Unitarity methods for scattering in two dimensions

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based on work with Lorenzo Bianchi and Ben Hoare

Emmy Noether-Programm
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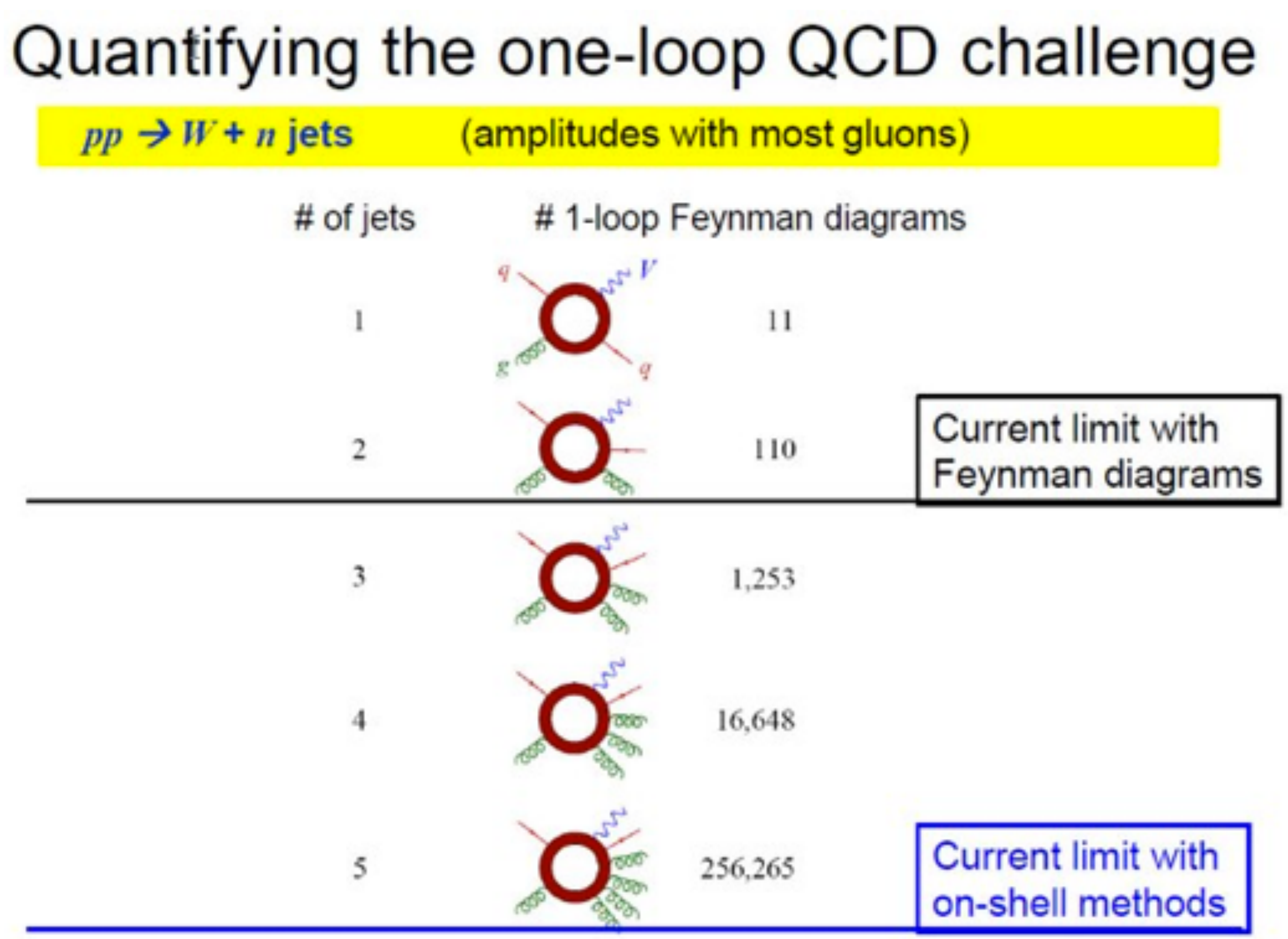
Emmy Noether Research Group
Gauge Fields from Strings

L. Bianchi, V.Forini, B.Hoare, arXiv: 1304.1798
O. T. Engelund, R. W. McKeown, R. Roiban, arXiv: 1304.4281
L. Bianchi, B. Hoare, arXiv: 1405.7947

Itzykson “Amplitudes” Conference, Paris, May 27 2014

Calculating scattering amplitudes efficiently

Remarkable efficiency of unitarity-based methods [Bern, Dixon, Dunbar, Kosower, 1994] for calculation of amplitudes in various qft's and various dimensions (non-abelian gauge theories, Chern-Simons theories, supergravity).



[from a L. Dixon talk]

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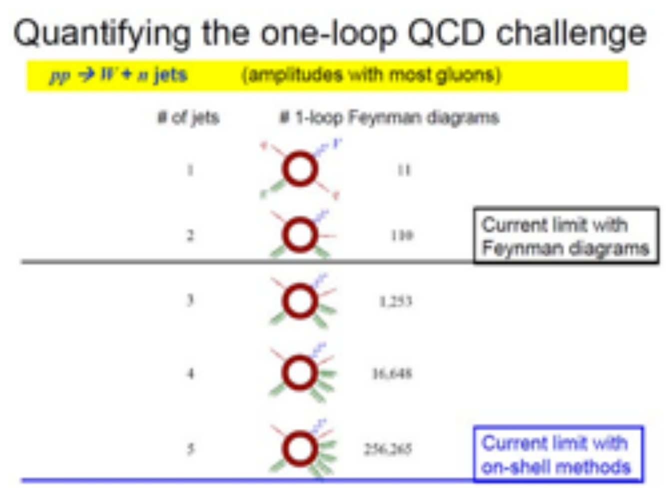


Goal is apply to evaluation amplitudes of two-dimensional cases of interest, where:

1. Feynman diagram calculations **are problematic** (divergencies do not cancel).
2. Need of **perturbative checks** for integrability-based proposal.
3. Need of **alternative strategies** to get (phases of) S-matrices.

1+1 dimensional world: simpler

- Highly constrained scattering kinematics, no phase space for 2->2 amplitudes.
- Natural set for the phenomenon of **integrability**, strong selection rules at work.

Existence of local higher rank conserved charges

$\Downarrow$ [Zamolodchikov & Zamolodchikov, 79]

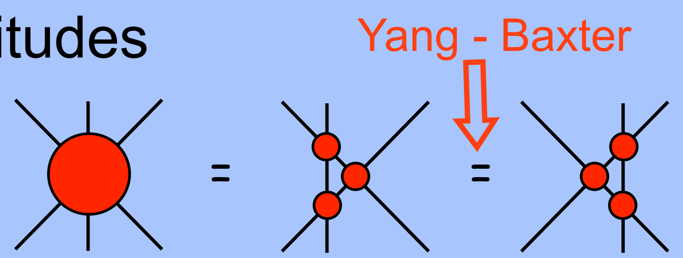
i) No particle production or annihilation

$$n_{\text{in}} = n_{\text{out}} = n$$

ii) Conservation of set of momenta

$$\{p_1^{\text{in}}, \ldots, p_n^{\text{in}}\} = \{p_1^{\text{out}}, \ldots, p_n^{\text{out}}\}$$

iii) Factorization of $n \to n$ amplitudes into
products of $2 \to 2$ amplitudes



Cut-constructibility might be expected!

1+1 dimensional world: non-trivial

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    > allows for existence of momentum configurations potentially singular

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- AdS/CFT- related models (string world-sheet) are non-trivial

  > UV divergencies appear in standard calculations of 4-point amplitudes

  > no clear symmetry-preserving regularization scheme [McLoughlin, Roiban, unpublished]

  > power-counting non-renormalizable

  > non-relativistic

  > coupling constant subject to finite renormalization
  (for target space with less-then-maximal susy)

  > non-supersymmetric as world-sheet theories

  > massive

  > integrability phenomenon: **solid fact only classically**! [Bena, Polchinski, Roiban 2003]

  infinite set of nonlocal charges (Z_4 automorphism of the coset action)

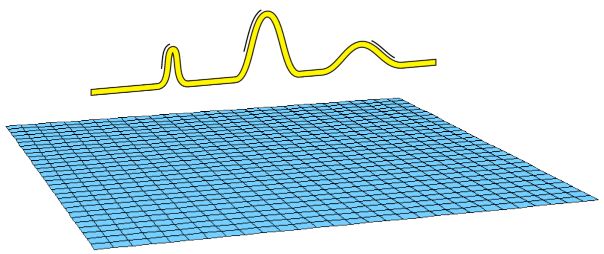
  quantum checks are only

  >> in simplifying limits
  >> in pure spinor language

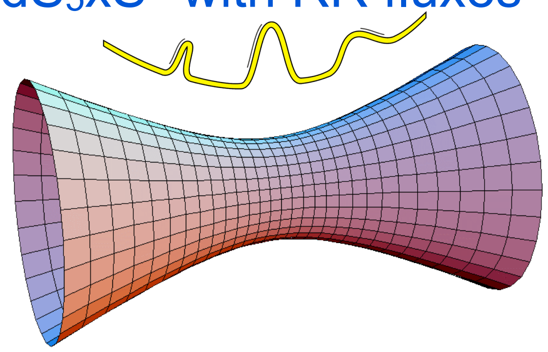
String worldsheet scattering

- Worldsheet amplitudes $(N \to \infty$, free strings), scattering of the (2d) lagrangean excitations. Non-trivial interactions due to highly non trivial background.

