^{\text{one-loop}} = \int d^{-2\epsilon}\mu \int d^4 q \; A_n(q,\mu^2)\,, \qquad A_n(q,\mu^2) \equiv \frac{\mathcal{N}_n(q,\mu^2)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{n-1}} \qquad \bar{D}_i = (\bar{q}+p_i)^2 - m_i^2 = (q+p_i)^2 - m_i^2 - \mu^2$$

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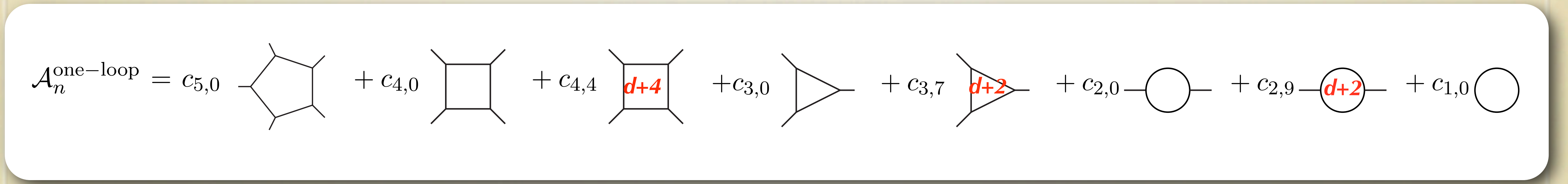
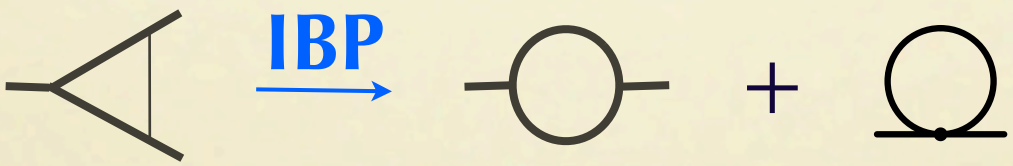


Ex: QED-like kinematic



Solution:
Integration-by-Parts Id's
@ integrand level

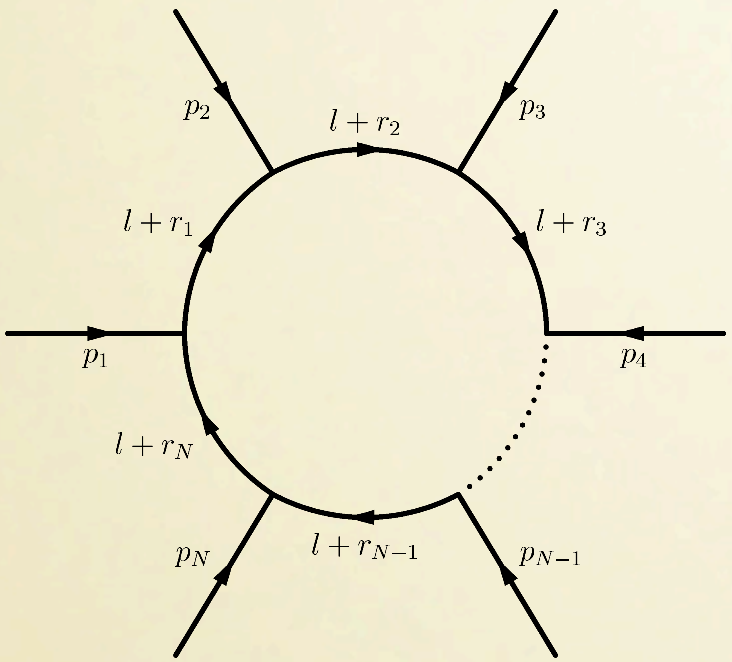
Ossola, Peraro, & ***P.M***.

Accessing the ***reducibility power*** of IBP-id's within the integrand

Let's begin with 1-Loop

1-Loop:
Dimensional-Recurrence from **IBP-id's**

Tarasov; Bern-Dixon-Kosower;
Duplancic-Nizic; Denner-Dittmaier;
Binoth-Guillet-Heinrich; ... ; Lee;



$$I_0^N(D;\{\nu_i\}) \equiv (\mu^2)^{2-D/2} \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{A_1^{\nu_1} A_2^{\nu_2} \cdots A_N^{\nu_N}} \quad .$$

$$0 \equiv \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\partial}{\partial l^\mu} \left(\frac{z_0 l^\mu + \sum_{i=1}^N z_i r_i^\mu}{A_1^{\nu_1} \cdots A_N^{\nu_N}} \right)$$

$$\begin{aligned} C\, I_0^N(D-2;\{\nu_k\}) \quad &= \quad \sum_{i=1}^{N} z_i I_0^N(D-2;\{\nu_k - \delta_{ki}\}) \\ &\quad + (4\pi\mu^2)(D-1-\sum_{j=1}^{N}\nu_j) z_0 I_0^N(D;\{\nu_k\}), \end{aligned}$$

Can we understand/obtain it @ integrand level?

1-Loop: Shifted-D Integrals

D = 4 - 2e

Loop Momentum Decomposition:

Mahlon; Bern-Morgan

$$\bar{q} = q + \mu \ , \qquad \bar{q}^2 = q^2 - \mu^2 \ ,$$

$\bar{q}$ in D-dimensions
q in 4-dimensions
μ in (-2ϵ)-dimensions

$$\int d^D\bar{q} \equiv \int d^{-2\epsilon}\mu \int d^4q = \int d\Omega_{-1-2\epsilon} \int_0^\infty d\mu^2 (\mu^2)^{-1-\epsilon} \ , \qquad \Omega_n \equiv \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$

$$I_n^D[f(q,\mu,p_i)] \equiv \int d^Dq \frac{f(q,\mu,p_i)}{D_1 \cdots D_n}$$

Dimension-raising @ Int'nd level

- From $D \to D+2$: integrand generation of $I_n^{6-2\epsilon}$:

$$I_n^{4-2\epsilon}[\mu^2] = (-\epsilon) I_n^{6-2\epsilon} \ , \qquad \frac{1}{(v_{\perp,1} \cdot v_{\perp,2})} I_n^{4-2\epsilon}[(v_{\perp,1} \cdot q)(v_{\perp,2} \cdot q)] = -\frac{1}{2} I_n^{6-2\epsilon} \qquad (v_{\perp,i} \cdot p_j = 0)$$

$$\text{(tadpole)} \quad I_1^{4-2\epsilon}[q^2] = -2 I_1^{6-2\epsilon}$$

1-Loop:
Dimensional-Recurrence from **Integrand Reduction**

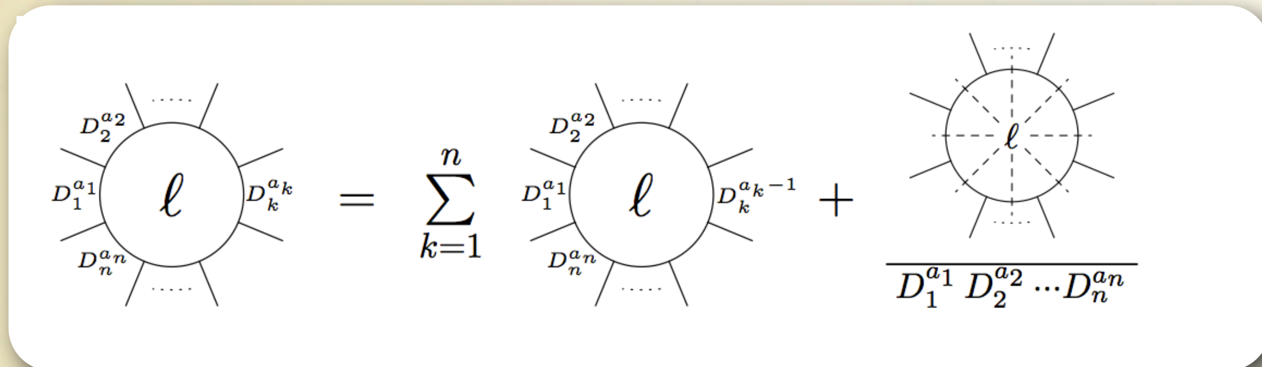
$$I_n^{D=6-2\epsilon} = \frac{1}{(n-5+2\epsilon)c_0}\left[2I_n^{D=4-2\epsilon} - \sum_{i=1}^{n} c_i I_{n-1}^{(i),D=4-2\epsilon}\right]$$

Proposition.
@ 1-Loop: Dimensional-Recurrence for I_n^D

- generated from the relation between μ^2 and $\frac{(v_{\perp,1}\cdot q)(v_{\perp,2}\cdot q)}{(v_{\perp,1}\cdot v_{\perp,2})}$ and D_i's

...Divide and Conquer

Mirabella, Ossola, Peraro, & ***P.M.*** (2013)



remainder = residue

$$\mathcal{I}_{\underbrace{i_1\cdots i_1}_{a_1}\cdots\underbrace{i_n\cdots i_n}_{a_n}} = \sum_{k=1}^{n} \mathcal{I}_{\underbrace{i_1\cdots i_1}_{a_1}\cdots\underbrace{i_k\cdots i_k}_{a_k-1}\cdots\underbrace{i_n\cdots i_n}_{a_n}} + \frac{\Delta_{i_1\cdots i_1\cdots i_n\cdots i_n}}{D_{i_1}^{a_1}\cdots D_{i_n}^{a_k}},$$

n-denominator integrand

(n-1)-denominator integrand

just apply the ***polynomial division***
to the integrand you want to reduce:
analytic/algebraic reduction

Pentagons

We start with the 5-point one-loop integrand $\mathcal{I}_{01234} = \frac{\mu^2}{D_0 D_1 D_2 D_3 D_4}$

Integrand decomposition

whose decomposition reads

$$\begin{aligned}
\mu^2 = {} & c_0^{(01234)} \\
& + \left(c_0^{(0123)} + c_1^{(0123)}(q \cdot v_\perp^{(0123)})\right) D_4 \\
& + \left(c_0^{(0124)} + c_1^{(0124)}(q \cdot v_\perp^{(0124)})\right) D_3 \\
& + \left(c_0^{(0134)} + c_1^{(0134)}(q \cdot v_\perp^{(0134)})\right) D_2 \\
& + \left(c_0^{(0234)} + c_1^{(0234)}(q \cdot v_\perp^{(0234)})\right) D_1 \\
& + \left(c_0^{(1234)} + c_1^{(1234)}((q + p_1) \cdot v_\perp^{(1234)})\right) D_0
\end{aligned}$$

Integration

$$\begin{aligned}
\mathcal{I}_{01234}[\mu^2] = -\epsilon\, \mathcal{I}_{01234}^{6-2\epsilon} = {} & c_0^{01234} \mathcal{I}_{01234} + \\
& + c_0^{(0123)}\, \mathcal{I}_{0123} + c_0^{(0124)}\, \mathcal{I}_{0124} + c_0^{(0134)}\, \mathcal{I}_{0134} \\
& + c_0^{(0234)}\, \mathcal{I}_{0234} + c_0^{(1234)}\, \mathcal{I}_{1234}.
\end{aligned}$$

Boxes

$$\mathcal{I}_{0123} = \frac{1}{v_\perp^2} \frac{(q \cdot v_\perp)^2}{D_0 D_1 D_2 D_3},$$

Integrand decomposition

$$\begin{aligned}
\frac{(q \cdot v_\perp)^2}{v_\perp^2} = {} & c_0^{(0123)} + \mu^2 \\
& + \left(c_0^{(0123)} + c_1^{(012)}(q \cdot e_3^{(012)}) + c_4^{(012)}(q \cdot e_4^{(012)})\right) D_3 \\
& + \left(c_0^{(013)} + c_1^{(013)}(q \cdot e_3^{(013)}) + c_4^{(013)}(q \cdot e_4^{(013)})\right) D_2 \\
& + \left(c_0^{(023)} + c_1^{(023)}(q \cdot e_3^{(023)}) + c_4^{(023)}(q \cdot e_4^{(023)})\right) D_1 \\
& + \left(c_0^{(123)} + c_1^{(123)}(q \cdot e_3^{(123)}) + c_4^{(123)}(q \cdot e_4^{(123)})\right) D_0.
\end{aligned}$$

Integration

$$\frac{1}{v_\perp^2}\mathcal{I}_n[(q.v_\perp)^2] - \mathcal{I}[\mu^2] = \frac{1}{2}(-1 + 2\epsilon)\, \mathcal{I}_{0123}^{6-2\epsilon} = c_0^{(0123)}\, \mathcal{I}_{0123} + \sum_{ijk} c_0^{(ijk)} \mathcal{I}_{ijk}.$$

☑

Triangles

$$\mathcal{I}_{012} = \frac{1}{(e_3 \cdot e_4)} \frac{(q \cdot e_3)(q \cdot e_4)}{D_0 D_1 D_2},$$

Integrand decomposition

$$\frac{(q \cdot e_3)(q \cdot e_4)}{(e_3 \cdot e_4)} = c_0^{(0123)} + \frac{1}{2}\mu^2 + \text{scalar bubbles} + \text{linear bubbles} + \text{tadpoles}.$$

$$= c_0^{(0123)} + \frac{1}{2}\mu^2 + \text{scalar bubbles}.$$

Integration

$$\frac{1}{4}(-2 + 2\,\epsilon)\,\mathcal{I}_{0123}^{d=6-2\epsilon} = c_0^{(0123)} \mathcal{I}_{0123} + \sum_{ij} c_{ij} \mathcal{I}_{ij}.$$

☑

Bubbles

$$\mathcal{I}_{01} = \frac{1}{(e_3 \cdot e_4)} \frac{(q \cdot e_3)(q \cdot e_4)}{D_0 D_1},$$

