Mathematical structures and tools for Feynman amplitudes

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Ideal and realistic scattering amplitudes

- N=4 SYM is ‘hydrogen atom of quantum field theory’

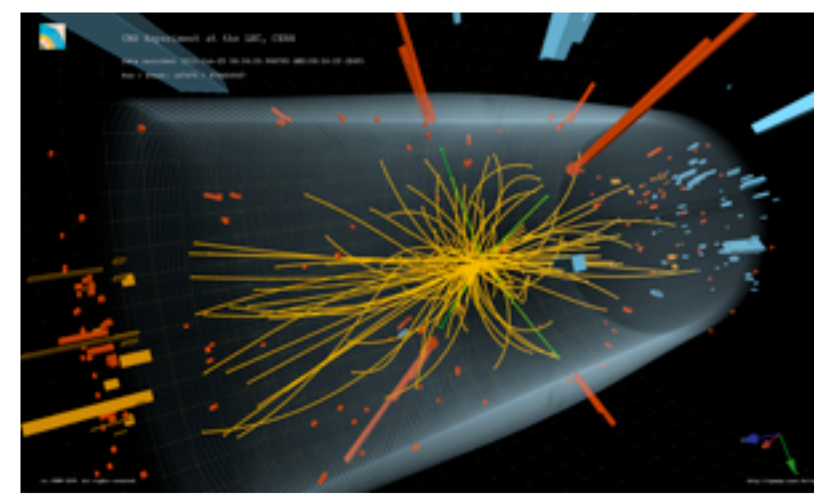
  dualities, dual conformal/Yangian symmetry, AdS/CFT, integrability, twistor space, simple integrands...

  what further surprises does it hold for us?

  what can we learn from it for QCD?

- Higgs discovery at the LHC

more theory predictions needed for precision measurements!



this talk: tools for loop-level scattering amplitudes
(building on tree/integrand insights)

Techniques for loop integrands

- unitarity cut based techniques [Bern, Dixon, Dunbar, Kosower] [...]

  cf. talks by Badger, Britto, Larsen, Mastrolia, Penante, Roiban, Yang,...

  used for many 1-loop phenomenological studies;
  multi-loop in super Yang-Mills and supergravity

- on-shell recursion relations and diagrams

  [Britto, Cachazo, Feng, (Witten)] [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka]

  very useful, compact answers, helps make symmetries manifest

  challenges for the future: apply to N<4 SYM, non-planar, D=4-2eps

  recent development: 'amplituhedron' [Arkani-Hamed, Trnka]

- physical properties of loop integrands

  make infrared (IR) properties manifest
  [Arkani-Hamed et al.] [Drummond, J.M.H.][Bourjaily, DiRe, Shaikh, Spradlin, Volovich]
  closely related to (generalized) cut structure

  UV properties, anomalies [Chen, Huang, McGady]

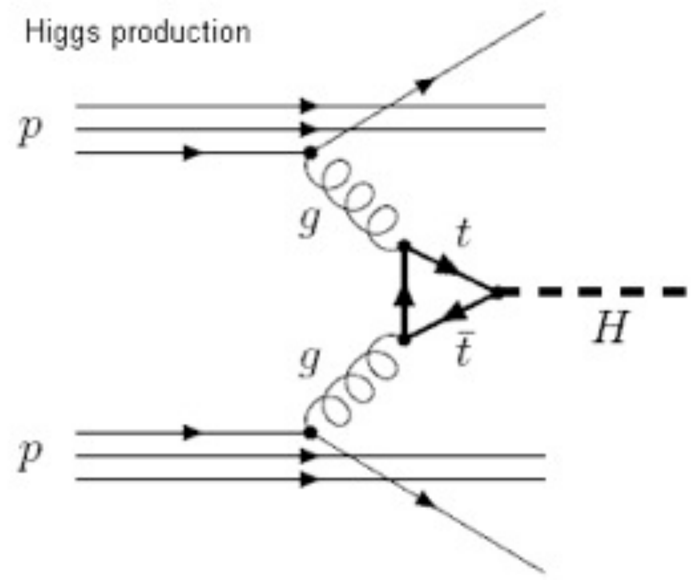
Scattering amplitudes at loop level

- What functions are needed to describe them?
- Example: integral appearing in Higgs production

$$\int \frac{d^4k}{i\pi^2} \frac{1}{(m_t^2 - k^2)(m_t^2 - (k+p_1)^2)(m_t^2 - (k-p_2)^2)} =$$

$$= -\frac{1}{2s} \log^2 \left(\frac{\sqrt{1 - 4m_t^2/s} - 1}{\sqrt{1 - 4m_t^2/s} + 1} \right)$$

$$s = (p_1 + p_2)^2$$



- multivalued function; two-particle threshold
- more generally: need integrals in a Laurent series about D = 4 - 2 eps.

- At one loop, only logarithm and dilogarithm needed

$$\log z = \int_1^z \frac{dt}{t} \qquad \mathrm{Li}_2(z) = \int_0^z \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1 - t_2}$$

- what functions will appear at higher loops?
- how to compute them in an efficient way?

Feynman integrals as iterated integrals (1)

- Logarithm and dilogarithm are first examples of iterated integrals with special ``d-log`` integration kernels

$$\frac{dt}{t} = d\log t \qquad \frac{-dt}{1-t} = d\log(1-t) \qquad \frac{dt}{1+t} = d\log(1+t)$$

- these are called harmonic polylogarithms (HPL) [Remiddi, Vermaseren]

e.g. $H_{1,-1}(x) = \int_0^x \frac{dx_1}{1-x_1} \int_0^{x_1} \frac{dx_2}{1+x_2}$

  - shuffle product algebra
  - coproduct structure
  - Mathematica implementation [Maitre]

- weight: number of integrations

- special values related to multiple zeta values (MZV) [cf. Duhr's talk]

$$\zeta_{i_1,i_2,\ldots,i_k} = \sum_{a_1>a_2>\ldots a_k\geq 1} \frac{1}{a_1^{i_1} a_2^{i_2} \ldots a_k^{i_k}}$$

cf. e.g. [Bluemlein, Broadhurst, Vermaseren]

e.g. $H_{0,1}(1) = \mathrm{Li}_2(1) = \zeta_2$

Feynman integrals as iterated integrals (2)

- Natural generalization: multiple polylogarithms [also called hyperlogarithms; Goncharov polylogarithms]

  allow kernels $w = d\log(t-a)$

  $$G_{a_1,\dots a_n}(z) = \int_0^z \frac{dt}{t-a_1} G_{a_2,\dots,a_n}(t)$$

  numerical evaluation: GINAC [Vollinga, Weinzierl]

- Chen iterated integrals

  $$\int_C \omega_1 \omega_2 \dots \omega_n \qquad C : [0,1] \longrightarrow M$$ (space of kinematical variables)

  Alphabet: set of differential forms $\omega_i = d\log\alpha_i$

  integrals we discuss will be monodromy invariant on $M \setminus S$

  S (set of singularities)

  more flexible than multiple polylogarithms!

