The superstring 3-loop amplitude

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Introduction

- Compute the N-point superstring amplitude for any number of loops
- The 4-point amplitude at 1- and 2-loops already computed by Berkovits with his pure spinor formalism (Berkovits '04, '05)
- The 4-point 3-loop amplitude was the next step. Developed some techniques, learned a few tricks and we should be better prepared for the difficulties ahead
- The overall coefficient was determined from first principles and compared with a prediction by Green and Vanhove from 2005 (based on S-duality arguments)
- Naive mismatch by a factor 3.

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Pure spinor formalism (Berkovits '00, '05)

$$S = \frac{1}{2\pi\alpha'} \int_{\Sigma_g} d^2z \left(\partial x^m \overline{\partial} x_m + \alpha' p_\alpha \overline{\partial}\theta^\alpha - \alpha' w_\alpha \overline{\partial}\lambda^\alpha - \alpha' \overline{w}^\alpha \overline{\partial\lambda_\alpha} + \alpha' s^\alpha \overline{\partial} r_\alpha\right)$$

- Pure spinors λ^α, $\overline{\lambda}_\alpha$ with conjugate momenta w_α, $\overline{w}^\alpha$:

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$$

- r_α constrained by $\overline{\lambda}\gamma^m r = 0$
- Manifest spacetime SUSY, pure spinor superspace (PSS): $(\lambda^\alpha, \theta^\alpha)$
- Covariant BRST Quantization: $Q = \lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha$
- Free CFT, simple OPEs

Pure spinor formalism

- Massless vertex operators

$$V = \lambda^\alpha A_\alpha(x,\theta), \qquad QV = 0$$

$$U = \partial\theta^\alpha A_\alpha + A_m\Pi^m + d_\alpha W^\alpha + \frac{1}{2}N^{mn}\mathcal{F}_{mn}, \qquad QU = \partial V$$

- $[A_\alpha, A_m, W^\alpha, \mathcal{F}_{mn}]$ are 10D superfields describing gluon and gluino

D=10 SYM theory

- Simple spectrum: gluon A_m and gluino χ^α

$$S = \int F^2 + \chi^\alpha \gamma^m_{\alpha\beta} D_m \chi^\beta$$

- Compactification leads to lower-dim SYMs, eg $\mathcal{N} = 4$ in 4D
- Contained in the low-energy limit of the open superstring
- Long-term goal: derive all the $\mathcal{N} = 4$ and $\mathcal{N} = 8$ amplitudes from string theory
- Recall that the 4-point amplitudes at 1-loop in both $\mathcal{N} = 4$ and $\mathcal{N} = 8$ were firstly computed from string theory (Brink, Green, Schwarz '81)

D=10 SYM theory

- Covariant description using 10D superfields (Siegel '79, Witten '86)

$$A_\alpha(x,\theta), A_m(x,\theta), W^\alpha(x,\theta), \mathcal{F}_{mn}(x,\theta)$$

- Linearized equations of motion (for one-particle states)

$$D_\alpha A_\beta + D_\beta A_\alpha = \gamma^m_{\alpha\beta} A_m, \quad D_\alpha A_m = (\gamma_m W)_\alpha + \partial_m A_\alpha$$
$$D_\alpha W^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathcal{F}_{mn}, \quad D_\alpha \mathcal{F}_{mn} = 2\partial_{[m}(\gamma_{n]} W)_\alpha$$

where D_α is covariant derivative, $\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m$.

Pure spinor BRST blocks (CM, Schlotterer '14)

- In computing higher-point string amplitudes, need to evaluate OPEs among vertex operators
- Recall integrated vertex

$$U = \partial\theta^\alpha A_\alpha + A_m \Pi^m + d_\alpha W^\alpha + \frac{1}{2} N^{mn} F_{mn}$$

- The idea is to note that the simple pole in the OPE $U^1(z_1)U^2(z_2)$ can be written as

$$U_{12} \equiv \partial\theta^\alpha A^{12}_\alpha + \Pi^m A^{12}_m + d_\alpha W^\alpha_{12} + \frac{1}{2} N^{mn} \mathcal{F}^{12}_{mn}$$

