Some features of closed string scattering amplitudes

Michael B. Green
University of Cambridge

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CONSTRAINTS ON CLOSED STRING EFFECTIVE ACTION FROM AMPLITUDE CALCULATIONS

I will consider narrowly-focused aspects of the low energy effective string action obtained from closed string scattering amplitudes.

- FEATURES OF CLOSED STRING PERTURBATION THEORY:

  Comments on relation to supergravity field theory amplitudes.

- NON-PERTURBATIVE FEATURES - DUALITY:

  Connects perturbative with non-perturbative effects.

  Powerful constraints imposed by SUSY, Duality, Unitarity

  Connections with quantum eleven-dimensional supergravity.

- CONNECTIONS WITH BEAUTIFUL MATHEMATICS:

  Modular Forms; Automorphic forms for higher-rank groups; Multi-Zeta Values; …….

MBG, Stephen Miller, Pierre Vanhove	`arXiv:1404.2192`
MBG, Eric D'Hoker,	`arXiv:1308.4597`
MBG, Eric D'Hoker, Boris Pioline, Rudolfo Russo;	`arXiv:1405.6226`

THE LOW ENERGY EXPANSION OF STRING THEORY

- LOWEST ORDER TERM reproduces the results of classical supergravity

EINSTEIN-HILBERT

$\alpha' = \ell_s^2$

ℓ_s is STRING LENGTH SCALE

$$\frac{1}{\alpha'^4}\int d^{10}x\sqrt{-\det G}\, e^{-2\phi}\, R + \ldots$$

several other supergravity fields

METRIC – $G_{\mu\nu}$

SCALAR FIELD - DILATON

$e^{-\phi} = \frac{1}{g_s}$

STRING COUPLING CONSTANT

Expanding the curvature in small fluctuations of the metric around D=10 Minkowski space gives contributions to "classical" MULTI-GRAVITON scattering amplitudes.

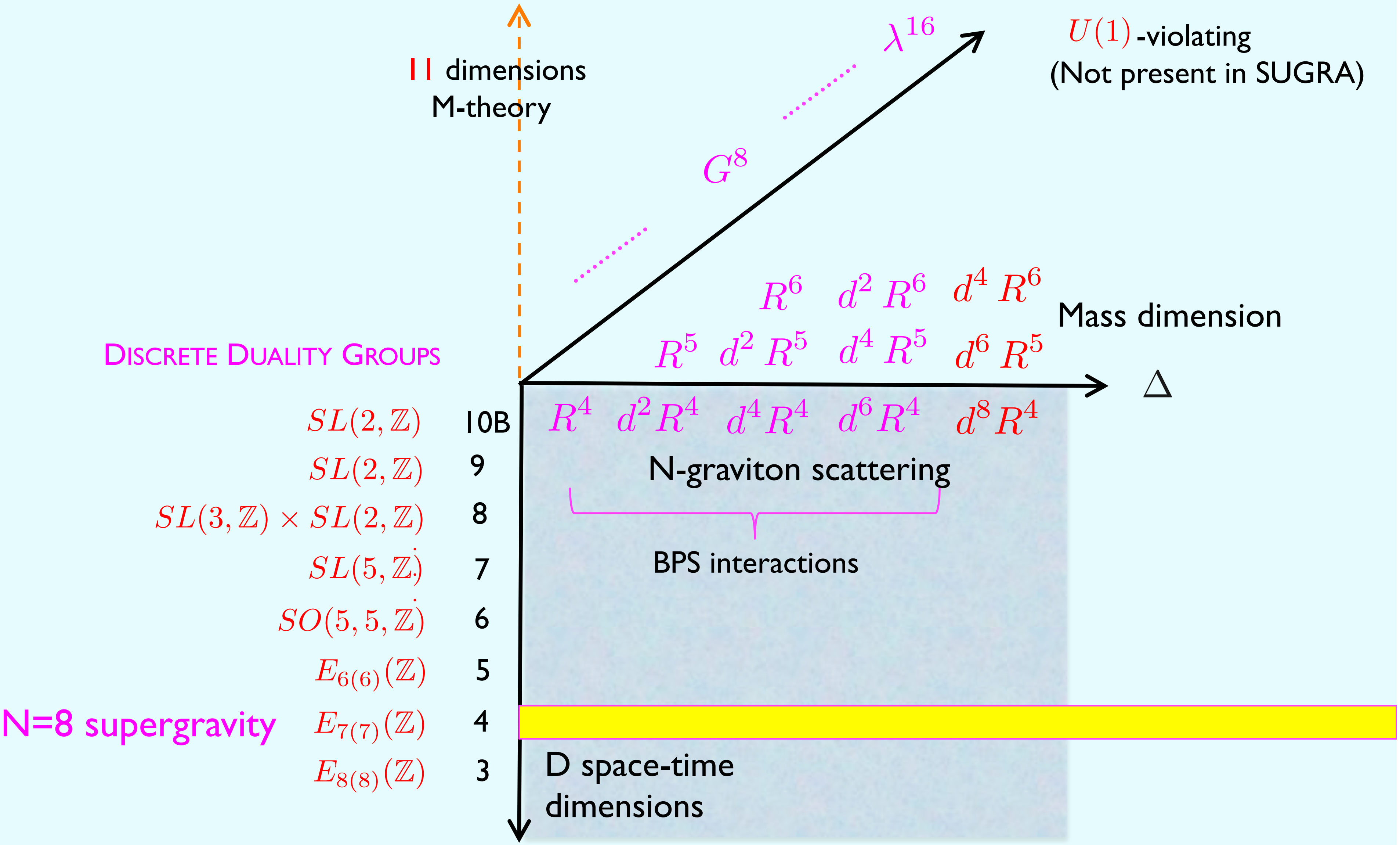
- HIGHER ORDER TERMS: $\frac{1}{\alpha'}\int d^{10}x\,\sqrt{-\det G}\,\mathcal{F}(\phi,\ldots)\,R^4 + \ldots$