- Uniform weight functions (pure functions):

  $\mathbb{Q}$ -linear combinations of functions of the same weight

Goncharov weight four conjecture

- rewrite any multiple polylogarithm in terms of function basis [Goncharov]

  e.g. at weight 4 (important for NNLO computations)

$$\{\log(x)\log(y)\log(z)\log(w), \log(x)\log(y)\mathrm{Li}_2(z),$$
$$\mathrm{Li}_2(x)\mathrm{Li}_2(y), \log(x)\mathrm{Li}_3(y), \mathrm{Li}_4(x), \mathrm{Li}_{2,2}(x,y)\}$$

  for set of arguments (to be found - symbol/coproduct provides guidance)

  [in N=4 SYM related to cluster coordinates? cf. Vergu's talk]

  minimal set of integration kernels vs. minimal set of function arguments

- practical tool: ``symbol`` useful projections [Goncharov, Spradlin, Vergu, Volovich]

  [Duhr, Gangl, Rhodes]

  [Brown] [Goncharov]

  e.g. project on $\mathrm{Li}_{2,2}(x,y)$ part

  e.g. project out all products

  lecture notes: [Vergu] [Brown][Zhao]

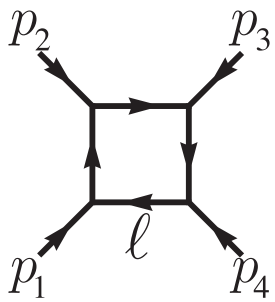
- ``symbol`` = Chen iterated integral without boundary information

  diff. eqs. or other information can be used to fix this

d-log representations

- Can we make it manifest when integrals evaluate to pure functions?

$$\mathcal{A}_4^{\ell=0} \times$$



$$= \mathcal{A}_4^{\ell=0} \times \int \frac{d^4\ell \ \ (p_1+p_2)^2 (p_1+p_3)^2}{\ell^2 (\ell+p_1)^2 (\ell+p_1+p_2)^2 (\ell-p_4)^2}$$

[Arkani-Hamed, Bourjaily, Cachzo, Goncharov, Postnikov, Trnka, 2012]

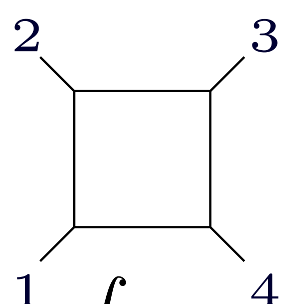
[Caron-Huot, talk at Trento, 2012]

[Lipstein, Mason, 2013]

$$\frac{d^4\ell \ \ (p_1+p_2)^2 (p_1+p_3)^2}{\ell^2 (\ell+p_1)^2 (\ell+p_1+p_2)^2 (\ell-p_4)^2}$$
$$= \mathrm{dlog}\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) \mathrm{dlog}\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) \mathrm{dlog}\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) \mathrm{dlog}\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)$$

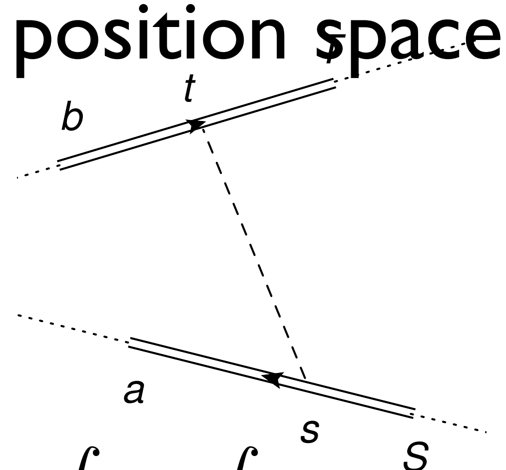
very suggestive! New ways of performing loop integrations?

- amplitude/Wilson loop duality: relation between momentum space spacetime integrals and position space line integrals



$$\int d^4 k$$

$$\longleftrightarrow$$



$$\int ds \int dt$$

right number of d-logs for weight 2 function

algorithm for evaluating (multiple) Wilson line integrals with any propagator exchanges

[J.M.H., Huber, 2013]

Cuts and integrated integrands

- discontinuities usually simpler than full answer   cf. talks by Badger, Britto, Mastrolia,...
- contain important information   dispersive representations, e.g. Mandelstam, optical theorem
- maximal cuts, leading singularities   [Cachzao, Skinner]

- integrals with simple cuts are expected to integrate to uniform weight functions

  idea: any cut that completely localizes the integral should give just a rational number

- use cuts of integrals as guiding principle for finding convenient integral basis   [J.M.H., 2013]

A word of caution: more exotic objects

• mathematicians like to consider single-scale Feynman integrals

• conjecture that certain periods only evaluate to multiple zeta values (MZV) appear disproven by [Brown, Schnetz]

• Elliptic functions

relevant e.g. in top quark physics Czakon et al.

also appear in massless N=4 SYM [Caron-Huot, Larsen]

recent work Elliptic polylogairthms [Brown, Levin]

[Bloch, Vanhove] [Vanhove] [Remiddi, Tancredi] [Adams, Bogner, Weinzierl]

Note: weight property generalizes weight n -> (n/2,n/2) mixed Hodge theory

systematic and practical way for dealing with them for practical applications?