Multiparticle SYM equations of motion

where

$$A^{12}_\alpha = -\frac{1}{2}\big[A^1_\alpha(k^1\cdot A^2) + A^1_m(\gamma^m W^2)_\alpha - (1\leftrightarrow 2)\big]$$
$$A^{12}_m = \frac{1}{2}\Big[A^1_p\mathcal{F}^2_{pm} - A^1_m(k^1\cdot A^2) + (W^1\gamma_m W^2) - (1\leftrightarrow 2)\Big]$$
$$W^\alpha_{12} = \frac{1}{4}(\gamma^{mn}W^2)^\alpha\mathcal{F}^1_{mn} + W^\alpha_2(k^2\cdot A^1) - (1\leftrightarrow 2)$$
$$\mathcal{F}^{12}_{mn} = k^{12}_m A^{12}_n - k^{12}_n A^{12}_m - (k^1\cdot k^2)(A^1_m A^2_n - A^1_n A^2_m).$$

satisfy the following equations of motion

$$2D_{(\alpha}A^{12}_{\beta)} = \gamma^m_{\alpha\beta}A^{12}_m + (k^1\cdot k^2)(A^1_\alpha A^2_\beta + A^1_\beta A^2_\alpha)$$
$$D_\alpha A^{12}_m = (\gamma_m W^{12})_\alpha + k^{12}_m A^{12}_\alpha + (k^1\cdot k^2)(A^1_\alpha A^2_m - A^2_\alpha A^1_m)$$
$$D_\alpha W^\beta_{12} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathcal{F}^{12}_{mn} + (k^1\cdot k^2)(A^1_\alpha W^\beta_2 - A^2_\alpha W^\beta_1)$$
$$D_\alpha F^{12}_{mn} = k^{12}_m(\gamma_n W^{12})_\alpha - k^{12}_n(\gamma_m W^{12})_\alpha + (k^1\cdot k^2)(A^1_\alpha\mathcal{F}^2_{mn} - A^2_\alpha\mathcal{F}^1_{mn})$$

Multiparticle SYM equations of motion and BRST blocks

- Straightforward to generalize (without computing OPEs)
- Example at rank-three

$$\hat{A}^{123}_\alpha = -\frac{1}{2}\big[A^{12}_\alpha(k^{12}\cdot A^3) + A^{12}_m(\gamma^m W^3)_\alpha - (12\leftrightarrow 3)\big]$$

$$\begin{aligned} D_\beta \hat{A}^{123}_\alpha + D_\alpha \hat{A}^{123}_\beta = {} & \gamma^m_{\alpha\beta}\hat{A}^{123}_m \\ & + (k^1\cdot k^2)\big[A^1_\alpha A^{23}_\beta + A^{13}_\alpha A^2_\beta - (1\leftrightarrow 2)\big] \\ & + (k^{12}\cdot k^3)\big[A^{12}_\alpha A^3_\beta - (12\leftrightarrow 3)\big] \end{aligned}$$

- Recursive construction of BRST blocks $K_B \in \{A^B_\alpha,\ A^m_B,\ W^\alpha_B,\ \mathcal{F}^{mn}_B\}$ with multiparticle label B obeying generalized SYM equations of motion

BRST blocks and BCJ identities

- The BRST blocks K_B satisfy Lie symmetries

$$
\begin{aligned}
0 &= K_{12} + K_{21}, \quad \text{(antisymmetry)} \\
0 &= K_{123} + K_{231} + K_{312}, \quad \text{(Jacobi identity)} \\
0 &= K_{1234} - K_{1243} + K_{3412} - K_{3421} \\
0 &= \text{general pattern known}
\end{aligned}
$$

- One can associate

$$K_{1234\ldots p} \leftrightarrow \; f^{12a_2} f^{a_2 3 a_3} f^{a_3 4 a_4} \ldots f^{a_{p-1} p a_p}$$

- Lie symmetries in the fundamentals of SYM theory!
- Connection with BCJ identities (Bern, Carrasco, Johansson '08)
- Hint of a kinematic algebra from OPEs of string theory?

