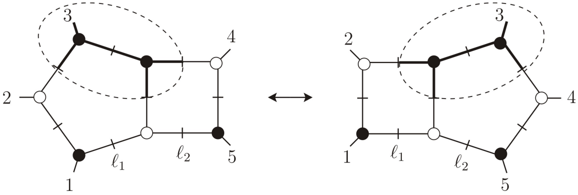
Cross-Order Integral Relations from Maximal Cuts

Kasper J. Larsen
Nikhef



Amplitudes 2014
IPhT CEA Saclay

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Based on 1108.1180, 1205.0801, 1208.1754 and work to appear (with S. Caron-Huot, H. Johansson, D. Kosower and M. Søgaard)

Part 1: Review of Maximal Unitarity at Two Loops

- Motivation and one-loop generalized unitarity
- Two-loop maximal cuts are contour integrals on algebraic varieties. How are the integration contours determined?
- One-to-one correspondence between basis integrals and integration contours

Motivation and approaches

Two motivations for studying two-loop amplitudes:

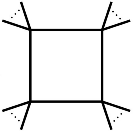
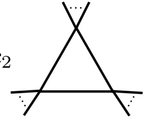
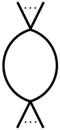
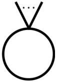
- **Precision LHC phenomenology**
  Quantitative estimates of QCD background: needed for precision measurements, uncertainty estimates of NLO calculations, and reducing renormalization scale dependence.
- **Geometric understanding of scattering amplitudes**
  Fascinating connection to algebraic geometry and multivariate complex analysis.

Our aim is to extend generalized unitarity to two loops so as to *automate* the computation of two-loop QCD amplitudes.

Other approaches:

- Integrand reduction [Mastrolia, Mirabella, Ossola, Peraro], 2011 and [Badger, Frellesvig, Zhang], 2012
- Spinor integration techniques [Feng, Zhen, Huang, Zhou], 2014
- Iterated cuts [Abreu, Britto, Duhr, Gardi], 2014

Use integrand reductions to write the one-loop amplitude as a linear combination of *basis integrals*

$A^{(1)} = c_1$  $+ \ c_2$  $+ \ c_3$  $+ \ c_4$ 

$$+ \ \text{rational terms}$$

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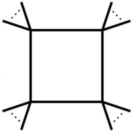
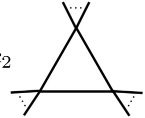
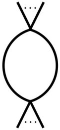
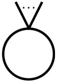
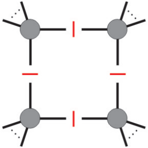
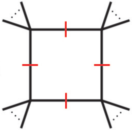
$A^{(1)} = c_1$  $+ c_2$  $+ c_3$  $+ c_4$ 

$+$ rational terms

Determine c_i by applying quadruple cuts [Britto, Cachazo, Feng]:

 $= c_1$  $\Longrightarrow \quad c_1 = \frac{1}{2} \sum_{\text{kin sols}} \prod_{j=1}^{4} A_j^{\text{tree}}$

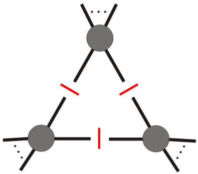
Generalized unitarity at one loop (2/2)

A triple cut will leave $4 - 3 = 1$ free *complex* parameter z.
Parametrizing the loop momentum,

$$\ell^\mu = \alpha_1 K_1^{\flat\mu} + \alpha_2 K_2^{\flat\mu} + \frac{z}{2}\langle K_1^{\flat-}|\gamma^\mu|K_2^{\flat-}\rangle + \frac{\alpha_4(z)}{2}\langle K_2^{\flat-}|\gamma^\mu|K_1^{\flat-}\rangle$$

one obtains an explicit formula for the triangle coefficient [Forde]

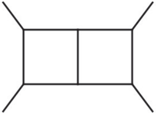
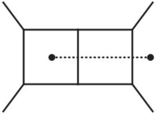
$$c_\triangle = \oint_{C(\infty)} \frac{dz}{z}$$



$$(z)$$

From trees to two loops

Expand the massless 4-point two-loop amplitude in a basis, e.g.

$A_4^{2-\text{loop}} = c_1(\epsilon)$  $+ \; c_2(\epsilon)$  $+$ ints with fewer props $+$ rational terms

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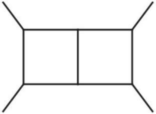
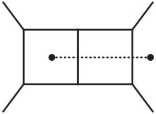
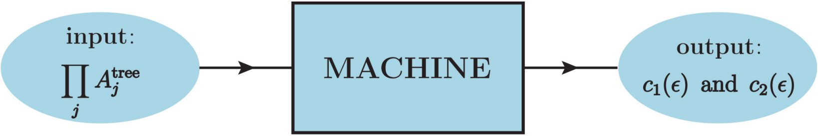
$A_4^{2-\text{loop}} = c_1(\epsilon)$  $+\ c_2(\epsilon)$  + ints with fewer props + rational terms