Integrand decomposition

$$\frac{(q \cdot e_3)(q \cdot e_4)}{(e_3 \cdot e_4)} = \frac{1}{2}\,\mu^2 \ + \text{scalar, linear and quadratic bubble} + \text{tadpoles.}$$

$$= \frac{1}{3}\,\mu^2 \ + \text{scalar bubble} + \text{tadpoles.}$$

Integration

$$\frac{1}{6}(-3+2\epsilon)\mathcal{I}_{01}^{6-2\epsilon} = c_0\ \mathcal{I}_{01} + \sum_i c_i\ \mathcal{I}_i$$

☑

Tadpoles

Integration

$$\frac{1}{e_3 \cdot e_4} \mathcal{I}_0[(q \cdot e_3)(q \cdot e_4)] = \frac{1}{4} \mathcal{I}[\mu^2] + \frac{1}{4} m_0^2 \mathcal{I}_0$$

$$\frac{1}{8}(-4 + 2\,\epsilon)\, \mathcal{I}_0^{d=6-2\epsilon} = \frac{1}{4} m_0^2 \mathcal{I}_0.$$ ☑

or simply from $\mathcal{I}_0[q^2] = \mathcal{I}_0[\mu^2] + m_0^2\, \mathcal{I}_0$

1-Loop:

Dimensional-Recurrence: got it!

$$I_n^{D=6-2\epsilon} = \frac{1}{(n-5+2\epsilon)}\left[c_{n,0}\; I_n^{D=4-2\epsilon} - \sum_{i=1}^{n} c_{n,i}\; I_{n-1}^{(i),D=4-2\epsilon}\right]$$

$$I_{n-1}^{D=6-2\epsilon} = \frac{1}{(n-6+2\epsilon)}\left[c_{n-1,0}\; I_{n-1}^{D=4-2\epsilon} - \sum_{i=1}^{n-1} c_{n-1,i}\; I_{n-2}^{(i),D=4-2\epsilon}\right]$$

$$\ldots = \ldots$$

$$I_2^{D=6-2\epsilon} = \frac{1}{(-3+2\epsilon)}\left[c_{2,0}\; I_2^{D=4-2\epsilon} - \sum_{i=1}^{2} c_{2,i}\; I_1^{(i),D=4-2\epsilon}\right]$$

$$I_1^{D=6-2\epsilon} = \frac{1}{(-4+2\epsilon)} c_{1,0}\; I_1^{D=4-2\epsilon}$$

Dimensional Recurrence
@ integrand level:
what we can do with it?

1-Loop:
IBP-*id's* from **Dimensional-Recurrence**

$$I_n^{D=6-2\epsilon} = \frac{1}{(n-5+2\epsilon)}\left[c_{n,0}\; I_n^{D=4-2\epsilon} - \sum_{i=1}^{n} c_{n,i}\; I_{n-1}^{(i),D=4-2\epsilon}\right]$$

$$I_{n-1}^{D=6-2\epsilon} = \frac{1}{(n-6+2\epsilon)}\left[c_{n-1,0}\; I_{n-1}^{D=4-2\epsilon} - \sum_{i=1}^{n-1} c_{n-1,i}\; I_{n-2}^{(i),D=4-2\epsilon}\right]$$

$$\ldots = \ldots$$

$$I_2^{D=6-2\epsilon} = \frac{1}{(-3+2\epsilon)}\left[c_{2,0}\; I_2^{D=4-2\epsilon} - \sum_{i=1}^{2} c_{2,i}\; I_1^{(i),D=4-2\epsilon}\right]$$

$$I_1^{D=6-2\epsilon} = \frac{1}{(-4+2\epsilon)} c_{1,0}\; I_1^{D=4-2\epsilon}$$

substitute them bottom-up!

Telescopic Identity

$$(n-1+D)I_n^{D+2} = \left[c_{n,0}I_n^{D} - \sum_{i=1}^{n} c'_{n,i}\; I_{n-1}^{(i),D+2} - \sum_{i=1}^{n-1} c'_{n-1,i}\; I_{n-2}^{(i)D+2} - \ldots - \sum_{i=1}^{n-1} c'_{n-1,i}\; I_1^{(i),D+2}\right]$$

Sending $D \to D-2$

$$(n-3+D)I_n^{D} = \left[c_{n,0}I_n^{D-2} - \sum_{i=1}^{n} c'_{n,i}\; I_{n-1}^{(i),D} - \sum_{i=1}^{n-1} c'_{n-1,i}\; I_{n-2}^{(i)D} - \ldots - \sum_{i=1}^{n-1} c'_{n-1,i}\; I_1^{(i),D}\right]$$

Telescopic Identity

$$(n-1+D) I_n^{D+2} = \left[c_{n,0} I_n^{D} - \sum_{i=1}^{n} c'_{n,i} \, I_{n-1}^{(i),D+2} - \sum_{i=1}^{n-1} c'_{n-1,i} \, I_{n-2}^{(i)D+2} - \ldots - \sum_{i=1}^{n-1} c'_{n-1,i} \, I_{1}^{(i),D+2} \right]$$

Sending $D \to D-2$

$$(n-3+D) I_n^{D} = \left[\cancel{c_{n,0} I_n^{D-2}} - \sum_{i=1}^{n} c'_{n,i} \, I_{n-1}^{(i),D} - \sum_{i=1}^{n-1} c'_{n-1,i} \, I_{n-2}^{(i)D} - \ldots - \sum_{i=1}^{n-1} c'_{n-1,i} \, I_{1}^{(i),D} \right]$$

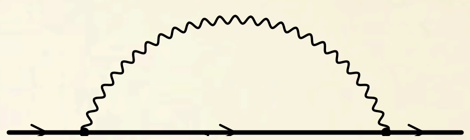
iff $c_{n,0} = 0$

$$(n-3+D) I_n^{D} = \left[- \sum_{i=1}^{n} c'_{n,i} \, I_{n-1}^{(i),D} - \sum_{i=1}^{n-1} c'_{n-1,i} \, I_{n-2}^{(i)D} - \ldots - \sum_{i=1}^{n-1} c'_{n-1,i} \, I_{1}^{(i),D} \right]$$

this is an IBP-id: I_n^D is reducible in terms of lower-point MI's (subtopologies).

Proposition.
$\forall n$, $c_{n,0}$ is found at the fist step of the integrand reduction, and it is not altered by the bottom-up recursive substitutions.
$\Rightarrow$ the integrand reduction can detect algebraically if I_n is MI or not.

7.1 Example: QED bubble



We consider a bubble $\mathcal{I}_{01}$ with the denominators

$$D_0 = q^2, \qquad D_1 = q^2 + 2(q \cdot p), \qquad (\text{i.e. } m_0^2 = p^2 - m_1^2 = 0).$$

- **Bubble rec. rel.** $$(1-d)\,\mathcal{I}_{01}^{(d+2)} = \mathcal{I}_1^d.$$

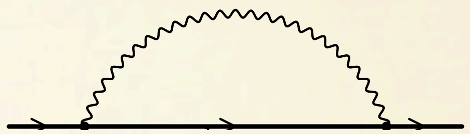
- **Tadpole rec. rel.** $$-d\,\mathcal{I}_1^{(d+2)} = 2m_e^2\,\mathcal{I}_1^{(d)}$$

- **Telescopic Identity** $$(1-d)\,\mathcal{I}_{01}^{(d+2)} = -\frac{1}{2\,m_1^2}\,d\,\mathcal{I}_1^{(d+2)},$$

shift $d \to d-2$

- **IBP-id** $$(3-d)\,\mathcal{I}_{01}^d = \frac{1}{2\,m_1^2}\,(2-d)\,\mathcal{I}_1^d.$$

7.1 Example: QED bubble



We consider a bubble $\mathcal{I}_{01}$ with the denominators

$$D_0 = q^2, \qquad D_1 = q^2 + 2(q \cdot p), \qquad (\text{i.e. } m_0^2 = p^2 - m_1^2 = 0).$$

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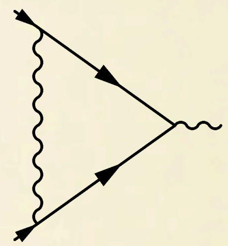
shift $d \to d-2$

- **IBP-id** $$(3-d)\,\mathcal{I}_{01}^d = \frac{1}{2\,m_1^2}\,(2-d)\,\mathcal{I}_1^d.$$

The reduction "knows" that the integral is reducible, at its first step

7.2 Example 2 (QED vertex)

We consider a triangle $\mathcal{I}_{012}$ with kinematics corresponding to the QED vertex



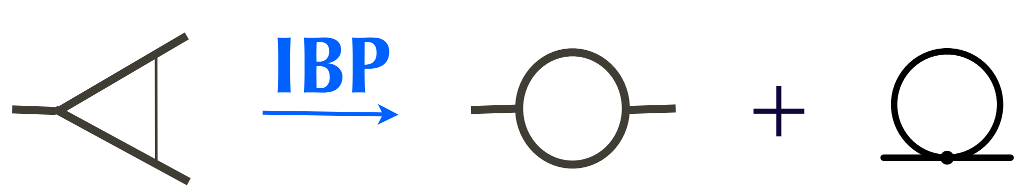
$$D_0 = \bar{q}^2, \qquad D_1 = (\bar{q}+k_1)^2 - m_e^2, \qquad D_1 = (\bar{q}-k_2)^2 - m_e^2,$$
$$\text{with} \quad m_0^2 = 0, \quad k_1^2 = k_2^2 = m_1^2 = m_2^2 = m_e^2, \quad (k_1+k_2)^2 = s.$$

- **Triangle rec. rel.** $\quad (2-d)\,\mathcal{I}_{012}^{(d+2)} = \mathcal{I}_{12}^{(d)}$
- **Bubble rec. rel.** $\quad (1-d)\,\mathcal{I}_{12}^{(d+2)} = \frac{4m_e^2 - s}{2}\mathcal{I}_{12}^{(d)} + \mathcal{I}_{1}^{(d)}$
- **Tadpole rec. rel.** $\quad -d\,\mathcal{I}_{1}^{(d+2)} = 2m_e^2\,\mathcal{I}_{1}^{(d)}$

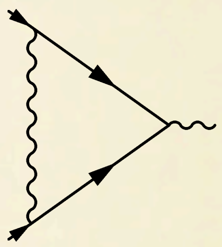
- **Telescopic Identity** $\quad (2-d)\,\mathcal{I}_{012}^{(d+2)} = \frac{2}{4m_e^2 - s}\left((1-d)\,\mathcal{I}_{12}^{(d+2)} + \frac{d}{2\,m_e^2}\,\mathcal{I}_{1}^{(d+2)}\right),$

shift $d \to d-2$

- **IBP-id** $\quad (4-d)\,\mathcal{I}_{012}^{(d)} = \frac{2}{4m_e^2 - s}\left((3-d)\,\mathcal{I}_{12}^{(d)} + \frac{d-2}{2\,m_e^2}\,\mathcal{I}_{1}^{(d)}\right).$



7.2 Example 2 (QED vertex)



We consider a triangle $\mathcal{I}_{012}$ with kinematics corresponding to the QED vertex

$$D_0 = \bar{q}^2, \qquad D_1 = (\bar{q} + k_1)^2 - m_e^2, \qquad D_1 = (\bar{q} - k_2)^2 - m_e^2,$$
$$\text{with} \quad m_0^2 = 0, \quad k_1^2 = k_2^2 = m_1^2 = m_2^2 = m_e^2, \quad (k_1 + k_2)^2 = s.$$

- **Triangle rec. rel.** $$(2-d)\,\mathcal{I}_{012}^{(d+2)} = \mathcal{I}_{12}^{(d)}$$
- **Bubble rec. rel.** $$(1-d)\,\mathcal{I}_{12}^{(d+2)} = \frac{4m_e^2 - s}{2}\mathcal{I}_{12}^{(d)} + \mathcal{I}_{1}^{(d)}$$
- **Tadpole rec. rel.** $$-d\,\mathcal{I}_{1}^{(d+2)} = 2m_e^2\,\mathcal{I}_{1}^{(d)}$$
- **Telescopic Identity** $$(2-d)\,\mathcal{I}_{012}^{(d+2)} = \frac{2}{4m_e^2 - s}\left((1-d)\,\mathcal{I}_{12}^{(d+2)} + \frac{d}{2\,m_e^2}\,\mathcal{I}_{1}^{(d+2)}\right),$$

shift $d \to d-2$

- **IBP-id** $$(4-d)\,\mathcal{I}_{012}^{(d)} = \frac{2}{4m_e^2 - s}\left((3-d)\,\mathcal{I}_{12}^{(d)} + \frac{d-2}{2\,m_e^2}\,\mathcal{I}_{1}^{(d)}\right).$$

The reduction "knows" that the integral is reducible, at its first step

Integrand Reduction@Shift-invariant monomials = Dimensional Recurrence ~ *IBP-id's*

mechanism

- From $D \to D+2$: integrand generation of $I_n^{6-2\epsilon}$:

$$I_n^{4-2\epsilon}[\mu^2] = (-\epsilon) I_n^{6-2\epsilon} \,, \qquad \frac{1}{(v_{\perp,1} \cdot v_{\perp,2})} I_n^{4-2\epsilon}[(v_{\perp,1} \cdot q)(v_{\perp,2} \cdot q)] = -\frac{1}{2} I_n^{6-2\epsilon}$$

$$\text{(tadpole)} \quad I_1^{4-2\epsilon}[q^2] = -2 I_1^{6-2\epsilon}$$

reducibility power of IBP-id's within the integrand: accessed!

How about 2-Loop, 3-Loop,...