Loop amplitude prescription

$$\mathcal{A}_g = S_g \kappa^4 e^{4\lambda} \int_{\mathcal{M}_g} \prod_{j=1}^{3g-3} d^2\tau_j \int_{\Sigma_4} \left| \langle \mathcal{N}(b,\mu_j)\, U^1(z_1) \ldots U^4(z_4) \rangle \right|^2$$

- b-ghost insertion

$$(b,\mu_j) = \frac{1}{2\pi} \int d^2 y_j b_{zz} \mu_{j\,\bar{z}}^{z}, \quad j = 1, \ldots, 3g-3$$

$$b = \frac{1}{2(\lambda\bar{\lambda})} \Pi^m (\bar{\lambda}\gamma_m d) + \frac{(\bar{\lambda}\gamma^{mnp} r)}{192(\lambda\bar{\lambda})^2} \frac{\alpha'}{2} (d\gamma_{mnp} d) + \ldots$$

- Zero-mode regulator

$$\mathcal{N} = \sum_{I=1}^{g} e^{-(\lambda\bar{\lambda}) - (w^I \overline{w}^I) - (r\theta) + (s^I d^I)}$$

Loop amplitude prescription

- Zero-mode integrations

$$\langle \ldots \rangle = \int [d\theta][dr][d\lambda][d\overline{\lambda}] \prod_{I=1}^{3} [dd^I][ds^I][d\overline{w}^I][dw^I]$$

- with measures (CM, Gomez '10)

$$[d\lambda]\, T_{\alpha_1 \ldots \alpha_5} = \left(\frac{\alpha'}{2}\right)^{-2} \frac{1}{11!} \left(\frac{A_g}{4\pi^2}\right)^{11/2} \epsilon_{\alpha_1 \ldots \alpha_{16}}\, d\lambda^{\alpha_6} \ldots d\lambda^{\alpha_{16}}$$

$$[d\overline{\lambda}]\, \overline{T}^{\alpha_1 \ldots \alpha_5} = \left(\frac{\alpha'}{2}\right)^{2} \frac{2^6}{11!} \left(\frac{A_g}{4\pi^2}\right)^{11/2} \epsilon^{\alpha_1 \ldots \alpha_{16}}\, d\overline{\lambda}_{\alpha_6} \ldots d\overline{\lambda}_{\alpha_{16}}$$

- Gaussian integration over pure spinors (Gomez '09)

$$\int [d\lambda][d\overline{\lambda}](\lambda\overline{\lambda})^n e^{-(\lambda\overline{\lambda})} = \left(\frac{A_g}{2\pi}\right)^{11} \frac{\Gamma(8+n)}{7!\, 60}$$

String amplitudes versus S-duality

4-point amplitudes at 0-, 1- and 2-loops (CM, Gomez '10):

$$\mathcal{A}_0 = \left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\kappa^4 e^{-2\lambda}\,\frac{\sqrt{2}}{2^{16}\pi^5}\left[\frac{3}{\sigma_3} + 2\zeta_3 + \zeta_5\sigma_2 + \frac{2}{3}\zeta_3^2\sigma_3 + \cdots\right.$$

$$\mathcal{A}_1 = \left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\kappa^4\,\frac{1}{2^{10}\,3\pi}\left[1 + \frac{\zeta_3}{3}\sigma_3 + \cdots\right.$$

$$\mathcal{A}_2 = \left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\kappa^4 e^{2\lambda}\,\frac{\sqrt{2}\pi^3}{2^6\,3^3\,5}\left[\sigma_2 + \cdots\right.$$

confirm the predictions based on S-duality (Green, Gutperle, Vanhove)

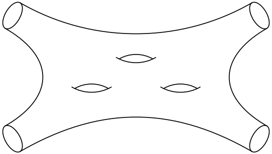
$$S^{\alpha'^3} = C_1\int d^{10}x\sqrt{-g}\,\mathcal{R}^4\left(2\zeta_3 e^{-2\phi} + \frac{2\pi^2}{3}\right)$$

$$S^{\alpha'^5} = C_2\int d^{10}x\sqrt{-g}\,D^4\mathcal{R}^4\left(2\zeta_5 e^{-2\phi} + \frac{8}{3}\zeta_4 e^{2\phi}\right)$$

String amplitudes versus S-duality

- Agreement up to 2-loops is very non-trivial
- What about 3-loops?