flat space



AdS_5xS^5 with RR fluxes



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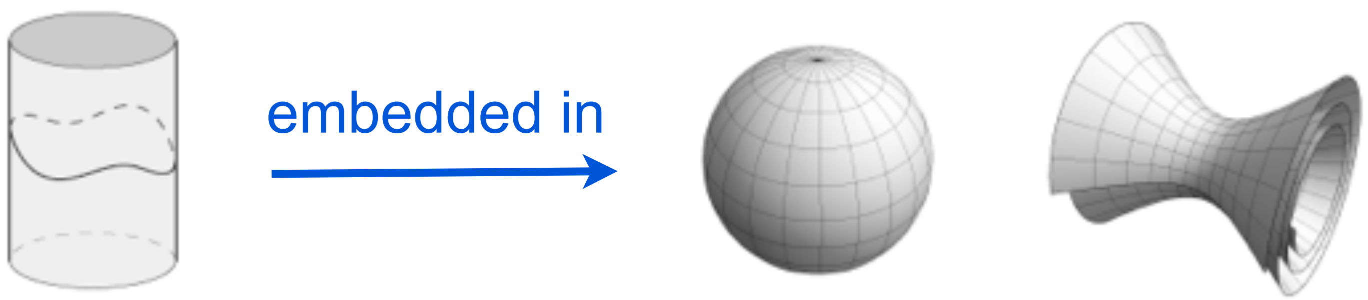

$AdS_5 \times S^5$

- Because of RR-background need a GS formulation

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \sqrt{-h}\, h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N + \text{fermions}$$

loop counting parameter $\hat{g} = \frac{2\pi}{\sqrt{\lambda}}$

- Work on a gauge-fixed sigma model (uniform light-cone gauge)

[Arutyunov, Frolov, Plefka, Zamaklar 2006]

$$H_{ws} = \int d\sigma\, \mathcal{H}_{ws} = -\int d\sigma\, p_- \equiv E - J$$

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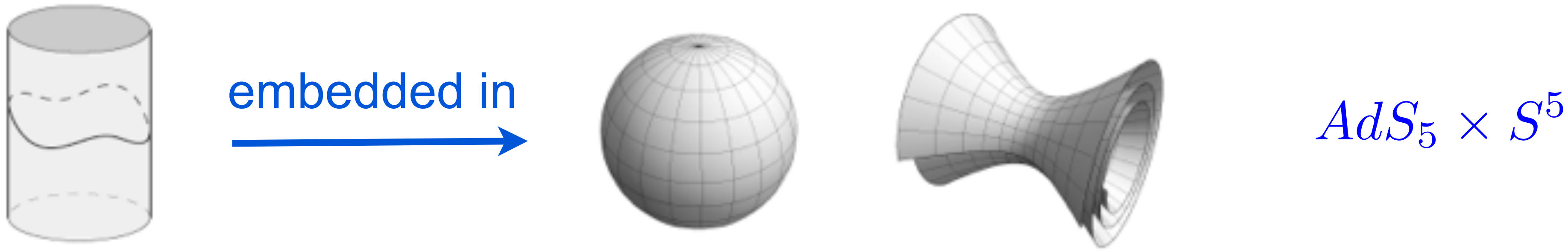

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- Gauge-fixed lagrangean involves rescaling $-\frac{\pi J_+}{\sqrt{\lambda}} < \sigma < \frac{\pi J_+}{\sqrt{\lambda}}$

Decompactification limit $\frac{J_+}{\sqrt{\lambda}} \to \infty$ and large tension expansion $\hat{g} \to \infty$

$\longrightarrow$ sensible definition of a **perturbative worldsheet S-matrix**

AdS/CFT (internal) S-matrix I

- This S-matrix is the perturbative expansion of the exact AdS_5/CFT_4 S-matrix *aka* "spin chain S-matrix" : ☆ the rhs of asymptotic Bethe eqs

[Staudacher 2004]
[Beisert Staudacher 2005]
[Beisert 2005]

$$
\begin{aligned}
1 &= \prod_{j}^{K_4} \frac{x_{4j}^+}{x_{4j}^-} \\
1 &= \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2k}-u_{2j}-i}{u_{2k}-u_{2j}+i} \prod_{j=1}^{K_3} \frac{u_{2k}-u_{3j}+\frac{i}{2}}{u_{2k}-u_{3j}-\frac{i}{2}} \\
1 &= \prod_{j=1}^{K_2} \frac{u_{3k}-u_{2j}+\frac{i}{2}}{u_{3k}-u_{2j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3k}-x_{4j}^+}{x_{3k}-x_{4j}^-} \\
\left(\frac{x_{4k}^+}{x_{4k}^-}\right)^L &= \prod_{j=1}^{K_4} \left(\frac{u_{4k}-u_{4j}+i}{u_{4k}-u_{4j}-i}\, e^{2i\theta(x_{4k},x_{4j})}\right) \prod_{j=1}^{K_3} \frac{x_{4k}^- - x_{3j}}{x_{4k}^+ - x_{3j}} \prod_{j=1}^{K_5} \frac{x_{4k}^- - x_{5j}}{x_{4k}^+ - x_{5j}} \\
1 &= \prod_{j=1}^{K_6} \frac{u_{5k}-u_{6j}+\frac{i}{2}}{u_{5k}-u_{6j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5k}-x_{4j}^+}{x_{5k}-x_{4j}^-} \\
1 &= \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6k}-u_{6j}-i}{u_{6k}-u_{6j}+i} \prod_{j=1}^{K_5} \frac{u_{6k}-u_{5j}+\frac{i}{2}}{u_{6k}-u_{5j}-\frac{i}{2}}
\end{aligned}
$$

$$x(u) = \tfrac{1}{2}u + \tfrac{1}{2}u\sqrt{1-2g^2/u^2}$$

$$x^\pm(u) = x(u \pm \tfrac{i}{2})$$

$$\delta D = \delta E = g^2 \sum_{j=1}^{K_4} \Big(\frac{i}{x_{4j}^+} - \frac{i}{x_{4j}^-}\Big)$$

$$g^2 = \frac{\lambda}{8\pi^2}$$

Describe the exact **asymptotic** spectrum of anomalous dimensions of local composite operators and energies of their dual string configurations.

☆ (not spacetime scattering! however **see Matthias talk**)

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- Back at the exact gauge-fixed lagrangean: hard to quantize, arbitrary order interactions, standard CFT methods do not help.

- **Integrability: most powerful tool to obtain string spectrum at finite coupling on non-trivial backgrounds.**

  (see also [2013: McEwan, Roiban] for very interesting lattice-discretization of GS string).

...together with being the way to the **first ever exact solution** of a four-dimensional interacting gauge theory.

AdS/CFT (internal) S-matrix II

Assuming integrability (consistency with Yang-Baxter equation) and using global symmetries one can:

- derive exact dispersion relation $\epsilon = \sqrt{1 + f(\lambda)\,\sin^2\frac{p}{2}}$ [Beisert 2006]

- derive two-particle S-matrix entering the Bethe equations ✪ [Staudacher 2004] [Beisert Staudacher 2005] [Beisert 2005]

$$S_{12} = S^0\,\mathbf{S_{12}}$$

✪ Various beautiful “upgrades” of Bethe equations exist: won’t play a role here.

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- **up to one (/more) scalar factor(/s)**, fixed with additional constraints like “crossing symmetry” and semiclassical string data. [Janik 2005]

Hardest thing to compute, crucial for the spectrum.

Cusp anomaly: impressive calculation establishing correctness of result for AdS_5xS^5.

[Beisert Eden Staudacher 2006]

[Bern Czakon Dixon Kosower Smirnov 2006]

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Necessity for new strategies in case of models relevant in AdS_3/CFT_2

Motivation

- Provide 2d scattering perturbation theory with efficient tools
- Provide tests of quantum integrability for certain string backgrounds.
- Extract information about the overall factors of scattering matrix.
  Expecially for worldsheet S-matrices for AdS_3 backgrounds
  where solutions to crossing-like equations are difficult to determine.

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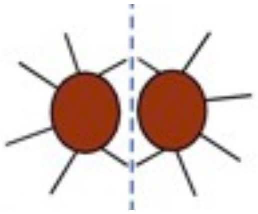
- Provide 2d scattering perturbation theory with efficient tools
- Provide tests of quantum integrability for certain string backgrounds.
- Extract information about the overall factors of scattering matrix. Expecially for worldsheet S-matrices for AdS_3 backgrounds where solutions to crossing-like equations are difficult to determine.
- Methodological: techniques never really applied in two dimensions.
- Apply to off-shell quantities

Initiate the use of unitarity-based methods for perturbative S-matrix in massive two-dimensional field theories.

