Unitarity methods for scattering in two dimensions

Valentina Forini
Humboldt University Berlin

based on work with Lorenzo Bianchi and Ben Hoare

L. Bianchi, V. Forini, B. Hoare, arXiv: 1304.1798
O. T. Engelund, R. W. McKeown, R. Roiban, arXiv: 1304.4281

Calculating scattering amplitudes efficiently


Quantifying the one-loop QCD challenge

\[ pp \rightarrow W + n \text{ jets} \quad \text{(amplitudes with most gluons)} \]

<table>
<thead>
<tr>
<th># of jets</th>
<th># 1-loop Feynman diagrams</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>2</td>
<td>110</td>
</tr>
<tr>
<td>3</td>
<td>1,253</td>
</tr>
<tr>
<td>4</td>
<td>16,648</td>
</tr>
<tr>
<td>5</td>
<td>256,265</td>
</tr>
</tbody>
</table>

Current limit with Feynman diagrams

Current limit with on-shell methods

[from a L. Dixon talk]
Calculating scattering amplitudes efficiently


Goal is apply to evaluation amplitudes of two-dimensional cases of interest, where:

1. Feynman diagram calculations are problematic (divergencies do not cancel).
2. Need of perturbative checks for integrability-based proposal.
3. Need of alternative strategies to get (phases of) S-matrices.
Highly constrained scattering kinematics, no phase space for 2->2 amplitudes.

Natural set for the phenomenon of **integrability**, strong selection rules at work.

**Existence of local higher rank conserved charges**

- **i)** No particle production or annihilation
  
  \[ n_{\text{in}} = n_{\text{out}} = n \]

- **ii)** Conservation of set of momenta
  
  \[ \{ p_{1}^{\text{in}}, \ldots, p_{n}^{\text{in}} \} = \{ p_{1}^{\text{out}}, \ldots, p_{n}^{\text{out}} \} \]

- **iii)** Factorization of \( n \rightarrow n \) amplitudes into products of \( 2 \rightarrow 2 \) amplitudes

\[ \text{Zamolodchikov & Zamolodchikov, 79} \]

**Cut-constructibility might be expected!**
1+1 dimensional world: non-trivial

- Highly constrained scattering kinematics
  - allows for existence of momentum configurations potentially singular
1+1 dimensional world: non-trivial

- Highly constrained scattering kinematics
  > allows for existence of momentum configurations potentially singular

- AdS/CFT- related models (string world-sheet) are non-trivial
  > **UV divergencies appear in standard calculations of 4-point amplitudes**
  > no clear symmetry-preserving regularization scheme
  > power-counting non-renormalizable
  > non-relativistic
  > coupling constant subject to finite renormalization
    (for target space with less-than-maximal susy)
  > non-supersymmetric as world-sheet theories
  > massive
  > integrability phenomenon: **solid fact only classically!**
    infinite set of nonlocal charges ($Z_4$ automorphism of the coset action)
    quantum checks are only
    >> in simplifying limits
    >> in pure spinor language

---

[McLoughlin, Roiban, unpublished]

[Bena, Polchinski, Roiban 2003]
String worldsheet scattering

- Worldsheet amplitudes \((N \to \infty\), free strings), scattering of the (2d) lagrangean excitations. Non-trivial interactions due to highly non trivial background.

\(\text{flat space} \quad \text{AdS}_5 \times \text{S}^5 \text{ with RR fluxes}\)
String worldsheet scattering

- Worldsheets amplitudes ($N \to \infty$, free strings), scattering of the (2d) Lagrangean excitations. Non-trivial interactions due to highly nontrivial background.

- Because of RR-background need a GS formulation

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2 \sigma \sqrt{-h} h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N + \text{fermions}$$

- Work on a gauge-fixed sigma model (uniform light-cone gauge)

$$H_{ws} = \int d\sigma \, \mathcal{H}_{ws} = -\int d\sigma \, p_- \equiv E - J$$

[Arutyunov, Frolov, Plefka, Zamaklar 2006]
String worldsheet scattering

- Worldsheets amplitudes \( (N \to \infty, \text{free strings}) \), scattering of the (2d) lagrangean excitations. Non-trivial interactions due to highly non-trivial background.

Because of RR-background need a GS formulation

\[
S = \frac{\sqrt{\lambda}}{4\pi} \int d^2 \sigma \sqrt{-h} \, h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N + \text{fermions}
\]

Work on a gauge-fixed sigma model (uniform light-cone gauge)

\[
H_{ws} = \int d\sigma \, \mathcal{H}_{ws} = -\int d\sigma \, p_- \equiv E - J
\]

Gauge-fixed lagrangean involves rescaling

\[
-\frac{\pi J_+}{\sqrt{\lambda}} < \sigma < \frac{\pi J_+}{\sqrt{\lambda}}
\]

Decompactification limit \( \frac{J_+}{\sqrt{\lambda}} \to \infty \) and large tension expansion \( \hat{g} \to \infty \)

sensible definition of a perturbative worldsheet S-matrix

\[ AdS_5 \times S^5 \]
This S-matrix is the perturbative expansion of the exact AdS$_5$/CFT$_4$
S-matrix aka “spin chain S-matrix”: ★ the rhs of asymptotic Bethe eqs

$$1 = \prod_{j}^{K_4} \frac{x_{4j}^+}{x_{4j}^-} \prod_{j \neq k}^{K_4} \frac{u_{2k} - u_{2j} - i}{u_{2k} - u_{2j} + i} \prod_{j = 1}^{K_3} \frac{u_{2k} - u_{3j} + \frac{1}{2}}{u_{2k} - u_{3j} - \frac{1}{2}}$$

$$1 = \prod_{j = 1}^{K_3} \frac{x_{3k} - x_{3j}^-}{x_{3k} - x_{3j}^+} \prod_{j = 1}^{K_4} \frac{x_{4k} - x_{4j}^-}{x_{4k} - x_{4j}^+}$$

$$(x_{4k}^-)^L \left( \frac{x_{4k}^-}{x_{4k}^+} \right) = \prod_{j = 1}^{K_4} \left( \frac{u_{4k} - u_{4j} + i}{u_{4k} - u_{4j} - i} e^{2i \theta(x_{4k}, x_{4j})} \right) \prod_{j = 1}^{K_3} \frac{x_{4k}^- - x_{3j}^-}{x_{4k}^- - x_{3j}^+} \prod_{j = 1}^{K_5} \frac{x_{4k}^- - x_{5j}^-}{x_{4k}^- - x_{5j}^+}$$

$$1 = \prod_{j = 1}^{K_5} \frac{u_{5k} - u_{6j} + \frac{1}{2}}{u_{5k} - u_{6j} - \frac{1}{2}} \prod_{j = 1}^{K_4} \frac{x_{5k}^- - x_{4j}^-}{x_{5k}^- - x_{4j}^+}$$

$$1 = \prod_{j = 1}^{K_6} \frac{u_{6k} - u_{6j} - i}{u_{6k} - u_{6j} + i} \prod_{j = 1}^{K_5} \frac{u_{6k} - u_{5j} + \frac{1}{2}}{u_{6k} - u_{5j} - \frac{1}{2}}$$

$x(u) = \frac{1}{2} u + \frac{1}{2} u \sqrt{1 - 2g^2/u^2}$

$x^\pm(u) = x(u \pm \frac{i}{2})$

$\delta D = \delta E = g^2 \sum_{j = 1}^{K_4} \left( \frac{i}{x_{4j}^+ - x_{4j}^-} \right)$

$g^2 = \frac{\lambda}{8\pi^2}$

Describe the **exact asymptotic** spectrum of anomalous dimensions of local composite operators and energies of their dual string configurations.