Finding out the integrands that control the dimension-shift...
...better if they are also loop-momentum shift invariant

Multi-Loop:
IBP-id's from Dimensional-Recurrence

Ossola, Peraro, & ***P.M.***

- **Schwinger Parametrization**

$$\frac{1}{(p_i^2)^{\nu_i}} = \frac{1}{\Gamma(\nu_i)} \int_0^\infty dt_i \, t_i^{\nu_i - 1} \exp(-t_i p_i^2),$$

$$I^{D=4-2\epsilon}[1] = \frac{1}{(4\pi)^D} \prod_{i=1}^{7} \int_0^\infty dt_i \; \Delta^{-\frac{D}{2}} \; e^{-Q/\Delta}$$

- **Gram Determinant as Gaussian Integrals**

$$\int \left(\prod_{i=1}^{l} \frac{d^{-2\epsilon} \vec{\mu}_i}{\pi^{-\epsilon}} \right) \exp\left(\sum_{i,j=1}^{l} A_{ij} \mu_{ij} \right) = \Delta^{\epsilon}.$$

$$\mu_{ij} \leftrightarrow \frac{\partial}{\partial A_{ij}}$$

Bern, De Freitas, Dixon
Weinzierl
....
Bern, Dennen, Davies, Huang
Badger, Frellesvig, Zhang

- **D-shift Operator (D --> D+2)**

$$\frac{\Delta^{-\frac{D}{2}}}{\Delta} = \Delta^{-\frac{D+2}{2}}$$

1-Loop

$$\Delta = -\det(A_{11}) = -A_{11}$$

$$\Delta^\epsilon = \int \exp\left(\sum_{ij} A_{ij}\mu_{ij}\right) = \int \exp(A_{11}\mu_{11}),$$

$$\frac{\partial}{\partial A_{11}}\Delta^\epsilon = -\epsilon\,\Delta^\epsilon = \int \mu_{11}\,\exp(\ldots),$$

$$\mathcal{I}[\mu_{11}] = -\epsilon\,\mathcal{I}^{(d+2)}.$$

2-Loop

$$\Delta = (-1)^2 \det\begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} = A_{11}A_{22} - A_{12}^2$$

$$\Delta^\epsilon = \int \exp\left(\sum_{ij} A_{ij}\mu_{ij}\right) = \int \exp(A_{11}\mu_{11} + A_{22}\mu_{22} + 2\,A_{12}\mu_{12}).$$

$$4\,\mathcal{I}[\mu_{11}\mu_{22} - \mu_{12}^2] = 2\epsilon(1+2\epsilon)\mathcal{I}^{(d+2)}.$$

Badger, Frellesvig, Zhang

3-Loop

$$\Delta = (-1)^3 \det\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

$$= A_{13}^2A_{22} - 2A_{12}A_{13}A_{23} + A_{11}A_{23}^2 + A_{12}^2A_{33} - A_{11}A_{22}A_{33}.$$

$$\Delta^\epsilon = \int \exp(A_{11}\mu_{11} + A_{22}\mu_{33} + 2A_{12}\mu_{12} + 2A_{13}\mu_{13} + A_{23}\mu_{23})$$

$$8\,\mathcal{I}[\mu_{13}^2\mu_{22} - 2\mu_{12}\mu_{13}\mu_{23} + \mu_{11}\mu_{23}^2 + \mu_{12}^2\mu_{33} - \mu_{11}\mu_{22}\mu_{33}] = 4\epsilon(1+\epsilon)(1+2\epsilon)\mathcal{I}^{(d+2)}.$$

4-Loop...

Multi-Loop Dimensional-Recurrence (Int'nd level)

Ossola, Peraro, & ***P.M.***

Gram-Determinants/Schouten Polynomials Remiddi, Tancredi

$$S(D;a) = a^2$$
$$S(D;a,b) = a^2 b^2 - (a\cdot d)^2$$
$$S(D;a;b,c) = a^2 b^2 c^2 - a^2 (b\cdot c)^2 - b^2 (a\cdot c)^2 - c^2 (a\cdot b)^2 + 2 (a\cdot b)^2 (b\cdot c)^2 (c\cdot a)^2$$
$$\ldots = \ldots$$

(-2e)-Schouten Polynomials *[loops dependent]*

$$S(-2\epsilon;\mu_1) = \mu_{11}$$
$$S(-2\epsilon;\mu_1,\mu_2) = \mu_{11}\mu_{22} - \mu_{12}^2$$
$$S(-2\epsilon;\mu_1,\mu_2,\mu_3) = \mu_{11}\mu_{22}\mu_{33} - \mu_{11}\mu_{23}^2 - \mu_{22}\mu_{13}^2 - \mu_{33}\mu_{12}^2 + 2\mu_{12}^2\mu_{13}^2\mu_{23}^2$$
$$\ldots = \ldots$$

(4D)-Schouten Polynomials *[loops & legs dependent]*

$$S(4;q_1)\ ,\quad S(4;q_1,p_1)\ ,\quad \ldots\ ,\quad S(4;q_1,p_1,...,p_{n-1})\ ,$$
$$S(4;q_1,q_2)\ ,\quad S(4;q_1,q_2,p_1)\ ,\quad \ldots\ ,\quad S(4;q_1,q_2,p_1...,p_{n-1})\ ,$$
$$S(4;q_1,q_2,q_3)\ ,\quad S(4;q_1,q_2,q_3,p_1)\ ,\quad \ldots\ ,\quad S(4;q_1,q_2,q_3,p_1...,p_{n-1})\ ,$$

Multi-Loop Dimensional-Recurrence (Int'nd level)

Ossola, Peraro, & ***P.M.***

Integrand decomposition

$$S(-2\epsilon; \ldots, \mu_i, \ldots) = a_1 S(4; \ldots, q_i, \ldots p_j, \ldots) \; + a_0 + D_i{}'\text{s} + \text{spurious}$$

Integration

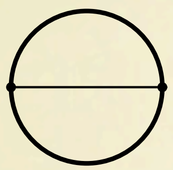
$$I_n^D[S(-2\epsilon; \ldots)] = c(\epsilon) \; I_n^{D+2} \; , \qquad I_n^D[S(4; \ldots)] = c_4 \; I_n^{D+2} \; ,$$

Dimensional Recurrence

$$\Big(c(\epsilon) - c_4 a_1\Big) I_n^{D+2} = a_0 \; I_n^D + \text{subdiagrams}$$

Proposition.

- @ All-Loop: The Dimensional-Recurrence for I_n^D is generated from the integrand relations between $S(-2\epsilon; \mu_{ij})$, $S(4; q_{ij}, p_{ij})$ and D_i's
- these relations capture the reducibility power of IBP-id's



$$\mathcal{I}_{123}[\mathcal{N}] = \frac{\mathcal{N}}{D_1 D_2 D_3}$$

$$D_1 = \bar{q}_1^2 - m^2 = q_1^2 - m^2 - \mu_{11}$$
$$D_2 = \bar{q}_2^2 - m^2 = q_2^2 - m^2 - \mu_{22}$$
$$D_3 = (\bar{q}_1 - \bar{q}_2)^2 = (q_1 - q_2)^2 - \mu_{11} - \mu_{22} + 2\,\mu_{12},$$

- **Integrand decomposition**

$$q_1^2 q_2^2 - (q_1 \cdot q_2)^2 = (\mu_{11}\mu_{22} - \mu_{12}^2) + m^2(\mu_1 - \mu_2)^2 + \frac{m^2}{2} D_3 + \text{spurious}$$

- **Integration**

$$\mathcal{I}_{123}[q_1^2 q_2^2 - (q_1 \cdot q_2)^2] = \mathcal{I}_{123}[S(4; q_1, q_2)] = 3\, \mathcal{I}_{123}^{d+2}$$

$$\mathcal{I}_{123}[\mu_{11}\mu_{22} - \mu_{12}^2] = \mathcal{I}_{123}[S(-2\epsilon; \mu_1, \mu_2)] = \frac{\epsilon}{2}(1 + 2\epsilon)\, \mathcal{I}_{123}^{d+2}$$

$$\mathcal{I}_{123}[(\mu_1 - \mu_2)^2] = \; = \frac{d-4}{d}\mathcal{I}_{12}$$

- **Dimensional Recurrence**

  - **2L-Vacuum**

$$-\frac{1}{4}(d-1)(d-8)\, \mathcal{I}_{123}^{d+2} = \frac{4m^2}{d}\mathcal{I}_{12}^{d}$$

  - **(1L-Tadpole)^2**

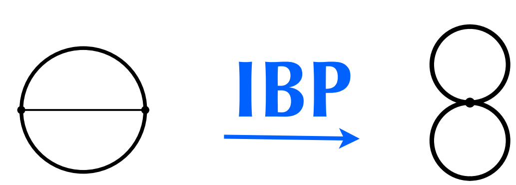
$$d^2 \mathcal{I}_{12}^{(d+2)} = 4m^4 \mathcal{I}_{12}^{(d)},$$

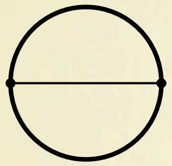
  - **Telescopic Id'y**

$$\mathcal{I}_{123}^{(d+2)} = \frac{d}{2m^2(d-1)}\mathcal{I}_{12}^{(d+2)}.$$

$$d \to d-2$$

$$\mathcal{I}_{123}^{d} = \frac{d-2}{2m^2(d-3)}\mathcal{I}_{12}^{d}.$$





$$\mathcal{I}_{123}[\mathcal{N}] = \frac{\mathcal{N}}{D_1 D_2 D_3}$$

$$D_1 = \bar{q}_1^2 - m^2 = q_1^2 - m^2 - \mu_{11}$$
$$D_2 = \bar{q}_2^2 - m^2 = q_2^2 - m^2 - \mu_{22}$$
$$D_3 = (\bar{q}_1 - \bar{q}_2)^2 = (q_1 - q_2)^2 - \mu_{11} - \mu_{22} + 2\,\mu_{12},$$

- **Integrand decomposition**

$$q_1^2 q_2^2 - (q_1 \cdot q_2)^2 = (\mu_{11}\mu_{22} - \mu_{12}^2) + m^2(\mu_1 - \mu_2)^2 + \frac{m^2}{2} D_3 + \text{spurious}$$

- **Integration**

$$\mathcal{I}_{123}[q_1^2 q_2^2 - (q_1 \cdot q_2)^2] = \mathcal{I}_{123}[S(4; q_1, q_2)] = 3\, \mathcal{I}_{123}^{d+2}$$

$$\mathcal{I}_{123}[\mu_{11}\mu_{22} - \mu_{12}^2] = \mathcal{I}_{123}[S(-2\epsilon; \mu_1, \mu_2)] = \frac{\epsilon}{2}(1 + 2\epsilon)\, \mathcal{I}_{123}^{d+2}$$

$$\mathcal{I}_{123}[(\mu_1 - \mu_2)^2] = = \frac{d-4}{d} \mathcal{I}_{12}$$

- **Dimensional Recurrence**
  - **2L-Vacuum**

$$-\frac{1}{4}(d-1)(d-8)\, \mathcal{I}_{123}^{d+2} = \frac{4m^2}{d} \mathcal{I}_{12}^{d}$$

  - **(1L-Tadpole)^2**

$$d^2 \mathcal{I}_{12}^{(d+2)} = 4m^4 \mathcal{I}_{12}^{(d)},$$

  - **Telescopic Id'y**

$$\mathcal{I}_{123}^{(d+2)} = \frac{d}{2m^2(d-1)} \mathcal{I}_{12}^{(d+2)}.$$

$$d \to d-2 \qquad \mathcal{I}_{123}^{d} = \frac{d-2}{2m^2(d-3)} \mathcal{I}_{12}^{d}.$$

The reduction "knows" that the integral is reducible, at its first step

D-Shifting Operator

Tarasov; Lee;
Ossola, Peraro, Remiddi, Schubert, Tancredi, & ***P.M***.

L-loops, m-legs, n-denominators, q_i's loop momenta, p_i's external momenta;
$\vec{q} \equiv \{q_1, \ldots, q_L\}$, $\vec{p} \equiv \{p_1, \ldots, p_{m-1}\}$, $\vec{a} \equiv \{a_1, \ldots, a_n\}$

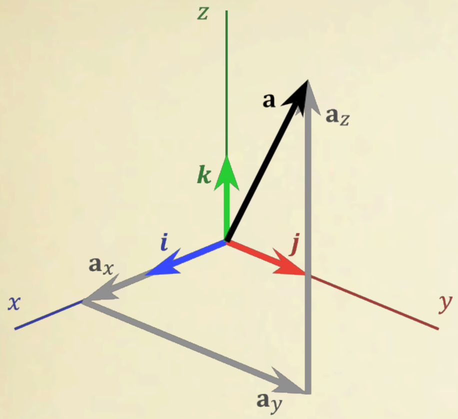
$$I^D_{m,n}[f(q_i;p_i);\vec{a}] \equiv \int d^D q_1 \cdots d^D q_L \frac{f(q_i;p_i)}{D_1^{a_1} \cdots D_n^{a_n}}$$

$$I^D_{m,n}[S(D;\vec{q},\vec{p}) f(q_i;p_i);\vec{a}] \equiv \int d^D q_1 \cdots d^D q_L \frac{S(D;\vec{q},\vec{p})\ f(q_i;p_i)}{D_1^{a_1} \cdots D_n^{a_n}}$$

$$= \text{coeff} \times I^{D+2}_{m,n}[f(q_i;p_i);\vec{a}]$$

Hence $S(D;\vec{q},\vec{p})$ plays the role of the $\mathbf{D}^+$ operator, raising $D \to D+2$.

- ☑ Easy to implement: just a polynomial in terms of of q's and p's (Gram Determinant)
- ☑ *S* is shift-invariant under redefinition of loop momentum (preserving mom. cons.)