Compute $c_1(\epsilon)$ and $c_2(\epsilon)$ according to



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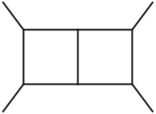
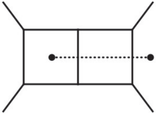
$A_4^{2-\text{loop}} = c_1(\epsilon)$  $+\ c_2(\epsilon)$  $+$ ints with fewer props $+$ rational terms

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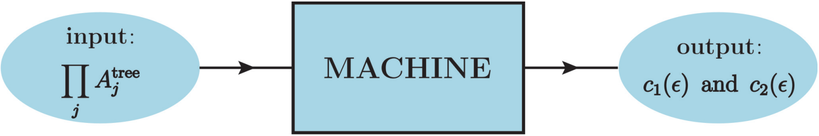
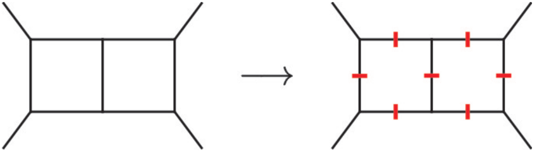


The machinery: *contour integrals* $\oint_{\Gamma_j}(\cdots)$

The philosophy: basis integral $I_j \longleftrightarrow$ unique Γ_j producing c_j

The anatomy of two-loop maximal cuts

Cutting all seven visible propagators in the double-box integral,



produces (cf. [Buchbinder, Cachazo] and [Kosower, KJL]), setting $\chi \equiv \frac{t}{s}$,

$$\int d^4p\, d^4q \prod_{i=1}^{7} \frac{1}{\ell_i^2} \longrightarrow \int d^4p\, d^4q \prod_{i=1}^{7} \delta^{\mathbb{C}}(\ell_i^2) = \oint_\Gamma \frac{dz}{z(z+\chi)}\,,$$

a contour integral in the complex plane.

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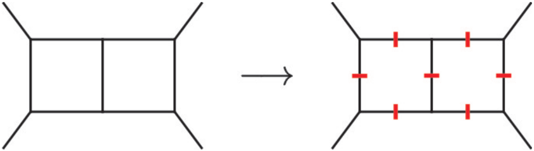

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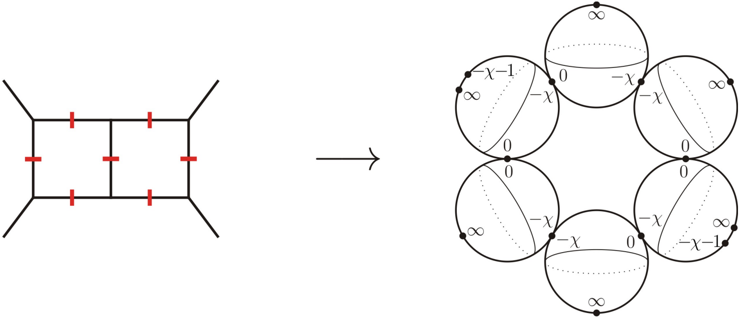
a contour integral in the complex plane.

Jacobian poles $z = 0$ and $z = -\chi$: composite leading singularities

encircle $z = 0$ and $z = -\chi$ with $\Gamma = \omega_1 C_\epsilon(0) + \omega_2 C_\epsilon(-\chi)$
$\longrightarrow$ freeze z ("8^{th} cut")

Choosing contours: *die Qual der Wahl*

Six inequivalent classes of solutions to on-shell constraints



4 massless external states $\longrightarrow$ 8 independent leading singularities

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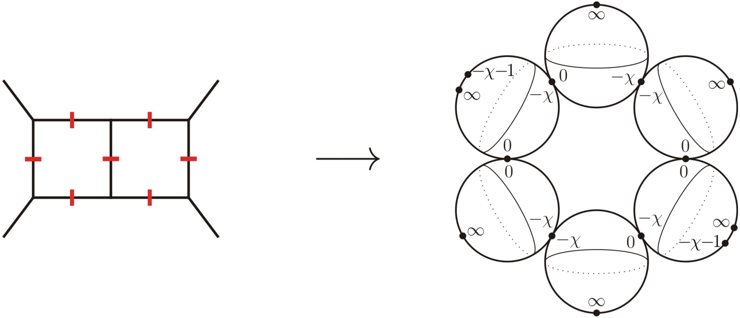

4 massless external states $\longrightarrow$ 8 independent leading singularities

How do we select contours within this variety of possibilities?

Principle for selecting contours

To fix the contours, insist that

vanishing Feynman integrals must have vanishing heptacuts.