★ (not spacetime scattering! however see Matthias talk)
This S-matrix is the perturbative expansion of the exact AdS$_5$/CFT$_4$
S-matrix *aka* “spin chain S-matrix” : the rhs of asymptotic Bethe eqs

Back at the exact gauge-fixed lagrangean: hard to quantize, arbitrary order interactions, standard CFT methods do not help.

**Integrability: most powerful tool to obtain string spectrum at finite coupling on non-trivial backgrounds.**

(see also [2013: McEwan, Roiban] for very interesting lattice-discretization of GS string).

...together with being the way to the **first ever exact solution** of a four-dimensional interacting gauge theory.
Assuming integrability (consistency with Yang-Baxter equation) and using global symmetries one can:

- derive exact dispersion relation \( \epsilon = \sqrt{1 + f(\lambda) \sin^2 \frac{p}{2}} \)

- derive two-particle S-matrix entering the Bethe equations \( S_{12} = S^0 S_{12} \)

\( S_{12} \) is the internal S-matrix. Various beautiful “upgrades” of Bethe equations exist: won’t play a role here.
Assuming integrability (consistency with Yang-Baxter equation) and using global symmetries one can:

- derive exact dispersion relation
  \[ \epsilon = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \]

- derive two-particle S-matrix entering the Bethe equations

\[ S_{12} = S^0 S_{12} \]

Various beautiful “upgrades” of Bethe equations exist: won’t play a role here.
Assuming integrability (consistency with Yang-Baxter equation) and using global symmetries one can:

- derive exact dispersion relation
  \[ \epsilon = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \]  
  [Beisert 2006]

- derive two-particle S-matrix entering the Bethe equations
  \[ S_{12} = S^0 S_{12} \]  
  [Staudacher 2004]
  [Beisert Staudacher 2005]
  [Beisert 2005]

- up to one (more) scalar factor(s), fixed with additional constraints like “crossing symmetry” and semiclassical string data.
  [Janik 2005]

Hardest thing to compute, crucial for the spectrum.

Cusp anomaly: impressive calculation establishing correctness of result for AdS$_5 \times$S$^5$.

[Beisert Eden Staudacher 2006]
[Bern Czakon Dixon Kosower Smirnov 2006]
Assuming integrability (consistency with Yang-Baxter equation) and using global symmetries one can:

- derive exact dispersion relation \( \epsilon = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \) [Beisert 2006]

- derive two-particle S-matrix entering the Bethe equations

\[ S_{12} = S^0 S_{12} \]

- up to one (more) scalar factor(s), fixed with additional constraints like “crossing symmetry” and semiclassical string data. [Janik 2005]

Hardest thing to compute, crucial for the spectrum.

Cusp anomaly: impressive calculation establishing correctness of result for AdS5xS5. [Beisert Eden Staudacher 2006] [Bern Czakon Dixon Kosower Smirnov 2006]

Necessity for new strategies in case of models relevant in AdS3/CFT2
Provide 2d scattering perturbation theory with efficient tools.

Provide tests of quantum integrability for certain string backgrounds.

Extract information about the overall factors of scattering matrix. Especially for worldsheet S-matrices for AdS$_3$ backgrounds where solutions to crossing-like equations are difficult to determine.
Motivation

- Provide 2d scattering perturbation theory with efficient tools
- Provide tests of quantum integrability for certain string backgrounds.
- Extract information about the overall factors of scattering matrix. Especially for worldsheet S-matrices for AdS$_3$ backgrounds where solutions to crossing-like equations are difficult to determine.
- Methodological: techniques never really applied in two dimensions.
- Apply to off-shell quantities

Initiate the use of unitarity-based methods for perturbative S-matrix in massive two-dimensional field theories.
Construct one-loop $2 \rightarrow 2$ scattering amplitude with standard unitarity directly from the corresponding on-shell tree-level amplitudes.
Outline

1. Method of unitarity cuts in d=2, general formula
2. Applications (mainly string worldsheet) and features
3. Concluding remarks
Two-dimensional scattering

Two-body scattering process of a theory invariant under space and time translations described via the four-point amplitude

$$
\langle \Phi^P(p_3) \Phi^Q(p_4) | \mathcal{S} | \Phi_M(p_1) \Phi_N(p_2) \rangle = (2\pi)^2 \delta^{(d)}(p_1 + p_2 - p_3 - p_4) \mathcal{A}^{PQ}_{MN}(p_1, p_2, p_3, p_4)
$$

For $d=2$ and in the single mass case, scattering $2 \rightarrow 2$ is simple. Particles either preserve or exchange their momenta

$$
\delta^{(2)}(p_1 + p_2 - p_3 - p_4) = J(p_1, p_2) \left( \delta(p_1 - p_3)\delta(p_2 - p_4) + \delta(p_1 - p_4)\delta(p_2 - p_3) \right)
$$

The Jacobian $J(p_1, p_2) = 1/(\partial \epsilon_{p_1}/\partial p_1 - \partial \epsilon_{p_2}/\partial p_2)$ depends on dispersion relation.
Two-dimensional scattering

Two-body scattering process of a theory invariant under space and time translations

\[ \Phi^P(p_3) \Phi^Q(p_4) |S| \Phi_M(p_1) \Phi_N(p_2) \rangle = (2\pi)^2 \delta^{(d)}(p_1 + p_2 - p_3 - p_4) A_{MN}^{PQ}(p_1, p_2, p_3, p_4) \]

For \( d=2 \) and in the single mass case, scattering \( 2 \rightarrow 2 \) is simple.

Particles either preserve or exchange their momenta

\[ \delta^{(2)}(p_1 + p_2 - p_3 - p_4) = J(p_1, p_2) \left( \delta(p_1 - p_3)\delta(p_2 - p_4) + \delta(p_1 - p_4)\delta(p_2 - p_3) \right) \]

The Jacobian \( J(p_1, p_2) = 1/(\partial \epsilon_{p_1}/\partial p_1 - \partial \epsilon_{p_2}/\partial p_2) \) depends on dispersion relation.