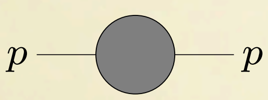


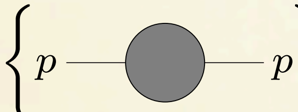
Basis :: Magnus Expansion for Feynman Integrals

$$(\exp X)(\exp Y) = \exp(X + Y + (1/2)[X,Y] + (1/12)[X,[X,Y]] - (1/12)[Y,[X,Y]] + \ldots).$$

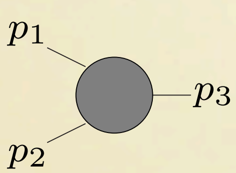
Differential Equations for Master Integrals

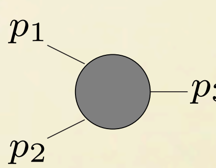
Kotikov; Remiddi;
Caffo, Czyc, Remiddi:
Gehrmann, Remiddi;
Bonciani, Remiddi, **P.M.**;
Argeri, Bonciani, Ferroglia, Remiddi, **P.M**.
Actis, Czakon, Gluza, Riemann;
...
Gehrmann, vonManteuffel, Tancredi
Henn;
Henn, Smirnov & Smirnov
Henn, Melnikov, Smirnov
Caron-Huot, Henn

$p^2 \frac{\partial}{\partial p^2} \Big\{$ 

$\Big\} = \frac{1}{2} p_\mu \frac{\partial}{\partial p_\mu} \Big\{$ 

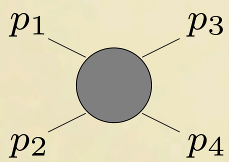
$\Big\}$

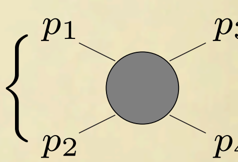
$P^2 \frac{\partial}{\partial P^2} \Big\{$ 

$\Big\} = \left[A \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} \right) + B \left(p_{1,\mu} \frac{\partial}{\partial p_{2,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{1,\mu}} \right) \right] \Big\{$ 

$\Big\}$

$$P = p_1 + p_2,$$

$P^2 \frac{\partial}{\partial P^2} \Big\{$ 

$\Big\} = \left[C \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} - p_{3,\mu} \frac{\partial}{\partial p_{3,\mu}} \right) + D p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} + E (p_{1,\mu} + p_{3,\mu}) \left(\frac{\partial}{\partial p_{3,\mu}} - \frac{\partial}{\partial p_{1,\mu}} + \frac{\partial}{\partial p_{2,\mu}} \right) \right] \Big\{$ 

$\Big\}$

>>> *Henn's* talk

>>> *Smirnov's* talk

>>> *Caron-Huot's* talk

Quantum Mechanics

- **Schroedinger Eq'n (*eps-linear Hamiltonian*)**

$$i\hbar\, \partial_t |\Psi(t)\rangle = H(\epsilon,t)|\Psi(t)\rangle \;, \qquad H(\epsilon,t) = H_0(t) + \epsilon H_1(t)$$

- **Interaction Picture**

$$A(t) = B(t)A_I(t)B^\dagger(t)$$

- **t-Evolution**

$$i\hbar\, \partial_t U_I(t) = \epsilon\, H_{1,I}(t)U_I(t) + \Big(H_{0,I}(t) - i\hbar\, B^\dagger(t)\; \partial_t B(t)\Big)\, U_I(t) \stackrel{!}{=} \epsilon\, H_{1,I}(t)U_I(t),$$

- **Schroedinger Eq'n (*canonical form*)**

$$i\hbar\, \partial_t |\Psi_I(t)\rangle = \epsilon\, H_{1,I}(t)|\Psi_I(t)\rangle \,,$$

- **Matrix Transform**

$$i\hbar\, \partial_t B(t) = H_0(t)B(t) \qquad\qquad B(t) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} d\tau H_0(\tau)}$$

Magnus Expansion

Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

System of 1st ODE

$$\partial_x Y(x) = A(x) Y(x)\ , \quad Y(x_0) = Y_0\ . \qquad A(x) \text{ non-commutative}$$

solution: Matrix Exponential & Iterated Integrals $Y(x) = e^{\Omega(x,x_0)}\, Y(x_0) \equiv e^{\Omega(x)}\, Y_0\,,$

$$\Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x)\ .$$

$$\Omega_1(x) = \int_{x_0}^{x} d\tau_1 A(\tau_1)\ ,$$

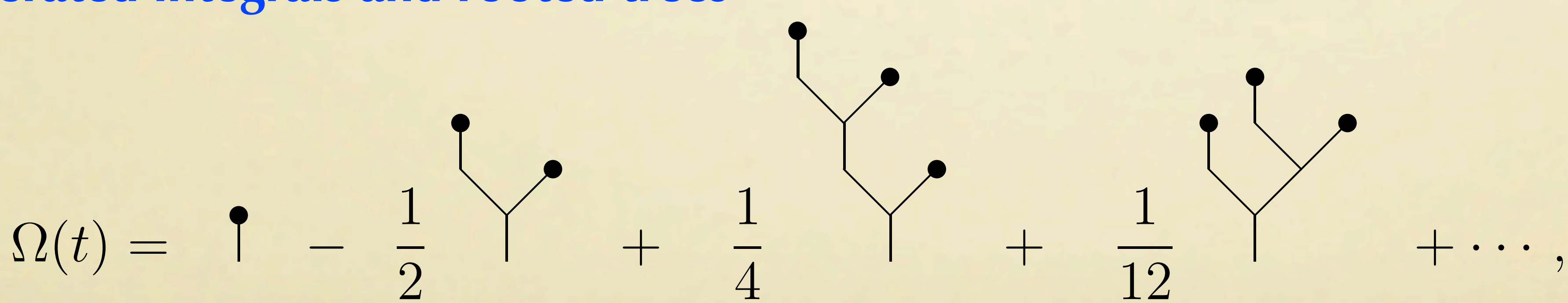
$$\Omega_2(x) = \frac{1}{2} \int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2\ [A(\tau_1), A(\tau_2)]\ ,$$

$$\Omega_3(x) = \frac{1}{6} \int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3\ [A(\tau_1),[A(\tau_2),A(\tau_3)]] + [A(\tau_3),[A(\tau_2),A(\tau_1)]]\ .$$

....

BHC-formula

Iterated integrals and rooted trees



Magnus & Dyson Series

Magnus

$$Y(x) = e^{\Omega(x,x_0)}\, Y(x_0) \equiv e^{\Omega(x)}\, Y_0\,,$$

Dyson

$$Y(x) = Y_0 + \sum_{n=1}^{\infty} Y_n(x)\ , \qquad Y_n(x) \equiv \int_{x_0}^{x} d\tau_1 \ldots \int_{x_0}^{\tau_{n-1}} d\tau_n\ A(\tau_1)A(\tau_2)\cdots A(\tau_n)$$

$$\sum_{j=1}^{\infty} \Omega_j(x) = \log\left(Y_0 + \sum_{n=1}^{\infty} Y_n(x)\right)$$

$$\begin{aligned}
Y_1 &= \Omega_1\ , \\
Y_2 &= \Omega_2 + \frac{1}{2!}\Omega_1^2\ , \\
Y_3 &= \Omega_3 + \frac{1}{2!}(\Omega_1\Omega_2 + \Omega_2\Omega_1) + \frac{1}{3!}\Omega_1^3\ , \\
&\vdots \qquad \vdots \\
Y_n &= \Omega_n + \sum_{j=2}^{n} \frac{1}{j} Q_n^{(j)}\,.
\end{aligned}$$

$$Q_n^{(j)} = \sum_{m=1}^{n-j+1} Q_m^{(1)} Q_{n-m}^{(j-1)}\ , \quad Q_n^{(1)} \equiv \Omega_n\ , \quad Q_n^{(n)} \equiv \Omega_1^n\ .$$

Argeri, Di Vita, Mirabella, Schlenk, Schubert, Tancredi, **P.M.** (2014)

- Linear-**eps** Matrix

$$\partial_x f(\epsilon, x) = A(\epsilon, x)\ f(\epsilon, x)\ , \qquad A(\epsilon, x) = A_0(x) + \epsilon A_1(x)\ ,$$

- change of basis :: Magnus #1

$$f(\epsilon, x) = B_0(x)\ g(\epsilon, x)\ , \qquad B_0(x) \equiv e^{\Omega[A_0](x,x_0)}\ . \qquad \partial_x B_0(x) = A_0(x) B_0(x)\ ,$$

- *Canonical form* Henn (2013)

$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x) \qquad \hat{A}_1(x) = B_0^{-1}(x) A_1(x) B_0(x)\ .$$

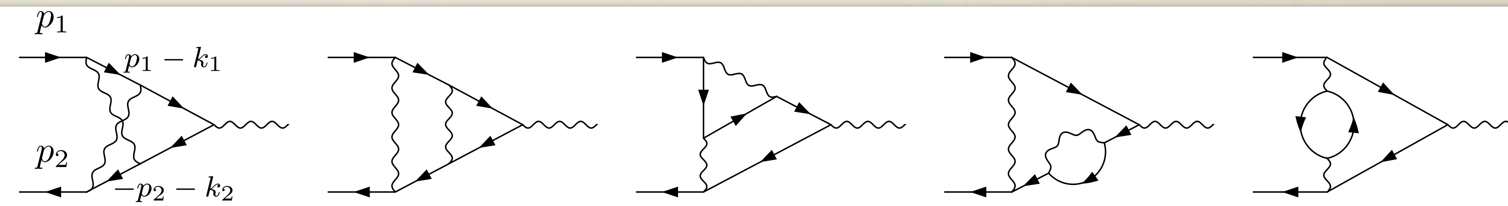
- Solution :: Magnus #2 (or Dyson)

$$g(\epsilon, x) = B_1(\epsilon, x) g_0(\epsilon)\ , \qquad B_1(\epsilon, x) = e^{\Omega[\epsilon \hat{A}_1](x,x_0)}$$

Uniform Transcendentality!

Feynman integrals can be determined from differential equations that looks like gauge transformations

QED (massive) Form Factors

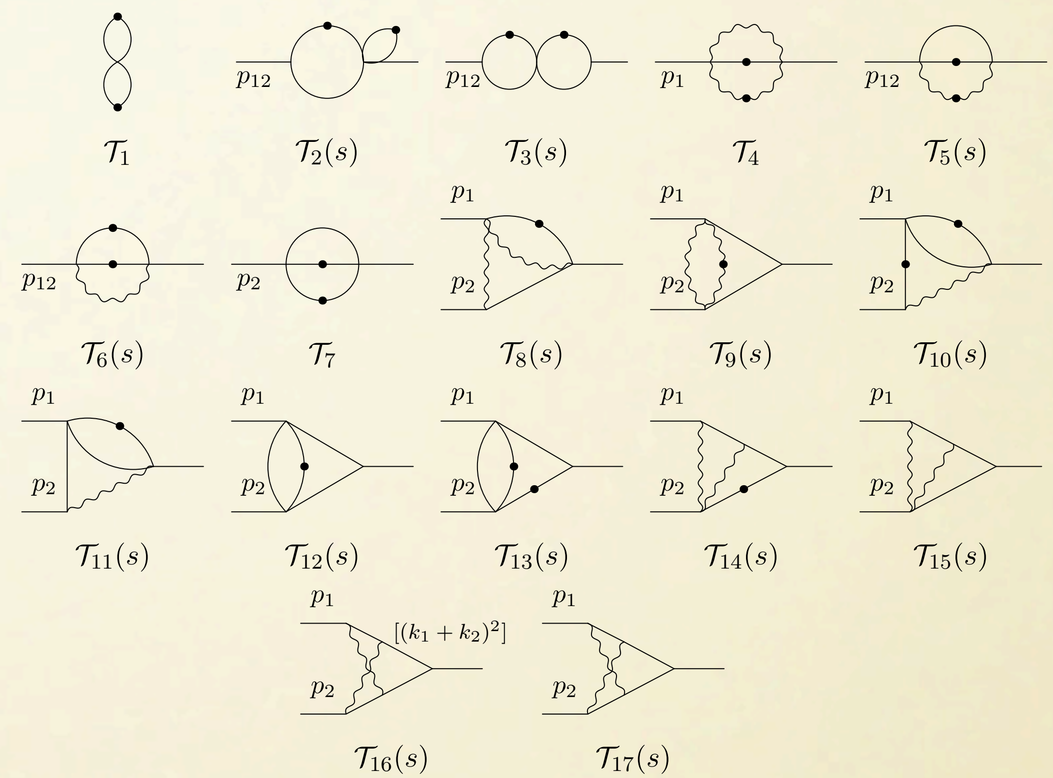


Bonciani,Remiddi, & **P.M.**(2003)

Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

- initial set of MI's

$$f_1 = \epsilon^2 \mathcal{T}_1\,, \quad f_2 = \epsilon^2 \mathcal{T}_2\,, \quad f_3 = \epsilon^2 \mathcal{T}_3\,, \quad f_4 = \epsilon^2 \mathcal{T}_4\,, \quad f_5 = \epsilon^2 \mathcal{T}_5\,,$$
$$f_6 = \epsilon^2 \mathcal{T}_6\,, \quad f_7 = \epsilon^2 \mathcal{T}_7\,, \quad f_8 = \epsilon^3 \mathcal{T}_8\,, \quad f_9 = \epsilon^3 \mathcal{T}_9\,, \quad f_{10} = \epsilon^2 \mathcal{T}_{10}\,,$$
$$f_{11} = \epsilon^3 \mathcal{T}_{11}\,, \quad f_{12} = \epsilon^3 \mathcal{T}_{12}\,, \quad f_{13} = \epsilon^2 \mathcal{T}_{13}\,, \quad f_{14} = \epsilon^3 \mathcal{T}_{14}\,, \quad f_{15} = \epsilon^4 \mathcal{T}_{15}\,,$$
$$f_{16} = \epsilon^4 \mathcal{T}_{16}\,, \quad f_{17} = \epsilon^4 \mathcal{T}_{17}\,,$$