The 3-loop amplitude



The pure spinor formalism:

$$\mathcal{A}_3 = \frac{1}{3}\kappa^4 e^{4\lambda} \int_{\mathcal{M}_3} \prod_{j=1}^{6} d^2\tau_j \int_{\Sigma_4} \left|\langle \mathcal{N}(b,\mu_j)\, U^1(z_1)\ldots U^4(z_4)\rangle\right|^2$$

S-duality (Green, Vanhove '05)

$$S^{\alpha'^6} = C_3 \int d^{10}x\sqrt{-g}\, D^6\mathcal{R}^4(4\zeta_3^2 e^{-2\phi} + 8\zeta_2\zeta_3 + \frac{48}{5}\zeta_2^2 e^{2\phi} + \frac{8}{9}\zeta_6 e^{4\phi})$$

The 3-loop amplitude

- The computation is guided by the saturation of the $(16, 16, 16)$ d_α zero-modes

$$d_\alpha(z) = \hat{d}_\alpha(z) + d^I_\alpha w_I(z), \quad I = 1, 2, 3$$

- s^α has $(11, 11, 11)$ zero modes (instead of 16) and appears only in $\mathcal{N}$

$$\int \prod_{I=1}^{3} [ds^I]\, e^{-(d^I s^I)} = \left(\frac{\alpha'}{2}\right)^6 \frac{(2\pi)^{33/2} Z_3^{11}}{R^3 2^{18} (11!\,5!)^3 (\lambda\bar{\lambda})^9} \prod_{I=1}^{3} (\epsilon \cdot T \cdot d^I)$$

$$Z_g = \frac{1}{\sqrt{\det(2\mathrm{Im}\Omega)}}, \quad T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = (\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}$$

- Brings down $(11, 11, 11)$ zero modes of d_α
- Need to saturate the remaining $(5, 5, 5)$ d zero modes

The b-ghost sector

Two possible contributions from $\int \prod_{j=1}^{6} d^2\tau_j \Big|(b,\mu_j)\Big|^2$:

- $(4,4,4)$:

$$= c_{b_1}^2 \int d^2\Omega_{IJ} \left| \frac{B_{(4,4,4)}}{(\lambda\bar{\lambda})^{12}} \right|^2$$

- $(3,4,4)$, $(4,3,4)$ or $(4,4,3)$:

$$= c_{b_2}^2 \int d^2\Omega_{IJ} \left| \frac{1}{(\lambda\bar{\lambda})^{11}} \big(\Pi^1_m B^m_{(3,4,4)} + \Pi^2_m B^m_{(4,3,4)} + \Pi^3_m B^m_{(4,4,3)}\big) \right|^2$$

The b-ghost sector

$$
\begin{aligned}
B_{(4,4,4)} &= (\bar\lambda r d^1 d^1)(\bar\lambda r d^1 d^2)(\bar\lambda r d^1 d^3)(\bar\lambda r d^2 d^2)(\bar\lambda r d^2 d^3)(\bar\lambda r d^3 d^3) \\
B^m_{(3,4,4)} &= +2(\bar\lambda\gamma^m d^1)(\bar\lambda r d^1 d^2)(\bar\lambda r d^1 d^3)(\bar\lambda r d^2 d^2)(\bar\lambda r d^2 d^3)(\bar\lambda r d^3 d^3) \\
&\quad - (\bar\lambda\gamma^m d^2)(\bar\lambda r d^1 d^1)(\bar\lambda r d^1 d^3)(\bar\lambda r d^2 d^2)(\bar\lambda r d^2 d^3)(\bar\lambda r d^3 d^3) \\
&\quad + (\bar\lambda\gamma^m d^3)(\bar\lambda r d^1 d^1)(\bar\lambda r d^1 d^2)(\bar\lambda r d^2 d^2)(\bar\lambda r d^2 d^3)(\bar\lambda r d^3 d^3)
\end{aligned}
$$

where

$$(\bar\lambda r d^I d^J) \equiv (\bar\lambda\gamma^{mnp} r)(d^I \gamma_{mnp} d^J)$$

- Teichmüller parameters $\longrightarrow$ period matrix

$$\int d^2 z\, w_I(z)\, w_J(z) \mu_i(z) = \frac{\delta\Omega_{IJ}}{\delta\tau_i}$$

$$\int \prod_{j=1}^{6} d^2\tau_j \left|\epsilon_{i_1\ldots i_6} \frac{\delta\Omega_{11}}{\delta\tau_{i_1}} \cdots \frac{\delta\Omega_{33}}{\delta\tau_{i_6}}\right|^2 = \int \prod_{I\leq J}^{3} d^2\Omega_{IJ}$$