This ensures that

$$\mathrm{I}_1 = \mathrm{I}_2 \quad \Longrightarrow \quad \mathsf{cut}\,(\mathrm{I}_1) = \mathsf{cut}\,(\mathrm{I}_2)\,.$$

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Origin of terms with vanishing $\mathbb{R}^D \times \mathbb{R}^D$ integration:
reduction of Feynman diagram expansion to a *basis of integrals* (including use of integration-by-parts identities [Chetyrkin and Tkachov], 1981).

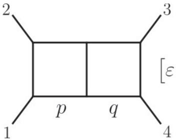
Remarkable simplification:

- 4 massless external states: 22 $\longrightarrow$ 2 double-box integrals
- 5 massless external states: 160 $\longrightarrow$ 2 "turtle-box" integrals
- 5 massless external states: 76 $\longrightarrow$ 1 pentagon-box integral

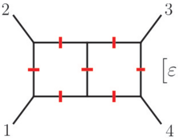
Contour constraints, part 1/2

There are two classes of constraints on Γ's:

1) Levi-Civita integrals. For example,



$$\left[\varepsilon(p,1,2,4)\right] = 0 \quad \Longrightarrow$$



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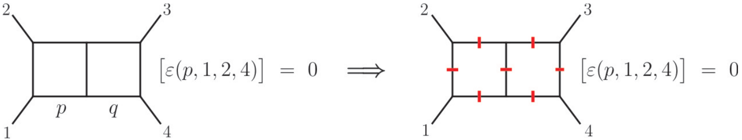
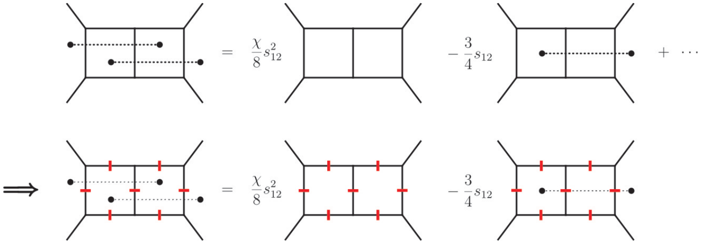
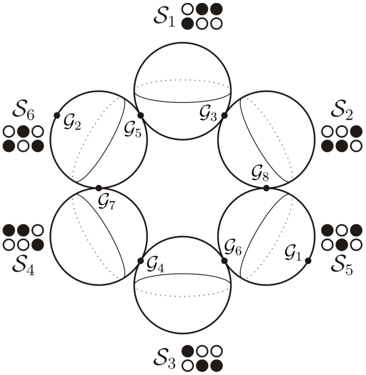


2) integration-by-parts (IBP) identities must be preserved. For example,



Contour constraints, part 2/2

The constraints in the case of four massless external momenta:

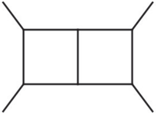
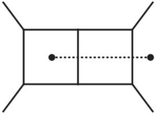


$$
\begin{aligned}
\omega_1 - \omega_2 &= 0 \\
\omega_3 - \omega_4 &= 0 \\
\omega_5 - \omega_6 &= 0 \\
\omega_7 - \omega_8 &= 0 \\
\omega_3 + \omega_4 - \omega_5 - \omega_6 &= 0 \\
\omega_1 + \omega_2 - \omega_5 - \omega_6 + \omega_7 + \omega_8 &= 0
\end{aligned}
$$

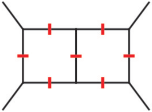
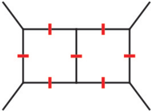
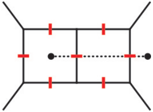
leaving $8 - 4 - 2 = 2$ free winding numbers.

Master contours: the concept

Going back to the two-loop basis expansion

$A_4^{2-\mathrm{loop}} = c_1(\epsilon)$  $+\ c_2(\epsilon)$  $+$ ints with fewer props $+$ rational terms

and applying a heptacut one finds

 $\left[\prod_{j=1}^{6} A_j^{\mathrm{tree}}\right] = c_1(\epsilon)$  $+\ c_2(\epsilon)$ 

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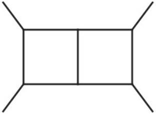
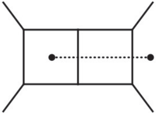
$A_4^{2-\text{loop}} = c_1(\epsilon)$  $+\ c_2(\epsilon)$  $+$ ints with fewer props $+$ rational terms

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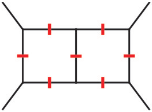
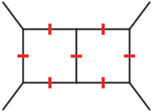
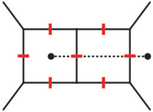
 $\left[\prod_{j=1}^{6} A_j^{\text{tree}}\right] = c_1(\epsilon)$  $+\ c_2(\epsilon)$ 

Exploit free parameters $\longrightarrow \exists$ contours with

$$P_1 : \ \big(\text{cut}(\mathrm{I}_1), \text{cut}(\mathrm{I}_2)\big) = (1,0)$$
$$P_2 : \ \big(\text{cut}(\mathrm{I}_1), \text{cut}(\mathrm{I}_2)\big) = (0,1)\,.$$

We call such P_i *master contours* (or projectors).