- after *Magnus' rotation*

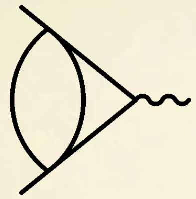
$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x)\ , \qquad \hat{A}_1(x) = \frac{M_1}{x} + \frac{M_2}{1+x} + \frac{M_3}{1-x}\,,$$

$$M_1 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&5&-6&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&-4&0&-2&0&-2&0&0&0&0&0&0&0&0&0\\
0&0&0&-2&0&0&0&0&2&0&0&0&0&0&0&0&0\\
-\frac{1}{2}&0&0&0&1&-2&-3&0&0&3&3&0&0&0&0&0&0\\
0&0&0&0&1&-1&2&0&0&-2&-2&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&-1&-1&0&0&0&0\\
0&-1&0&0&0&0&-3&0&0&0&0&3&3&0&0&0&0\\
0&-1&0&0&1&-\frac{1}{2}&0&2&2&0&0&0&0&2&2&0&0\\
0&0&0&0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0&0&0&0&-1&-1&0&0\\
-\frac{1}{2}&0&0&-2&-1&0&-2&1&0&2&0&-2&0&0&-2&-2&2\\
0&0&0&0&-1&\frac{1}{2}&0&3&-2&0&-6&-2&0&0&-4&-4&4
\end{pmatrix}$$

$$M_2 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-4&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-6&3&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-4&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&\frac{1}{2}&0&0&0&-4&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&-6&0&0&0&0&0&-2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-4&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-4
\end{pmatrix}$$

$$M_3 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-6&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-12&0&0&0&0&0&4&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&-4&0&0&-4&-2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4
\end{pmatrix}$$

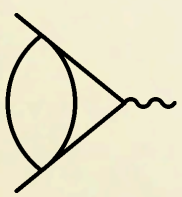
Bonciani, Remiddi, **P.M.** (2003)



$$M_{-2} = \frac{1}{2},$$

$$M_{-1} = \frac{5}{2} - \left[1 - \frac{2}{(1-x)}\right] H(0,x),$$

$$M_0 = \frac{19}{2} + \zeta(2) + \left[1 - \frac{2}{(1-x)}\right]\left[\zeta(2) - 5H(0,x) + 2H(-1,0,x)\right] + \frac{2}{(1-x)} H(0,0,x) + \left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\left[\zeta(2) H(0,x) + H(0,0,0,x)\right].$$

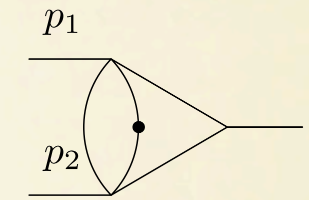
 $(p_2 \cdot k_1)$

$$\frac{N_{-2}}{a} = \frac{1}{8} + \frac{1}{16}\left[x + \frac{1}{x}\right],$$

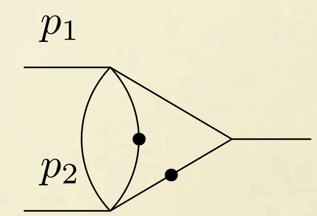
$$\frac{N_{-1}}{a} = \frac{9}{32}\left[2 + x + \frac{1}{x}\right] - \frac{1}{8}\left[4 + x - \frac{1}{x}\right] H(0,x) + \frac{1}{(1-x)} H(0,x),$$

$$\frac{N_0}{a} = \frac{63}{32} + \frac{\zeta(2)}{2} + \frac{63}{64}\left[\left(1 + \frac{16}{63}\zeta(2)\right) x + \frac{1}{x}\right] - \frac{\zeta(2)}{(1-x)} - \frac{1}{16}\left[32 + 9x - \frac{9}{x}\right] H(0,x) + \frac{(16+\zeta(2))}{4(1-x)} H(0,x) - \frac{\zeta(2)}{4(1+x)} H(0,x) - \frac{1}{4}\left[2 - \frac{1}{x} - \frac{4}{(1-x)}\right] H(0,0,x) + \frac{1}{4}\left[4 + x - \frac{1}{x} - \frac{8}{(1-x)}\right] H(-1,0,x) + \frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0,0,0,x).$$

AdVMMSST (2014)

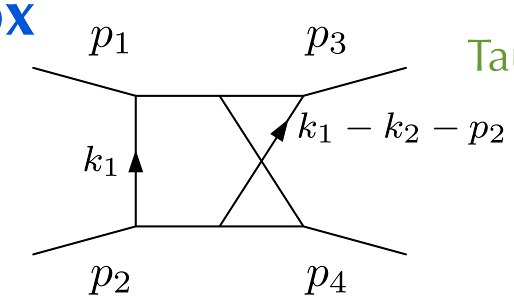


$$g_{12}^{(0)} = 0,$$
$$g_{12}^{(1)} = 0,$$
$$g_{12}^{(2)} = 0,$$
$$g_{12}^{(3)} = -\,\mathrm{H}(0,0,0;x) - \zeta_2 \mathrm{H}(0;x),$$
$$g_{12}^{(4)} = -2\,\mathrm{H}(-1,0,0,0;x) + 2\,\mathrm{H}(0,-1,0,0;x) + 2\,\mathrm{H}(0,0,-1,0;x) - 3\,\mathrm{H}(0,0,0,0;x) - 4\,\mathrm{H}(0,1,0,0;x) + \zeta_2(-2\,\mathrm{H}(-1,0;x) + 6\,\mathrm{H}(0,-1;x) - \mathrm{H}(0,0;x)) + 2\,\zeta_3\,\mathrm{H}(0;x) + \frac{\zeta_4}{4},$$



$$g_{13}^{(0)} = 0,$$
$$g_{13}^{(1)} = 0,$$
$$g_{13}^{(2)} = \mathrm{H}(0,0;x) + \frac{3\,\zeta_2}{2},$$
$$g_{13}^{(3)} = -2\,\mathrm{H}(-1,0,0;x) - 2\,\mathrm{H}(0,-1,0;x) + 4\,\mathrm{H}(0,0,0;x) + 4\,\mathrm{H}(1,0,0;x) + \zeta_2(-6\,\mathrm{H}(-1;x) + 2\,\mathrm{H}(0;x) - 3\log 2) - \frac{\zeta_3}{4},$$
$$g_{13}^{(4)} = 4\,\mathrm{H}(-1,-1,0,0;x) + 4\,\mathrm{H}(-1,0,-1,0;x) - 8\,\mathrm{H}(-1,0,0,0;x) - 8\,\mathrm{H}(-1,1,0,0;x) + 4\,\mathrm{H}(0,-1,-1,0;x) - 8\,\mathrm{H}(0,-1,0,0;x) - 8\,\mathrm{H}(0,0,-1,0;x) + 10\,\mathrm{H}(0,0,0,0;x) + 12\,\mathrm{H}(0,1,0,0;x) - 8\,\mathrm{H}(1,-1,0,0;x) - 8\,\mathrm{H}(1,0,-1,0;x) + 16\,\mathrm{H}(1,0,0,0;x) + 16\,\mathrm{H}(1,1,0,0;x) + 12\,\mathrm{Li}_4\frac{1}{2} + \frac{\log^4 2}{2} + 2\,\zeta_2\,(12\log 2\,\mathrm{H}(-1;x) + 12\log 2\,\mathrm{H}(1;x) + 6\,\mathrm{H}(-1,-1;x) - 2\,\mathrm{H}(-1,0;x) - 8\,\mathrm{H}(0,-1;x) + \mathrm{H}(0,0;x) - 12\,\mathrm{H}(1,-1;x) + 4\,\mathrm{H}(1,0;x) + 3\log^2 2) - 2\,\zeta_3(5\,\mathrm{H}(-1;x) + 4\,\mathrm{H}(0;x) + 11\,\mathrm{H}(1;x)) - \frac{47\,\zeta_4}{4},$$

2-Loop massless cross-box

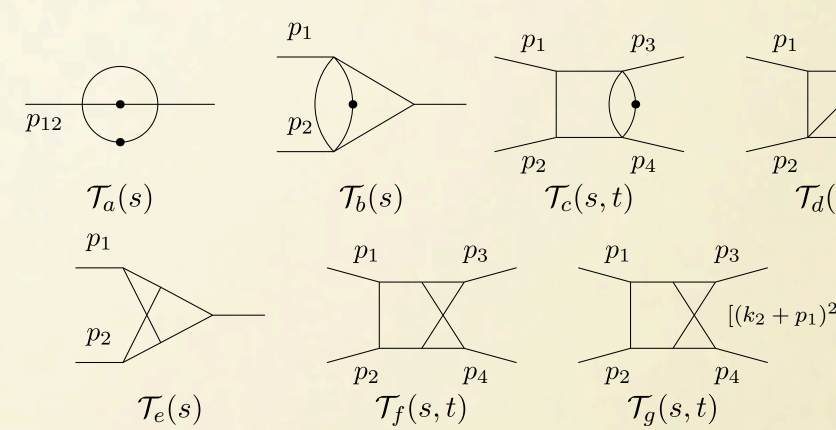


Tausk (1999) Anastasiou, Gehrmann, Oleari, Remiddi, Tausk (2000)

Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

- initial set of MI's

$$
\begin{aligned}
&f_1 = \epsilon^2\, s\, \mathcal{T}_a(s)\,, \quad f_2 = \epsilon^2\, t\, \mathcal{T}_a(t)\,, \quad f_3 = \epsilon^2\, u\, \mathcal{T}_a(u)\,,\\
&f_4 = \epsilon^3\, s\, \mathcal{T}_b(s)\,, \quad f_5 = \epsilon^3\, s\, t\, \mathcal{T}_c(s,t)\,, \quad f_6 = \epsilon^3\, s\, u\, \mathcal{T}_c(s,u)\,,\\
&f_7 = \epsilon^4\, u\, \mathcal{T}_d(s,t)\,, \quad f_8 = \epsilon^4\, s\, \mathcal{T}_d(t,u)\,, \quad f_9 = \epsilon^4\, t\, \mathcal{T}_d(u,s)\,,\\
&f_{10} = \epsilon^4\, s^2\, \mathcal{T}_e(s)\,,\\
&f_{11} = \epsilon^4\, s\, t\, u\, \mathcal{T}_f(s,t) - \frac{3}{4\, s\, (4\epsilon+1)} \left[\epsilon^2 \left(s^2\, \mathcal{T}_a(s) + t^2\, \mathcal{T}_a(t) + u^2\, \mathcal{T}_a(u)\right)\right.\\
&\qquad \left. - 4\epsilon^4 \left(u^2\, \mathcal{T}_d(s,t) + s^2\, \mathcal{T}_d(t,u) + t^2\, \mathcal{T}_d(u,s)\right)\right]\,,\\
&f_{12} = \epsilon^4\, s\, t\, \mathcal{T}_g(s,t) - \frac{3}{8\, u\, (4\epsilon+1)} \left[\epsilon^2 \left(s^2\, \mathcal{T}_a(s) + t^2\, \mathcal{T}_a(t) + u^2\, \mathcal{T}_a(u)\right)\right.\\
&\qquad \left. - 4\epsilon^4 \left(u^2\, \mathcal{T}_d(s,t) + s^2\, \mathcal{T}_d(t,u) + t^2\, \mathcal{T}_d(u,s)\right)\right]\,,
\end{aligned}
$$

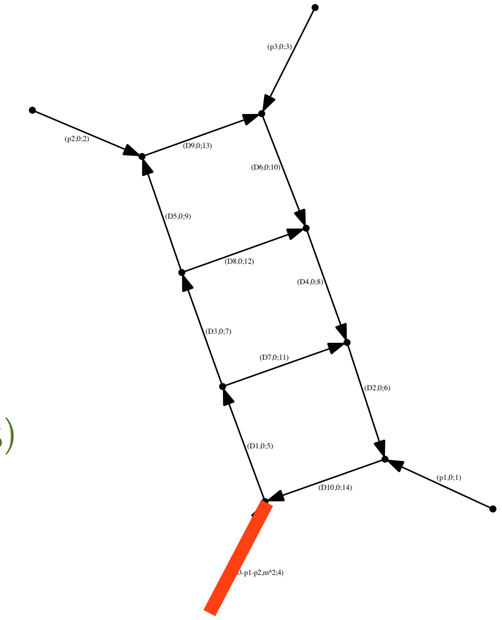


- after *Magnus' rotation*

$$
\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x)\ , \qquad \hat{A}(x) = \frac{M_1}{x} + \frac{M_2}{1-x}\,,
$$