The 3-loop amplitude

Up to now

$$A_{12} = c_{12} \int \frac{d^2\Omega_{IJ}}{Z_3^{22}} \int_{\Sigma_4} \langle \prod_{j=1}^{4} \mathrm{e}^{ik^j\cdot x^j} \rangle \Big| \langle D_{(15,15,15)} U^1 U^2 U^3 U^4 \rangle_{(-9)} \Big|^2$$

$$A_{11} = c_{11} \int \frac{d^2\Omega_{IJ}}{Z_3^{22}} \int_{\Sigma_4} \langle \prod_{j=1}^{4} \mathrm{e}^{ik^j\cdot x^j} \rangle$$

$$\times \Big| \langle (\Pi^1_m D^m_{(14,15,15)} + \Pi^2_m D^m_{(15,14,15)} + \Pi^3_m D^m_{(15,15,14)}) U^1 U^2 U^3 U^4 \rangle_{(-8)} \Big|^2$$

with coefficients

$$c_{12} = \frac{\sqrt{2}\, 2^{-101} \kappa^4 e^{4\lambda}}{\pi^{42}\, 3^{13} (11!\, 5!)^6} \left(\frac{\alpha'}{2}\right)^{24}$$

$$c_{11} = \frac{\sqrt{2}\, 2^{-93} \kappa^4 e^{4\lambda}}{\pi^{42}\, 3^{11} (11!\, 5!)^6} \left(\frac{\alpha'}{2}\right)^{22}$$

The b-ghost sector

- The b-ghost sector with $D_{(15,15,15)}$ requires $(1,1,1)$ d zero modes from the vertices; $U_1 U_2 U_3 U_4 =$

$$= \left(\frac{\alpha'}{2}\right)^4 \left[(dW_{12})(dW_3)(dW_4)\,\eta_{12} + (1,2|1,2,3,4)\right.$$
$$+ \left(\frac{\alpha'}{2}\right)^3 \sum_{I=1}^{3} \Pi_I^m w_I(z_1) A_m^1 (dW^2)(dW^3)(dW^4) + (1 \leftrightarrow 2,3,4)$$

- W_{ij} is a rank-two BRST block and η_{ij} is proportional to the worldsheet derivative of the Green's function

3-loop kinematic factors

Use the operator $D^m_{(p+11,q+11,r+11)} \equiv \int \prod_{I=1}^3 [dd^I](\epsilon \cdot T \cdot d^I) B^m_{(p,q,r)}$ to integrate remaining $(1,1,1)$ zero modes to get

$$D_{(15,15,15)}(dW_{ij})(dW_k)(dW_l) = (11!\,5!)^3 96^3\ c_d^3\ T_{ij,k,l}(\lambda, \overline{\lambda}, r)\Delta(z_j; z_k; z_l)$$
$$D_{(15,15,15)}(dW_i)(dW_j)(dW_k)A^m_l = (11!\,5!)^3 96^3\ c_d^3\ L^m_{ijkl}(\lambda, \overline{\lambda}, r)\Delta(z_j; z_k; z_l)$$

where $\Delta(z_i; z_j; z_k) \equiv \epsilon^{IJK} w_I(z_i) w_J(z_j) w_K(z_k)$ and

$$T_{ij,k,l}(\lambda, \overline{\lambda}, r) = (\overline{\lambda}\gamma^{abc} r)(\overline{\lambda}\gamma^{def} r)(\overline{\lambda}\gamma^{ghi} r)(\overline{\lambda}\gamma^{mnp} r)(\overline{\lambda}\gamma^{qrs} r)(\overline{\lambda}\gamma^{tuv} r)$$
$$\times (\lambda\gamma^{adefm}\lambda)(\lambda\gamma^{bghit}\lambda)(\lambda\gamma^{uqrsn}\lambda)(\lambda\gamma^c W_{ij})(\lambda\gamma^p W_k)(\lambda\gamma^v W_l)\,,$$
$$L^m_{ijkl}(\lambda, \overline{\lambda}, r) = (\overline{\lambda}\gamma^{abc} r)(\overline{\lambda}\gamma^{def} r)(\overline{\lambda}\gamma^{ghi} r)(\overline{\lambda}\gamma^{mnp} r)(\overline{\lambda}\gamma^{qrs} r)(\overline{\lambda}\gamma^{tuv} r)$$
$$\times (\lambda\gamma^{adefm}\lambda)(\lambda\gamma^{bghit}\lambda)(\lambda\gamma^{uqrsn}\lambda)(\lambda\gamma^c W_i)(\lambda\gamma^p W_j)(\lambda\gamma^v W_k)A^m_l$$