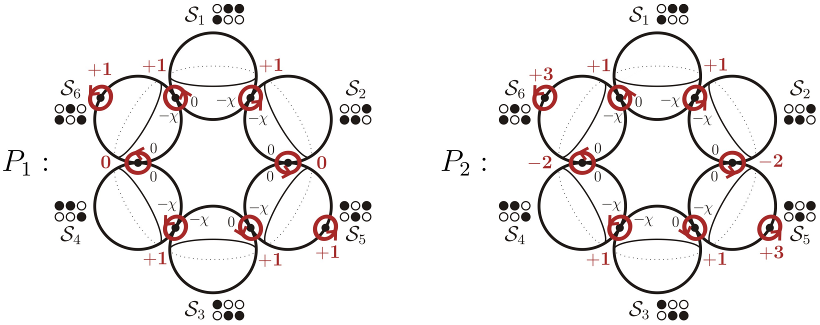
Master contours: results

With four massless external states,

$$c_1 = \frac{i\chi}{8} \oint_{P_1} \frac{dz}{z(z+\chi)} \prod_{j=1}^{6} A_j^{\text{tree}}(z)$$

$$c_2 = -\frac{i}{4s_{12}} \oint_{P_2} \frac{dz}{z(z+\chi)} \prod_{j=1}^{6} A_j^{\text{tree}}(z)$$

With our choice of basis integrals, the P_i are



n = winding number

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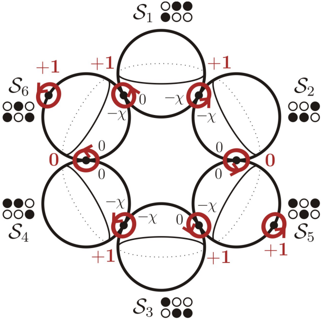
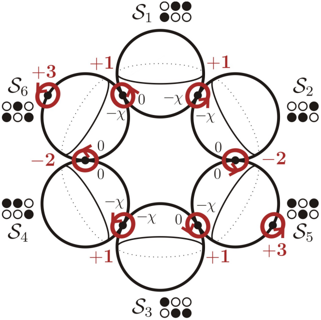


P_2 :

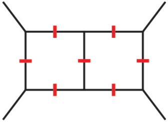


n = winding number

(Generalizations: [Søgaard; 1306.1496] and [Søgaard, Zhang; 1310.6006])

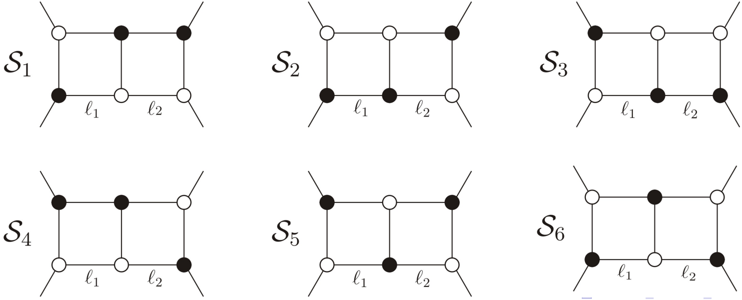
Characterizing the on-shell solutions

There are six solutions for the heptacut loop momenta

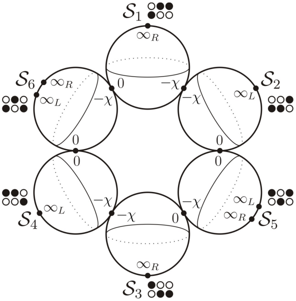


Set $k_i^\mu = \lambda_i \sigma^\mu \widetilde{\lambda}_i$ and classify each vertex according to

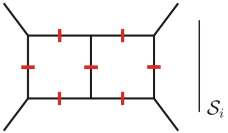
$$\widetilde{\lambda}_a \propto \widetilde{\lambda}_b \propto \widetilde{\lambda}_c \quad (\text{MHV}) \quad \longrightarrow \quad \circ$$
$$\lambda_a \propto \lambda_b \propto \lambda_c \quad (\overline{\text{MHV}}) \quad \longrightarrow \quad \bullet$$



Physical interpretation of nodal points



$\longleftarrow$



$$= \oint_{\Gamma_i} \frac{dz}{z(z+\chi)}$$

points $\in \mathcal{S}_i \cap \mathcal{S}_j \longrightarrow$ no notion of ● or ○ $\longrightarrow$ resp. prop. is soft
also: $\mathcal{S}_i \cap \mathcal{S}_j \subset \{\text{leading singularities}\}$!