Construct one-loop $2 \rightarrow 2$ scattering amplitude with <u>standard</u> unitarity



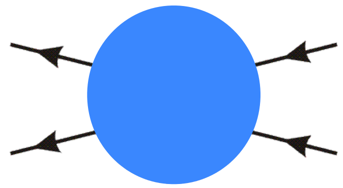
directly from the corresponding on-shell tree-level amplitudes.

Outline

- 1. Method of unitarity cuts in d=2, general formula
- 2. Applications (mainly string worldsheet) and features
- 3. Concluding remarks

Two-dimensional scattering

Two-body scattering process of a theory invariant under space and time translations



described via the four-point amplitude

$$\langle \Phi^P(p_3)\Phi^Q(p_4)\,|\mathbb{S}|\,\Phi_M(p_1)\Phi_N(p_2)\rangle = (2\pi)^2\delta^{(d)}(p_1+p_2-p_3-p_4)\,\mathcal{A}_{MN}^{PQ}(p_1,p_2,p_3,p_4)$$

For d=2 and in the single mass case, scattering 2 → 2 is simple.

Particles either preserve or exchange their momenta

$$\delta^{(2)}(p_1+p_2-p_3-p_4) = J(p_1,p_2)\,\big(\delta(\mathrm{p}_1-\mathrm{p}_3)\delta(\mathrm{p}_2-\mathrm{p}_4)+\delta(\mathrm{p}_1-\mathrm{p}_4)\delta(\mathrm{p}_2-\mathrm{p}_3)\big)$$

The Jacobian $J(p_1,p_2) = 1/(\partial\epsilon_{\mathrm{p}_1}/\partial\mathrm{p}_1 - \partial\epsilon_{\mathrm{p}_2}/\partial\mathrm{p}_2)$ depends on dispersion relation.

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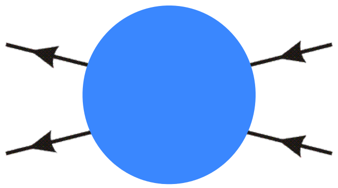

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S-matrix element defined by

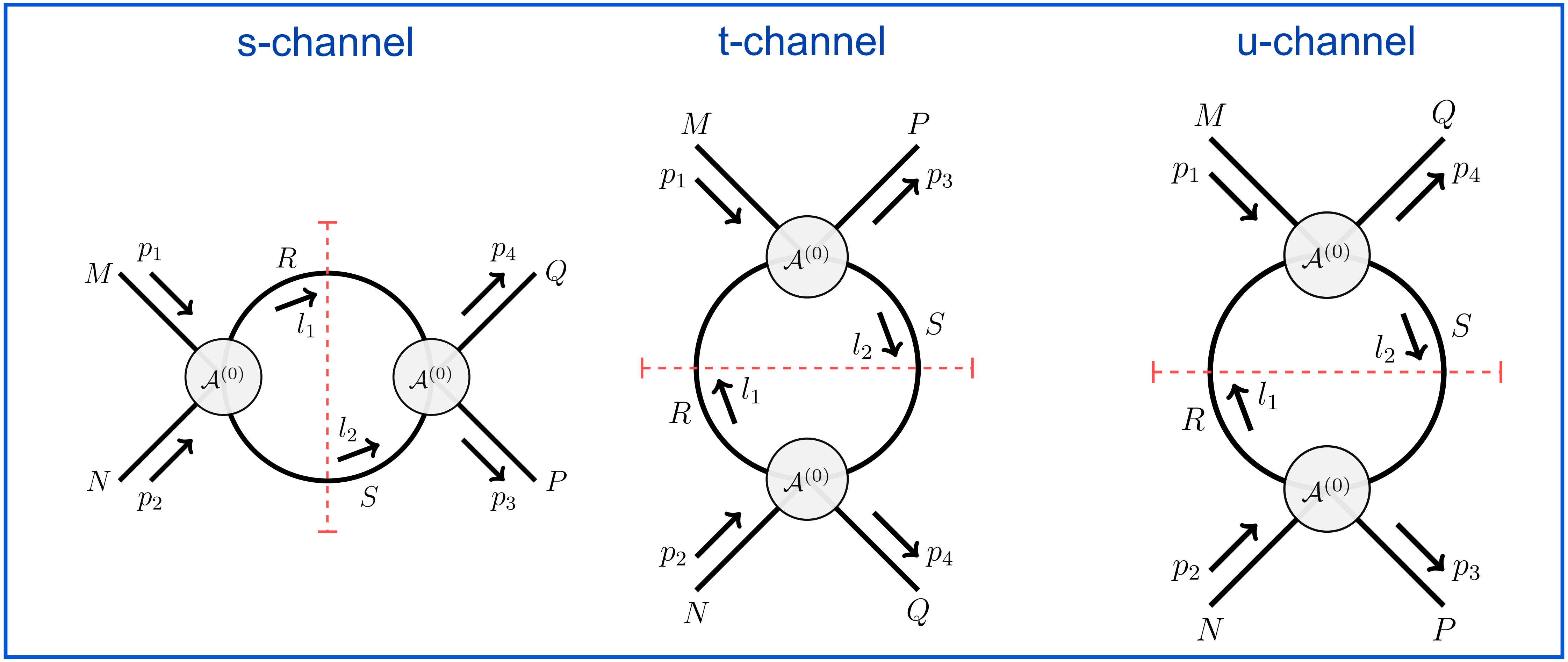
$$S^{PQ}_{MN}(p_1,p_2) \equiv \frac{J(p_1,p_2)}{4\epsilon_1\epsilon_2}\mathcal{A}^{PQ}_{MN}(p_1,p_2,p_1,p_2)$$

Dispersion relation for asymptotic states (equal masses =1): $\epsilon_i^2 = 1+\mathrm{p}_\mathrm{i}^2$

Fix ordering of incoming states $\mathrm{p}_1 > \mathrm{p}_2$.

Scattering in d=2: unitarity cuts (1)

One-loop result from unitarity techniques: contributions from three cut-diagrams

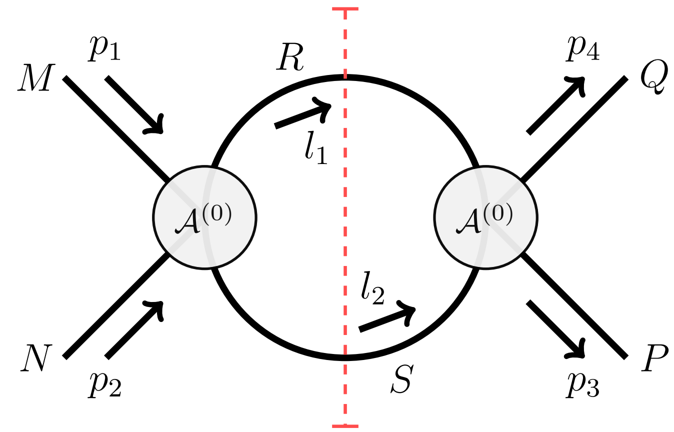


Example: s-cut contribution.

$$\mathcal{A}^{(1)}{}^{PQ}_{MN}(p_1,p_2,p_3,p_4)|_{s-cut} = \frac{1}{2}\int \frac{d^2 l_1}{(2\pi)^2}\int \frac{d^2 l_2}{(2\pi)^2}\; i\pi\delta^+({l_1}^2-1)\; i\pi\delta^+(l_2^2-1)$$
$$\times\,\mathcal{A}^{(0)}{}^{RS}_{MN}(p_1,p_2,l_1,l_2)\mathcal{A}^{(0)}{}^{PQ}_{SR}(l_2,l_1,p_3,p_4)$$

Glue tree-amplitudes and **uplift**: $i\pi\delta^+(l_1^2-1) \;\longrightarrow\; \dfrac{1}{l_1^2-1}$

Scattering in d=2: unitarity cuts (2)



- Use 2-momentum conservation at the first vertex

$$\widetilde{\mathcal{A}}^{(1)}{}^{PQ}_{MN}(p_1,p_2,p_3,p_4)|_{s-cut} = \frac{1}{2}\int \frac{d^2 l_1}{(2\pi)^2}\, i\pi\delta^+({l_1}^2-1)\, i\pi\delta^+((l_1-p_1-p_2)^2-1)$$
$$\times \widetilde{\mathcal{A}}^{(0)}{}^{RS}_{MN}(p_1,p_2,l_1,-l_1+p_1+p_2)\, \widetilde{\mathcal{A}}^{(0)}{}^{PQ}_{SR}(-l_1+p_1+p_2,l_1,p_3,p_4)$$

- Use the zeroes of δ - functions in the $\widetilde{\mathcal{A}}^{(0)}$ (like $f(x)\,\delta(x) = f(0)\,\delta(x)$)

- Restore loop momentum off-shell $\quad i\pi\delta^+(l_1^2-1) \longrightarrow \dfrac{1}{l_1^2-1}$

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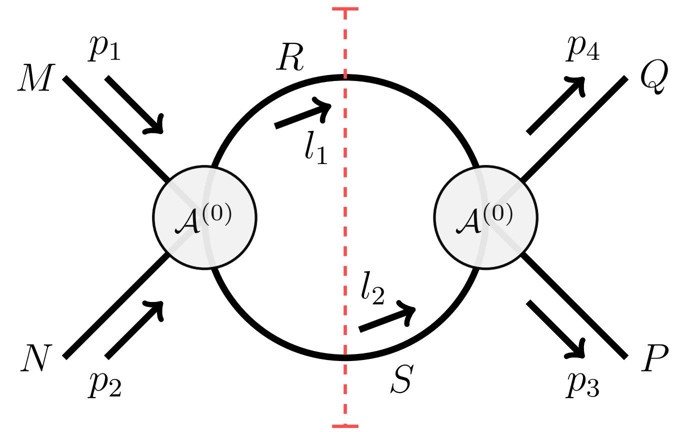

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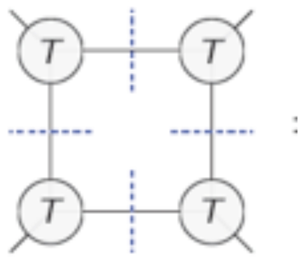
Two-particle cuts in d=2 at one loop are maximal cuts.