Moduli-dependent coefficient

- Expansion in powers of $\alpha' R\,,\quad \alpha' D^2\,,\quad \ldots$

THE LOW ENERGY EXPANSION OF (TYPE IIB) STRING THEORY

HIGHER DERIVATIVE CORRECTIONS to Einstein theory



11 dimensions
M-theory

λ^{16}

$U(1)$ -violating
(Not present in SUGRA)

G^8

R^6 $d^2\,R^6$ $d^4\,R^6$

R^5 $d^2\,R^5$ $d^4\,R^5$ $d^6\,R^5$

Mass dimension

Δ

DISCRETE DUALITY GROUPS

$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

R^4 d^2R^4 d^4R^4 d^6R^4 d^8R^4

N-graviton scattering

BPS interactions

N=8 supergravity

D space-time
dimensions

SCALAR FIELDS (MODULI) AND DUALITY

groups in E_n series (real split forms) (Cremmer, Julia)

SUPERGRAVITY (low energy limit of string theory):

Scalar fields parameterize a symmetric space $G(\mathbb{R})/K(\mathbb{R})$

STRING THEORY:

Discrete identifications of scalar fields $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$

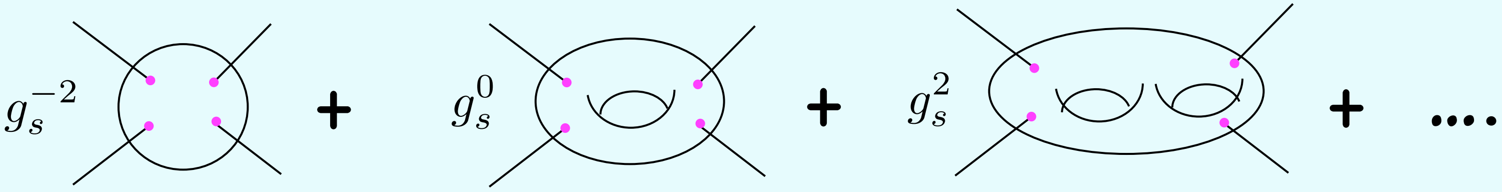
DUALITY GROUP $G(\mathbb{Z})$

Only a discrete arithmetic subgroup of $G(\mathbb{R})$ is symmetry of string theory – even at tree level.

STRING PERTURBATION THEORY: Expansion around boundary of moduli space.

e.g. in powers of $g_s = e^{\phi} \ll 1$ (c.f. FEYNMAN DIAGRAMS of quantum field theory) :

Sum of functional integrals over Riemann surfaces



g_s^{2h-2} × (genus-h Riemann surface)

HOW POWERFUL ARE THE CONSTRAINTS IMPOSED BY SUSY, DUALITY AND UNITARITY ??

The aim is to investigate the exact moduli dependence of low lying terms in the low energy expansion.

Duality relates different regions of moduli space –
connects perturbative and non-perturbative features in a highly nontrivial manner.

e.g.

FOUR-GRAVITON SCATTERING IN TYPE II STRING THEORY

moduli

$$A_D(s,t,u;\mu_D) = R^4\, T_D(s,t,u;\mu_D)$$

R Linearized curvature $\sim k_\mu k_\nu \zeta_{\rho\sigma}$

Symmetric function of Mandelstam invariants s,t,u (with $s+t+u=0$).

Has an expansion in power series of $\sigma_2 = s^2+t^2+u^2$ and $\sigma_3 = s^3+t^3+u^3$.

(non-analytic pieces are essential, but will be ignored here)

$$T_D(s,t,u;\mu_D) = \sum_{p,q} \mathcal{E}^{(D)}_{(p,q)}(\mu_D)\, \sigma_2^p \sigma_3^q$$

$\sim s^{2p+3q} + \dots$

Coefficients are duality invariant functions of scalar fields (moduli, or coupling constants).

TO WHAT EXTENT CAN WE DETERMINE THESE COEFFICIENTS?