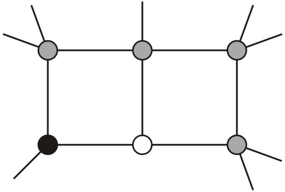
two-loop leading singularities $\longrightarrow$ IR singularities of integral

Classification of heptacut solutions

Arbitrary # of external states. Define

$$\mu_i \equiv \begin{cases} \text{m} & \text{if } i^{\text{th}} \text{ vertical prop.} \in \text{3-pt. vertex} \\ \text{M} & \text{if } i^{\text{th}} \text{ vertical prop.} \notin \text{3-pt. vertex} \end{cases}$$

e.g.,



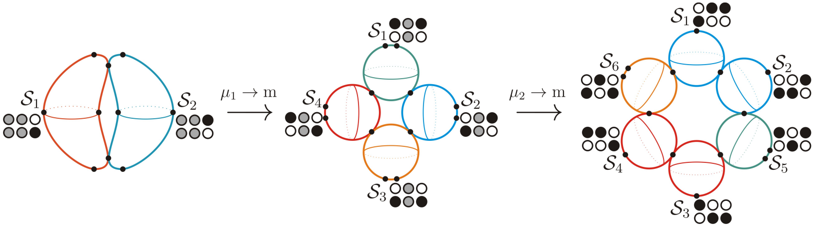
is (m,m,M)

The solution to $\ell_i^2 = 0$, $i = 1, \ldots, 7$ is

- case 1 (M,M,M): 1 torus
- case 2 (M,M,m) etc.: 2 $\mathbb{CP}^1$ with $\mathcal{S}_i \longleftrightarrow$ distrib. of ●, ○
- case 3 (M,m,m) etc.: 4 $\mathbb{CP}^1$ with $\mathcal{S}_i \longleftrightarrow$ distrib. of ●, ○
- case 4 (m,m,m): 6 $\mathbb{CP}^1$ with $\mathcal{S}_i \longleftrightarrow$ distrib. of ●, ○

Uniqueness of master contours

Limits $\mu_i \to \mathrm{m} \implies$ chiral branchings: torus $\overset{\mu_3 \to \mathrm{m}}{\longrightarrow}$



Each torus-pinching: new IR-pole + new residue thm

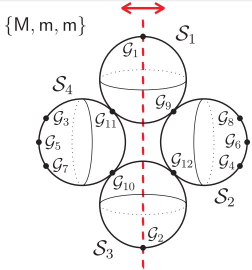
$\implies$ # of independent poles same in all cases

In all cases: **# of master Γ's = # of basis integrals**

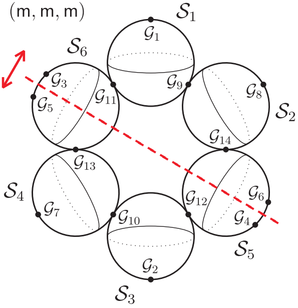
$\implies$ all linear relations are preserved

$\implies$ perfect analogy with one-loop generalized unitarity

Symmetries and systematics of IBP constraints



The IBP constraints are invariant under flips.



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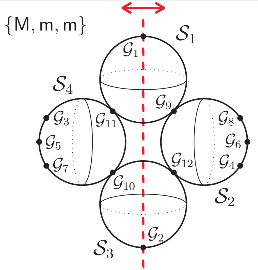


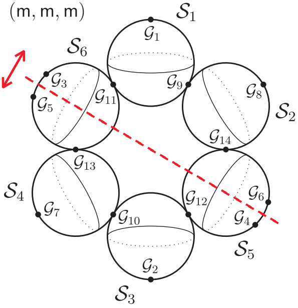
The IBP constraints are invariant under flips.

Reverse logic $\longrightarrow$ demand constraints to be invariant under flips and π-rotations.

{M,m,m} case: choose basis, e.g. $\omega_{1,2,5,6} = 0$

$$r_1^{(\mathrm{b})}(\omega_3 + \omega_4 + \omega_7 + \omega_8) + r_2^{(\mathrm{b})}(\omega_9 + \omega_{10} - \omega_{11} - \omega_{12}) = 0$$

where, in fact, $r_1^{(\mathrm{b})} = r_2^{(\mathrm{b})} \neq 0$.



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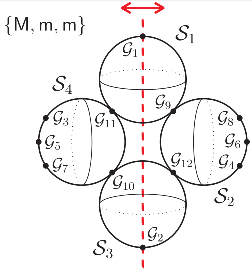

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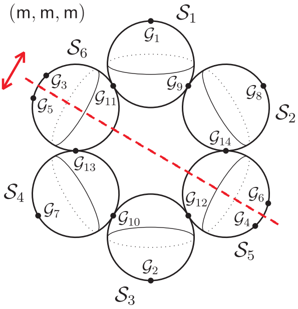


(m,m,m) case:

1) constraint from {M,m,m} case inherited.
2) new flip symmetry $\longrightarrow$ new constraint:

$$r_1^{(c)}(\omega_3 + \omega_4) + r_2^{(c)}(\omega_{11} + \omega_{12} - \omega_{13} - \omega_{14}) = 0$$

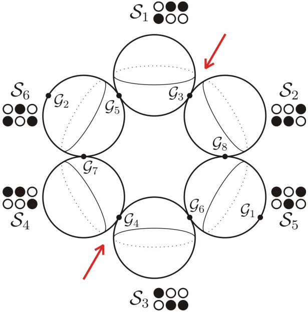
as expressed in the basis $\omega_{1,2,5,6,7,8} = 0$.
In fact, $r_1^{(c)} = -r_2^{(c)} \neq 0$.