S-matrix element defined by

\[ S_{MN}^{PQ}(p_1, p_2) \equiv \frac{J(p_1, p_2)}{4\epsilon_1 \epsilon_2} A_{MN}^{PQ}(p_1, p_2, p_1, p_2) \]

Dispersion relation for asymptotic states (equal masses =1): \( \epsilon_i^2 = 1 + p_i^2 \)

Fix ordering of incoming states \( p_1 > p_2 \).
Scattering in $d=2$: unitarity cuts (1)

One-loop result from unitarity techniques: contributions from three cut-diagrams

Example: s-cut contribution.

$$\mathcal{A}^{(1)}_{MN} (p_1, p_2, p_3, p_4) |_{s\text{-cut}} = \frac{1}{2} \int \frac{d^2 l_1}{(2\pi)^2} \int \frac{d^2 l_2}{(2\pi)^2} \ i\pi \delta^+ (l_1^2 - 1) \ i\pi \delta^+ (l_2^2 - 1)$$

\[ \times A^{(0)}_{RS} (p_1, p_2, l_1, l_2) A^{(0)}_{SP} (l_2, l_1, p_3, p_4) \]

Glue tree-amplitudes and **uplift**: $i\pi \delta^+ (l_1^2 - 1) \rightarrow \frac{1}{l_1^2 - 1}$
Scattering in $d=2$: unitarity cuts (2)

- Use 2-momentum conservation at the first vertex

$$\tilde{A}^{(1)}_{MN}(p_1, p_2, p_3, p_4)|_{s-cut} = \frac{1}{2} \int \frac{d^2 l_1}{(2\pi)^2} \frac{i\pi \delta^+(l_1^2 - 1) i\pi \delta^+((l_1 - p_1 - p_2)^2 - 1)}{2\pi} \times \tilde{A}^{(0)}_{MN}(p_1, p_2, l_1, -l_1 + p_1 + p_2) \tilde{A}^{(0)}_{SR}(l_1 + p_1 + p_2, l_1, p_3, p_4)$$

- Use the zeroes of $\delta$-functions in the $\tilde{A}^{(0)}$ (like $f(x) \delta(x) = f(0) \delta(x)$)

- Restore loop momentum off-shell

$$i\pi \delta^+(l_1^2 - 1) \rightarrow \frac{1}{l_1^2 - 1}$$
Scattering in d=2: unitarity cuts (2)

- Use 2-momentum conservation at the first vertex

\[ \tilde{A}^{(1)}_{MN}(p_1, p_2, p_3, p_4)|_{s-cut} = \frac{1}{2} \int \frac{d^2l_1}{(2\pi)^2} i\pi \delta^+(l_1^2 - 1) i\pi \delta^+((l_1 - p_1 - p_2)^2 - 1) \times \tilde{A}^{(0)}_{MN}(p_1, p_2, l_1, -l_1 + p_1 + p_2) \tilde{A}^{(0)}_{SR}(l_1 + p_1 + p_2, l_1, p_3, p_4) \]

- Use the zeroes of \( \delta \) - functions in the \( \tilde{A}^{(0)} \) (like \( f(x) \delta(x) = f(0) \delta(x) \))

- Restore loop momentum off-shell \( i\pi \delta^+(l_1^2 - 1) \rightarrow \frac{1}{l_1^2 - 1} \)

Two-particle cuts in d=2 at one loop are maximal cuts.

Expect same as quadrupole cuts in d=4:

\[ A_{4}^{1-loop} = \sum (A_{4}^{tree})^4 I_{box} \]
4-points amplitude at one-loop

Tree-level amplitudes can be pulled out of the integral, evaluated at those zeroes.

A simple sum over discrete solutions of the on-shell conditions

$$\tilde{A}^{(1)}_{MN}(p_1, p_2, p_3, p_4) = \frac{I(p_1 + p_2)}{2} \left[ \tilde{A}^{(0)}_{RS}(p_1, p_2, p_1, p_2) \tilde{A}^{(0)}_{SR}(p_2, p_1, p_3, p_4) + \tilde{A}^{(0)}_{RS}(p_1, p_2, p_2, p_1) \tilde{A}^{(0)}_{SR}(p_1, p_2, p_3, p_4) \right]$$

$$+ I(p_1 - p_3) \tilde{A}^{(0)}_{SP}(p_1, p_3, p_1, p_3) \tilde{A}^{(0)}_{SN}(p_1, p_2, p_3, p_4)$$

$$+ I(p_1 - p_4) \tilde{A}^{(0)}_{SQ}(p_1, p_4, p_1, p_4) \tilde{A}^{(0)}_{SN}(p_1, p_2, p_4, p_3)$$

weighted by scalar “bubble” integrals

$$I(p) = \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(q^2 - 1 + i\epsilon)((q - p)^2 - 1 + i\epsilon)}$$

Inherently FINITE formula.
Final formula for the S-matrix (choose $p_3 = p_1$, $p_4 = p_2$)

$$S^{(1)}_{MN}^{PQ}(p_1, p_2) = \frac{1}{4(\epsilon_2 p_1 - \epsilon_1 p_2)} \left[ \tilde{S}^{(0)}_{RS}^{MN}(p_1, p_2) \tilde{S}^{(0)}_{RS}^{PQ}(p_1, p_2) I(p_1 + p_2) \\
+ \tilde{S}^{(0)}_{SP}^{MN}(p_1, p_1) \tilde{S}^{(0)}_{SN}^{RQ}(p_1, p_2) I(0) \\
+ \tilde{S}^{(0)}_{SQ}^{MR}(p_1, p_2) \tilde{S}^{(0)}_{SR}^{PR}(p_1, p_2) I(p_1 - p_2) \right],$$

where

$$\tilde{S}^{(0)}(p_1, p_2) = 4(\epsilon_2 p_1 - \epsilon_1 p_2) S^{(0)}(p_1, p_2)$$

Sum of products of two tree-level amplitudes weighted by scalar bubble integrals

$$I_s \equiv I(p_1 + p_2) = \frac{1}{\epsilon_2 p_1 - \epsilon_1 p_2} - \frac{\text{arsinh}(\epsilon_2 p_1 - \epsilon_1 p_2)}{4\pi i (\epsilon_2 p_1 - \epsilon_1 p_2)}$$

$$I_t \equiv I(0) = \frac{1}{4\pi i}$$

$$I_u \equiv I(p_1 - p_2) = \frac{\text{arsinh}(\epsilon_2 p_1 - \epsilon_1 p_2)}{4\pi i (\epsilon_2 p_1 - \epsilon_1 p_2)}$$

Possible absence of rational terms: formula cannot be completely general!

Needs to be tested on various examples.
Final formula for the S-matrix

Final formula for the S-matrix (choose $p_3 = p_1$, $p_4 = p_2$)

\[
S^{(1)PQ}_{MN}(p_1, p_2) = \frac{1}{4(\epsilon_2 p_1 - \epsilon_1 p_2)} \left[ \tilde{S}^{(0)RS}_{MN}(p_1, p_2) \tilde{S}^{(0)PQ}_{RS}(p_1, p_2) I(p_1 + p_2) \\
+ (-1)^{[P][S]+[R][S]} \tilde{S}^{(0)SP}_{MR}(p_1, p_1) \tilde{S}^{(0)RQ}_{SN}(p_1, p_2) I(0) \\
+ (-1)^{[P][R]+[Q][S]+[R][S]+[P][Q]} \tilde{S}^{(0)SQ}_{MR}(p_1, p_2) \tilde{S}^{(0)PR}_{SN}(p_1, p_2) I(p_1 - p_2) \right]
\]

where

\[
\tilde{S}^{(0)}(p_1, p_2) = 4(\epsilon_2 p_1 - \epsilon_1 p_2) S^{(0)}(p_1, p_2)
\]

Sum of products of two tree-level amplitudes weighted by scalar bubble integrals

\[
I_s \equiv I(p_1 + p_2) = \frac{1}{\epsilon_2 p_1 - \epsilon_1 p_2} - \frac{\text{arsinh}(\epsilon_2 p_1 - \epsilon_1 p_2)}{4\pi i (\epsilon_2 p_1 - \epsilon_1 p_2)}
\]

\[
I_t \equiv I(0) = \frac{1}{4\pi i}
\]

\[
I_u \equiv I(p_1 - p_2) = \frac{\text{arsinh}(\epsilon_2 p_1 - \epsilon_1 p_2)}{4\pi i (\epsilon_2 p_1 - \epsilon_1 p_2)}
\]

Possible absence of rational terms: formula cannot be completely general!