For now focus on the ten-dimensional cases with one modulus:

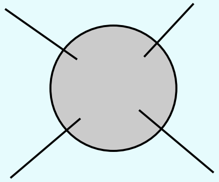
Type IIA: $\Omega = g_A^{-1} = e^{-\phi_A}$

Type IIB: $\Omega = \Omega_1 + i\Omega_2$ $SL(2,\mathbb{Z})$ duality

inverse string coupling constant $\Omega_2 = g_B^{-1} = e^{-\phi_B}$

BOUNDARY DATA: STRING PERTURBATION THEORY

TREE-LEVEL: (VIRASORO AMPLITUDE)



Polarisation tensor

dilaton

coupling $g = e^{\phi}$

$$A_0^{(4)}(\epsilon_r, k_r; \phi) = e^{-2\phi} R^4 T_0^{(4)}(s,t,u)$$

$$T_0^{(4)} = \frac{4}{stu} \frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)}$$

$s^k R^4 \sim d^{2k} R^4$

Tree-level SUPERGRAVITY

$$= \frac{3}{\sigma_3} + \overset{R^4}{2\zeta(3)} \alpha'^3 + \zeta(5)\alpha'^5 \overset{d^4 R^4}{\sigma_2} + \frac{2\zeta(3)^2}{3}\alpha'^6 \overset{d^6 R^4}{\sigma_3} + \frac{\zeta(7)}{2}\alpha'^7 \overset{d^8 R^4}{\sigma_2^2}$$

$$+ \frac{2\zeta(3)\zeta(5)}{3}\alpha'^8 \underset{d^{10} R^4}{\sigma_2\sigma_3} + \frac{\zeta(9)}{4}\alpha'^8 \zeta_2^3 + \frac{2}{27}(2\zeta(3)^3 + \zeta(9))\alpha'^9 \sigma_3^2 + \ldots$$

$d^{12} R^4$

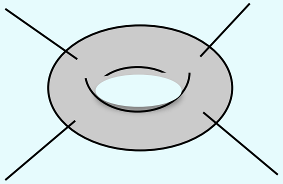
$\sigma_2 = s^2 + t^2 + u^2$

$\sigma_3 = s^3 + t^3 + u^3 = 3stu$

INFINITE SERIES of $d^{2k} R^4$ terms. Coefficients are powers of ζ values with rational coefficients – as in loop amplitudes in quantum field theory

GENUS ONE

moduli $\in SO(d,d)/(SO(d)\times SO(d))$



$$A_1^{(4)}(\epsilon_r, k_r; \phi, \rho_d) = \frac{\pi}{16} R^4 \int_{\mathcal{M}_1} \frac{|d\tau|^2}{(\mathrm{Im}\,\tau)^2} \mathcal{B}_1(s,t,u;\tau)\, \Gamma_{d,d,1}(\rho_d;\tau)$$

Genus-one lattice factor for d-torus; moduli ρ_d

Integral over complex structure

$$\mathcal{B}_1(s,t,u;\tau) = \int_{\Sigma^4} \frac{\prod_{i=1}^{i=4} d^2 z}{(\mathrm{Im}\,\tau)^4} \exp\left(-\frac{\alpha'}{2} \sum_{i<j} k_i \cdot k_j\, G(z_i,z_j)\right)$$

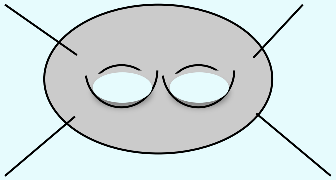
Low energy expansion - integrate powers of the genus-one Green function over the torus and over the modulus of the torus – difficult!

e.g $d=0\,,\quad D=10$ $\quad A^{(4)}_{1\ an} = \left(\frac{\pi}{3} + 0\,\sigma_2 + \frac{\pi\zeta(3)}{9}\,\sigma_3 + \ldots\right) R^4$ (MBG, Russo, Vanhove)

These coefficients look analogous to the tree-level coefficients:

WHAT IS THE CONNECTION BETWEEN THEM??

GENUS TWO :



$$A_2^{(4)}(\epsilon_r, k_r; \phi, \rho_d) = \frac{\pi}{64}\, e^{2\phi}\, R^4 \int_{\mathcal{M}_2} d\mu_2\, \mathcal{B}_2(s,t,u;\Omega)\, \Gamma_{d,d,2}(\rho_d;\Omega)$$

$$\mathcal{B}_2(s,t,u;\Omega) = \int_{\Sigma^4} \frac{|\mathcal{Y}_S|^2}{(\det Y)^2} \exp\left\{ -\frac{\alpha'}{2} \sum_{i<j} k_i \cdot k_j\, G(z_i, z_j) \right\}$$

Genus-two Green function

$Sp(4,\mathbb{Z})$-invariant measure proportional to $|\,\omega(z_1)\,\omega(z_2)\,\omega(z_3)\,\omega(z_4)\,|^2$ and $O(s^2)$ where $\omega(z)$ is holomorphic abelian differential

Expand in powers of α' :

Lowest-order term $O(s^2)$

Next term $O(s^3)$

$$A_2^{(4)} = g_s^2 \Big(\frac{4}{3}\zeta(4)\, \sigma_2 R^4 \qquad + \qquad 64 \int_{\mathcal{M}_2} d\mu_2\, \varphi\, \sigma_3\, R^4 + \ldots \Big)$$

Proportional to volume of genus-two moduli space

D'Hoker, Gutperle, Phong

$$\varphi(\Sigma) = -\frac{1}{8} \int_{\Sigma^2} P(x,y)\, G(x,y)$$

An invariant of genus-h Riemann surface defined by Zhang and Kawazumi.