Part 2: Integral identities from maximal cuts

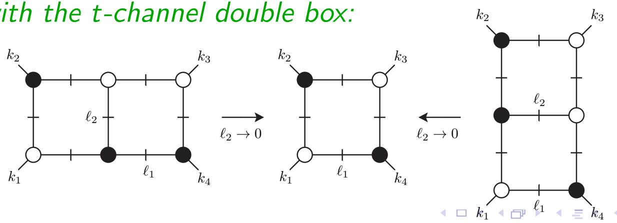
- Loop integrals share leading singularities
- Consequences of shared leading singularities: integral relations

s- and t-channel integrals share global poles

At the nodal points, the rung momenta become soft.

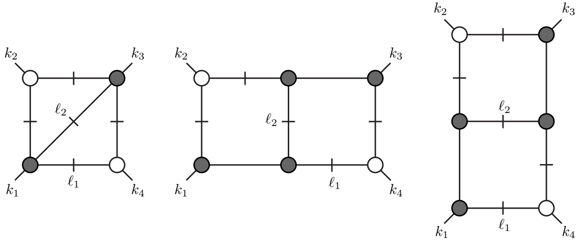


At $\mathcal{G}_4$ (and $\mathcal{G}_3$), the middle rung goes soft. These poles are *shared with the t-channel double box:*



Consequences of global pole sharing

The two double boxes can be embedded in a single *pentacut:*



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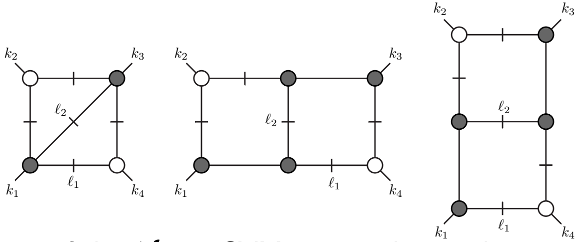

The pentacut of the $\mathcal{N}=4$ SYM amplitude vanishes,

$$\left(s_{12}^2 s_{23} I_{\text{HDB}} + s_{23}^2 s_{12} I_{\text{VDB}}\right)\Big|_{\mathcal{S}_1}$$

$$= s_{12}s_{23} \oint d^3z\, J(z) \frac{s_{12}(\ell_1+k_1)^2(\ell_1+\ell_2-k_4)^2 + s_{23}(\ell_1+\ell_2)^2(\ell_1-K_{34})^2}{(\ell_1+k_1)^2(\ell_1-K_{34})^2(\ell_1+\ell_2)^2(\ell_1+\ell_2-k_4)^2}\Bigg|_{\mathcal{S}_1} = 0$$

because the numerator insertion vanishes identically on $\mathcal{S}_1$ (consistent with $G(1,2,3,\ell_1,\ell_2)=0$).

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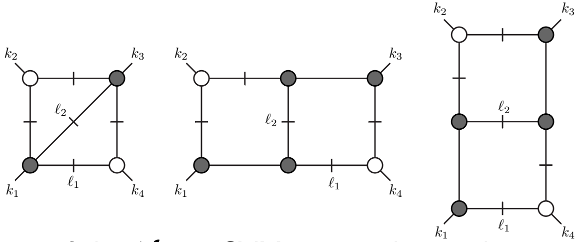

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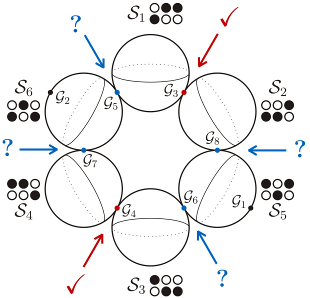
because the numerator insertion vanishes identically on $\mathcal{S}_1$ (consistent with $G(1,2,3,\ell_1,\ell_2) = 0$).

Despite its $\mathcal{N} = 4$ SYM origin, the identity $\left(s_{12}^2 s_{23} I_{\rm HDB} + s_{23}^2 s_{12} I_{\rm VDB}\right)\Big|_{\mathcal{S}_1} = 0$ is theory independent.

Consequences of global pole sharing

The blue poles are also shared between s- and t-channel double boxes.

But unlike the red ones, they are not accessible through any maximal cut (e.g., a pentacut) encompassing both DBs.