Expect same as quadrupole cuts in d=4: $A_4^{1-loop} = \sum (A_4^{tree})^4\, I_{box}$



4-points amplitude at one-loop

Tree-level amplitudes can be pulled out of the integral, evaluated at those zeroes.

A simple sum over discrete solutions of the on-shell conditions

$$
\begin{aligned}
\widetilde{\mathcal{A}}^{(1)}{}^{PQ}_{MN}(p_1,p_2,p_3,p_4) = \quad & \frac{I(p_1+p_2)}{2}\Big[\widetilde{\mathcal{A}}^{(0)}{}^{RS}_{MN}(p_1,p_2,p_1,p_2)\widetilde{\mathcal{A}}^{(0)}{}^{PQ}_{SR}(p_2,p_1,p_3,p_4) \\
& + \widetilde{\mathcal{A}}^{(0)}{}^{RS}_{MN}(p_1,p_2,p_2,p_1)\widetilde{\mathcal{A}}^{(0)}{}^{PQ}_{SR}(p_1,p_2,p_3,p_4)\Big] \\
+ \quad & I(p_1-p_3)\widetilde{\mathcal{A}}^{(0)}{}^{SP}_{MR}(p_1,p_3,p_1,p_3)\widetilde{\mathcal{A}}^{(0)}{}^{RQ}_{SN}(p_1,p_2,p_3,p_4) \\
+ \quad & I(p_1-p_4)\widetilde{\mathcal{A}}^{(0)}{}^{SQ}_{MR}(p_1,p_4,p_1,p_4)\widetilde{\mathcal{A}}^{(0)}{}^{RP}_{SN}(p_1,p_2,p_4,p_3)
\end{aligned}
$$

weighted by scalar “bubble” integrals

$$
I(p) = \int \frac{d^2q}{(2\pi)^2}\frac{1}{(q^2-1+i\epsilon)((q-p)^2-1+i\epsilon)}
$$

Inherently FINITE formula.

Final formula for the S-matrix

Final formula for the S-matrix (choose $p_3 = p_1, \quad p_4 = p_2$)

$$S^{(1)}{}^{PQ}_{MN}(p_1,p_2) = \frac{1}{4(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}\Big[\tilde{S}^{(0)}{}^{RS}_{MN}(p_1,p_2)\tilde{S}^{(0)}{}^{PQ}_{RS}(p_1,p_2)I(p_1+p_2)$$
$$+\tilde{S}^{(0)}{}^{SP}_{MR}(p_1,p_1)\tilde{S}^{(0)}{}^{RQ}_{SN}(p_1,p_2)I(0)$$
$$+\tilde{S}^{(0)}{}^{SQ}_{MR}(p_1,p_2)\tilde{S}^{(0)}{}^{PR}_{SN}(p_1,p_2)I(p_1-p_2)\Big]\,,$$

where $$\tilde{S}^{(0)}(p_1,p_2) = 4(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)S^{(0)}(p_1,p_2)$$

Sum of products of two tree-level amplitudes weighted by scalar bubble integrals

$$I_s \equiv I(p_1+p_2) = \frac{1}{\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2} - \frac{\operatorname{arsinh}(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}{4\pi i\,(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}$$
$$I_t \equiv I(0) = \frac{1}{4\pi i}$$
$$I_u \equiv I(p_1-p_2) = \frac{\operatorname{arsinh}(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}{4\pi i\,(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}$$

Possible absence of rational terms: formula cannot be completely general !
Needs to be tested on various examples.

Final formula for the S-matrix

Final formula for the S-matrix (choose $p_3 = p_1, \quad p_4 = p_2$)

$[M] = 0$ bosons
$[M] = 1$ fermions

$$S^{(1)}{}_{MN}^{PQ}(p_1, p_2) = \frac{1}{4(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}\Big[\tilde{S}^{(0)}{}_{MN}^{RS}(p_1,p_2)\tilde{S}^{(0)}{}_{RS}^{PQ}(p_1,p_2)I(p_1+p_2)$$
$$+(-1)^{[P][S]+[R][S]}\,\tilde{S}^{(0)}{}_{MR}^{SP}(p_1,p_1)\tilde{S}^{(0)}{}_{SN}^{RQ}(p_1,p_2)I(0)$$
$$+(-1)^{[P][R]+[Q][S]+[R][S]+[P][Q]}\,\tilde{S}^{(0)}{}_{MR}^{SQ}(p_1,p_2)\tilde{S}^{(0)}{}_{SN}^{PR}(p_1,p_2)I(p_1-p_2)\Big]$$

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$$I_s \equiv I(p_1+p_2) = \frac{1}{\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2} - \frac{\mathrm{arsinh}(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}{4\pi i\,(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}$$
$$I_t \equiv I(0) = \frac{1}{4\pi i}$$
$$I_u \equiv I(p_1-p_2) = \frac{\mathrm{arsinh}(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}{4\pi i\,(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}$$

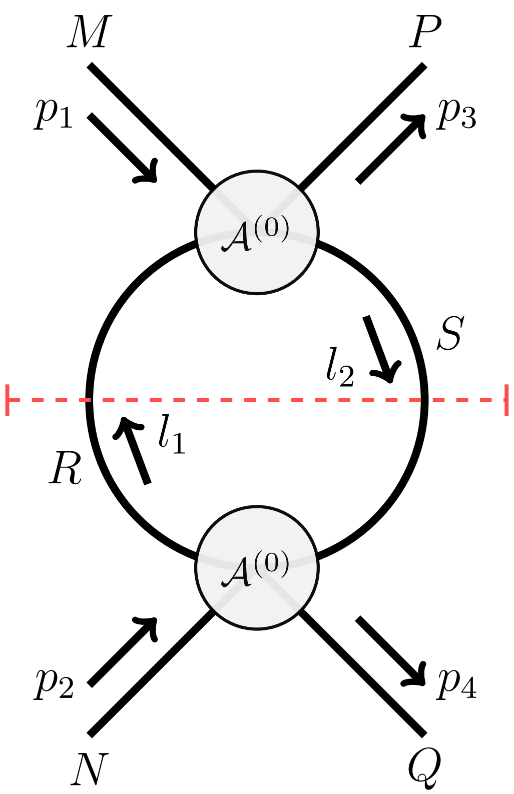
Possible absence of rational terms: formula cannot be completely general !
Needs to be tested on various examples.

Remarks

- The t-channel cut is special.

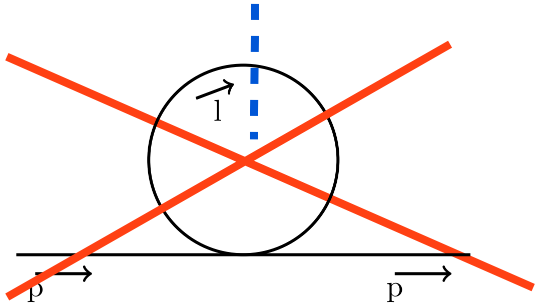
  - Using first $\delta(\mathrm{p}_1 - \mathrm{p}_3)\delta(\mathrm{p}_2 - \mathrm{p}_4)$
    makes it ill-defined and requires a prescription:
    use delta-function **only at the end** of the calculation

  - Asymmetrical wrt choice of the vertex
    used to solve momenta:
    leads to a consistency condition



$$\tilde{S}^{(0)}{}^{SP}_{MR}(p_1,p_1)\,\tilde{S}^{(0)}{}^{RQ}_{SN}(p_1,p_2) \;=\; \tilde{S}^{(0)}{}^{PS}_{MR}(p_1,p_2)\,\tilde{S}^{(0)}{}^{QR}_{SN}(p_2,p_2)$$

- We are NOT including contributions from tadpoles (no physical cuts)



- A inherently finite result says **nothing** about UV-finiteness or renormalizability.
  Might be missing rational terms following from regularization procedure.