• Here: cases where Chen iterated integrals are sufficient

differential equations, uniform weight basis

Strategy for computing Feynman integrals using differential equations

cf. Smirnov's talk

- Useful facts:

  (1) For a given problem, one can choose a finite basis of Feynman integrals

  cf. Zhang's talk

  (2) Basis integrals satisfy coupled first-order differential equations

  (3) many classes of Feynman integrals evaluate to iterated integrals

- Idea: choose basis such that the differential equations are simple, and such that (3) is made obvious

Key points of the method

[JMH, PRL 110 (2013) 25]

- differential equations for master integrals $\vec{f}$
- crucial: choose convenient basis (systematic procedure)

$\longrightarrow$ makes solution trivial to obtain

- elegant description: Feynman integrals specified by:

(1) set of 'letters' (related to singularities x_k)

(2) set of constant matrices A_k

Example: one dimensionless variable x; $D = 4 - 2\epsilon$

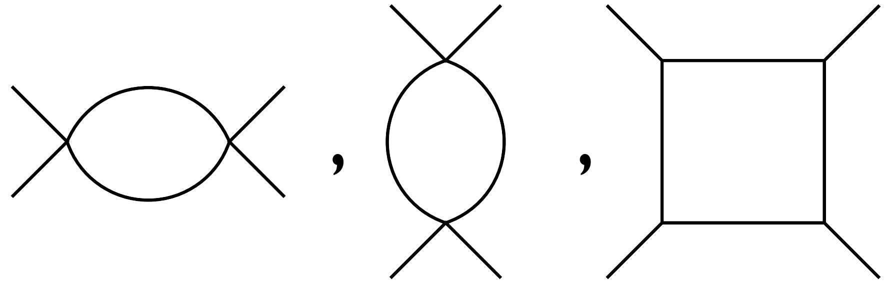
$$\partial_x \vec{f}(x;\epsilon) = \epsilon \sum_k \frac{A_k}{x - x_k} \vec{f}(x;\epsilon)$$

- expansion to any order in ϵ is linear algebra

answer: multiple polylogarithms of uniform weight ('transcendentality')

- asymptotic behavior $\vec{f}(x;\epsilon) \sim (x - x_k)^{\epsilon A_k} \vec{f}_0(\epsilon)$
- natural extension to multi-variable case

Example: massless 2 to 2 scattering

- basis $f = \{$  $\}$

$x = t/s$

$D = 4 - 2\epsilon$

differential equations

$$\partial_x f = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f \qquad a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

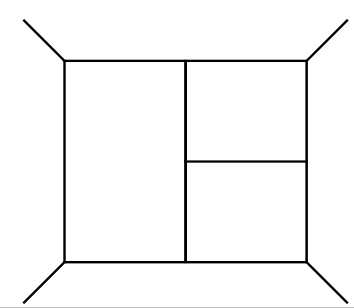
- (regular) singular points $\quad s = 0\,, \quad t = 0\,, \quad u = -s - t = 0$
- asymptotic behavior governed by matrices a, b

- Solution: expand to any order in ϵ

$f = \sum_{k \geq 0} \epsilon^k f^{(k)}$ $\qquad f^{(k)}$ is k-fold iterated integral (uniform weight)

alphabet $\{d \log x, d \log(1+x)\}$ or equivalently $\{x, 1+x\}$

- same eqs. at 2,3 loops, only bigger matrices a,b (!)



Multi-variable case and the alphabet

- Natural generalization to multi-variable case

$$d\vec{f}(\vec{x};\epsilon) = \epsilon\, d\left[\sum_k A_k \log \alpha_k(\vec{x})\right] \vec{f}(\vec{x};\epsilon)$$

constant matrices (A_k) — letters (alphabet) ($\alpha_k(\vec{x})$)

- Examples of alphabets:

4-point on-shell $\alpha = \{x, 1+x\}$

two-variable example (from 1-loop Bhabha scattering): $\alpha = \{x\,, 1 \pm x\,, y\,, 1 \pm y\,, x+y\,, 1+xy\}$

[J.M.H., Smirnov]

``hexagon functions`` in N=4 SYM $\alpha = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$

[Goncharov, Spradlin, Vergu, Volovich] [Caron-Huot, He]

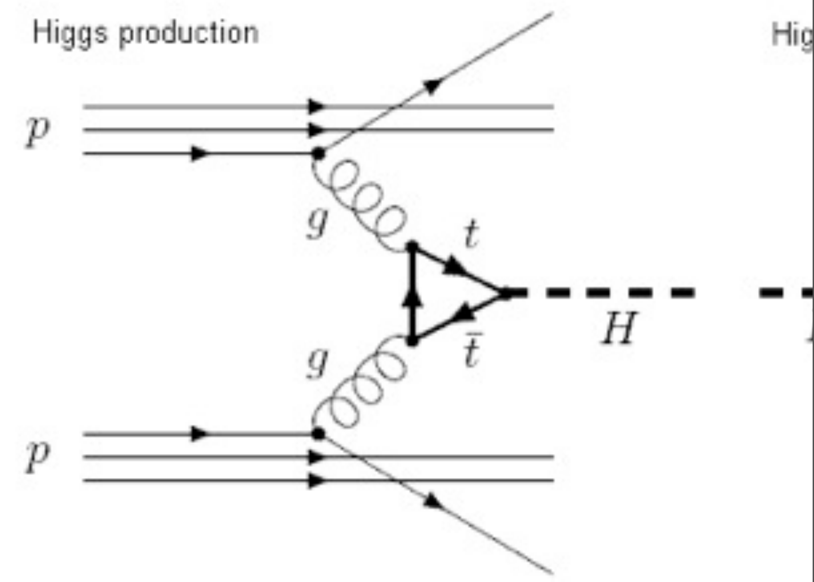
[Dixon, Drummond, J.M.H.] [Dixon et al.] [cf. Dixon's talk]

- Matrices and letters determine solution
- Immediate to solve in terms of Chen iterated integrals

The alphabet and perfect bricks (1)

Can we parametrize variables such that alphabet is rational?

Not essential, but nice feature.



- Example: Higgs production

encounter $\sqrt{1-4m^2/s}$

choose $-m^2/s = x/(1-x)^2$

$\alpha = \{x, 1-x, 1+x\}$ (to two loops)

Note: this is a purely kinematical question. Independent of basis choice.