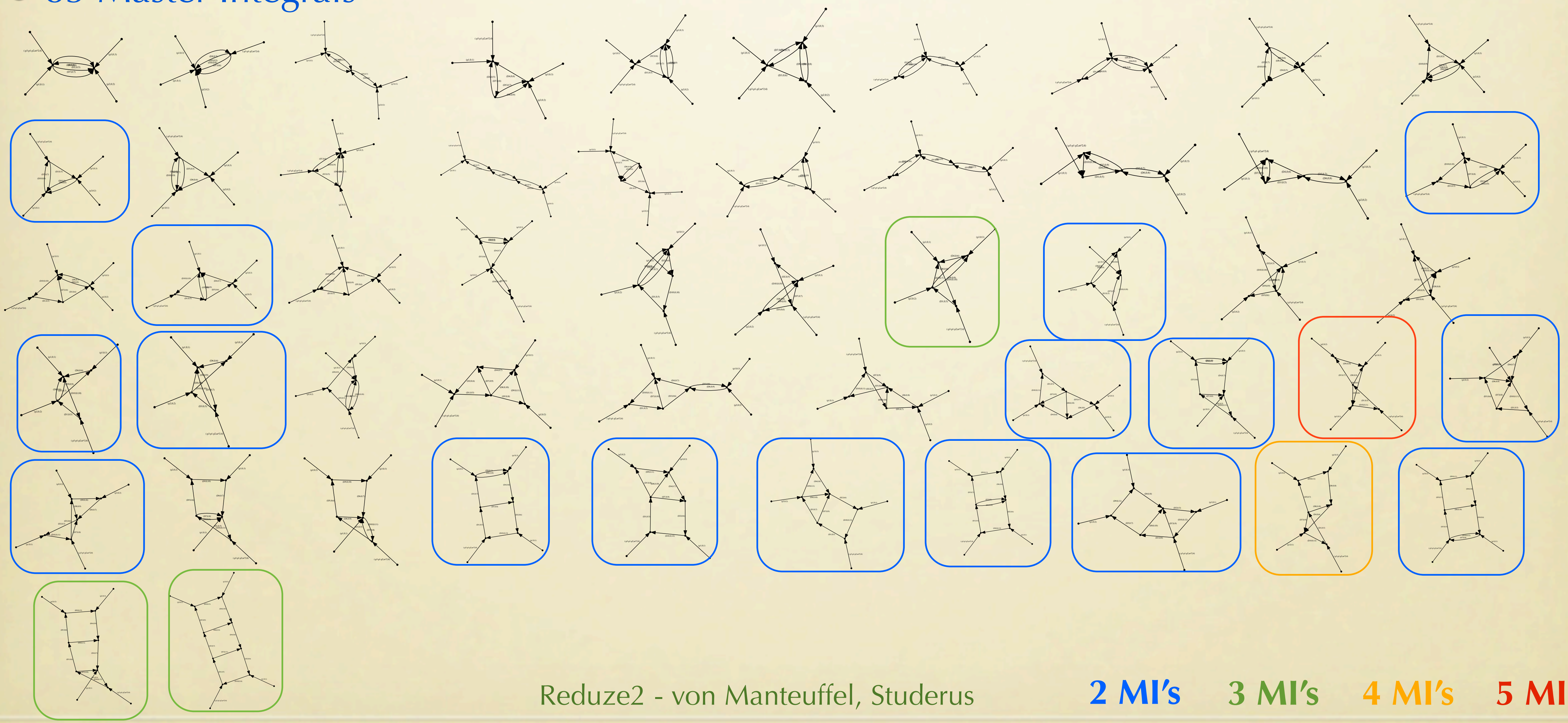
$$
M_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{3}{2} & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{3}{2} & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-6 & -6 & -\frac{9}{2} & 0 & -4 & -2 & -18 & -12 & -12 & 1 & 1 & -2\\
\frac{3}{4} & \frac{9}{4} & -\frac{21}{4} & 3 & 2 & -3 & 12 & -6 & -18 & 0 & 0 & -2
\end{pmatrix},
\qquad
M_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{3}{2} & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{3}{2} & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0\\
\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-6 & -6 & -\frac{9}{2} & 0 & -4 & -2 & -18 & -12 & -12 & 1 & 1 & -2\\
-\frac{21}{4} & \frac{9}{4} & -\frac{27}{4} & -6 & 2 & -4 & 12 & -6 & -24 & 1 & -1 & 0
\end{pmatrix}
$$

3-Loop 1-mass Ladder Box



Di Vita, Schubert, Yundin, **P.M.** (in progress)

- 85 Master Integrals



Reduze2 - von Manteuffel, Studerus

2 MI's 3 MI's 4 MI's 5 MI's

- eps-linear basis

$$\partial_x\ f(x,y,\epsilon) = \Big(A_{10}(x,y) + \epsilon A_{11}(x,y)\Big)\ f(x,y,\epsilon)$$
$$\partial_y\ f(x,y,\epsilon) = \Big(A_{20}(x,y) + \epsilon A_{21}(x,y)\Big)\ f(x,y,\epsilon)$$

- canonical form: Magnus #1

$$\partial_x\ g(x,y,\epsilon) = \epsilon \hat{A}_1(x,y)\ g(x,y,\epsilon)$$
$$\partial_y\ g(x,y,\epsilon) = \epsilon \hat{A}_2(x,y)\ g(x,y,\epsilon)$$

- dLog

$$dg(x,y,\epsilon) = \epsilon\ d\hat{\mathcal{A}}(x,y)\ g(x,y,\epsilon)\ , \qquad d\hat{\mathcal{A}} \equiv \hat{A}_1 dx + \hat{A}_2 dy$$

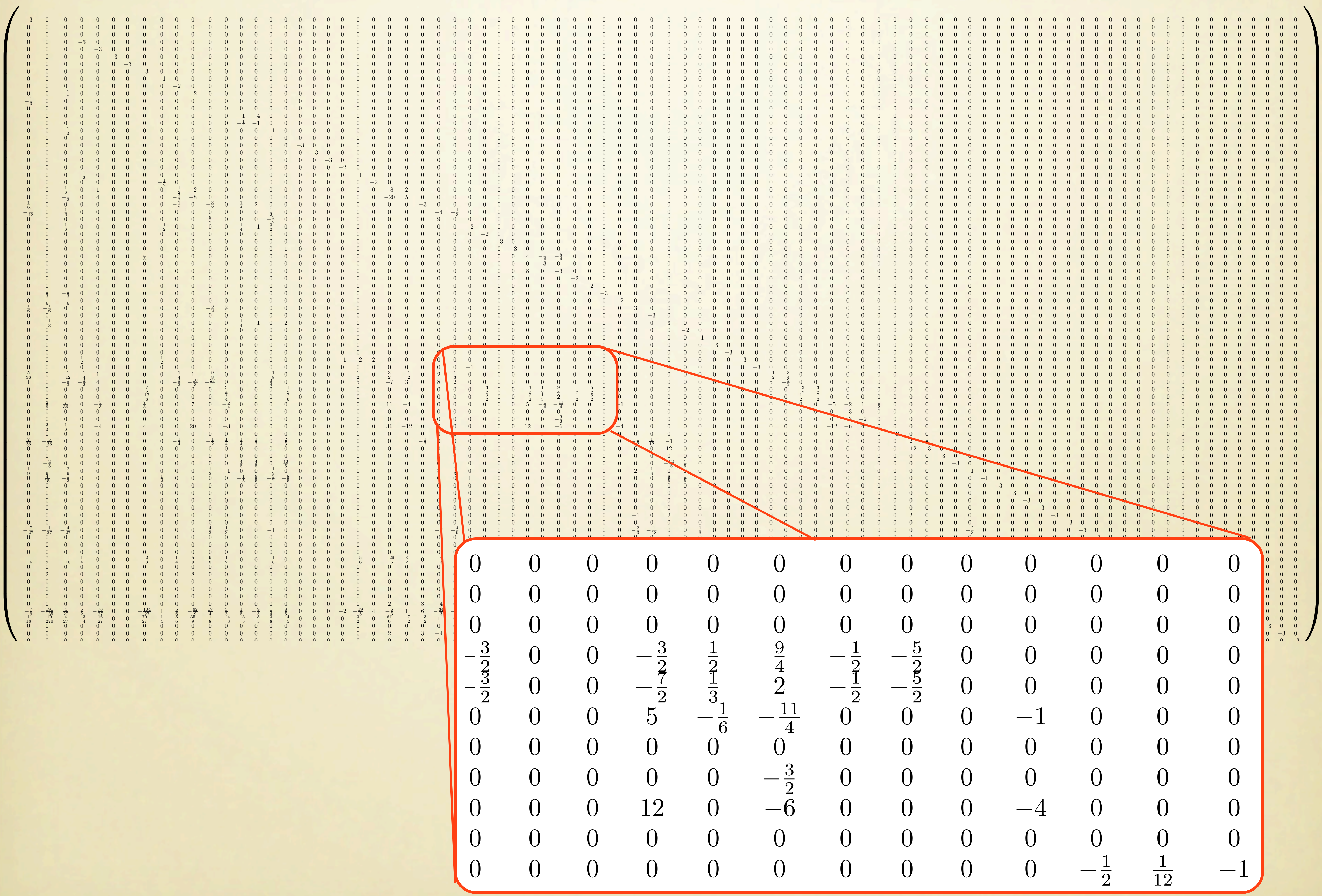
- alphabet

$$\{x, 1-x, y, 1-y, 1-x-y, x+y\}$$

$1/x$

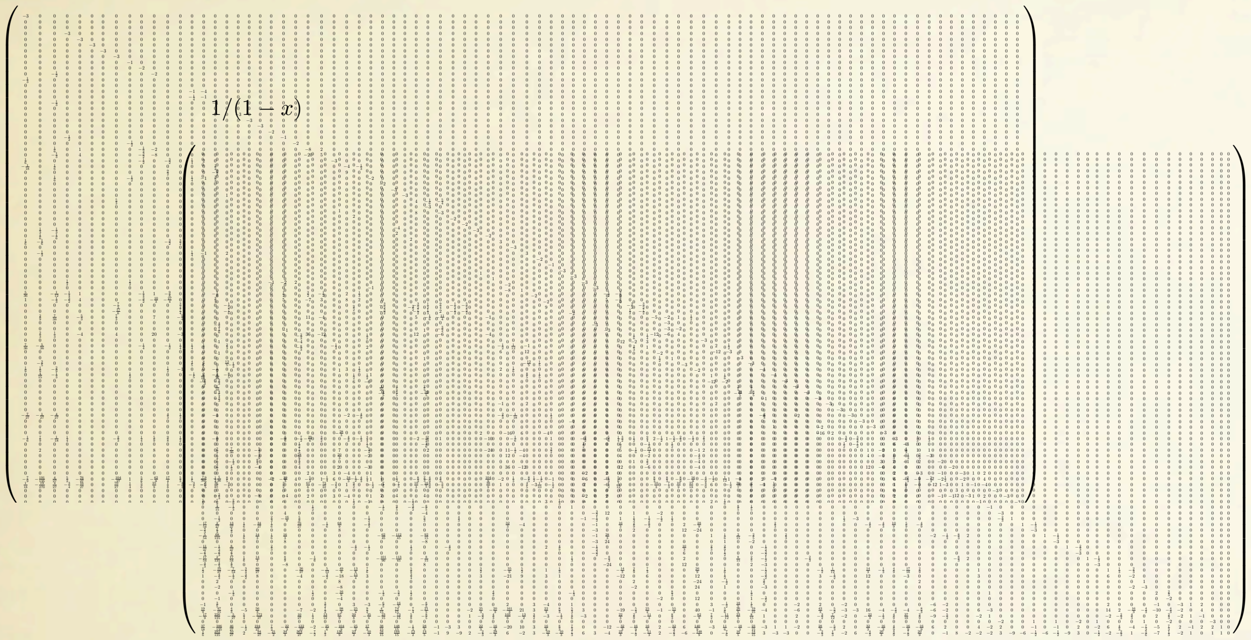
[illegible]