Cut-constructibility to be always checked

Final formula for the S-matrix II

Define contractions in field space

$$
\begin{aligned}
(A \textcircled{s} B)^{PQ}_{MN}(p,p') &= A^{RS}_{MN}(p,p')B^{PQ}_{RS}(p,p') \,, \\
(A \textcircled{u} B)^{PQ}_{MN}(p,p') &= (-1)^{([P]+[S])([Q]+[R])} A^{SQ}_{MR}(p,p')B^{PR}_{SN}(p,p') \,, \\
(A \underset{\leftarrow}{\textcircled{t}} B)^{PQ}_{MN}(p,p') &= (-1)^{[P][S]+[R][S]} A^{SP}_{MR}(p,p)B^{RQ}_{SN}(p,p') \,, \\
(A \underset{\rightarrow}{\textcircled{t}} B)^{PQ}_{MN}(p,p') &= (-1)^{[Q][R]+[R][S]} A^{PS}_{MR}(p,p')B^{QR}_{SN}(p',p') \,,
\end{aligned}
$$

use $\quad I_s = \frac{1}{J}\left(1 - \frac{\theta}{i\pi}\right) \qquad I_t = \frac{1}{4\pi i} \qquad I_u = \frac{1}{J}\frac{\theta}{i\pi}$ $\quad$ with $\quad \theta \equiv \operatorname{arcsinh}(e'p - ep')$

$J = \frac{1}{4(e'p - ep')}$ $\quad$ and $\quad S = \mathbf{1} + i\sum_{n=1} \hat{g}^{-n} T^{(n-1)}$

To rewrite the one-loop unitarity cut result as (assuming $T^{(0)}$ real)

$$
T^{(1)} = \underbrace{\frac{\theta}{2\pi}\left(T^{(0)} \textcircled{u} T^{(0)} - T^{(0)} \textcircled{s} T^{(0)}\right)}_{\text{logarithmic terms}} + \underbrace{\frac{i}{2} T^{(0)} \textcircled{s} T^{(0)}}_{\text{rational imaginary}} + \underbrace{\frac{1}{16\pi}\left(\tilde{T}^{(0)} \underset{\leftarrow}{\textcircled{t}} T^{(0)} + T^{(0)} \underset{\rightarrow}{\textcircled{t}} \tilde{T}^{(0)}\right)}_{\text{rational real}}
$$

Another approach (Engelund, McKeown and Roiban 1304.4281)

- Ignore potentially singular cuts.

  (In the Green Schwarz worldsheet string, no regularization is known which preserves the symmetry of the theory: kappa-symmetry, Weyl-symmetry, integrability...)

- The t-channel bubble integral is a constant (rational): ignore and determine all rational terms from symmetry considerations.

- Determine **logarithmic** terms, **up to two loops**, staying in d=2.

Relativistic models

- Bosonic models:
  - generalized sine-Gordon: gauged WZW model for a coset G/H = SO(n + 1)/SO(n) plus an integrable potential (n=1: sine-Gordon, n=2: complex sine-Gordon)

The method works <u>up to a finite shift in the coupling</u>.

- Supersymmetric generalizations (``Pohlmeyer reductions'' of string theories):
  - $\mathcal{N} = 1, 2$ supersymmetric sine-Gordon

The method reproduces the <u>full</u> result.

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- Supersymmetric generalizations (``Pohlmeyer reductions'' of string theories):
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The method reproduces the <u>full</u> result.

In two cases (complex sine-Gordon and Pohlmeyer-reduced AdS_3xS^3 theory) cut-constructibility is highly non trivial!

Theory only integrable at classical level. Quantum counterterms restoring various properties of integrability (e.g. Yang-Baxter equation).

It is this "quantum integrable" result that the unitarity method gives.

AdS/CFT S-matrix: exact and perturbative structure

In the asymptotic case, symmetry algebra is a (centrally extended) PSU(2|2)^2

From symmetries and integrability follows a **group factorization**

$$\mathbb{S} = \quad e^{i\,\theta}\ \hat{S}^{PSU(2|2)} \otimes \hat{S}^{PSU(2|2)}$$

$$\hat{S}_{AB}^{CD} = \begin{cases} A\delta_a^c\delta_b^d + B\delta_a^d\delta_b^c & \\ D\delta_\alpha^\gamma\delta_\beta^\delta + E\delta_\alpha^\delta\delta_\beta^\gamma & \\ C\epsilon_{ab}\epsilon^{\gamma\delta} & F\epsilon_{\alpha\beta}\epsilon^{cd} \\ G\delta_a^c\delta_\beta^\delta & H\delta_a^d\delta_\beta^\gamma \\ L\delta_\alpha^\gamma\delta_b^d & K\delta_\alpha^\delta\delta_b^c \end{cases}$$

Each factor has manifest $SU(2) \times SU(2)$ invariance

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Each factor has manifest $SU(2) \times SU(2)$ invariance

Logarithms

Matrix structure,
rational dependence on
momenta

The phase

Bilinear of local charges

$$\theta = \sum_{n=0}^{\infty} \hat{g}^{1-n}\theta_{12}^{(n)}$$

$$\begin{aligned}\theta^{(n)} &= \chi^{(n)}(x_1^+, x_2^+) - \chi^{(n)}(x_1^+, x_2^-) - \chi^{(n)}(x_1^-, x_2^+) + \chi^{(n)}(x_1^-, x_2^-) \\ &- \chi^{(n)}(x_2^+, x_1^+) + \chi^{(n)}(x_2^+, x_1^-) + \chi^{(n)}(x_2^-, x_1^+) - \chi^{(n)}(x_2^-, x_1^-)\end{aligned}$$

with Zhukovsky variables encoding dispersion relation

$$x_p^{\pm} = \frac{\pi e^{\pm \frac{i}{2}p}}{\sqrt{\lambda}\sin\frac{p}{2}}\left(1 + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}\right)$$

Beyond one loop each contribution is rational.

$$\begin{aligned}\chi^{(1)}(x_1, x_2) &= -\frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} - \frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} \\ &+\frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} + \frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}}\end{aligned}$$

Crossing equation

$$i\theta(x_j, x_k) + i\theta(1/x_j, x_k) = 2\log h(x_j, x_k)$$

$$h(x_j, x_k) = \frac{x_k^-}{x_k^+}\frac{(1 - \frac{1}{x_j^- x_k^-})(x_j^- - x_k^+)}{(1 - \frac{1}{x_j^+ x_k^-})(x_j^+ - x_k^+)}$$

Perturbative structure of worldsheet S-matrix

Expansion of symmetry-determined and phase parts ($\theta^{(0)}$ absorbed in $T^{(0)}$)

$$\hat{S} = \mathbf{1} + i \sum_{n=1} g^{-n} \hat{T}^{(n-1)} \qquad\qquad \theta = \sum_{n=1}^{\infty} g^{-n} \theta^{(n-1)}$$

requires one-loop logarithms to contribute only to the diagonal terms

$$S = \mathbf{1} + \frac{i}{g}\hat{T}^{(0)} + \frac{i}{g^2}\left(\hat{T}^{(1)} + \theta^{(1)}\,\mathbf{1}\right) + \frac{1}{g^3}\left(\hat{T}^{(2)} + \frac{i}{2}\theta^{(1)}\hat{T}^{(0)} + \theta^{(2)}\,\mathbf{1}\right)$$

(and two-loop logarithms to be proportional to the tree-level S-matrix
- just the effect of two loop exponentiation - as $\theta^{(2)}$ has no logs)

Goal: compute one loop worldsheet S-matrix
"bootstrapping" it from tree level.

sigma-model

Superstring action

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \sqrt{-h}\, h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N + \text{fermions}$$

- Green-Schwarz formulation for fermions $\qquad \varrho_a = \partial_a x^\mu E_\mu{}^A \Gamma_A$

  quadratic part $\qquad L_F = i(\sqrt{-g} g^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ}) \bar{\theta}^I \varrho_a D_b \theta^J$

$$D_a \theta^I = \left(\partial_a + \frac{1}{4} \partial_a x^\mu \omega_\mu{}^{AB} \Gamma_{AB} \right) \theta^I + \frac{1}{2} \varrho_a \Gamma_{01234} \epsilon^{IJ} \theta^J$$

- Classical limit: $\lambda \to \infty$ $\quad$ Sigma model coupling constant: $\hat{g} = \dfrac{2\pi}{\sqrt{\lambda}}$

- Gauge-fixed lagrangean involves rescaling $\quad -\dfrac{\pi J_+}{\sqrt{\lambda}} < \sigma < \dfrac{\pi J_+}{\sqrt{\lambda}}$

  Decompactification limit $\dfrac{J_+}{\sqrt{\lambda}} \to \infty$ and large tension expansion $\hat{g} \to \infty$

Gauge fixing

Use an interpolating lightcone -gauge

[Arutyunov Frolov Plefka Zamaklar 06]

$$X^+ = (1+a)\,t + a\,\varphi \equiv \tau + a\,\sigma$$

AdS5 S5

$a = 1/2$ light-cone gauge

$a = 0$ temporal gauge

Transverse coordinates $z^\mu, \quad \mu = 1\ldots 4 \qquad y^m, \quad m = 1\ldots 4$

$$ds^2 = \underbrace{-G_{tt}(z)dt^2 + G_{zz}(z)dz^2}_{\text{AdS}_5} + \underbrace{G_{\varphi\varphi}(y)d\varphi^2 + G_{yy}(y)dy^2}_{S^5}$$

$$G_{tt} = \left(\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}}\right)^2, \quad G_{zz} = \frac{1}{(1-\frac{z^2}{4})^2}, \quad G_{\varphi\varphi} = \left(\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}}\right)^2, \quad G_{yy} = \frac{1}{\left(1+\frac{y^2}{4}\right)^2}.$$

Gauge choice preserves SO(8) at quadratic level, broken by interactions.