D'Hoker, MBG

Bi-form (projection operator) contains $|\,\omega(x)\,\omega(y)\,|^2$

$d^4\, R^4$

$d^6\, R^4$

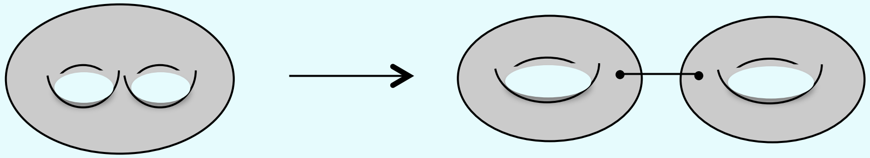
Recently evaluated. D'Hoker, MBG, Pioline, R.Russo

Strikingly, it turns out that : $(\Delta_{Sp(4)} - 5)\,\varphi = 0$

So integrate by parts: $\int_{\mathcal{M}_2} d\mu_2\,\varphi = \frac{1}{5}\int_{\mathcal{M}_2} d\mu_2\,\Delta_{Sp(4)}\,\varphi$

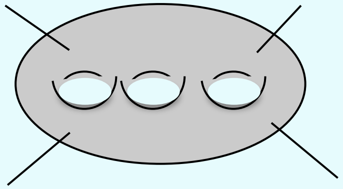
Using expression for φ

Integral picks up non-zero boundary contribution from the limit in which the genus-two surface degenerates into the union of two genus one surfaces



Result: $A_2^{(4)} = g_s^2\Big(\underbrace{\frac{4}{3}\zeta(4)\,\sigma_2 R^4}_{d^4\,R^4} + \underbrace{4\zeta(4)\sigma_3 R^4}_{d^6\,R^4} + \ldots\Big)$

GENUS THREE AND HIGHER



Technical difficulties analysing 3-loops. Recently, Gomez and Mafra constructed the genus-three amplitude using Berkovits' PURE SPINOR FORMALISM. They evaluated the leading low energy behaviour, giving,

Alternative to supermoduli space of RNS formalism

$$A_{10}^{3-loop} = g^4 \left(\frac{4}{27} \zeta(6) \overset{d^6\, R^4}{\downarrow}\sigma_3 + \ldots \right) R^4$$

BUT! There may be a spurious factor of 3

- Problems with singularities in the pure spinor formalism at genus > 4 (for four-graviton amplitude) remain to be resolved.

- New issues for genus > 4 (for four-graviton amplitude) in Ramond-Neveu-Schwarz formalism (integration over super-Riemann surfaces). Superspace is non-projected so cannot express the amplitude as an integral over bosonic moduli. (Donagi, Witten)

EXTENSIONS TO N-PARTICLE AMPLITUDES

Very brief summary

- OPEN-STRING TREES: For $N > 4$ coefficients of higher derivative interactions involve (Yang-Mills) non-trivial multi-zeta values (MZV's) (Mafra, Schlotterer, Stieberger)

First case is $\zeta(5,3) + \ldots$ weight $w = 8$

$$\zeta(n_1 \ldots, n_r) = \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} k_l^{-n_l}$$

$$\text{weight},\ w = \sum_i n_i;\ \ \text{depth} = r$$

- CLOSED-STRING TREE amplitudes (gravity)

(N-3)! - component basis vector of colour-ordered Yang-Mills tree amplitudes.

$$A_N^{tree} = A_{YM}^{\pi}\, S_{\pi\sigma}^{tree}(\{s_{ij})\})\, \tilde{A}_{YM}^{\sigma}$$

$(N-3)! \times (N-3)!$ matrix

Non-trivial MZV's with odd weights arise, starting at weight

$w = 11$ $\zeta(5,3,3) + \ldots$

- CLOSED-STRING 1-LOOP 5-POINT AMPLITUDE (MBG, Mafra, Schlotterer)

Pure spinor formalism

$$A_5^{1-loop} = A_{YM}^{\pi}\, S_{\pi\sigma}^{1-loop}(\{s_{ij})\})\, \tilde{A}_{YM}^{\sigma}$$

2×2 matrix

"DOUBLING" OF YANG-MILLS THEORY TREE AMPLITUDES

...............AND MUCH MORE

NON-PERTURBATIVE EXTENSION

Duality, supersymmetry and unitarity constraints

Focus here on the simplest nontrivial duality group $SL(2,\mathbb{Z})$

Type IIB in D=10 dimensions

TYPE IIB SUPERGRAVITY

Ten-dimensional supersymmetric extension of Einstein theory.

- Scalars $\Omega = \Omega_1 + i\Omega_2 \, ,$ $\Omega_2 = e^{-\phi} = g_B^{-1}$ $($string coupling$)^{-1}$

  span coset space $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/U(1)$

- Fields carry $U(1)$ charges u_Φ :

	$\partial_\mu\Omega/\Omega_2$		$\Omega_2^{-\frac{1}{2}}(F_{\mu\nu\rho} + i\Omega_2 H_{\mu\nu\rho})$			
	$P_\mu \, ,$	$\lambda \, ,$	$G_{\mu\nu\rho} \, ,$	$\psi_\mu ,$	$g_{\mu\nu} \, ,$	F_5
	dilaton	dilatino	3-form	gravitino	metric	5-form
u_Φ :	$-2 \, ,$	$-3/2 \, ,$	$-1 \, ,$	$-1/2 \, ,$	$0 \, ,$	0

- $SL(2,Z)$ duality symmetry $\Omega \to \frac{a\Omega+b}{c\Omega+d}$ $a, b, c, d \in \mathbb{Z}$ $ad - bc = 1$

- Pattern of u_Φ non-conserving higher-order interactions.