After analogous treatment of the $b(11)$ sector:

$$\mathcal{A}_3 = c \int_{\mathcal{M}_3} \frac{d^2\Omega_{IJ}}{(\det(2\,\mathrm{Im}\,\Omega))^5} \int_{\Sigma_4} \left[\langle|\mathcal{F}|^2\rangle + \langle|\mathcal{T}|^2\rangle\right] \mathcal{I}(s_{ij})$$

$$c = (2\pi)^{10}\delta^{(10)}(k)\frac{\kappa^4 e^{4\lambda}}{2^{31}3^9\,5^2\,7^2}\left(\frac{\alpha'}{2}\right)^5$$

$$\begin{aligned}\mathcal{F} = {} & T_{12,3,4}\Delta(z_2;z_3;z_4)\,\eta_{12} + T_{13,2,4}\Delta(z_3;z_2;z_4)\,\eta_{13} + T_{14,2,3}\Delta(z_4;z_2;z_3)\,\eta_{14} \\ & + T_{23,1,4}\Delta(z_3;z_1;z_4)\,\eta_{23} + T_{24,1,3}\Delta(z_4;z_1;z_3)\,\eta_{24} + T_{34,1,2}\Delta(z_4;z_1;z_2)\,\eta_{34} \\ \mathcal{T} = {} & T^m_{1234}\Delta^m(z_1,z_2;z_3;z_4) + T^m_{1324}\Delta^m(z_1,z_3;z_2;z_4) + T^m_{1423}\Delta^m(z_1,z_4;z_2;z_3 \\ & + T^m_{2314}\Delta^m(z_2,z_3;z_1;z_4) + T^m_{2413}\Delta^m(z_2,z_4;z_1;z_3) + T^m_{3412}\Delta^m(z_3,z_4;z_1;z_2\end{aligned}$$

$$T^m_{1234} = L^m_{1342} + L^m_{2341} + \frac{5}{2}S^m_{1234}$$

Low energy limit

- Kinematic factors $|\mathcal{F}|^2$ and $|\mathcal{T}|^2$ of order D^8R^4 and D^6R^4 respectively
- Integrals inside $|\mathcal{F}|^2$ have kinematic poles. Integration by parts identities generate D^6R^4 contributions.
- Set Koba-Nielsen factor to 1 and use Riemann relations

$$\int_{\Sigma_4} \langle |\mathcal{F}|^2 \rangle \mathcal{I}(s_{ij}) = -\pi \left(\frac{\alpha'}{2}\right) \langle \mathcal{K} \rangle \int_{\Sigma_4} \Omega_{12} \Delta(z_2; z_3; z_4) \overline{\Delta}(z_1; z_3; z_4)$$
$$= -36\pi \left(\frac{\alpha'}{2}\right) \langle \mathcal{K} \rangle \det(2\,\mathrm{Im}\,\Omega)$$

where

$$\mathcal{K} = \frac{|T_{23,1,4}|^2 + |T_{14,2,3}|^2}{s_{23}} + \frac{|T_{24,1,3}|^2 + |T_{13,2,4}|^2}{s_{24}} + \frac{|T_{34,1,2}|^2 + |T_{12,3,4}|^2}{s_{34}}$$

- This is how one gets D^6R^4 contributions from the "D^8R^4" sector!