- Related to diophantine equations

e.g. find rational solutions to equations such as

$$1 + 4\,a = b^2$$

here we found a 1-parameter solution

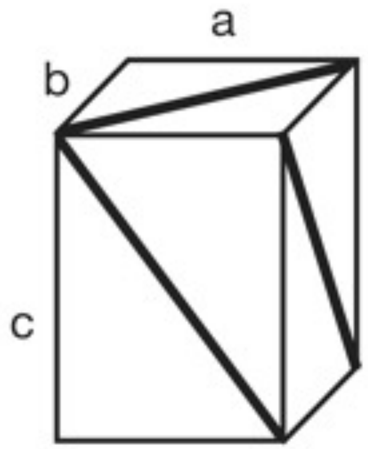
$$a = \frac{x}{(1-x)^2} \qquad b = \frac{1+x}{1-x}$$

The alphabet and perfect bricks (2)

- Classic example: Euler brick problem

Find a brick with sides a, b, c and diagonals d, e, f integers

$$a^2 + b^2 = d^2 \,,$$
$$a^2 + c^2 = e^2 \,,$$
$$b^2 + c^2 = f^2 \,.$$



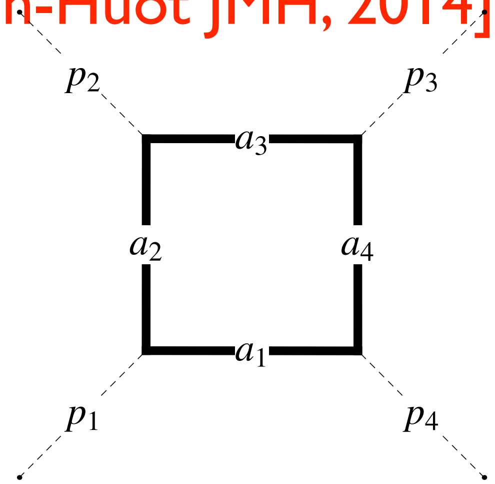
smallest solution (P. Halcke):

(a,b,c)=(44,117,240)

Perfect cuboid (add eq. $a^2 + b^2 + c^2 = g^2$): open problem in mathematics!

- Similar equations for particle kinematics

[Caron-Huot JMH, 2014]



e.g encountered in 4-d light-by-light scattering

$$u = -4m^2/s \qquad v = -4m^2/t$$

$$\beta_u = \sqrt{1+u},\ \beta_v = \sqrt{1+v},\ \beta_{uv} = \sqrt{1+u+v}$$

Need two-parameter solution to

$$\beta_u^2 + \beta_v^2 = \beta_{uv}^2 + 1$$

e.g. $$\beta_u = \frac{1-wz}{w-z}\,,\quad \beta_v = \frac{w+z}{w-z}\,,\quad \beta_{uv} = \frac{1+wz}{w-z}\,.$$

more roots in D-dim and at 3 loops! - in general alphabet changes with the loop order!

Find such solutions systematically? Minimal polynomial order?

Equivalent representations

- version 1: Chen iterated integrals

[most compact]

[flexible: analytic continuation, limits]

[easy to see DE, cuts]

[ideas for numerics: J.M.H., Caron-Huot]

$$g_6 = \int_\gamma d\,\log\frac{\beta_u-1}{\beta_u+1}\,d\log\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}+\int_\gamma d\,\log\frac{\beta_v-1}{\beta_v+1}\,d\,\log\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}\,.$$

[2 loops: 10 terms]

- version 2: Goncharov polylogarithms

(if alphabet rational in at least one variable)

$$g_6 = -G_{-1,0}(w)+G_{0,-1}(w)-G_{0,1}(w)+G_{1,0}(w)+H_{-1,0}(z)-H_{0,-1}(z)-H_{0,1}(z) + H_{1,0}(z)-G_0(w)H_{-1}(z)+G_{-1}(w)H_0(z)-G_1(w)H_0(z)-G_0(w)H_1(z)\,.$$

[2 loops: 2-3 pages]

[longer expressions; requires rational alphabet; GINAC numerical evaluation]

- version 3: minimal function basis $g_6 = -\beta_{uv}/2I_1$

$$I_1 = \frac{2}{\beta_{uv}}\Big\{2\log^2\left(\frac{\beta_{uv}+\beta_u}{\beta_{uv}+\beta_v}\right)+\log\left(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}\right)\log\left(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}\right)-\frac{\pi^2}{2} + \sum_{i=1,2}\Big[2\,\mathrm{Li}_2\left(\frac{\beta_i-1}{\beta_{uv}+\beta_i}\right)-2\,\mathrm{Li}_2\left(-\frac{\beta_{uv}-\beta_i}{\beta_i+1}\right)-\log^2\left(\frac{\beta_i+1}{\beta_{uv}+\beta_i}\right)\Big]\Big\}.$$

[arbitraryness; usually long expressions; good at low weight; fast numerical evaluation]

[2 loops: several pages]

$$\beta_u=\sqrt{1+u},\ \beta_v=\sqrt{1+v},\ \beta_{uv}=\sqrt{1+u+v}$$

- some examples from literature: [Goncharov et al.] [Duhr] [Gehrmann et al.] ...

Important points differential equations

- Uniform weight basis can be found systematically using cuts
(related to d-log representations) [Arkani-Hamed et al.] [J.M.H.]

  other algebraic ideas [Mastrolia et al.] [Caron-Huot, J.M.H.] [Gehrmann et al.]

- DE provide information about integrals in compact form
  (alphabet, matrices)

- contain more information than epsilon expansion: exact limits
- boundary conditions often for free (e.g. finiteness in certain limits)

[applications to single-scale integrals: cf. Smirnov's talk]

- Chen iterated integrals give most compact form of answer
- To given weight, answer can be rewritten in terms of minimal function basis [Goncharov]

On the QCD cusp anomalous dimension

based on work in progress with

A. Grozin

G. Korchemsky

P. Marquard

Cusp anomalous dimension

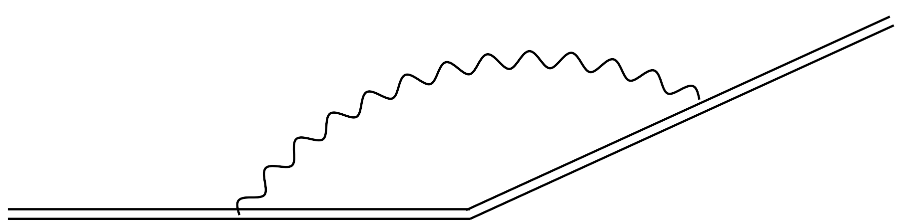
- Cusp anomalous dimension describes infrared divergences

[cf. L. Magnea's talk on Friday]

- $\Gamma_{\text{cusp}}(\phi)$ governs UV divergences at cusp

[Polyakov; 1 loop]

[2 loops: Korchemsky, Radyushkin (1987)]



$$\langle W \rangle \sim e^{-\left| \ln \frac{\mu_{UV}}{\mu_{IR}} \right| \, \Gamma_{\text{cusp}}}$$

- relation to light-like anomalous dimension K

[Korchemsky et al]

$$x = e^{i\phi} \qquad \lim_{x \to 0} \Gamma_{\text{cusp}} = -K \log x + \mathcal{O}(x^0)$$

- N=4 SYM susy/non-susy Wilson loop operator

$$\xi = \frac{\cos\theta - \cos\phi}{i \sin\phi} \qquad \theta = \frac{\pi}{2} \quad \longrightarrow \quad \xi = \frac{1+x^2}{1-x^2}$$

Beautiful answers

- Observation: constants in N=4 SYM anomalous dimensions have uniform ‘transcendentality’ [Kotikov, Lipativ, Velizhanin]

- generalize: pure functions of uniform weight (UT)

- suggests iterative differential structure

- what about QCD?

ref. [JMH, PRL 110 (2013)] suggests QCD integrals can also be chosen UT

do physical results look nice when expressed in a good basis?