Needs to be tested on various examples.
Remarks

- The t-channel cut is special.

- Using first \( \delta(p_1 - p_3) \delta(p_2 - p_4) \) makes it ill-defined and requires a prescription: use delta-function only at the end of the calculation.

- Asymmetrical wrt choice of the vertex used to solve momenta: leads to a consistency condition

\[
\tilde{S}^{(0)}_{MR}(p_1, p_1) \tilde{S}^{(0)}_{SN}(p_1, p_2) = \tilde{S}^{(0)}_{PS}(p_1, p_2) \tilde{S}^{(0)}_{QR}(p_2, p_2)
\]

- We are NOT including contributions from tadpoles (no physical cuts)

- A inherently finite result says nothing about UV-finiteness or renormalizability. Might be missing rational terms following from regularization procedure.

Cut-constructibility to be always checked
Define contractions in field space

\[
\begin{align*}
(A \otimes B)^{PQ}_{MN}(p, p') &= A^{RS}_{MN}(p, p') B^{PQ}_{RS}(p, p') , \\
(A \oplus B)^{PQ}_{MN}(p, p') &= (-1)^{([P]+[S])([Q]+[R])} A^{SQ}_{MR}(p, p') B^{PR}_{SN}(p, p') , \\
(A \ominus B)^{PQ}_{MN}(p, p') &= (-1)^{[P][S]+[R][S]} A^{SP}_{MR}(p, p') B^{RQ}_{SN}(p, p') , \\
(A \oslash B)^{PQ}_{MN}(p, p') &= (-1)^{[Q][R]+[R][S]} A^{PS}_{MR}(p, p') B^{QR}_{SN}(p, p') ,
\end{align*}
\]

use

\[
\begin{align*}
I_s &= \frac{1}{J} \left( 1 - \frac{\theta}{i \pi} \right) , \\
I_t &= \frac{1}{4 \pi i} , \\
I_u &= \frac{1}{J} \frac{\theta}{i \pi} ,
\end{align*}
\]

with \( \theta \equiv \text{arcsinh}(e'p - ep') \)

\[
J = \frac{1}{4(e'p - ep')} \quad \text{and} \quad S = 1 + i \sum_{n=1}^{\infty} \hat{g}^{-n} T^{(n-1)}
\]

To rewrite the one-loop unitarity cut result as (assuming \( T^{(0)} \) real)

\[
T^{(1)} = \frac{\theta}{2 \pi} (T^{(0)} \oslash T^{(0)} - T^{(0)} \otimes T^{(0)}) + \frac{i}{2} T^{(0)} \otimes T^{(0)} + \frac{1}{16 \pi} (\widetilde{T}^{(0)} \oslash T^{(0)} + T^{(0)} \oslash \widetilde{T}^{(0)})
\]

logarithmic terms | rational imaginary | rational real
Ignore potentially singular cuts.

(In the Green Schwarz worldsheet string, no regularization is known which preserves the symmetry of the theory: kappa-symmetry, Weyl-symmetry, integrability...)

The t-channel bubble integral is a constant (rational): ignore and determine all rational terms from symmetry considerations.

Determine logarithmic terms, up to two loops, staying in d=2.
Bosonic models:

- Generalized sine-Gordon: gauged WZW model for a coset $G/H = \text{SO}(n + 1)/\text{SO}(n)$ plus an integrable potential ($n=1$: sine-Gordon, $n=2$: complex sine-Gordon)

  The method works up to a finite shift in the coupling.

Supersymmetric generalizations (``Pohlmeyer reductions'' of string theories):

- $\mathcal{N} = 1, 2$ supersymmetric sine-Gordon

  The method reproduces the full result.
Relativistic models

- Bosonic models:
  - generalized sine-Gordon: gauged WZW model for a coset $G/H = \text{SO}(n + 1)/\text{SO}(n)$ plus an integrable potential ($n=1$: sine-Gordon, $n=2$: complex sine-Gordon)

    **The method works up to a finite shift in the coupling.**

- Supersymmetric generalizations (``Pohlmeyer reductions” of string theories):
  - $\mathcal{N} = 1, 2$ supersymmetric sine-Gordon

    **The method reproduces the full result.**

In two cases (complex sine-Gordon and Pohlmeyer-reduced AdS$_3 \times$S$^3$ theory) cut-constructibility is highly non trivial!

Theory only integrable at classical level. Quantum counterterms restoring various properties of integrability (e.g. Yang-Baxter equation).

It is this “quantum integrable” result that the unitarity method gives.
In the asymptotic case, symmetry algebra is a (centrally extended) $\text{PSU}(2|2)^2$

From symmetries and integrability follows a group factorization

$$S = e^{i\theta} \hat{S}^{\text{PSU}(2|2)} \otimes \hat{S}^{\text{PSU}(2|2)}$$

Each factor has manifest $SU(2) \times SU(2)$ invariance
AdS/CFT S-matrix: exact and perturbative structure

In the asymptotic case, symmetry algebra is a (centrally extended) $\text{PSU}(2|2)^2$

From symmetries and integrability follows a **group factorization**

$$S = e^{i \theta} \hat{S}^{\text{PSU}(2|2)} \otimes \hat{S}^{\text{PSU}(2|2)}$$

Each factor has manifest $SU(2) \times SU(2)$ invariance

Matrix structure, rational dependence on momenta
Bilinear of local charges

\[ \theta = \sum_{n=0}^{\infty} \hat{g}^{1-n} \theta_{12}^{(n)} \]
\[ \theta^{(n)} = \chi^{(n)}(x_1^+, x_2^+) - \chi^{(n)}(x_1^-, x_2^-) - \chi^{(n)}(x_1^-, x_2^+) + \chi^{(n)}(x_1^+, x_2^-) - \chi^{(n)}(x_2^+, x_1^+) + \chi^{(n)}(x_2^-, x_1^-) + \chi^{(n)}(x_2^-, x_1^+) - \chi^{(n)}(x_2^+, x_1^-) \]

with Zhukovsky variables encoding dispersion relation

\[ x_p^\pm = \frac{\pi e^{\mp i p}}{\sqrt{\lambda \sin \frac{p}{2}}} \left( 1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \right) \]

Beyond one loop each contribution is rational.

\[ \chi^{(1)}(x_1, x_2) = -\frac{1}{2\pi} \text{Li}_2 \frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} - \frac{1}{2\pi} \text{Li}_2 \frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} + \frac{1}{2\pi} \text{Li}_2 \frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} + \frac{1}{2\pi} \text{Li}_2 \frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} \]