Interacting lagrangean

- Bosonic lagrangean to quartic order in the fields $X = (Y, Z)$

$$L = \frac{1}{2}\,(\partial_{\rm a} X)^2 - \frac{1}{2}\,X^2 + \frac{1}{4}\,Z^2\,(\partial_{\rm a} Z)^2 - \frac{1}{4}\,Y^2\,(\partial_{\rm a} Y)^2 + \frac{1}{4}\,(Y^2 - Z^2)\left(\dot{X}^2 + \acute{X}^2\right)$$
$$- \frac{1-2a}{8}\,(X^2)^2 + \frac{1-2a}{4}\,(\partial_{\rm a} X \cdot \partial_{\rm b} X)^2 - \frac{1-2a}{8}\,\left[(\partial_{\rm a} X)^2\right]^2 \ .$$

Lorentz invariance (quadratic part) broken by interactions.

Massive states with relativistic dispersion relation $\epsilon = \sqrt{1+p^2}$

$$\epsilon = \sqrt{1 + \frac{\lambda}{\pi^2}\,\sin^2\frac{p}{2}}$$

loop corrections

Exact dispersion relation known via symmetries

p

(Scattering ws particles, for parametrically large momentum, become solitonic solutions - giant magnons - with $\epsilon \sim \frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2}$)

- Bosonic part invariant under $SO(4) \times SO(4)$.

Worldsheet fields

- Worldsheet fields (embedding coordinates in AdS_5xS^5)

$$T,\ \Phi,\ Y^m,\ Z^m,\ \text{fermions}$$

can be represented as bispinors $SO(4) \simeq (SU(2) \times SU(2))/\mathbb{Z}_2$

$$Y_{a\dot{a}} = (\sigma_m)_{a\dot{a}}\, Y^m, \qquad Z_{\alpha\dot{\alpha}} = (\sigma_\mu)_{\alpha\dot{\alpha}}\, Z^\mu \qquad a, \dot{a}, \alpha, \dot{\alpha} = 1, 2$$

- Bosons and fermions form bi-fundamental representation of $\mathrm{PSU}(2|2)_L \times \mathrm{PSU}(2|2)_R$

$$\begin{tikzcd} & Y_{a\dot{a}} \arrow[dl] \arrow[dr] & \\ \Upsilon_{\alpha\dot{a}} \arrow[dr] & & \Psi_{a\dot{\alpha}} \arrow[dl] \\ & Z_{\alpha\dot{\alpha}} & \end{tikzcd}$$

$$\mathrm{PSU(2,2)}_L \times \mathrm{PSU(2,2)}_R$$

$$\underset{a}{\mathrm{SU(2)}} \times \underset{\dot{a}}{\mathrm{SU(2)}} \times \underset{\alpha}{\mathrm{SU(2)}} \times \underset{\dot{\alpha}}{\mathrm{SU(2)}}$$

- Formal definition of a bi-fundamental supermultiplet $\Phi_{A\dot{A}}\,,\ A = (a|\alpha)\ \ \dot{A} = (\dot{a}|\dot{\alpha})$ providing a basis for the definition of the S-matrix.

Worldsheet S-matrix

- Two-particle S-matrix is 256 x 256

$$\mathbb{S}\,|\Phi_{A\dot{A}}(p)\Phi_{B\dot{B}}(p')\rangle = |\Phi_{C\dot{C}}(p)\Phi_{D\dot{D}}(p')\rangle\,\mathbb{S}^{C\dot{C}D\dot{D}}_{A\dot{A}B\dot{B}}(p,p')$$

- Integrability predicts

$$\mathbb{S} = \mathbf{S}\otimes\mathbf{S} \quad , \quad \mathbb{S}^{C\dot{C}D\dot{D}}_{A\dot{A}B\dot{B}}(p,p') = \mathbf{S}^{CD}_{AB}(p,p')\mathbf{S}^{\dot{C}\dot{D}}_{\dot{A}\dot{B}}(p,p')$$

- S-matrices parametrized in terms of the basic SU(2) invariants

$$S^{CD}_{AB} = \begin{cases} A\delta^c_a\delta^d_b + B\delta^d_a\delta^c_b & \\ D\delta^\gamma_\alpha\delta^\delta_\beta + E\delta^\delta_\alpha\delta^\gamma_\beta & \\ C\epsilon_{ab}\epsilon^{\gamma\delta} & F\epsilon_{\alpha\beta}\epsilon^{cd} \\ G\delta^c_a\delta^\delta_\beta & H\delta^d_a\delta^\gamma_\beta \\ L\delta^\gamma_\alpha\delta^d_b & K\delta^\delta_\alpha\delta^c_b \end{cases}$$

and similar for the dotted one.

Worldsheet S-matrix: explicit perturbative evaluation

- Expansion of worldsheet S-matrix in coupling: defines the T-matrix

$$\mathbb{S} = \mathbb{1} + \frac{1}{\hat{g}}\mathbb{T}^{(0)} + \frac{1}{\hat{g}^2}\mathbb{T}^{(1)} + \ldots = \mathbb{1} + \mathbb{T} \qquad \hat{g} = \frac{\sqrt{\lambda}}{2\pi}$$

- **Tree level result**: first non trivial order in the perturbative expansion
  Obtained applying LSZ reduction to quartic vertices of the lagrangean.

[Klose McLoughlin Roiban Zarembo 06]

$$\mathbb{T}^{(0)}_{YY\to YY} = \frac{1}{2}\left[(1-2a)(\varepsilon' p - \varepsilon p') + \frac{(p-p')^2}{\varepsilon' p - \varepsilon p'}\right]\mathbb{1}\otimes\mathbb{1} + \frac{pp'}{\varepsilon' p - \varepsilon p'}(\mathbb{1}\otimes P + P\otimes\mathbb{1})$$

✓ Coincide with the related expansion of the exact spin chain S-matrix.

✓ A test of group factorization

- **One-loop result** via standard Feynman diagrammatics: not existing! unsuccessful attempts (non-cancellation of UV divergences).

[McLoughlin Roiban 07]

Worldsheet S-matrix at one loop via unitarity cuts: result

$$S_{AB}^{CD}(\mathrm{p}_1,\mathrm{p}_2)=\exp\left(i\varphi_a(\mathrm{p}_1,\mathrm{p}_2)\right)\ \tilde{S}_{AB}^{CD}$$
$$=\exp\Big(\underline{-\frac{i}{2\hat{g}}(e_2\mathrm{p}_1-e_1\mathrm{p}_2)(a-\tfrac{1}{2})}+\frac{i}{\hat{g}^2}\tilde{\varphi}(\mathrm{p}_1,\mathrm{p}_2)\Big)\ \tilde{S}_{AB}^{CD}+\mathcal{O}\Big(\frac{1}{\hat{g}^3}\Big)$$

where

Ex. $\quad A^{(1)}=1+\frac{i}{4\,\hat{g}}\frac{(\mathrm{p}_1-\mathrm{p}_2)^2}{\epsilon_2\mathrm{p}_1-\epsilon_1\mathrm{p}_2}+\frac{1}{4\,\hat{g}^2}\Big(\mathrm{p}_1\mathrm{p}_2-\frac{(\mathrm{p}_1+\mathrm{p}_2)^4}{8(\epsilon_2\mathrm{p}_1-\epsilon_1\mathrm{p}_2)^2}\Big)$

and

$$\tilde{\varphi}(\mathrm{p}_1,\mathrm{p}_2)=\frac{1}{2\pi}\frac{\mathrm{p}_1^2\mathrm{p}_2^2\big((\epsilon_2\mathrm{p}_1-\epsilon_1\mathrm{p}_2)-(\epsilon_1\epsilon_2-\mathrm{p}_1\mathrm{p}_2)\,\mathrm{arsinh}[\epsilon_2\mathrm{p}_1-\epsilon_1\mathrm{p}_2]\big)}{(\epsilon_2\mathrm{p}_1-\epsilon_1\mathrm{p}_2)^2}$$

- ✓ All logarithmic dependence encoded in the scalar factor (as required from integrability!)
- ✓ All gauge dependence encoded in the scalar factor (as required from physical arguments!)
- ✓ All rational dependence coincides with related expansion of EXACT worldsheet S-matrix

Other string backgrounds: $AdS_3 \times S^3 \times M^4$

- Three light-cone gauge-fixed string theories (Type IIB)

  - $AdS_3 \times S^3 \times T^4$ supported by pure RR flux
  - $AdS_3 \times S^3 \times S^3 \times S^1$ supported by pure RR flux
  - $AdS_3 \times S^3 \times T^4$ supported by a mix RR and NS NS fluxes

  relevant for the AdS_3/CFT_2 correspondence, interesting physics (e.g. BTZ black holes)

  Useful for connecting different working methods (CFT, integrability).

- Super-coset sigma models

$$\frac{PSU(1,1|2) \times PSU(1,1|2)}{SL(2) \times SU(2)} \times U(1)^4 \qquad \frac{D(2,1;\alpha) \times D(2,1;\alpha)}{SL(2) \times SU(2) \times SU(2)} \times U(1)$$

  with Z_4 automorphism -> classical integrability.