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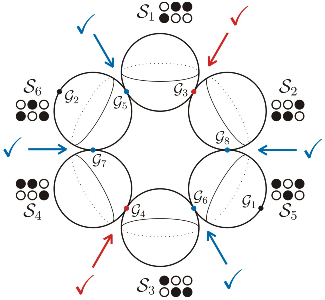
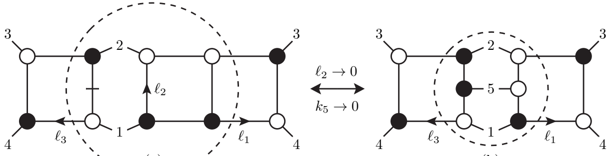


The sharing of the blue poles thus has no implications for the amplitude. But it does for the *cross section:* the residues at $\mathcal{G}_{5,6,7,8}$

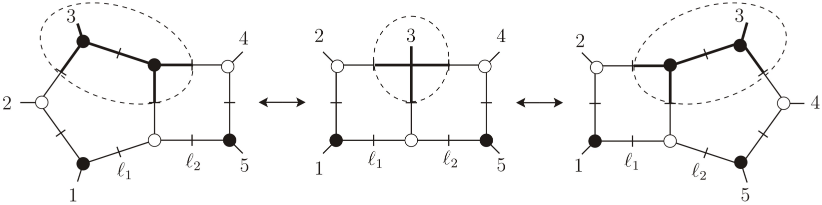


(a) (b)

cancel between the virtual and real contributions, cf. the KLN thm.

Global poles shared between pentaboxes

Pentaboxes embedded in the same "turtle-box" cut share poles:



as the "merge-and-split" move preserves the global pole location.

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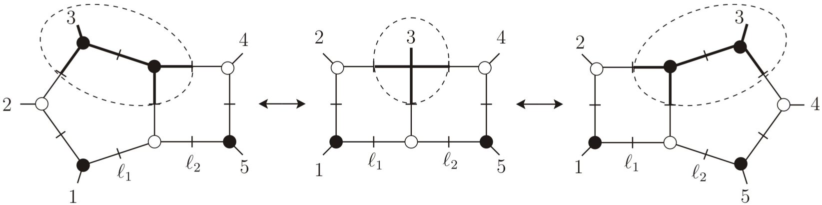


as the “merge-and-split” move preserves the global pole location.

Moreover, the "turtle-box octacut” residues cancel:

$$\left(s_{12}s_{23}s_{45}P^{**}_{3,2;\sigma_1}[(\ell_1+k_5)^2] + s_{34}s_{45}s_{12}P^{**}_{3,2;\sigma_3}[(\ell_2+k_1)^2]\right)\Big|_{8-\mathrm{cut}} = 0$$

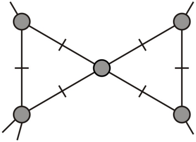
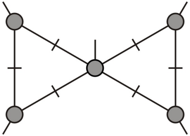
There are 10 such cancelations; can be summarized in 5 identities

$$\left(\sum_{\mathrm{cyclic}} s_{12}s_{23}s_{45}P^{**}_{3,2;\sigma_1}[(\ell_1+k_5)^2]\right)\Bigg|_{8-\mathrm{cut}} = 0$$

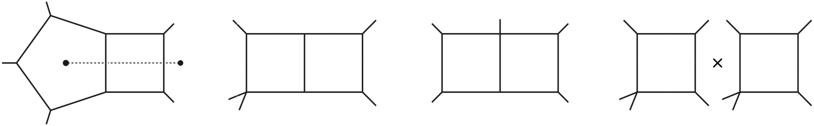
Thus, this cyclic sum of DC pentaboxes can be expressed in terms of simpler integrals.

ABDK/BDS relation from maximal cuts

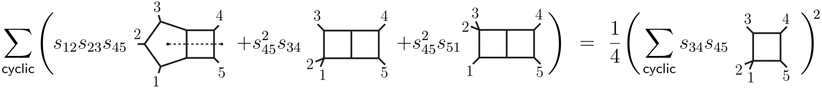
Reveal candidate integrals by dropping cut constraints. Consider

The integrals sharing these cuts are



as well as factorized two-loop integrals. Fix their coefficients by demanding equal residues on any parity-even contour, yielding



which is precisely the parity-even part of the ABDK/BDS relation.

Conclusions and outlook

- Programme aiming towards fully automated computation of two-loop amplitudes in generic gauge theories
- Integration-by-parts identities $\longrightarrow$ reduce # of Feynman integrals by factor of 10-100
- One-to-one correspondence between two-loop master contours and master integrals

- Sharing of leading singularities between loop integrals allow us to infer integral relations

- Underlying algebraic geometry $\longrightarrow$ deeper understanding of maximal cuts (i.e., contour constraints)
- Integrals with fewer propagators

Backup slides

Integrals and integral bases

- ideal two-loop basis: chiral integrals
- evaluate 4-point chiral integrals analytically

Maximally IR-finite basis

The two-loop integral coefficients c_i have $\mathcal{O}(\epsilon)$ corrections.