Crossing equation

\[ i\theta(x_j, x_k) + i\theta(1/x_j, x_k) = 2 \log h(x_j, x_k) \]
\[ h(x_j, x_k) = \frac{x_k^- (1 - \frac{1}{x_j x_k})(x_j^- - x_k^+)}{x_k^+ (1 - \frac{1}{x_j x_k})(x_j^+ - x_k^-)} \]
Expansion of symmetry-determined and phase parts ($\theta^{(0)}$ absorbed in $T^{(0)}$)

$$\hat{S} = 1 + i \sum_{n=1}^{\infty} g^{-n} \hat{T}^{(n-1)}$$

$\theta = \sum_{n=1}^{\infty} g^{-n} \theta^{(n-1)}$

requires one-loop logarithms to contribute only to the diagonal terms

$$S = 1 + \frac{i}{g} \hat{T}^{(0)} + \frac{i}{g^2} (\hat{T}^{(1)} + \theta^{(1)} \mathbf{1}) + \frac{1}{g^3} (\hat{T}^{(2)} + \frac{i}{2} \theta^{(1)} \hat{T}^{(0)} + \theta^{(2)} \mathbf{1})$$

(and two-loop logarithms to be proportional to the tree-level S-matrix - just the effect of two loop exponentiation - as $\theta^{(2)}$ has no logs)

**Goal:** compute one loop worldsheet S-matrix “bootstrapping” it from tree level.
Superstring action

\[ S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \sqrt{-h} h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N + \text{fermions} \]

- Green-Schwarz formulation for fermions

\[ \mathcal{Q}_a = \partial_a x^\mu E_\mu^A \Gamma_A \]

quadratic part

\[ L_F = i(\sqrt{-g} g^{ab} \delta^{IJ} - \epsilon^{ab} \epsilon^{IJ}) \bar{\theta}^I \mathcal{Q}_a D_b \theta^J \]

\[ D_a \theta^I = \left( \partial_a + \frac{1}{4} \partial_a x^\mu \omega_\mu^{AB} \Gamma_{AB} \right) \theta^I + \frac{1}{2} \mathcal{Q}_a \Gamma_{0123} \epsilon^{IJ} \theta^J \]

- Classical limit: \( \lambda \to \infty \)  
  Sigma model coupling constant: \( \hat{g} = \frac{2\pi}{\sqrt{\lambda}} \)

- Gauge-fixed lagrangean involves rescaling

\[ -\frac{\pi J_+}{\sqrt{\lambda}} < \sigma < \frac{\pi J_+}{\sqrt{\lambda}} \]

Decompactification limit

\[ \frac{J_+}{\sqrt{\lambda}} \to \infty \quad \text{and large tension expansion} \quad \hat{g} \to \infty \]
Gauge fixing

Use an interpolating lightcone -gauge

\[ X^+ = (1 + a) t + a \varphi \equiv \tau + a \sigma \]

\[ a = 1/2 \quad \text{light-cone gauge} \]

\[ a = 0 \quad \text{temporal gauge} \]

AdS$_5$ \[ \rightarrow \] S$^5$

Transverse coordinates \[ z^\mu, \ \mu = 1 \ldots 4 \quad y^m, \ m = 1 \ldots 4 \]

\[ ds^2 = -G_{tt}(z)dt^2 + G_{zz}(z)dz^2 + G_{\varphi\varphi}(y)d\varphi^2 + G_{yy}(y)dy^2 \]

\[ G_{tt} = \left(1 + \frac{z^2}{4}\right)^2, \quad G_{zz} = \frac{1}{(1 - \frac{z^2}{4})^2}, \quad G_{\varphi\varphi} = \left(1 - \frac{y^2}{4}\right)^2, \quad G_{yy} = \frac{1}{\left(1 + \frac{y^2}{4}\right)^2}. \]

Gauge choice preserves SO(8) at quadratic level, broken by interactions.
Bosonic lagrangean to quartic order in the fields  \( X = (Y, Z) \)

\[
L = \frac{1}{2} \left( \partial_a X \right)^2 - \frac{1}{2} X^2 + \frac{1}{4} Z^2 \left( \partial_a Z \right)^2 - \frac{1}{4} Y^2 \left( \partial_a Y \right)^2 + \frac{1}{4} \left( Y^2 - Z^2 \right) \left( \dot{X}^2 + \dot{Y}^2 \right) \\
- \frac{1 - 2a}{8} (X^2)^2 + \frac{1 - 2a}{4} (\partial_a X \cdot \partial_b X)^2 - \frac{1 - 2a}{8} [(\partial_a X)^2]^2.
\]

Lorentz invariance (quadratic part) broken by interactions.
Massive states with relativistic dispersion relation  \( \epsilon = \sqrt{1 + p^2} \)

\[
\epsilon = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}
\]

Exact dispersion relation known via symmetries
(Scattering with particles, for parametrically large momentum, become solitonic solutions - giant magnons - with  \( \epsilon \sim \frac{\sqrt{\lambda}}{\pi} \sin \frac{p}{2} \))

Bosonic part invariant under \( SO(4) \times SO(4) \).
Worldsheet fields (embedding coordinates in AdS$_5 \times $S$^5$)

\[ T, \Phi, Y^m, Z^m, \text{ fermions} \]

can be represented as bispinors

\[ SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2 \]

\[ Y_{a\dot{a}} = (\sigma_m)_{a\dot{a}} Y^m, \quad Z_{\alpha\dot{\alpha}} = (\sigma_\mu)_{\alpha\dot{\alpha}} Z^\mu \quad a, \dot{a}, \alpha, \dot{\alpha} = 1, 2 \]

Bosons and fermions form bi-fundamental representation of \( \text{PSU}(2|2)_L \times \text{PSU}(2|2)_R \)

Formal definition of a bi-fundamental supermultiplet providing a basis for the definition of the S-matrix.

\[ \Phi_{a\dot{a}}, A = (a|\alpha) \quad \dot{A} = (\dot{a}|\dot{\alpha}) \]
Worldsheet S-matrix

- Two-particle S-matrix is 256 x 256

\[ S |\Phi_{A\dot{A}}(p)\Phi_{\dot{B}B}(p')\rangle = |\Phi_{C\dot{C}}(p)\Phi_{D\dot{D}}(p')\rangle S_{A\dot{A}B\dot{B}}^{C\dot{C}D\dot{D}}(p,p') \]

- Integrability predicts

\[ S = S \otimes S, \quad S_{A\dot{A}B\dot{B}}^{C\dot{C}D\dot{D}}(p,p') = S_{AB}^{CD}(p,p') S_{\dot{A}B}^{\dot{C}\dot{D}}(p,p') \]

- S-matrices parametrized in terms of the basic SU(2) invariants

\[
S_{AB}^{CD} = \begin{cases} 
A\delta^c_a \delta^d_b + B\delta^d_a \delta^c_b \\
D\delta_\alpha^\gamma \delta_\beta^\delta + E\delta_\alpha^\delta \delta_\beta^\gamma \\
C\epsilon_{ab} \epsilon^{\gamma\delta} + F\epsilon_{\alpha\beta} \epsilon^{cd} \\
G\delta_\alpha^\gamma \delta_\beta^{\delta} + H\delta^d_a \delta^\gamma_b \\
L\delta_\alpha^\gamma \delta_\beta^d + K\delta_\alpha^\delta \delta_\beta^c 
\end{cases}
\]

and similar for the dotted one.
Tree level result: first non trivial order in the perturbative expansion
Obtained applying LSZ reduction to quartic vertices of the lagrangean.