Perturbative results in N=4 SYM

- 1 loop $\quad A^{(1)}(\phi) = -\xi \log x$
- 2 loops

[Makeenko, Oleson, Semenoff (2006)]
[Drukker, Forini (2012)]

$$A^{(2)}(\phi) = \frac{1}{3}\xi\left[\pi^2 \log x + \log^3 x\right]$$
$$- \xi^2 \left[\zeta_3 + \zeta_2 \log x + \frac{1}{3}\log^3 x + \log x \mathrm{Li}_2(x^2) - \mathrm{Li}_3(x^2)\right] .$$

- bosonic Wilson loop in N=4 SYM, 2 loops

$$\Gamma^{(2)g}_{\text{cusp}}(\phi) = A^{(2)}(\phi) - A^{(2)}(0) + B^{(2)}(\phi) - B^{(2)}(0) , \qquad \theta = \frac{\pi}{2}$$
$$B^{(2)}(\phi) = \left[\log^2 x + \frac{1}{3}\pi^2\right] - \xi\left[\zeta_2 + \log^2 x + 2\log x \mathrm{Li}_1(x^2) - \mathrm{Li}_2(x^2)\right] .$$

- 3 loops; ξ term at any loop order [Correa, JMH, Maldacena, Sever (2012)]

- 4 loops planar; nonplanar ξ^4 term;
- d-log algorithm for ladder integrals

[JMH, Huber (2013)]

A new look at two loops in QCD

- QCD result

[Korchemsky, Radyushkin (1987)]
nf [Braun, Beneke, 1995]
[Kidonakis (2009)]

$$\Gamma^{(1)} = C_F \left[A^{(1)}(\phi) - A^{(1)}(0) \right]$$

$$\Gamma^{(2)} = C_F C_A \left[A^{(2)}(\phi) - A^{(2)}(0) + B^{(2)}(\phi) - B^{(2)}(0) \right]$$
$$+ \left(-\frac{5}{9} C_F T_F n_f - \frac{67}{36} C_F C_A \right) \left[A^{(1)}(\phi) - A^{(1)}(0) \right] .$$

Only functions from N=4 SYM needed!

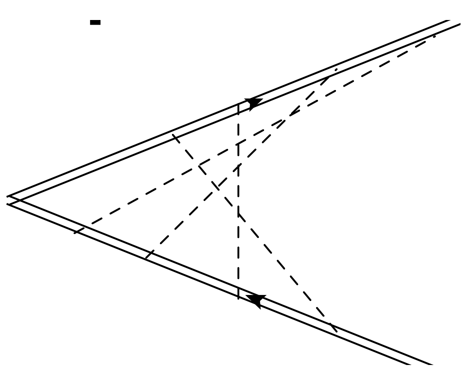
- $A^{(1)}$ uniform weight 1 : from susy WL
- $B^{(2)}$ uniform weight 2 : from bosonic WL
- $A^{(2)}$ uniform weight 3 : from susy WL

- what happens at 3 loops?
- why functions of uniform weight?

Why should we get pure functions?

- For Wilson line integrals, this is easy to see [JMH, Huber, JHEP 1309 (2013) 147]
  - key: 'd-log representations'
  - make it obvious that result is given by pure functions
  - provides algorithm for computing the answer

$$
\begin{aligned}
I_{\mathrm{NP,four-loop}}(x) = & -2\zeta_2(18H_{1,1,1,2} + 24H_{1,1,2,1} + 18H_{1,2,1,1} + 30H_{1,1,1,1,1}) \\
& + 48H_{1,1,1,4} + 64H_{1,1,2,3} + 64H_{1,1,3,2} + 48H_{1,2,1,3} \\
& + 48H_{1,2,2,2} + 80H_{1,1,1,1,3} + 80H_{1,1,1,2,2} + 24H_{1,1,1,3,1} \\
& + 64H_{1,1,2,1,2} + 32H_{1,1,2,2,1} + 32H_{1,1,3,1,1} + 48H_{1,2,1,1,2} \\
& + 24H_{1,2,1,2,1} + 24H_{1,2,2,1,1} + 62H_{1,1,1,1,1,2} + 40H_{1,1,1,1,2,1} \\
& + 22H_{1,1,1,2,1,1} + 8H_{1,1,2,1,1,1} + 6H_{1,2,1,1,1,1} + H_{1,1,1,1,1,1,1}
\end{aligned}
$$



  - note: implies that all functions of this family have this property!

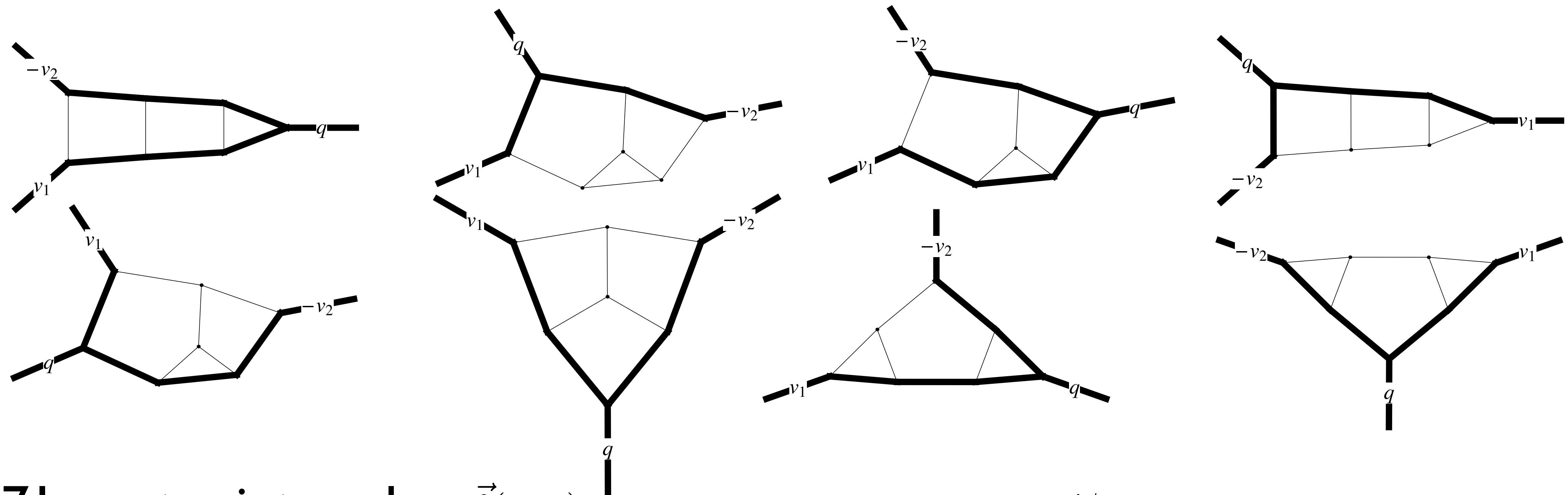
see this more generally: [JMH, PRL 110 (1013) 25]