Important to know, as the integrals have poles in ϵ.

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IR-finite integrals $\longrightarrow$ $\mathcal{O}(\epsilon)$ corrections not needed for amplitude

Candidates: num. insertions $\to 0$ in collinear int. regions, e.g.

$$\begin{aligned} I_{++} &\equiv I\big[[1|\not\ell_1|2\rangle\langle 3|\not\ell_2|4]\big] \times [2\,3]\langle 1\,4\rangle \\ I_{+-} &\equiv I\big[[1|\not\ell_1|2\rangle\langle 4|\not\ell_2|3]\big] \times [2\,4]\langle 1\,3\rangle \end{aligned}$$

Essentially the chiral integrals of [Arkani-Hamed et al.]

I_{++} and I_{+-} lin. independent $\longrightarrow$ use in *any gauge theory*

Philosophy: maximally IR-finite basis
$\longrightarrow$ minimize need for cuts in $D = 4 - 2\epsilon$

Evaluation of chiral integrals (1/3)

$I_{+\pm}$ are finite $\longrightarrow$ can be computed in $D = 4$

1) Feynman parametrize

$$I_{++} = -\chi^2 \left(1 + (1+\chi)\frac{\partial}{\partial\chi}\right) I_1(\chi) \quad \text{and} \quad I_{+-} = -(1+\chi)^2 \left(1 + \chi\frac{\partial}{\partial\chi}\right) I_1(\chi)$$

where

$$I_1(\chi) = \int \frac{d^3a\; d^3b\; dc\; c\; \delta\left(1 - c - \sum_i a_i - \sum_i b_i\right)\left(\sum_i a_i \sum_i b_i + c\left(\sum_i a_i + \sum_i b_i\right)\right)^{-1}}{\left(a_1 a_3 \left(c + \sum_i b_i\right) + (a_1 b_4 + a_3 b_6 + a_2 b_5 \chi)c + b_4 b_6 \left(c + \sum_i a_i\right)\right)^2}$$

2) "Projectivize"

$$I_1(\chi) = 6 \int_1^\infty dc \int_0^\infty \frac{d^7(a_1 a_2 a_3 a_{\mathcal{I}} b_1 b_2 b_3 b_{\mathcal{I}})}{\text{vol}(\text{GL}(1))} \frac{1}{(cA^2 + A.B + B^2)^4}$$

Evaluation of chiral integrals (2/3)

3) Obtain symbol

Integrate projective form one variable at the time, at the level of the symbol.

$$\mathcal{S}[I_1(\chi)] = \frac{2}{\chi}[\chi \otimes \chi \otimes (1+\chi) \otimes (1+\chi)] - \frac{2}{1+\chi}[\chi \otimes \chi \otimes (1+\chi) \otimes \chi]$$

4) "Integrate" symbol, using

a) I_1 has transcendentality 4 (fact, not a conjecture)

b) I_1 has no u-channel discontinuity

c) Regge limits:

$$I_1(\chi) \to \frac{\pi^2}{6}\log^2\chi + \left(4\zeta(3) - \frac{\pi^2}{3}\right)\log\chi + \mathcal{O}(1) \quad \text{as} \quad \chi \to 0$$

$$I_1(\chi) \to 6\zeta(3)\frac{\log\chi}{\chi} + \mathcal{O}(\chi^{-1}) \quad \text{as} \quad \chi \to \infty$$

Evaluation of chiral integrals (3/3)

In conclusion, for the "chiral" integrals

$$I_{++} \equiv I\big[[1|\ell_1|2\rangle\langle 3|\ell_2|4]\big] \times [2\,3]\langle 1\,4\rangle$$
$$I_{+-} \equiv I\big[[1|\ell_1|2\rangle\langle 4|\ell_2|3]\big] \times [2\,4]\langle 1\,3\rangle$$

we find the results

$$I_{++}(\chi) = 2H_{-1,-1,0,0}(\chi) - \frac{\pi^2}{3}\mathrm{Li}_2(-\chi) + \left(\frac{\pi^2}{2}\log(1+\chi) - \frac{\pi^2}{3}\log\chi + 2\zeta(3)\right)\log(1+\chi) - 6\chi\zeta(3)$$

$$I_{+-}(\chi) = 2H_{0,-1,0,0}(\chi) - \pi^2\mathrm{Li}_2(-\chi) - \frac{\pi^2}{6}\log^2\chi - 4\zeta(3)\log\chi - \frac{\pi^4}{10} - 6(1+\chi)\zeta(3)$$

Actual chiral integrals: transcendentality-breaking terms cancel.