\[ S = 1 + \frac{1}{\hat{g}} T^{(0)} + \frac{1}{\hat{g}^2} T^{(1)} + \ldots = 1 + T \]
\[ \hat{g} = \frac{\sqrt{\lambda}}{2\pi} \]

One-loop result via standard Feynman diagrammatics: not existing!
unsuccessful attempts (non-cancellation of UV divergences).

\[ T_{YY\to YY}^{(0)} = \frac{1}{2} \left[ (1 - 2a)(\varepsilon'p - \varepsilon p') + \frac{(p - p')^2}{\varepsilon'p - \varepsilon p'} \right] 1 \otimes 1 + \frac{pp'}{\varepsilon'p - \varepsilon p'} (1 \otimes P + P \otimes 1) \]
Worldsheet S-matrix at one loop via unitarity cuts: result

\[ S_{AB}^{CD}(p_1, p_2) = \exp \left( i \varphi_a(p_1, p_2) \right) \tilde{S}_{AB}^{CD} \]

\[ = \exp \left( - \frac{i}{2\hat{g}}(e_2p_1 - e_1p_2)(a - \frac{1}{2}) + \frac{i}{\hat{g}^2}\tilde{\varphi}(p_1, p_2) \right) \tilde{S}_{AB}^{CD} + \mathcal{O}\left( \frac{1}{\hat{g}^3} \right) \]

where

\[ A^{(1)} = 1 + \frac{i}{4\hat{g}} \frac{(p_1 - p_2)^2}{\epsilon_2 p_1 - \epsilon_1 p_2} + \frac{1}{4\hat{g}^2} \left( p_1 p_2 - \frac{(p_1 + p_2)^4}{8(\epsilon_2 p_1 - \epsilon_1 p_2)^2} \right) \]

and

\[ \tilde{\varphi}(p_1, p_2) = \frac{1}{2\pi} \frac{p_1^2 p_2^2 ((\epsilon_2 p_1 - \epsilon_1 p_2) - (\epsilon_1 \epsilon_2 - p_1 p_2) \arcsinh[\epsilon_2 p_1 - \epsilon_1 p_2])}{(\epsilon_2 p_1 - \epsilon_1 p_2)^2} \]

\[ \checkmark \text{ All logarithmic dependence encoded in the scalar factor (as required from integrability!)} \]

\[ \checkmark \text{ All gauge dependence encoded in the scalar factor (as required from physical arguments!)} \]

\[ \checkmark \text{ All rational dependence coincides with related expansion of EXACT worldsheet S-matrix} \]
Other string backgrounds: $AdS_3 \times S^3 \times M^4$

- Three light-cone gauge-fixed string theories (Type IIB)
  - $AdS_3 \times S^3 \times T^4$ supported by pure RR flux
  - $AdS_3 \times S^3 \times S^3 \times S^1$ supported by pure RR flux
  - $AdS_3 \times S^3 \times T^4$ supported by a mix RR and NS NS fluxes

relevant for the $AdS_3/CFT_2$ correspondence, interesting physics (e.g. BTZ black holes)

Useful for connecting different working methods (CFT, integrability).

- Super-coset sigma models

\[
\frac{PSU(1,1|2) \times PSU(1,1|2)}{SL(2) \times SU(2)} \times U(1)^4 \quad \frac{D(2,1; \alpha) \times D(2,1; \alpha)}{SL(2) \times SU(2) \times SU(2)} \times U(1)
\]

with $Z_4$ automorphism -> classical integrability.

[Cagnazzo Zarembo 2011]
[Hoare Tseytlin 2012]
New features

1. **Multiple masses** (also massless) in the asymptotic spectrum.

<table>
<thead>
<tr>
<th>Theory</th>
<th>Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AdS_3 \times S^3 \times T^4$ (RR flux)</td>
<td>$(4+4) \times 1$, $(4+4) \times 0$</td>
</tr>
<tr>
<td>$AdS_3 \times S^3 \times S^3 \times S^1$ (RR flux)</td>
<td>$(2+2) \times 1$, $(2+2) \times \alpha$ $(2+2) \times 1 - \alpha$, $(2+2) \times 0$</td>
</tr>
<tr>
<td>$AdS_3 \times S^3 \times T^4$ (mixed flux)</td>
<td>$(4+4) \times \sqrt{1-q^2}$, $(4+4) \times 0$</td>
</tr>
</tbody>
</table>

2. Expansion of the light-cone lagrangian contains **odd powers** in the field terms.

3. Dispersion relation is in terms of a function non trivially related to string tension:

$$\epsilon = \sqrt{m^2 + 4f^2(\lambda) \sin^2 \frac{p}{2}}$$
Exact S-matrices have been proposed, with conjectures for the phase $S$

$$S_{MN}^{PQ}(p, p') = \exp[i\omega_{\sigma_M \sigma_N}(p, p')] \hat{S}_{MN}^{PQ}(p, p')$$

[Roiban Engelund 2013]

Under global U(1) symmetry groups excitations are particles (+) or antiparticles (-), $++$, $+-$, $-+$, $--$.

Non-trivial modification of the ``crossing equations'', which relate all of them.

Logarithmic contributions for these phases were evaluated with unitarity-cuts up to two loops in [Roiban Engelund 2013]
Enlarging unitarity 2d program: different masses

- Working with a tree-level integrable S-matrix (always verified)
  - > outgoing two-momenta = permutation of incoming two-momenta
  - > scattering preserves these U(1) charges \( \sigma_M = \sigma_P \)
  - Ignore massless particles in the loop if considers only external massive legs
  - Focus on sectors ++, +- [Bianchi Hoare, 2014]

- Compact formula for the one-loop contribution (explicitating bubble integrals)

\[
\theta \equiv \text{arcsinh}\left(\frac{e'p - ep'}{mm'}\right) \quad e = \sqrt{p^2 + m^2} \quad e' = \sqrt{p'^2 + m'^2}
\]

\[
T^{(1)} = \frac{\theta}{2\pi} \left( T^{(0)} \square T^{(0)} - T^{(0)} \otimes T^{(0)} \right) + \frac{i}{2} T^{(0)} \otimes T^{(0)} + \frac{1}{16\pi} \left( \frac{1}{m^2} \tilde{T}^{(0)} \leftrightarrow T^{(0)} + \frac{1}{m'^2} T^{(0)} \leftrightarrow \tilde{T}^{(0)} \right)
\]

- Logarithmic terms
- Rational imaginary
- Rational real
Unitarity-cut result and the Yang-Baxter equation

Cubic matrix equation that is necessarily satisfied by integrable theories

\[ S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}, \]

Classical Yang-Baxter

\[
\begin{align*}
[T^{(0)}_{12}, T^{(0)}_{13}] + [T^{(0)}_{12}, T^{(0)}_{23}] + [T^{(0)}_{13}, T^{(0)}_{23}] &= 0 \\
[T^{(0)}_{12}, T^{(1)}_{13}] + [T^{(0)}_{12}, T^{(1)}_{23}] + [T^{(0)}_{13}, T^{(1)}_{23}] - [T^{(0)}_{13}, T^{(1)}_{12}] - [T^{(0)}_{23}, T^{(0)}_{12}] - [T^{(0)}_{23}, T^{(1)}_{13}] &= T^{(0)}_{23}T^{(0)}_{13}T^{(0)}_{12} - T^{(0)}_{12}T^{(0)}_{13}T^{(0)}_{23}
\end{align*}
\]
Unitarity-cut result and the Yang-Baxter equation