HIGHER-DERIVATIVE INTERACTIONS

Consider a composite operator $\mathcal{P}^{(u)}_{2n+2}$ of $U(1)$ charge u, dimension $\Delta = 2n+2\,$.

e.g. $R^4 \;:\quad u=0\,,\quad \Delta = 8\,;$ $\qquad \lambda^{16} \;:\quad u=-24\,,\quad \Delta=8\,;$

$(G\bar{G})^p\, R^4 \;:\quad u=0\,,\quad \Delta = 2p+8$

$SL(2,\mathbb{Z})$-invariant action (Einstein frame)

$$S^{(n)} = \ell_s^{2n-8} \sum_{u,i} \int d^{10}x\, e\, \mathcal{F}_n^{(u)\,i}(\Omega)\, \mathcal{P}^{(-u)\,i}_{2n+2}$$

$\mathcal{F}_n^{(u)\,i}$ has holomorphic and anti-holomorphic weights $\pm u/2$

Index i labels degeneracy of the term.

$$\mathcal{F}_n^{(u)\,i}(\Omega,\bar\Omega) \to \left(\frac{c\bar\Omega+d}{c\Omega+d}\right)^{u/2} \mathcal{F}_n^{(u)\,i}(\Omega,\bar\Omega) \qquad\qquad \Omega \to \frac{a\Omega+b}{c\Omega+d}$$

HOW IS $\mathcal{F}_n^{(u)\,i}$ CONSTRAINED BY SUPERSYMMETRY?

CONSEQUENCES OF SUPERSYMMETRY

Invariance of action

$$\sum_{m=0}^{\infty} \delta^{(m)} \sum_{n=0}^{\infty} S^{(n)} = 0$$

i.e,

$$(\delta^{(0)} + \alpha'^3 \delta^{(3)} + \dots)(S^{(0)} + \alpha'^3 S^{(3)} + \dots) = 0$$

On-shell algebra

$$[\delta\,,\delta]\Phi = [\delta^{(0)} + \alpha'^3 \delta^{(3)} + \dots\,, \delta^{(0)} + \alpha'^3 \delta^{(3)} + \dots]\Phi$$
$$= a \cdot P\,\Phi + \Phi \text{ eqn. of motion} + \delta_{gauge}\Phi$$

Strongly constrains the form of $S^{(n)}$ $\delta^{(n)}$

Difficult to implement in detail in absence of off-shell superspace formalism.

Leads to expression of general form : (suppressing superscripts and coefficients)

$$\mathcal{D}\,\mathcal{F}_n = \mathcal{F}_n + \mathcal{F}_{m_1}\,\mathcal{F}_{n-m_1} + \mathcal{F}_{m_1}\,\mathcal{F}_{m_2}\,\mathcal{F}_{n-m_2-m_1}$$
$$+ \dots + \mathcal{F}_{m_1}\,\mathcal{F}_{m_2}\dots\mathcal{F}_{n-m_1-\dots-m_{n-1}} \;+ \dots$$

Modular covariant derivative $\mathcal{D} = i\Omega_2 \frac{\partial}{\partial\Omega} - \frac{u}{4}$

(a) Simple examples non-degenerate examples (index i on $\mathcal{F}_n^{(u)\,i}$ is redundant) :

$$\mathcal{D}\,\mathcal{F}_n^{(u)} = c_u\,\mathcal{F}_n^{(u+2)} \qquad\qquad \bar{\mathcal{D}}\,\mathcal{F}_n^{(u+2)} = \bar{c}_{u+2}\,\mathcal{F}_n^{(u+2)}$$

Implies LAPLACE EIGENVALUE EQUATION :

$$\bar{\mathcal{D}}\mathcal{D}\,\mathcal{F}_n^{(u)} = c_u\bar{c}_{u+2}\,\mathcal{F}_n^{(u)}$$

i) $U(1)$ preserving: e.g. R^4 $d^4\,R^4$

$u = 0\,,$ $c_0\,\bar{c}_2 = s(s-1)$ where $n = 2s = \frac{1}{2}\Delta - 1$

$$\Delta_\Omega\,\mathcal{F}_n^{(0)} = s(s-1)\,\mathcal{F}_n^{(0)} \qquad\qquad \Delta_\Omega = 4\Omega_2^2\,\partial_\Omega\,\partial_{\bar{\Omega}}$$

Solution is NON-HOLOMORPHIC EISENSTEIN SERIES

$$E_s(\Omega) = \sum_{\gcd(p,q)=1} \frac{\Omega_2^s}{|p+q\Omega|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} (\mathrm{Im}\,\gamma\Omega)^s$$

Poincare series – manifest $SL(2,\mathbb{Z})$

Parabolic subgroup

NON-HOLOMORPHIC EISENSTEIN SERIES

- $SL(2,\mathbb{Z})$ - INVARIANT (generalises to higher rank duality groups)