- algorithm also works for the multi-line case.

other method:
[cf. E. Gardi (2014)]

Master integrals

- abelian eikonal exponentiation: need only planar integrals



- 71 master integrals $\vec{f}(x;\epsilon)$ $\quad D = 4 - 2\epsilon \quad x = e^{i\phi}$
- differential equations in suitable basis

[method: see JMH, PRL 110 (1013) 25]

$$\partial_x \vec{f}(x;\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{x-1} + \frac{c}{x+1} \right] \vec{f}(x;\epsilon)$$

a, b, c constant 71x71 matrices

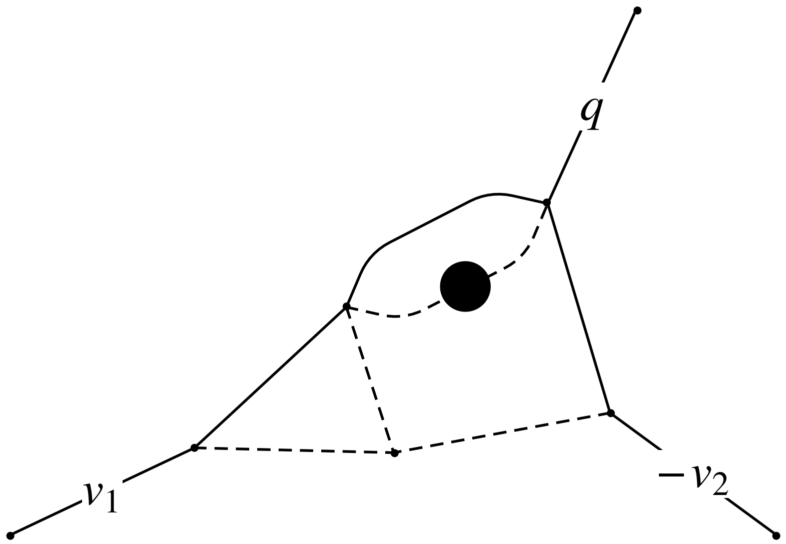
- boundary conditions trivially from $x = 1$

one integral: [Chetyrkin, Grozin, NP B666 (2003)]

- solution in terms of harmonic polylogarithms

[cf. V. Smirnov's talk for applications to multi-scale cases]

Example



$$f_{44} = \epsilon^5 \frac{1-x^2}{x} G_{1,0,1,0,1,0,1,1,2,0,1,0}$$

$$x = e^{i\phi}$$

$$f_{44} = \epsilon^4 \Big[-\frac{1}{6}\pi^2 H_{0,0}(x) - \frac{2}{3}\pi^2 H_{1,0}(x) - 4H_{0,-1,0,0}(x) + 2H_{0,0,-1,0}(x) + 2H_{0,1,0,0}(x) - 4H_{1,0,0,0}(x) + 4\zeta_3 H_0(x) - \frac{17\pi^4}{360} \Big] + \mathcal{O}(\epsilon^5)$$

- all basis integrals are pure functions of uniform weight
- numerical checks with FIESTA
- confirmed previously known `N=4 SYM` result

Calculation at three loops

(1) compute proper vertex function

(2) take into account renormalization of Lagrangian

(3) compute vertex renormalization

(4) extract Gamma cusp $\Gamma_{\rm cusp} = \dfrac{\partial}{\partial \log \mu} \log Z$

- color structures $\Gamma^{(3)}_{\rm cusp} : c_1 C_F C_A^2 + c_2 C_F (T_f n_f)^2 + c_3 C_F^2 T_f n_f + c_4 C_F C_A T_F n_f$

$C_F (T_F n_f)^2$ [Braun, Beneke, 1995]

$C_F^2 T_F n_f$, $C_F C_A T_F n_f$ } this talk

$C_F C_A^2$ stay tuned!

Results $\Gamma^{(3)}_{\rm cusp} : c_1 C_F C_A^2 + c_2 C_F (T_f n_f)^2 + c_3 C_F^2 T_f n_f + c_4 C_F C_A T_F n_f$

$$c_2 = -\frac{1}{27} A^{(1)} \qquad c_3 = \left(\zeta_3 - \frac{55}{48}\right) A^{(1)}$$

$$c_4 = -\frac{5}{9}\left(A^{(2)} + B^{(2)}\right) - \frac{1}{6}\left(7\zeta_3 + \frac{209}{36}\right) A^{(1)}$$

$$A = A(\phi) - A(0)$$

Only functions from N=4 SYM needed!