Cubic matrix equation that is necessarily satisfied by integrable theories

$$S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12},$$

- $$[T_{12}^{(0)}, T_{13}^{(0)}] + [T_{12}^{(0)}, T_{23}^{(0)}] + [T_{13}^{(0)}, T_{23}^{(0)}] = 0$$

  **Classical Yang-Baxter**

- $$[T_{12}^{(0)}, T_{13}^{(1)}] + [T_{12}^{(0)}, T_{23}^{(1)}] + [T_{13}^{(0)}, T_{23}^{(1)}] - [T_{13}^{(0)}, T_{12}^{(1)}] - [T_{23}^{(0)}, T_{12}^{(1)}] - [T_{23}^{(0)}, T_{13}^{(1)}] = T_{23}^{(0)} T_{13}^{(0)} T_{12}^{(0)} - T_{12}^{(0)} T_{13}^{(0)} T_{23}^{(0)}$$

\[
T^{(1)} = \frac{\theta}{2\pi} (T^{(0)} \boxtimes T^{(0)} - T^{(0)} \boxtimes T^{(0)}) + \frac{i}{2} T^{(0)} \boxtimes T^{(0)} + \frac{1}{16\pi} (\frac{1}{m^2} \tilde{T}^{(0)} \boxleft T^{(0)} + \frac{1}{m'^2} T^{(0)} \boxright \tilde{T}^{(0)})
\]

- logarithmic terms
- rational imaginary
- rational real
Unitarity-cut result and the Yang-Baxter equation

Cubic matrix equation that is necessarily satisfied by integrable theories

\[ S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12} , \]

\[
\begin{array}{c}
\text{Classical Yang-Baxter} \\

\bullet \quad [T^{(0)}_{12}, T^{(0)}_{13}] + [T^{(0)}_{12}, T^{(0)}_{23}] + [T^{(0)}_{13}, T^{(0)}_{23}] = 0 \\

\bullet \quad [T^{(0)}_{12}, T^{(1)}_{13}] + [T^{(0)}_{12}, T^{(1)}_{23}] + [T^{(0)}_{13}, T^{(1)}_{23}] - [T^{(0)}_{13}, T^{(1)}_{12}] - [T^{(0)}_{23}, T^{(1)}_{12}] - [T^{(0)}_{23}, T^{(1)}_{13}] = 0
\end{array}
\]

\[ T^{(1)} = \frac{\theta}{2\pi} (T^{(0)} \ominus T^{(0)} - T^{(0)} \otimes T^{(0)}) + \]

\[ \frac{1}{16\pi} \left( \frac{1}{m^2} \tilde{T}^{(0)} \ominus T^{(0)} + \frac{1}{m'^2} T^{(0)} \otimes \tilde{T}^{(0)} \right) \]

logarithmic terms

rational real
Unitarity-cut result and the Yang-Baxter equation

Cubic matrix equation that is necessarily satisfied by integrable theories

\[ S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}, \]

\[ \begin{array}{c}
\text{Classical Yang-Baxter} \\
\end{array} \]

- \[ [T_{12}^{(0)}, T_{13}^{(0)}] + [T_{12}^{(0)}, T_{23}^{(0)}] + [T_{13}^{(0)}, T_{23}^{(0)}] = 0 \]

- \[ [T_{12}^{(0)}, T_{13}^{(1)}] + [T_{12}^{(0)}, T_{23}^{(1)}] + [T_{13}^{(0)}, T_{23}^{(1)}] - [T_{13}^{(0)}, T_{12}^{(1)}] - [T_{23}^{(0)}, T_{12}^{(1)}] - [T_{23}^{(0)}, T_{13}^{(1)}] = 0 \]

\[ T^{(1)} = \frac{\theta}{2\pi} (T^{(0)} \boxplus T^{(0)} - T^{(0)} \boxotimes T^{(0)}) + \]

logarithmic terms

\[ \frac{1}{16\pi} \left( \frac{1}{m^2} \tilde{T}^{(0)} \boxotimes T^{(0)} + \frac{1}{m'^2} T^{(0)} \boxotimes \tilde{T}^{(0)} \right) \]

rational real

All satisfy this property, except \( AdS_3 \times S^3 \times S^3 \times S^1 \): rational remainder \( \sim \left( \frac{p^2}{m} + \frac{p'^2}{m'} \right) T^{(0)} \)
Contributions from external leg corrections

\[ Z^4 \begin{array}{c} S \end{array} = Z^4 \left( 1 + \frac{1}{\hat{g}} T^{(0)} + \frac{1}{\hat{g}^2} T^{(1)} + \ldots \right) \rightarrow T_{ext}^{(1)} = (\Sigma_1^{(1)}(p) + \Sigma_1^{(1)}(p'))T^{(0)} \]

\[ = \frac{i}{p^2 - m^2 - \Sigma(p)} = \frac{i}{p^2 - m^2 - \hat{g}^{-1}\Sigma^{(1)}(p)} + O(\hat{g}^{-2}) \]

Residue of first perturbation correction of self-energy around the on-shell condition
Contributions from external leg corrections

\[
\sqrt{Z^4} = \sqrt{Z^4} \left( 1 + \frac{1}{\hat{g}} T^{(0)} + \frac{1}{\hat{g}^2} T^{(1)} + \ldots \right) \quad \rightarrow \quad T^{(1)}_{\text{ext}} = (\Sigma_1^{(1)}(p) + \Sigma_1^{(1)}(p')) T^{(0)}
\]

\[
\frac{i(1 + g^{-1}\Sigma_1^{(1)}(p))}{p^2 - m^2 - g^{-1}\Sigma_0^{(1)}(p)}
\]
External legs correction

- Contributions from external leg corrections

\[ \sqrt{Z}^4 \left( 1 + \frac{1}{\hat{g}} T^{(0)} + \frac{1}{\hat{g}^2} T^{(1)} + \ldots \right) \Rightarrow T_{ext}^{(1)} = (\Sigma_1^{(1)}(p) + \Sigma_1^{(1)}(p')) T^{(0)} \]

- In AdS$_3 \times S^3 \times S^3 \times S^1$ model cubic vertices

- Self-energy via "fusing" two tree-level form factors

\[ \Sigma^{(1)}(p)|_{cut} = \int \frac{d^2 l_1}{(2\pi)^2} i\pi \delta^+ (l_1^2 - m_1) i\pi \delta^+ ((l_1 - p)^2 - m_2) \mathcal{F}_{RS}^{(0)}(p, l_1, l_1 - p) \mathcal{F}_{RS}^{(0)\dagger}(p, l_1, l_1 - p) \]