- Solution of LAPLACE EIGENVALUE EQN. (consequence of maximal supersymmetry)

$$\Delta_\Omega \, E_s(\Omega) = s(s-1) \, E_s(\Omega) \qquad\qquad \Delta_\Omega = \Omega_2^2(\partial^2_{\Omega_1} + \partial^2_{\Omega_2})$$

- FOURIER SERIES $E_s(\Omega) = 2\sum_{k=0}^{\infty} \mathcal{F}_k(\Omega_2) \cos(2\pi i k \Omega_1)$

- ZERO MODE $k=0$ - *TWO POWER-BEHAVED TERMS* (perturbative) :

$$\mathcal{F}_0 = \Omega_2^s + \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\zeta(2s)\Gamma(s)} \, \Omega_2^{1-s}$$

- NON-ZERO MODES $k > 0$ - *D-INSTANTON SUM*

K-Bessel function
$K_s(y) \underset{y>>1}{\sim} e^{-y} \, y^{-1/2}$

$$\mathcal{F}_k = \frac{2\pi^s}{\zeta(2s)\Gamma(s)} \, |k|^{s-\frac{1}{2}} \, \sigma_{2s-1}(k) \, \Omega_2^{\frac{1}{2}} \, K_{s-\frac{1}{2}}(2\pi|k|\Omega_2)$$

$$\sim \frac{\pi^{s-\frac{1}{2}}}{\zeta(2s)\Gamma(s)} \, |k|^{s-1} \, \sigma_{2s-1}(k) \, e^{-2\pi|k|\Omega_2}$$

divisor sum
$\sigma_n(k) = \sum_{p|k} p^n$

ILLUSTRATED BY FOUR-GRAVITON AMPLITUDE $\mathcal{E}^{(D)}_{(p,q)}(\Omega)\,\sigma_2^p\sigma_3^q\,R^4$ (Einstein frame)

D=10 examples (in string frame):

$$\Omega_2^{\frac{1}{2}}\,\mathcal{E}_{(0,0)}\,R^4 \qquad \mathcal{E}_{(0,0)} = 2\zeta(3)\,E_{\frac{3}{2}}(\Omega)$$

TREE + 1-LOOP

$1/2$-BPS $\quad \Delta = 8\,, \quad s = \frac{3}{2}$

$$\Omega_2^{\frac{3}{2}}\,\mathcal{E}_{(1,0)}\,d^4R^4 \qquad \mathcal{E}_{(1,0)} = \zeta(5)\,E_{\frac{5}{2}}(\Omega)$$

TREE + 2-LOOP

$1/4$-BPS $\quad \Delta = 12\,, \quad s = \frac{5}{2}$

NON-RENORMALIZATION AT HIGHER LOOPS

$$2\zeta(2s)\,E_s(\Omega) \sim 2\zeta(2s)\,\Omega_2^s + (\dots)\,2\zeta(2s-1)\,\Omega_2^{1-s} + \sum_{k=1}^{\infty}\mu(k,s)\,\left(e^{2\pi i k\Omega} + c.c.\right)\left(1 + O(\Omega_2^{-1})\right)$$

TREE-LEVEL $\qquad$ GENUS-$(s-1/2)$ $\qquad$ D-INSTANTONS

ii) $U(1)$ - violating processes at order $n = 3$:

Modular covariant derivative

$$\mathcal{F}_3^{(u)} = \mathcal{D}^u\,\mathcal{F}_3^{(0)} = \mathcal{D}^u\,E_{\frac{3}{2}}$$

examples : $\quad \mathcal{F}_3^{(8)}\,G^8\,, \qquad \mathcal{F}_3^{(24)}\,\lambda^{16}$

A NOTE ON THE $AdS_5 \times S^5$ CORRESPONDENCE.

Type IIB STRING THEORY in
D=5 Anti de-Sitter space

$\Leftrightarrow$

D=4 SU(N) YANG-MILLS
on boundary of AdS$_5$

AdS/CFT
dictionary

Inverse string
coupling

$$\Omega_2 \equiv e^{-\varphi} = \frac{4\pi}{g^2_{YM}}$$

YM coupling

AdS length scale

$$\frac{\alpha'^2}{L^4} = \frac{1}{g^2_{YM}N} \equiv \frac{1}{\lambda}$$

't Hooft coupling

Effective R^4 string action

$$\frac{1}{\alpha'}\int d^{10}x\sqrt{-\det G}\,\Omega_2^{-\frac{1}{2}}\,E_{\frac{3}{2}}(\Omega)\,R^4$$

$\Leftrightarrow$ Coefficient of gauge invariant Yang-Mills correlator, e.g. $\langle O(x_1)\ldots O(x_4)\rangle$

$N\to\infty$

$1 \ll \lambda \ll N$

$$N^{\frac{1}{2}}\left(2\zeta(3)g_s^{-3/2} + 4\zeta(2)g_s^{\frac{1}{2}} + 2\sqrt{\pi}\sum_{k\neq 0}|k|\sigma_2(k)e^{-2\pi|k|/g_s+2\pi ik\Omega_1}\right)$$

$$= 2\zeta(3)\,N^2\,\lambda^{-\frac{3}{2}} + 4\zeta(2)\,N^0\,\lambda^{\frac{1}{2}} + 2\sqrt{\pi N}\sum_k|k|\sigma_2(k)e^{-2\pi|k|/g^2_{YM}+2\pi ik\Omega_1}$$