[Cagnazzo Zarembo 2011]
[Hoare Tseytlin 2012]

Other string backgrounds: $AdS_3 \times S^3 \times M^4$ II

New features

1. **Multiple masses** (also massless) in the asymptotic spectrum.

Theory	Spectrum	
$AdS_3 \times S^3 \times T^4$ (RR flux)	$(4+4) \times 1$	$(4+4) \times 0$
$AdS_3 \times S^3 \times S^3 \times S^1$ (RR flux)	$(2+2) \times 1$	$(2+2) \times \alpha$
	$(2+2) \times 1 - \alpha$	$(2+2) \times 0$
$AdS_3 \times S^3 \times T^4$ (mixed flux)	$(4+4) \times \sqrt{1-q^2}$	$(4+4) \times 0$

2. Expansion of the light-cone lagrangian contains **odd powers** in the field terms.

3. Dispersion relation is in terms of a function non trivially related to string tension:

$$\epsilon = \sqrt{m^2 + 4f^2(\lambda)\sin^2\frac{p}{2}}$$

Other string backgrounds: $AdS_3 \times S^3 \times M^4$ II

Exact S-matrices have been proposed, with conjectures for the phase**S**

$$S^{PQ}_{MN}(p,p') = \exp[i\omega_{\sigma_M \sigma_N}(p,p')]\hat{S}^{PQ}_{MN}(p,p')$$

[Borsato, Ohlsson Sax, Sfondrini 2012, 2013]

Under global U(1) symmetry groups excitations are particles (+) or antiparticles (-),
$++, \quad +-, \quad -+, \quad --.$

Non-trivial modification of the ``crossing equations'', which relate all of them.

Logarithmic contributions for these phases were evaluated with unitarity-cuts up to two loops in [Roiban Engelund 2013]

Enlarging unitarity 2d program: different masses

- Working with a tree-level **integrable** S-matrix (always verified) [Bianchi Hoare, 2014]

  > outgoing **two**-momenta = permutation of incoming **two**-momenta

  > scattering preserves these U(1) charges $\sigma_M = \sigma_P$

Ignore massless particles in the loop
if considers only external massive legs

Charge
conjugation
symmetry

focus on sectors $++, +-$

- Compact formula for the one-loop contribution (explicitating bubble integrals)

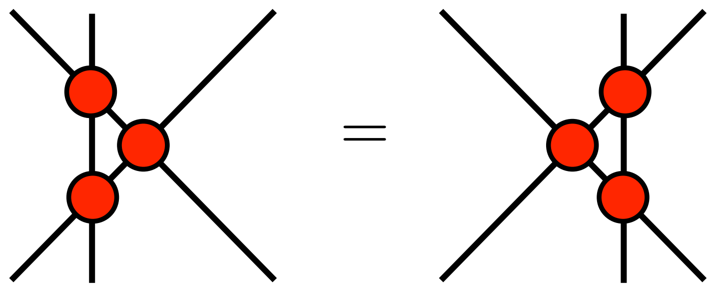
$$\theta \equiv \operatorname{arcsinh}\left(\frac{e'p - ep'}{mm'}\right) \qquad e = \sqrt{p^2+m^2}\,, \qquad e' = \sqrt{p'^2+m'^2}$$

$$T^{(1)} = \underbrace{\frac{\theta}{2\pi}\left(T^{(0)}\, u\, T^{(0)} - T^{(0)}\, s\, T^{(0)}\right)}_{\text{logarithmic terms}} + \underbrace{\frac{i}{2}T^{(0)}\, s\, T^{(0)}}_{\text{rational imaginary}} + \underbrace{\frac{1}{16\pi}\left(\frac{1}{m^2}\widetilde{T}^{(0)} \underset{\leftarrow}{t}\, T^{(0)} + \frac{1}{m'^2}T^{(0)} \underset{\rightarrow}{t}\, \widetilde{T}^{(0)}\right)}_{\text{rational real}}$$

Unitarity-cut result and the Yang-Baxter equation

Cubic matrix equation that is necessarily satisfied by integrable theories

$$\mathbb{S}_{12}\mathbb{S}_{13}\mathbb{S}_{23} = \mathbb{S}_{23}\mathbb{S}_{13}\mathbb{S}_{12} \ ,$$



- $[\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(0)}_{13}] + [\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(0)}_{23}] + [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(0)}_{23}] = 0$ Classical Yang-Baxter

- $$[\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(1)}_{13}] + [\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(1)}_{23}] + [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(1)}_{23}] - [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(1)}_{12}] - [\mathbb{T}^{(0)}_{23}, \mathbb{T}^{(1)}_{12}] - [\mathbb{T}^{(0)}_{23}, \mathbb{T}^{(1)}_{13}] = \mathbb{T}^{(0)}_{23}\mathbb{T}^{(0)}_{13}\mathbb{T}^{(0)}_{12} - \mathbb{T}^{(0)}_{12}\mathbb{T}^{(0)}_{13}\mathbb{T}^{(0)}_{23}$$

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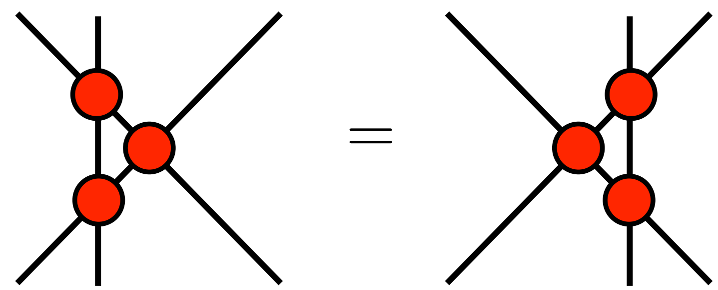

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$$\boxed{T^{(1)} = \underbrace{\frac{\theta}{2\pi}(T^{(0)} \textcircled{u}\, T^{(0)} - T^{(0)} \textcircled{s}\, T^{(0)})}_{\text{logarithmic terms}} + \underbrace{\frac{i}{2}T^{(0)} \textcircled{s}\, T^{(0)}}_{\text{rational imaginary}} + \underbrace{\frac{1}{16\pi}\left(\frac{1}{m^2}\widetilde{T}^{(0)} \underset{\leftarrow}{\textcircled{t}}\, T^{(0)} + \frac{1}{m'^2}T^{(0)} \underset{\rightarrow}{\textcircled{t}}\, \widetilde{T}^{(0)}\right)}_{\text{rational real}}}$$

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Cubic matrix equation that is necessarily satisfied by integrable theories

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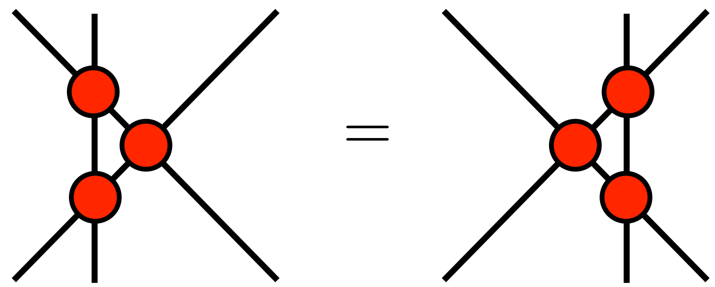

- $[\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(0)}_{13}] + [\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(0)}_{23}] + [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(0)}_{23}] = 0$ Classical Yang-Baxter
- $[\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(1)}_{13}] + [\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(1)}_{23}] + [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(1)}_{23}] - [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(1)}_{12}] - [\mathbb{T}^{(0)}_{23}, \mathbb{T}^{(1)}_{12}] - [\mathbb{T}^{(0)}_{23}, \mathbb{T}^{(1)}_{13}] = 0$

$$T^{(1)} = \underbrace{\frac{\theta}{2\pi}\left(T^{(0)} \textcircled{u}\, T^{(0)} - T^{(0)} \textcircled{s}\, T^{(0)}\right) +}_{\text{logarithmic terms}} \underbrace{\frac{1}{16\pi}\left(\frac{1}{m^2}\widetilde{T}^{(0)} \underset{\leftarrow}{\textcircled{t}}\, T^{(0)} + \frac{1}{m'^2} T^{(0)} \underset{\rightarrow}{\textcircled{t}}\, \widetilde{T}^{(0)}\right)}_{\text{rational real}}$$

Unitarity-cut result and the Yang-Baxter equation

Cubic matrix equation that is necessarily satisfied by integrable theories

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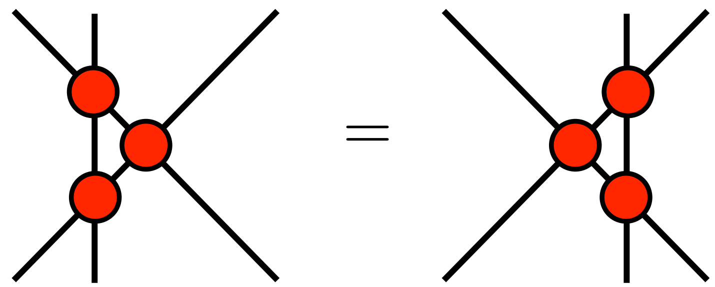