[Brandhuber Travaglini Yang 2012]
[Engelund Roiban 2013]
Contributions from external leg corrections

\[ \sqrt{Z^4} = \sqrt{Z^4} \left( 1 + \frac{1}{\hat{g}} T^{(0)} + \frac{1}{\hat{g}^2} T^{(1)} + \ldots \right) \rightarrow T_{\text{ext}}^{(1)} = (\Sigma_1^{(1)}(p) + \Sigma_1^{(1)}(p')) T^{(0)} \]

In \( AdS_3 \times S^3 \times S^3 \times S^1 \) model cubic vertices

Self-energy via ``fusing'' two tree-level form factors

\[ \Sigma^{(1)}(p) |_{\text{cut}} = \frac{1}{2} \left| \mathcal{F}^{(0)}_{RS}(p, l_1, l_2) \right|^2 I(p^2, m_1, m_2) = \Sigma_0^{(1)}(p) + \Sigma_1^{(1)}(p)(p^2 - m^2) + \mathcal{O}((p^2 - m^2)^2) \]
... leads to an agreement with the exact result where “integrable coupling” coincides with sigma model coupling constant.

\[ \epsilon = \sqrt{m^2 + 4f^2(\lambda) \sin^2 \frac{p}{2}} \]
... leads to an agreement with the exact result where “integrable coupling” coincides with sigma model coupling constant.

$$\epsilon = \sqrt{m^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}$$

The unitarity construction here misses a shift in the coupling.
Summary of models analyzed

One-loop logarithmic and rational terms of 2→2 amplitudes reproduced:

- ✓ Bosonic relativistic models  
- ✓ Supersymmetric relativistic models  
- ✓ AdS$_5 \times S^5$  
- ✓ AdS$_3 \times S^3 \times T^4$  
- ✓ AdS$_3 \times S^3 \times S^3 \times S^1$  

Shift in the coupling is not seen by integrability (indeed the result respects YB)!  
As expected.
A cut-constructibility criterion?

Not a "uniqueness result" proving cut constructibility a’ la [Bern, Dixon, Dunbar, Kosower 1994]

Power-counting criterion: the \( m \)-point integrals have at most \( m-2 \) powers of the loop momentum in the numerator of the integrand, and the two-point (bubble) integrals have at most one power of the loop momentum can be completely determined from its cuts.

for one-loop amplitudes in massless supersymmetric gauge theories.

> Here just scalar bubbles in the basis, and no \( m \)-point information yet.

> Here no 2d supersymmetry (GS string)

> Up to shift in the coupling: something missing is “expected”, as we are in a massive theory in \( d=2 \) (issues tadpole- and regularization-related).
A cut-constructibility criterion?

Not a "uniqueness result" proving cut constructibility a’ la [Bern, Dixon, Dunbar, Kosower 1994]

Power-counting criterion: the m-point integrals have at most m-2 powers of the loop momentum in the numerator of the integrand, and the two-point (bubble) integrals have at most one power of the loop momentum can be completely determined from its cuts.

for one-loop amplitudes in massless supersymmetric gauge theories.

> Here just scalar bubbles in the basis, and no m-point information yet.

> Here no 2d supersymmetry (GS string)

> Up to shift in the coupling: something missing is “expected”, as we are in a massive theory in d=2 (issues tadpole- and regularization-related).

Clear interplay (including external legs corrections) between cut-constructibility and integrability.
Concluding remarks and a wish list

- Enough evidence that for large class of 2-d models (relativistic and not) four-points one-loop amplitudes are cut-constructible
  - Standard unitarity (2-particle cuts) reproduces all rational terms, up to shifts in the coupling.
  - It bypasses issues of UV divergences emerging in Feynman diagrams.

- Efficient way for
  - Checks of quantum integrability aspects (e.g. group factorization).
  - Proposing/checking matrix structure and overall phases for models relevant for the AdS/CFT correspondence.

- Cut-constructibility “criterion”
  - Integrability is crucial asset
  - External leg corrections required to meet the exact result (up to shifts): off-shell continuation of the unitarity method to include form factors on either side of the cut.
Wish list

- Two loops rational terms (all logarithms reproduced in [Engelund McEwan Roiban 2013])

- Higher points: factorization should emerge

\[ S_{3\rightarrow 3} = (S_{2\rightarrow 2})^3 \]

- Extend to other interesting integrable string backgrounds
  (require a tree-level S-matrix! and massless modes treatment) [Basso 2010]

- Extend to off-shell objects, including correlation functions. [Klose McLoughlin 2012/2013] [Engelund McEwan Roiban 2013]
Other string backgrounds I: “ABJM” worldsheet S-matrix

No direct check of our method (also not in [Roiban Engelung 2013]).

- **Same** S-matrix as in $AdS_5 \times S^5$ *despite* elementary excitations (BMN modes) differ.


Modes are (4|4) *light* + (4|4) *heavy*. *dissolve in the continuum* [Zarembo 2009][Kalousios Vergu Volovich 2009]

Consistent with Bethe ansatz solution where heavy modes are interpreted as stacks of Bethe roots.

- A unitarity-cut check of recent proposal for $h(\lambda)$ in $\epsilon = \sqrt{1 + h(\lambda) \sin^2 \frac{p}{2}}$

  \[
  \lambda = \frac{\sinh(2\pi h)}{2\pi} \,_3 F_2 \left( \frac{1}{2}, \frac{1}{2}, 1; 1, 3; -\sinh^2(2\pi h) \right) \\
  h(\lambda) = \lambda - \frac{\pi^2 \lambda^3}{3} + \frac{5\pi^4 \lambda^5}{12} - \frac{893\pi^6 \lambda^7}{1260} + \mathcal{O}(\lambda^9), \\
  h(\lambda) = \sqrt{\frac{1}{2} \left( \lambda - \frac{1}{24} \right) - \frac{\log 2}{2\pi} + \mathcal{O} \left( e^{-\pi \sqrt{8\lambda}} \right)}, \\
  \]

  [Gromov Sizov 2014]

requires treatment of tadpoles.
Simply gluing T0 and T1 and uplifting seems to fail (you don’t capture all contrs that at the end give you the expected result in terms of logs or double logs)

Since there is no integral basis -> strategy of maximal cuts

Begin with the generalized cuts imposing $D L = 2 L = 4$ cut conditions
proceed by releasing the on-shell condition for one propagator at a time (near-maximal cuts).
\[ i T^{(2)} = \frac{1}{4} C_a \tilde{I}_a + \frac{1}{2} C_b \tilde{I}_b + \frac{1}{2} C_c \tilde{I}_c + \frac{1}{4} C_d \tilde{I}_d + \frac{1}{2} C_e \tilde{I}_e + \frac{1}{2} C_f \tilde{I}_f \]
\[ + \frac{1}{2} C_{s,\text{extra}} \tilde{I}_s + \frac{1}{2} C_{u,\text{extra}} \tilde{I}_u \]
\[ + \text{ rational} \]

Get coefficients by comparing with maximal cuts

Get coefficients by comparing with near-maximal (here 2) cuts

\[
C_{\ln^2} = \frac{1}{8\pi^2 J^2} \left( -2C_a + C_b + C_c - 2C_d + C_e + C_f \right) = 0
\]
\[
C_{\ln^1} = \frac{i}{4\pi^2 J^2} (2C_a - C_b - C_c) \propto -\frac{1}{2} T^{(0)}
\]

Successfully reproduces expected logarithms behavior

To be improved to catch all (+rational) terms.