- Checks: expected divergence structure

$$\log Z = -\frac{1}{2\epsilon}\left(\frac{\alpha_s}{\pi}\right)\Gamma^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{\beta_0}{16\epsilon^2}\Gamma^{(1)} - \frac{1}{4\epsilon}\Gamma^{(2)}\right] + \left(\frac{\alpha_s}{\pi}\right)^3\left[-\frac{\beta_0^2\Gamma^{(1)}}{96\epsilon^3} + \frac{\beta_1\Gamma^{(1)} + 4\beta_0\Gamma^{(2)}}{96\epsilon^2} - \frac{\Gamma^{(3)}}{6\epsilon}\right].$$

- Known limit $\lim_{x\to 0}\Gamma_{\rm cusp} = -K\log x + \mathcal{O}(x^0)$

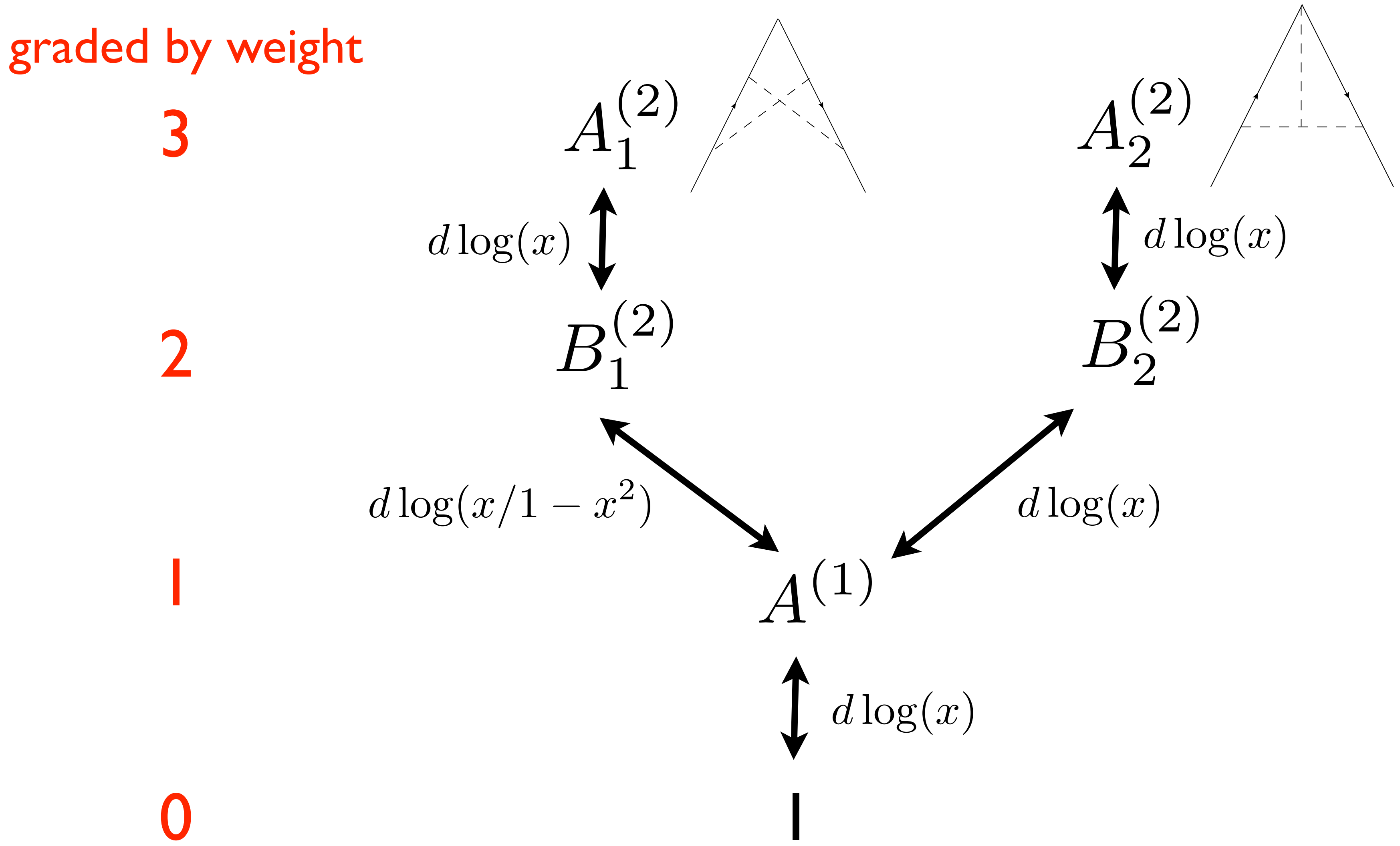
$$K^{(3)} = \frac{1}{4} C_F C_A^2\left(\frac{245}{24} - \frac{67}{9}\zeta_2 + \frac{11}{6}\zeta_3 + \frac{11}{5}\zeta_2^2\right) + C_F^2 n_f T_F\left(-\frac{55}{48} + \zeta_3\right) + \frac{1}{2} C_F C_A n_f T_F\left(-\frac{209}{108} + \frac{10}{9}\zeta_2 - \frac{7}{3}\zeta_3\right) + C_F n_f^2 T_F^2\left(-\frac{1}{27}\right)$$

[Vogt (2001)] [Berger (2002)] [Moch, Vermeaseren, Vogt (2004)]

Iterative structure of loop integrals

cf. [Caron-Huot, J.M.H. (2014)

- The physical result is finite as $D \to 4$
- Obtain it from a subset of finite integrals/functions?



- Note: functions appear already in `simpler` N=4 SYM calculations!
- top-down vs. bottom-up approach

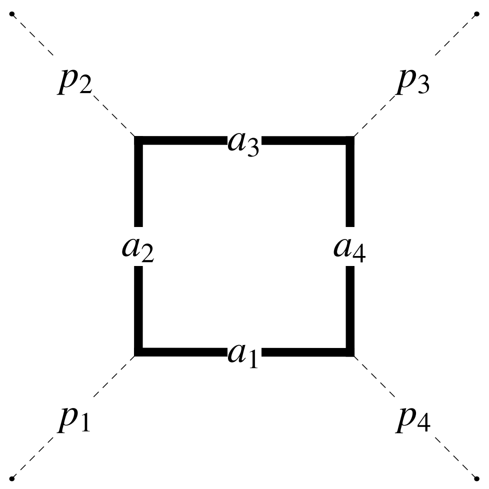
Massive scattering amplitudes in N=4 SYM

Massive scattering amplitudes in N=4 SYM

- define analog of light-by-light scattering [Alday, J.M.H., Plefka, Schuster]

$$s, t, m^2$$

[Schabinger]



- natural for dual conformal symmetry
- previously studied only in limits:

use mass as regulator $m^2 \ll s, t$

Regge limit $s \gg m^2, t$ related to cusp anomalous dimension

[J.M.H., Naculich, Schnitzer, Spradlin]

- Systematic analysis for generic kinematics [Caron-Huot, J.M.H., 2014]

one loop:

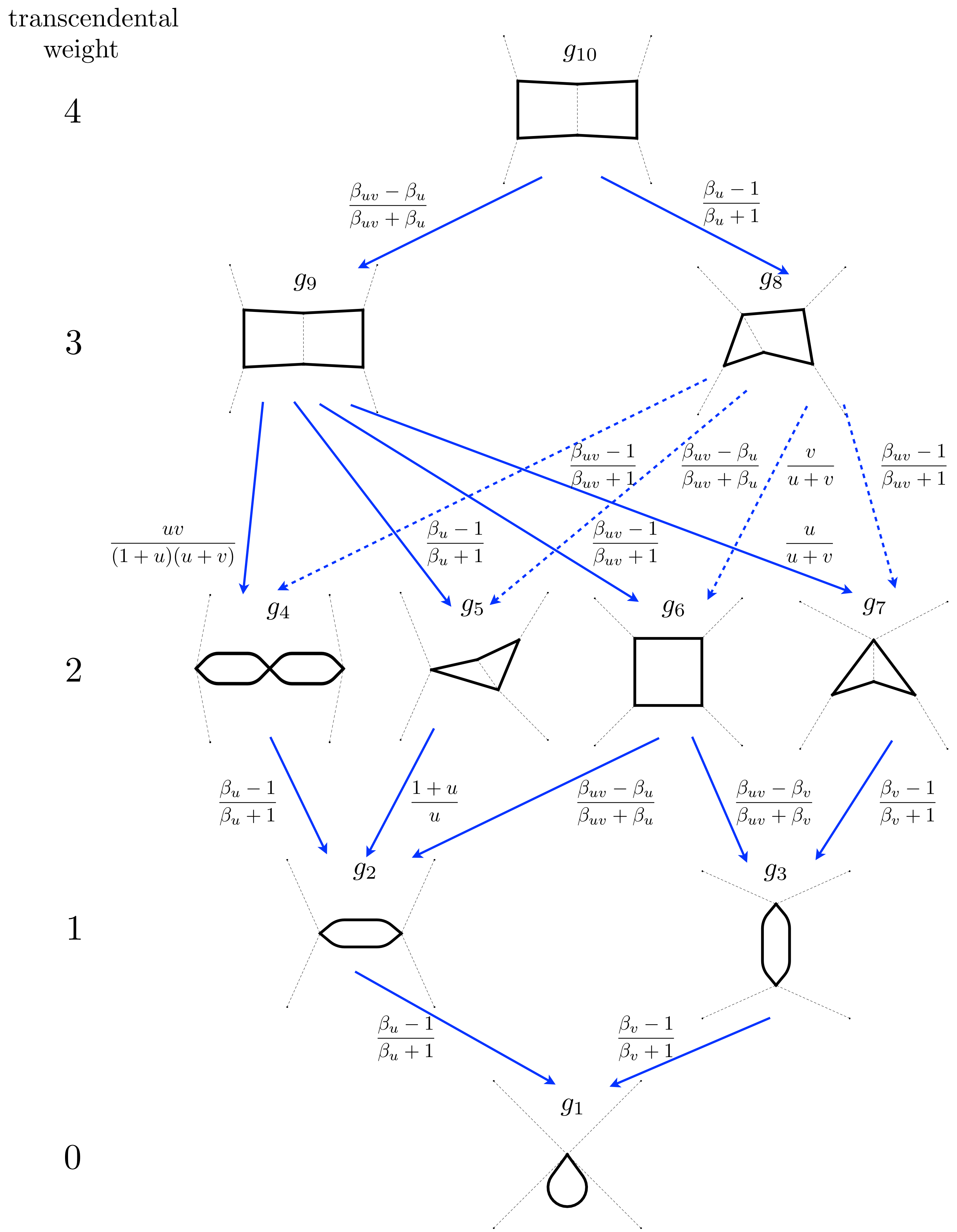
$$I_1 = \frac{2}{\beta_{uv}} \Big\{ 2\log^2\left(\frac{\beta_{uv}+\beta_u}{\beta_{uv}+\beta_v}\right) + \log\left(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}\right)\log\left(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}\right) - \frac{\pi^2}{2} + \sum_{i=1,2}\Big[2\,\mathrm{Li}_2\left(\frac{\beta_i-1}{\beta_{uv}+\beta_i}\right) - 2\,\mathrm{Li}_2\left(-\frac{\beta_{uv}-\beta_i}{\beta_i+1}\right) - \log^2\left(\frac{\beta_i+1}{\beta_{uv}+\beta_i}\right)\Big]\Big\}.$$

$$u = -4m^2/s \qquad v = -4m^2/t \qquad \beta_u = \sqrt{1+u},\ \beta_v = \sqrt{1+v},\ \beta_{uv} = \sqrt{1+u+v}$$

two-loop and three-loop answer now also known.

Iterative structure for finite loop integrals

[Caron-Huot, J.M.H. (2014)]



- block triangular matrix structure (weight grading)
- algorithm for finding this form

Discussion and outlook

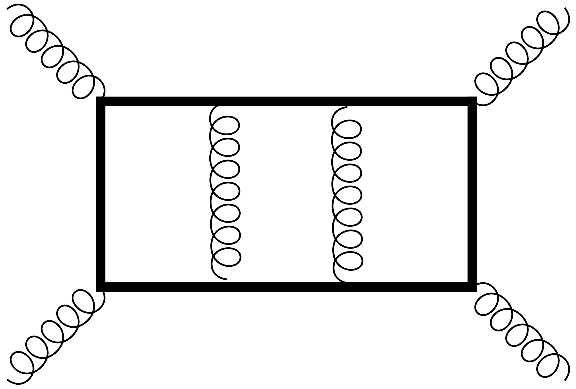
- iterative structure of finite loop integrals

  perfect for finite physical objects, e.g.

  - similar to structure for MHV and NMHV hexagon functions in N=4 SYM cf. [Dixon, Drummond, J.M.H. (2012)

  - possible application: correlation functions in CFT [cf. e.g. Sokatchev's talk]

- integrals and cross sections for light-by-like scattering in N=4 SYM



3 loops and 3 scales!

full calculation, no guesses

similar integrals appear in QCD for finite top quark mass

- results for Regge trajectories [Caron-Huot, J.M.H., to appear], [cf. Caron-Huot's talk]

- dual conformal symmetry is generalization of conservation of Laplace-Runge-Lenz-(Pauli) vector for hydrogen atom!

Conclusions

- exciting results and techniques

- some already applicable in QCD
(e.g. uniform weight basis, Chen iterated integrals)

- more work needed for elliptic functions and generalizations

- New results:
    - 3-loop QCD cusp anomalous dimension
    - 3-loop light-by-light scattering in N=4 SYM

Thank you!