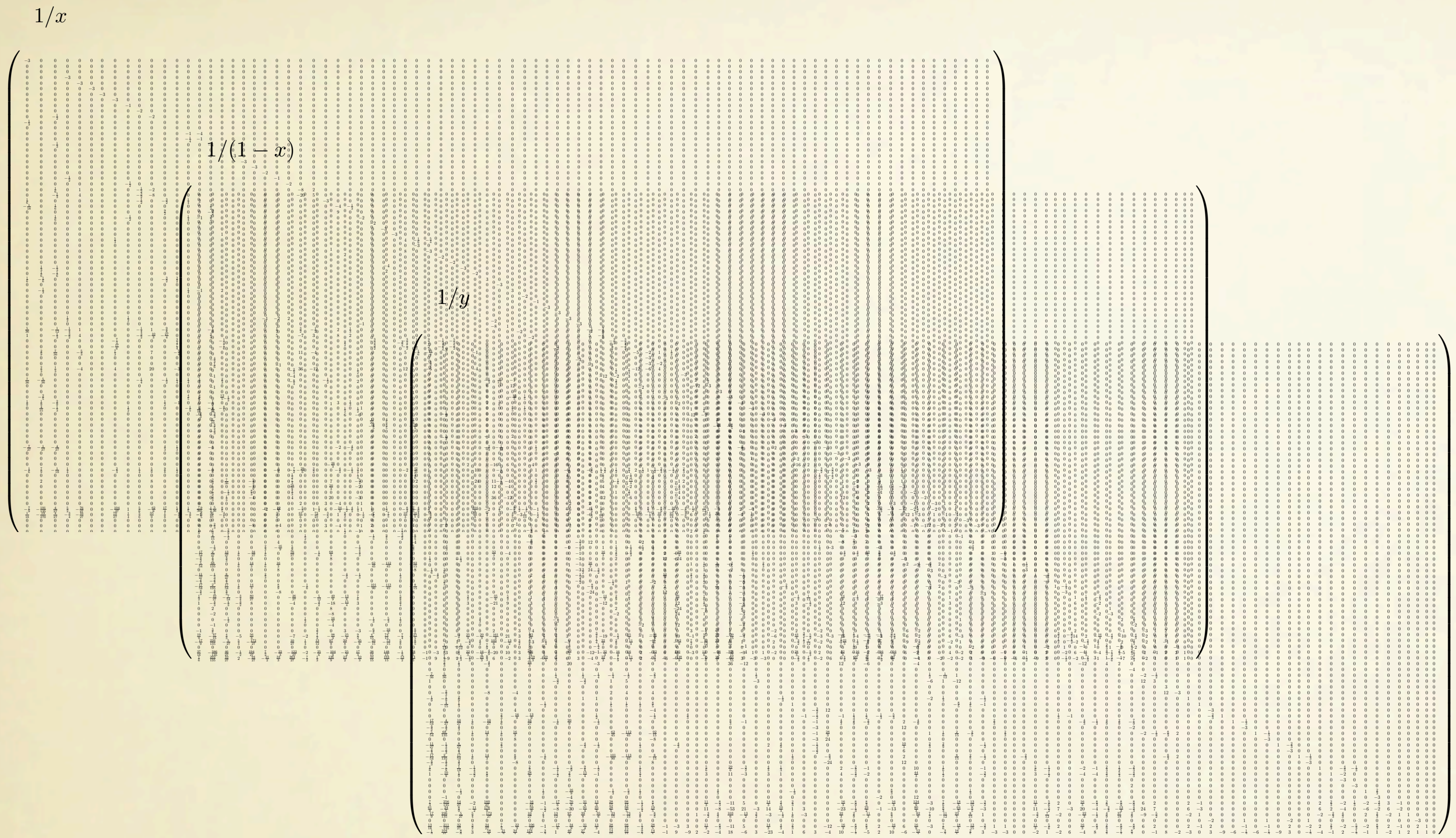
$1/x$

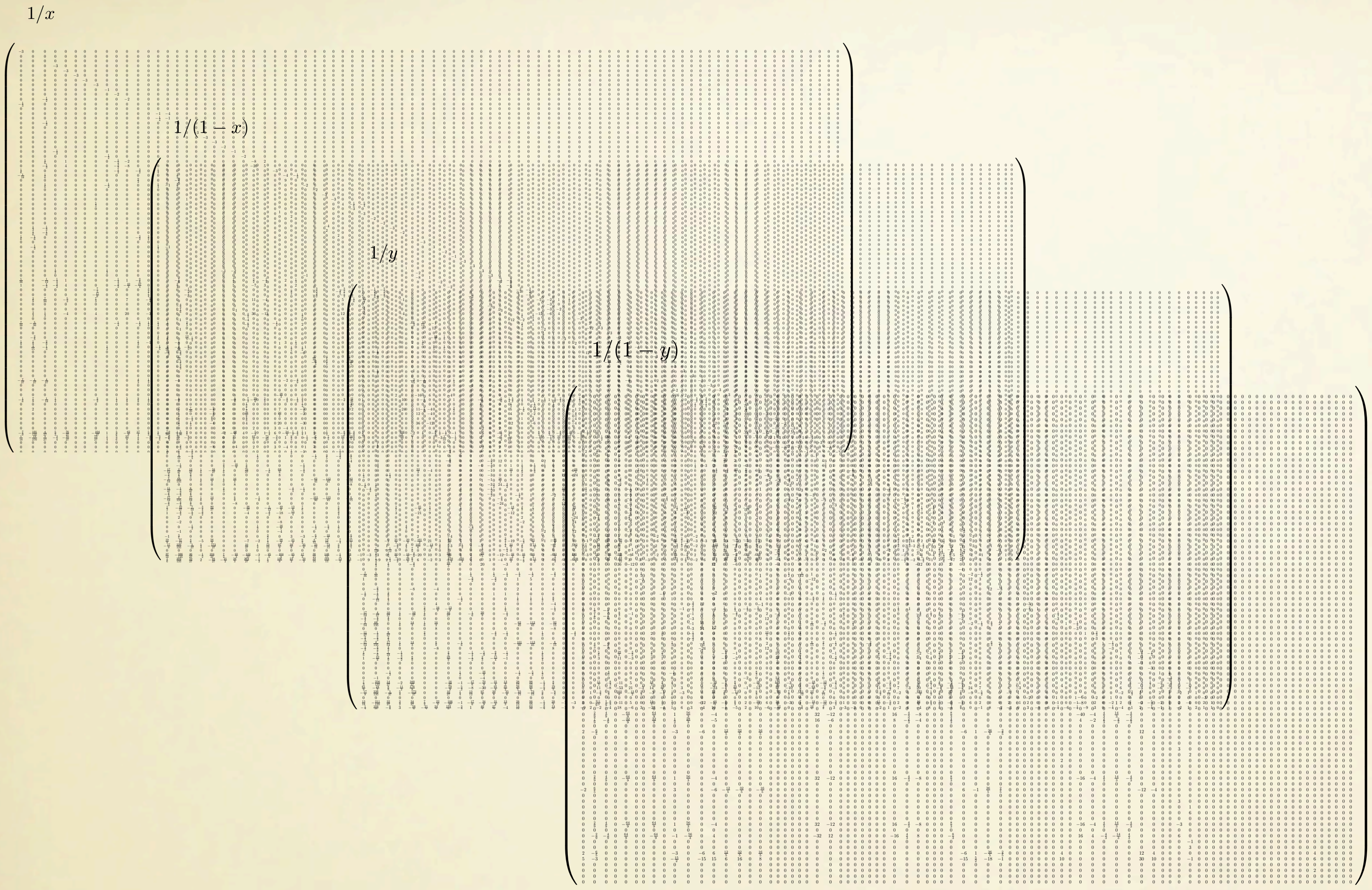

0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0
−3/2 0 0 −3/2 1/2 9/4 −1/2 −5/2 0 0 0 0 0
−3/2 0 0 −7/2 1/3 2 −1/2 −5/2 0 0 0 0 0
0 0 0 5 −1/6 −11/4 0 0 0 −1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 −3/2 0 0 0 0 0 0 0
0 0 0 12 0 −6 0 0 0 −4 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 −1/2 1/12 −1