PLANAR contribution

$\lambda \gg 1$

measure obtained from $SU(N)$ Yang-Mills
k-INSTANTON as $N\to\infty$

(Dorey, Hollowood, Khoze)

iii) HIGHER ORDER

Next order $\Omega_2^{-1}\,\mathcal{F}_6^{(0)}(\Omega)\,\sigma_3\,\overset{\overset{d^6\,R^4}{\downarrow}}{R^4}$ $(\mathcal{F}_6^{(0)}(\Omega) \equiv \mathcal{E}_{(0,1)}(\Omega))$

$1/8\text{-BPS} \quad (\Delta = 14\,, \quad n = 6\,, \quad u = 0)$

Expand integrand to next order in s, t, u, leads to an integral that satisfies

INHOMOGENEOUS LAPLACE EQUATION: (MBG, Vanhove)

$$(\Delta_\Omega - 12)\,\mathcal{F}_6^{(0)}(\Omega) = -\Big(2\zeta(3)\,E_{\frac{3}{2}}(\Omega)\Big)^2$$

The square of the coefficient of R^4

Detailed structure not yet derived in detail from supersymmetry but is based on duality with M-theory :

11-dimensional supergravity on two-torus = Type IIB on a circle

The inhomogeneous Laplace equation was obtained by evaluation of two-loop 11-dimensional supergravity compactified on two-torus.

SOLUTION OF THE INHOMOGENEOUS LAPLACE EQUATION

MBG, Miller, Vanhove

$$(\Delta_\Omega - 12)\, f(\Omega) = -\left(2\zeta(3)\, E_{\frac{3}{2}}(\Omega)\right)^2$$

FOURIER SERIES: $f(\Omega) \;=\; \sum_n \widehat{f}_n(\Omega_2)\, e^{2\pi i n \Omega_1}\,.$

EQUATION FOR FOURIER MODES : $(\Omega_2^2\, \partial^2_{\Omega_2} - 12 - 4\pi^2 n^2 \Omega_2^2)\, \widehat{f}(\Omega_2) = S_n(\Omega_2)$ Fourier mode of source

BOUNDARY CONDITIONS : $\widehat{f}_n(\Omega_2) = O(\Omega_2^3)\,,$ $\Omega_2 \to \infty$ Weak coupling

Weak coupling (TREE LEVEL) power behaviour

$\widehat{f}_n(\Omega_2) = O(\Omega_2^{-2})\,,$ $\Omega_2 \to 0$ Strong coupling

SUBTLE consequence of $SL(2,\mathbb{Z})$ invariance

These b.c.'s determine a unique solution by fixing the coefficient of the solution of the homogeneous equation, $\alpha_n \sqrt{y}\, K_{\frac{7}{2}}(2\pi|n|y)$, for each value of n

ZERO MODE - four power-behaved terms :

$$\widehat{f}_0(\Omega_2) = \frac{2\,\zeta(3)^2}{3}\,\Omega_2^3 + \frac{4\,\zeta(2)\,\zeta(3)}{3}\,\Omega_2 + \frac{4\,\zeta(4)}{\Omega_2} + \frac{4\,\zeta(6)}{27}\,\Omega_2^{-3} + \sum_{m\neq 0} \widehat{f}_0^{\,m}(\Omega_2)$$

GENUS 0 1 2 3 Non-Perturbative

- ALL PERTURBATIVE CONTRIBUTIONS AGREE WITH EXPLICIT CALCULATIONS

(although GENUS 3 string calculation needs RE-CHECKING)

- NON-PERTURBATIVE TERMS

2 X 2 matrix of polynomial coefficients

$$\widehat{f}_0^{\,m}(\Omega_2) = \frac{32\,\pi\,\sigma_2(|m|)^2}{315\,|m|^3} \sum_{i,j=0,1} r^{i,j}(\pi|m|\Omega_2)\,K_i(2\pi|m|\Omega_2)\,K_j(2\pi|m|\Omega_2)$$

Bilinear in $K_0\,,\ K_1$

$$\Omega_2 \to \infty \qquad \sim e^{-4\pi|m|\Omega_2}\left(\frac{\sigma_2(|m|)^2}{|m|^5\,\Omega_2^2} + O(\Omega_2^{-3})\right)$$

$$\sim\, e^{-4\pi m\Omega_2}$$

Behaviour suggestive of charge-zero *INSTANTON / ANTI-INSTANTON* pairs.

$$\Omega_2 \to 0 \qquad \sim \frac{945\,\zeta(3)^2\,\zeta(5)}{4\,\pi^5}\,\frac{1}{\Omega_2^2} + O(\log\Omega_2)$$

cancellation of Ω_2^{-3} term by infinite number of “instantons”.