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Low energy limit

Similarly,

$$\int_{\Sigma_4} \Delta^m(z_1, z_2; z_3; z_4)\overline{\Delta}^n(\bar{z}_1, \bar{z}_2; \bar{z}_3; \bar{z}_4) = -12\pi\eta^{mn}\Big(\frac{\alpha'}{2}\Big)\det(2\,\mathrm{Im}\,\Omega),$$

$$\int_{\Sigma_4} \Delta^m(z_1, z_2; z_3; z_4)\overline{\Delta}^n(\bar{z}_1, \bar{z}_3; \bar{z}_2; \bar{z}_4) = +12\pi\eta^{mn}\Big(\frac{\alpha'}{2}\Big)\det(2\,\mathrm{Im}\,\Omega),$$

$$\int_{\Sigma_4} \Delta^m(z_1, z_2; z_3; z_4)\overline{\Delta}^n(\bar{z}_3, \bar{z}_4; \bar{z}_1; \bar{z}_2) = 0$$

leads to

$$\int_{\Sigma_4} |\mathcal{T}|^2 = -36\pi\Big(\frac{\alpha'}{2}\Big)\det(2\,\mathrm{Im}\,\Omega)\,\mathcal{L}\cdot\tilde{\mathcal{L}}$$

where $\mathcal{L}^m$ is some linear combination of T^m_{ijkl}.

Low energy limit

- Factor of $\det(2\,\mathrm{Im}\,\Omega)$ leads to modular invariant measure with volume

$$\int_{\mathcal{M}_3} \frac{d^2\Omega_{IJ}}{(\det(2\,\mathrm{Im}\,\Omega))^4} = \frac{3^3\cdot 5\cdot 7}{2^6\cdot 3^6}\zeta_6$$

- Amplitude in low energy limit becomes

$$\mathcal{A}_3 = -(2\pi)^{10}\delta^{(10)}(k)\left(\frac{\alpha'}{2}\right)^6 \langle \mathcal{K} + \mathcal{L}\cdot\tilde{\mathcal{L}}\rangle\,\kappa^4 e^{4\lambda}\,\frac{\pi\zeta_6}{2^{35}\,3^{10}\,5^3\,7^2}$$

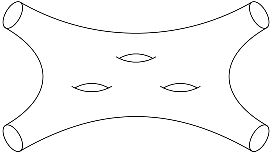
- Component expansion in pure spinor superspace gives

$$\langle \mathcal{K} + \mathcal{L}\cdot\tilde{\mathcal{L}}\rangle = -2^{35}\,3^7\,5^3\,7^2\,(s_{12}^3 + s_{13}^3 + s_{14}^3)\,K\overline{K}$$

where

$$K = -2^3\,2880\,A^{\mathrm{YM}}_{1234}\,s_{12}s_{23}$$

3-loop amplitude



$$= \frac{1}{3}(2\pi)^{10}\delta^{(10)}(k)\,\kappa^4 e^{4\lambda}\,\frac{\pi\,\zeta_6}{3^2}\left(\frac{\alpha'}{2}\right)^6(s_{12}^3 + s_{13}^3 + s_{14}^3)\,K\overline{K}$$

S-duality prediction versus amplitude calculation

- From the effective action, ratio of tree-level and 3-loop $D^6 R^4$ interaction:

$$\frac{2\zeta_6}{9\zeta_3^2}\, e^{6\phi} = \frac{\sqrt{2}2^{14}\pi^6\zeta_6}{9\zeta_3^2}\, e^{6\lambda}$$

- This ratio is equal to the ratio of the tree-level and 3-loop amplitudes

$$\frac{\mathcal{A}_3^{\alpha'^6}}{\mathcal{A}_0^{\alpha'^6}} = \frac{\sqrt{2}2^{14}\pi^6\zeta_6}{9\zeta_3^2}\, e^{6\lambda}$$

- provided one includes the $\frac{1}{3}$ symmetry factor

- The end

The Z_3 argument

- Recall that both the 1- and 2-loop amplitudes require a factor 1/2
- Every genus-1 and genus-2 surface is hyperelliptic

$$
\begin{aligned}
y^2 &= z(z-1)(z-a), & g &= 1, \\
y^2 &= z(z-1)(z-a_1)(z-a_2)(z-a_3), & g &= 2,
\end{aligned}
$$

- where a and a_1, a_2, a_3 parametrize the moduli space. These curves have the Z_2 symmetry $y \leftrightarrow -y$, implying the symmetry factor 1/2
- Every genus-three Riemann surface can be embedded in CP^2 as a quartic curve. The number of free parameters is (after some considerations) 6, therefore one is free to choose

$$
xy^3 + a_1 zy^3 + a_2 x^4 + a_3 x^3 z + a_4 x^2 z^2 + a_5 xz^3 + a_6 z^4 = 0,
$$

- Z_3 symmetry over y coordinate implying a 1/3 symmetry factor