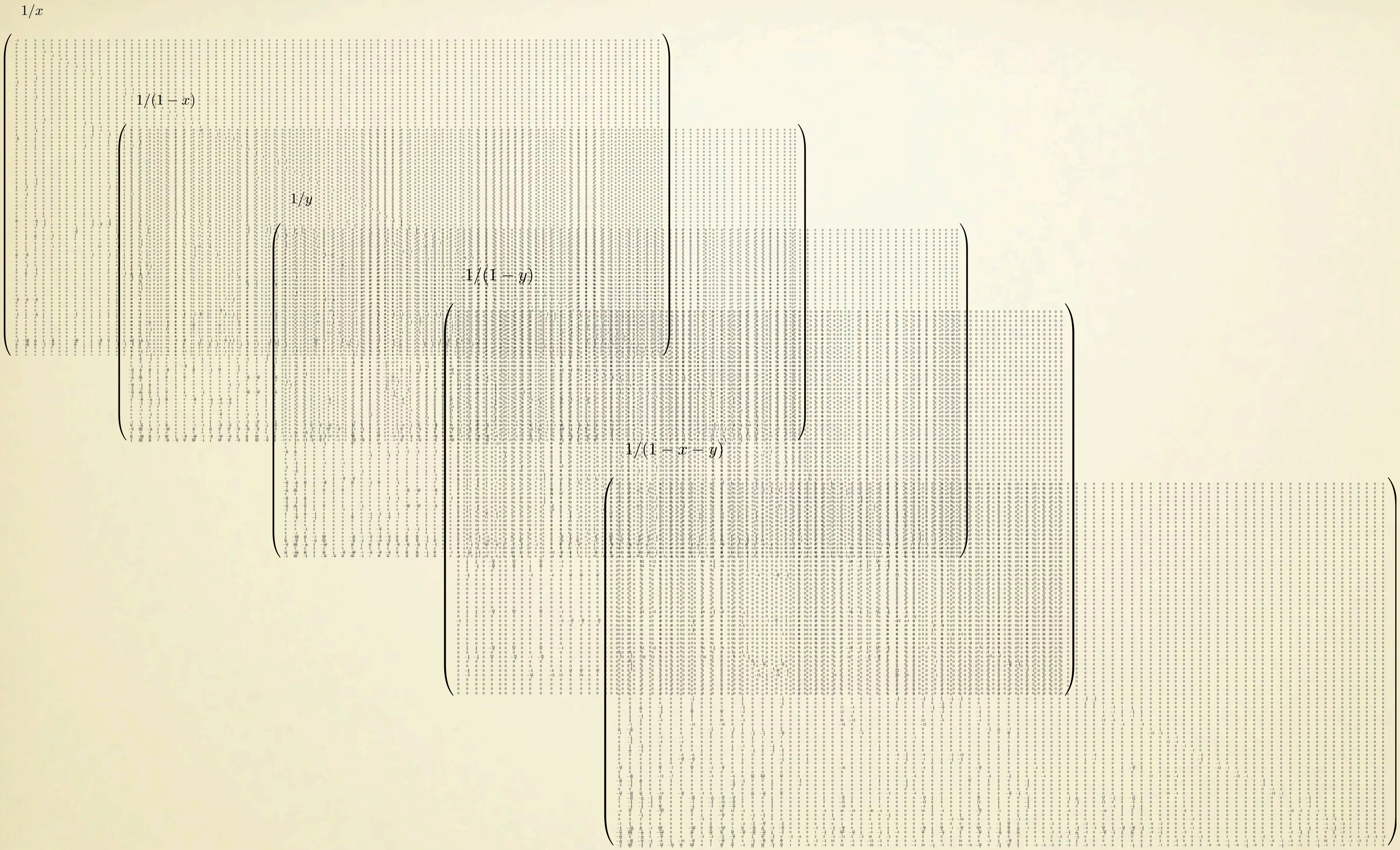
$1/x$

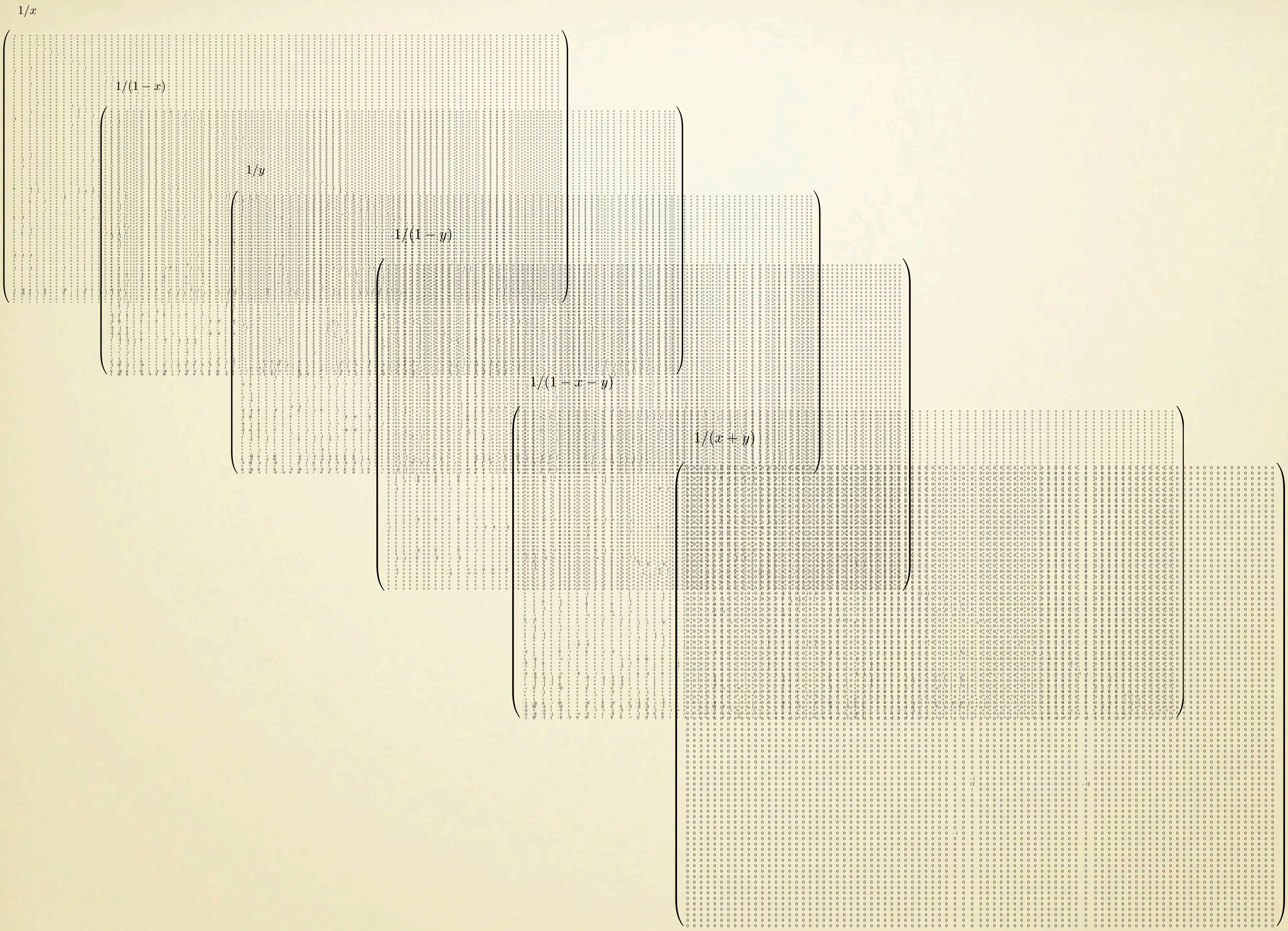
$1/(1-x)$

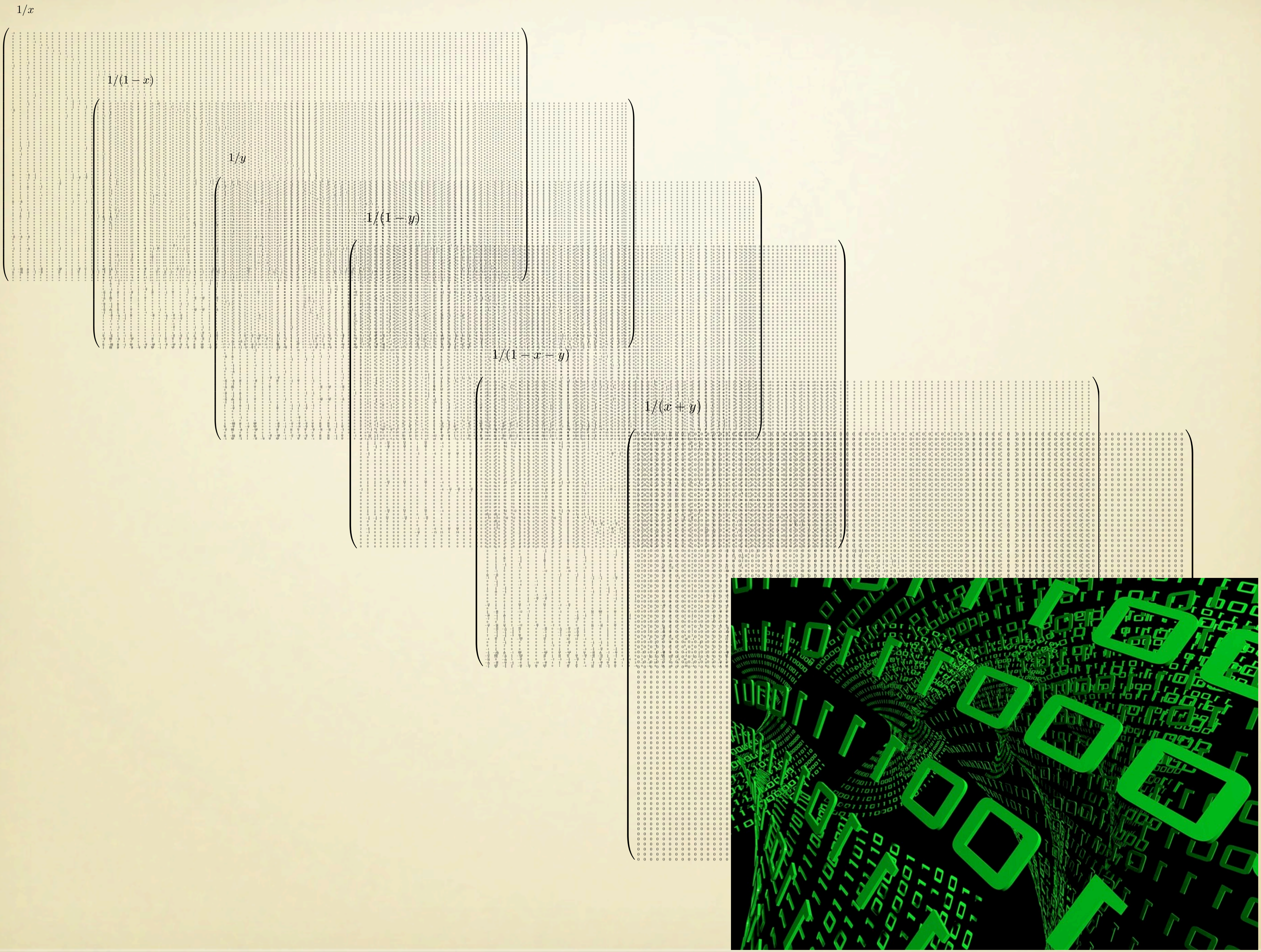



$1/x$
$1/(1-x)$
$1/y$

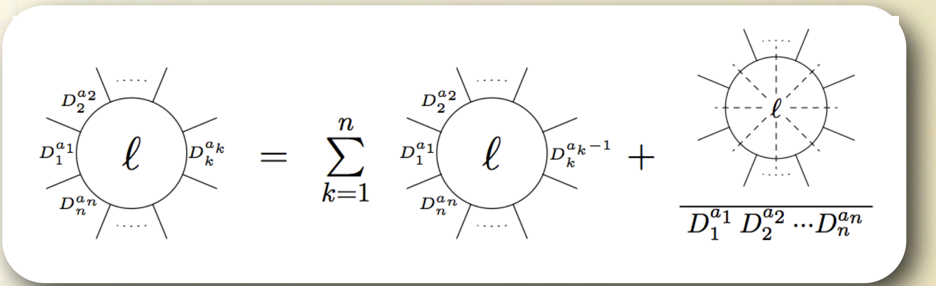

1/x
1/(1 − x)
1/y
1/(1 − y)


$1/x$
$1/(1-x)$
$1/y$
$1/(1-y)$
$1/(1-x-y)$


1/x
1/(1−x)
1/y
1/(1−y)
1/(1−x−y)
1/(x+y)

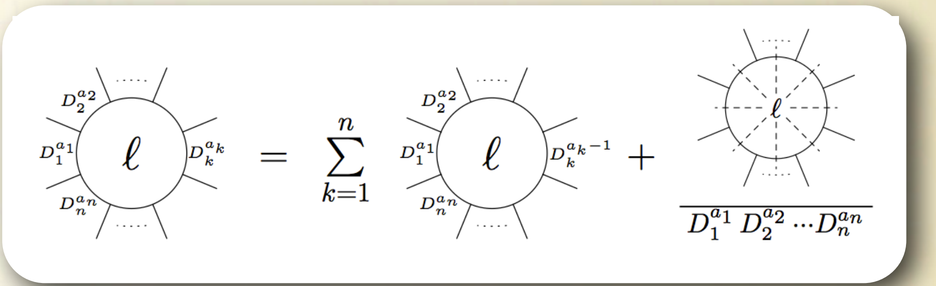

1/x
1/(1 − x)
1/y
1/(1 − y)
1/(1 − x − y)
1/(x + y)

Conclusions



- **a new tool for the Decomposition of Scattering Amplitudes**
  - Multivariate Polynomial Division
    - one ingredient: Feynman denominator
    - one operation: *partial fractioning*
  - Dimensional Recurrence at the integrand level
  - embedding: Unitarity, Factorization, and loop-momentum shift invariance
  - Minimal set of MI's
- **key ideas**
  - D-shifted Master Integrals
  - Schouten Polynomials/Gram-determinants in 4- and (-2e)-dimensions
- **results**
  - Purely algebraic procedure to detect MI's
  - A new, simple operator for Dimension-raising: Schouten Polynomials
- **geometry beneath**
  - Algebraic Geometry and Theory of Invariants
  - Gram-determinants ~ (iper)Volumes of polyhedra (<= Amplituhedron?)

Conclusions



- **a new tool for the Decomposition of Scattering Amplitudes**
  - Multivariate Polynomial Division
  - Dimensional Recurrence at the integrand level
  - embedding: Unitarity, Factorization, and loop-momentum shift invariance
  - Minimal set of MI's
  - Advantage: purely algebraic

- **key ideas**
  - D-shifted Master Integrals
  - Schouten Polynomials/Gram-determinants in 4- and (-2e)-dimensions

- **Differential Equations for MI's**
  - D-Linear systems + Magnus Exponential: Canonical Form
  - D-shifted MI's may play a role here as well
  - New application: 3-Loop 1-Mass Ladder (and its chain of subtopologies)

EXTRA Slides

- Residues

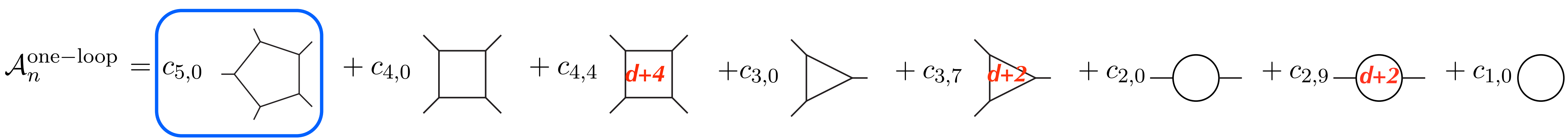
$$\boxed{\Delta_{i_1 i_2 i_3 i_4 i_5} = c_0}$$

$$\Delta_{i_1 i_2 i_3 i_4} = c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4)$$

$$\Delta_{i_1 i_2 i_3} = c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4)$$

$$\Delta_{i_1 i_2} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2$$

$$\Delta_{i_1} = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$$



- Residues

Samurai

Ossola, Reiter, Tramontano, & ***P.M.***

Ninja

Peraro

Mirabella, Peraro, & ***P.M.***

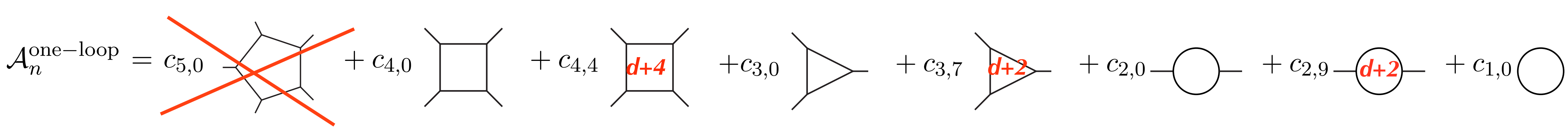
$$\boxed{\Delta_{i_1 i_2 i_3 i_4 i_5} = c_0 \, \mu^2}$$

$$\Delta_{i_1 i_2 i_3 i_4} = c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4)$$

$$\Delta_{i_1 i_2 i_3} = c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4)$$

$$\Delta_{i_1 i_2} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2$$

$$\Delta_{i_1} = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$$



- PV decomposition

$$I_n^{D=4-2\epsilon}[\bar{q}^\mu \bar{q}^\nu] = A_{2,0}\ \bar{g}^{\mu\nu} + \sum_{ij} A_{2,ij}\ p_i^\mu p_j^\nu$$

Contracting by $g^{\mu\nu}_{[-2\epsilon]}$:

$$I_n^{4-2\epsilon}[\mu^2] = A_{2,0}(2\epsilon) = (-\epsilon) I_n^{6-2\epsilon} \qquad \Rightarrow \qquad A_{2,0} = -\frac{1}{2} I_n^{6-2\epsilon}$$

Contracting by $v^\mu_{\perp,1} v^\nu_{\perp,2}$ with $(v_{\perp,i} \cdot p_j = 0)$:

$$I_n^{4-2\epsilon}[(v_{\perp,1} \cdot q)(v_{\perp,2} \cdot q)] = A_{2,0}(v_{\perp,1} \cdot v_{\perp,2})$$

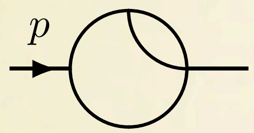
$$\Rightarrow \quad \frac{1}{(v_{\perp,1} \cdot v_{\perp,2})} I_n^{4-2\epsilon}[(v_{\perp,1} \cdot q)(v_{\perp,2} \cdot q)] = -\frac{1}{2} I_n^{6-2\epsilon}$$

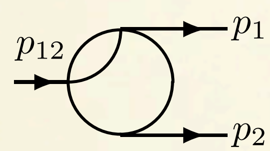
- From $D \to D+2$: integrand generation of $I_n^{6-2\epsilon}$:

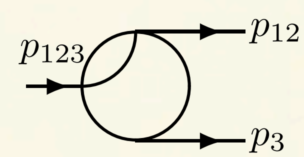
$$I_n^{4-2\epsilon}[\mu^2] = (-\epsilon) I_n^{6-2\epsilon}\ , \qquad \frac{1}{(v_{\perp,1} \cdot v_{\perp,2})} I_n^{4-2\epsilon}[(v_{\perp,1} \cdot q)(v_{\perp,2} \cdot q)] = -\frac{1}{2} I_n^{6-2\epsilon}$$

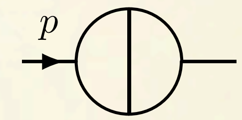
$$\text{(tadpole)} \quad I_1^{4-2\epsilon}[q^2] = -2 I_1^{6-2\epsilon}$$

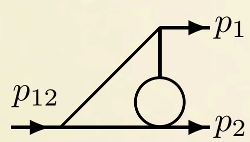
Examples of Reducible Integrals

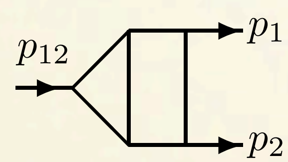


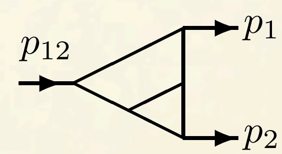


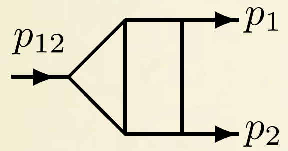












Schouten =

+ D6 * (1/8*mu12*mH^4 + 1/8*q1.e3*q2.e4*mH^4 + 1/8*q1.e4*q2.e3*mH^4)

+ D4 * (1/4*q1.e3*q2.e4*q2.k1*mH^2 + 1/4*q1.e4*q2.e3*q2.k1*mH^2 + 1/4*q2.k1*mu12*mH^2)

+ D3 * (- 1/8*mu12*mH^4 - 1/4*q1.e3*q2.e4*q2.k1*mH^2 - 1/8*q1.e3*q2.e4*mH^4 - 1/4*q1.e4*q2.e3*q2.k1*mH^2 - 1/8*q1.e4*q2.e3*mH^4 - 1/4*q2.k1*mu12*mH^2)

+ D3*D5 * (1/8*mu12*mH^2 + 1/8*q1.e3*q2.e4*mH^2 + 1/8*q1.e4*q2.e3*mH^2 + q1.k1*q2.k1 + 1/2*q2.k1*mH^2)

+ D3*D4*D5 * (- 1/2*q2.k1)

+ D2 * (- 1/8*q2.k2*mH^4)

+ D2*D6 * (- 1/16*mH^4)

+ D2*D5 * (1/16*mH^4)

+ D2*D4 * (- 1/8*q2.k1*mH^2)

+ D2*D3 * (- 3/16*mH^4 - 1/8*mu12*mH^2 - 1/8*q1.e3*q2.e4*mH^2 - 1/8*q1.e4*q2.e3*mH^2 - q1.k1*q2.k1 - 1/2*q1.k1*mH^2 - 3/8*q2.k1*mH^2 - 1/8*q2.k2*mH^2)

+ D2*D3*D4 * (1/4*mH^2 + 1/2*q2.k1)

+ D1*D5 * (- 1/8*mu12*mH^2 - 1/8*q1.e3*q2.e4*mH^2 - 1/8*q1.e4*q2.e3*mH^2)

+ D1*D4*D5 * (1/2*q2.k1)

+ D1*D2 * (1/8*mu12*mH^2 + 1/8*q1.e3*q2.e4*mH^2 + 1/8*q1.e4*q2.e3*mH^2 + 1/8*q2.k2*mH^2)

+ D1*D2*D4 * (- 1/4*mH^2 - 1/2*q2.k1);