NON-ZERO MODES:

$i.j = 0, 1$

2 X 2 matrix of polynomial coefficients

$$\widehat{f}_n(\Omega_2) = \alpha_n \sqrt{\Omega_2}\, K_{\frac{7}{2}}(2\pi|n|\Omega_2) + \sum_{\substack{n_1+n_2 = n \\ (n_1,n_2)\neq(0,0)}} M^{ij}_{n_1,n_2}(\pi|n|\Omega_2)\, K_i(2\pi|n_1|\Omega_2)\, K_j(2\pi|n_2|\Omega_2)$$

Constant α_n determined by cancellation of the Ω_2^{-3} term in the $\Omega_2 \to 0$ limit.

$$\underset{\Omega_2 >> 1}{\sim} e^{-2\pi(|n_1|+|n_2|)\Omega_2}$$

BPS INSTANTON PAIR if $|n_1| + |n_2| = |n| = |n_1 + n_2|$ $\qquad (\text{sign}\, n_1 = \text{sign}\, n_2)$

charge = action

"INSTANTON / ANTI-INSTANTON" pair if $|n_1| + |n_2| < |n|$ $\qquad (\text{sign}\, n_1 = -\text{sign}\, n_2)$

charge < action

$$\widehat{f}_n(\Omega_2) \underset{\Omega_2 >> 1}{\sim} e^{-2\pi|n|\Omega_2}\left(8\,\frac{\sigma_2(|n|)}{|n|^{5/2}}\,\zeta(3)\,\Omega_2^{1/2} + O(1)\right) + c\, e^{-2\pi(|n|+1)\Omega_2}(\ldots) + \ldots$$

- Solution can be expressed as a Poincare series:

$$f(\Omega) = \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z})} \Phi(\gamma\Omega)$$

where $\Phi(\Omega) = a_0(\Omega_2) + \sum_{n\neq 0} a_n(\Omega_2)\, e^{2\pi i n \Omega_1}$ ($a_n(\Omega_2)$ is linear in $K_0\,,\ K_1$)

- D-instantons contribute with distinctive leading powers of Ω_2 (g^{-1}) – origin not understood in detail.

(b) Higher order in α' (non-trivial degeneracy)

Laplace eigenvalue equation generalizes to inhomogeneous simultaneous equations :

$$(\delta_{ij}\, \bar{D}D - \lambda^{(u)}_{n;ij})\, \mathcal{F}^{(u)\,j}_n = \sum_{j,k,m,v} \mathcal{F}^{(v)\,j}_m\, \mathcal{F}^{(u-v)\,k}_{n-m} + \ldots$$

Lower order source coefficients

[Some examples (e.g. Basu + Sethi; MBG, Russo, Vanhove).]

HIGHER-RANK DUALITY GROUPS

Compactify M-theory on a d-torus to D=11-d dimensions

MBG, Miller, Russo, Vanhove

Pioline

Duality Group $G(\mathbb{Z})$	space-time dimension
1	10A
$SL(2,\mathbb{Z})$	10B
$SL(2,\mathbb{Z})$	9
$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	8
$SL(5,\mathbb{Z})$	7
$SO(5,5,\mathbb{Z})$	6
$E_{6(6)}(\mathbb{Z})$	5
$E_{7(7)}(\mathbb{Z})$	4
$E_{8(8)}(\mathbb{Z})$	3

Automorphic functions for higher-rank groups ;

Langlands Eisenstein series' associated with maximal parabolic subgroups of G.

$$E^G_{s_1,\dots,s_r}$$

rank r $\quad s_1, \dots, s_r \in \mathbb{C}$

labels associated with nodes of Dynkin diagram

$$E^G_{\frac{3}{2},0,\dots,0}\; R^4 \qquad E^G_{\frac{5}{2},0,\dots,0}\; D^4 R^4$$

$$\mathcal{E}^G_{(0,1)}\; D^6 R^4$$

Satisfies inhomogeneous
Laplace equation for G

- Encodes perturbative string results in compactified theories.
- D-INSTANTONS fill out expected fractional BPS orbits – minimal, next-to-minimal, ….

S-DUALITY OF N-PARTICLE AMPLITUDES

MBG, Mafra, Schlotterer

$D^2R^4\,,\ R^5$ are zero at all loops

Vanish at one loop in D=10

$$E_{\frac{5}{2}}(\Omega)\,(D^4R^4+D^2R^5+R^6)$$

$$\mathcal{E}_{(0,1)}(\Omega)\,(D^6R^4+D^4R^5+D^2R^6)$$

Detailed agreement at tree and one loop in any dimension.

$$\mathcal{E}_{(2,0)}(\Omega)\,(D^8R^4+D^6R^5+D^4R^6)$$

Non-BPS. Only partially understood. Is there a 5-loop contribution?

Modular coefficient $\mathcal{E}_{(2,0)}$ unknown but (rather impressively) the ratio of tree to one loop is the same in each case for terms with the same kinematic structure.

We know from perturbative information that there must be at least one new modular function for $D^6R^5\,,\ D^4R^6\,,$ that starts at one loop and has no tree contribution.

BUT

there must be at least one new coefficient function for $D^6R^5\,,\ D^4R^6\,,\ \ldots$ that starts at one loop and has no tree contribution.

COMMENTS:

- Some results on higher derivative interactions for N<8 SUSY e.g. Tourquine, Vanhove
- To what extent do string theory dualities constrain the structure of perturbative supergravity? – ultraviolet divergences??

Need more information regarding higher orders in the low energy expansion.

String theory is free of UV divergences, How do such divergences arise in the field theory limit?

What is the structure of $\mathcal{E}_{(2,0)}(\Omega)\ D^8 R^4$?

DOES IT HAVE A 5-LOOP CONTRIBUTION? (Is it protected from higher loop corrections)?