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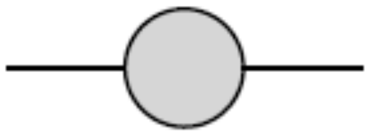
All satisfy this property, **except** $AdS_3 \times S^3 \times S^3 \times S^1$: rational remainder $\sim \left(\frac{p^2}{m} + \frac{p'^2}{m'}\right) T^{(0)}$

External legs correction

- Contributions from external leg corrections

residue of first pert. correction
of self-energy around the on-shell condition

$$\sqrt{Z}^4$$  $$= \sqrt{Z}^4 \left(1 + \frac{1}{\hat{g}} T^{(0)} + \frac{1}{\hat{g}^2} T^{(1)} + \ldots\right) \longrightarrow T^{(1)}_{ext} = (\Sigma_1^{(1)}(p) + \Sigma_1^{(1)}(p'))T^{(0)}$$

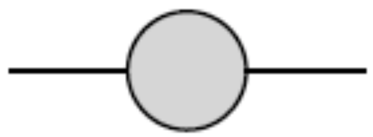
 $$= \frac{i}{\mathrm{p}^2 - m^2 - \Sigma(\mathrm{p})} = \frac{i}{\mathrm{p}^2 - m^2 - \hat{g}^{-1}\Sigma^{(1)}(\mathrm{p})} + \mathcal{O}(\hat{g}^{-2})$$

External legs correction

- Contributions from external leg corrections

$\sqrt{Z}^4$ 

$$= \sqrt{Z}^4\left(1+\frac{1}{\hat{g}}T^{(0)}+\frac{1}{\hat{g}^2}T^{(1)}+\dots\right) \longrightarrow T^{(1)}_{ext} = (\Sigma_1^{(1)}(p)+\Sigma_1^{(1)}(p'))T^{(0)}$$


$$= \frac{i(1+g^{-1}\Sigma_1^{(1)}(p))}{\mathrm{p}^2-m^2-g^{-1}\Sigma_0^{(1)}(p)}$$

External legs correction

- Contributions from external leg corrections

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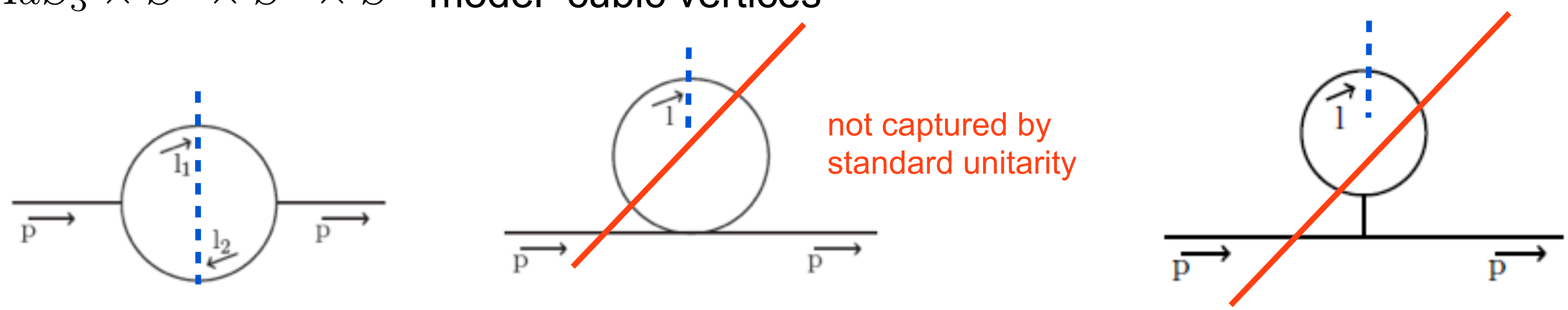
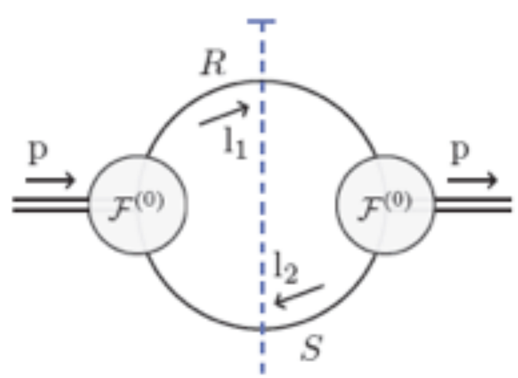
- In $AdS_3 \times S^3 \times S^3 \times S^1$ model cubic vertices



- Self-energy via ``fusing'' two tree-level form factors

[Brandhuber Travaglini Yang 2012]
[Engelund Roiban 2013]

$$\Sigma^{(1)}(\mathrm{p})|_{cut} =$$ 

$$= \int \frac{d^2 \mathrm{l}_1}{(2\pi)^2} i\pi\delta^+(\mathrm{l}_1{}^2 - m_1)\; i\pi\delta^+((\mathrm{l}_1 - p)^2 - m_2)\mathcal{F}^{(0)}_{RS}(\mathrm{p}, \mathrm{l}_1, \mathrm{l}_1 - \mathrm{p})\mathcal{F}^{(0)\dagger}_{RS}(\mathrm{p}, \mathrm{l}_1, \mathrm{l}_1 - \mathrm{p})$$

External legs correction

- Contributions from external leg corrections

$\sqrt{Z}^4$  $= \sqrt{Z}^4\left(1+\frac{1}{\hat{g}}T^{(0)}+\frac{1}{\hat{g}^2}T^{(1)}+\dots\right) \longrightarrow T^{(1)}_{ext} = (\Sigma_1^{(1)}(p)+\Sigma_1^{(1)}(p'))T^{(0)}$

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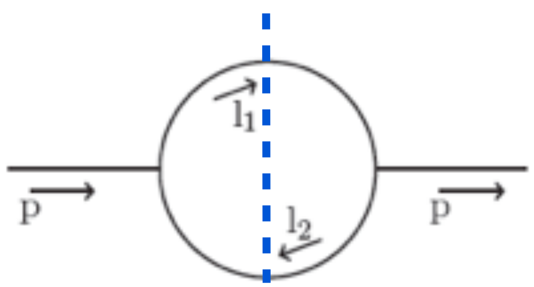
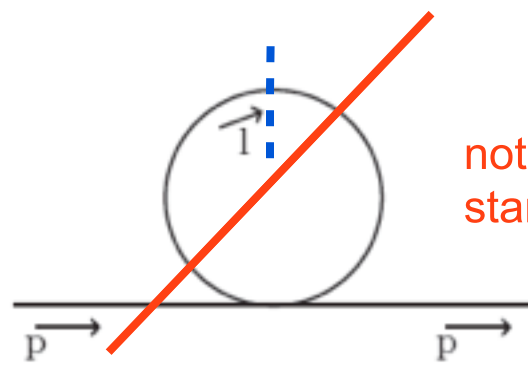
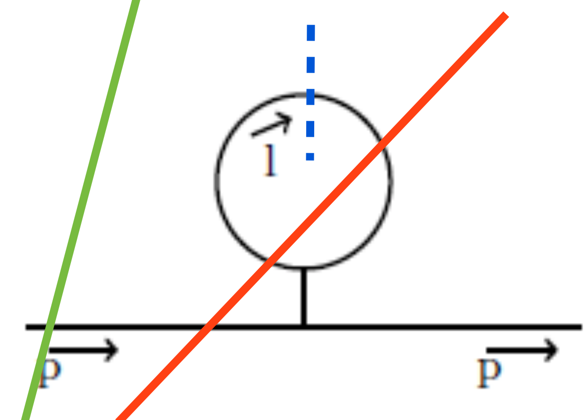
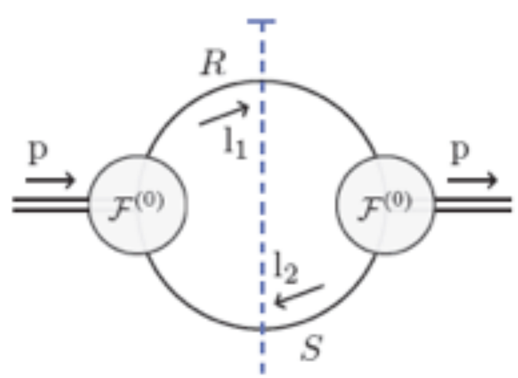






- Self-energy via ``fusing'' two tree-level form factors

$\Sigma^{(1)}(\mathrm{p})|_{cut} =$ 

$$= \frac{1}{2}\left|\mathcal{F}^{(0)}_{RS}(\mathrm{p},\mathrm{l}_*,\mathrm{l}'_*)\right|^2 I(\mathrm{p}^2,m_1,m_2) = \Sigma_0^{(1)}(p) + \Sigma_1^{(1)}(p)(\mathrm{p}^2-m^2) + \mathcal{O}((\mathrm{p}^2-m^2)^2)$$

... leads to an agreement with the exact result where "integrable coupling" coincides with sigma model coupling constant.

$$\epsilon = \sqrt{m^2 + 4f^2(\lambda)\sin^2\frac{p}{2}}$$

... leads to an agreement with the exact result where "integrable coupling" coincides with sigma model coupling constant.

$$\epsilon = \sqrt{m^2 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}$$

The unitarity construction here misses a shift in the coupling.

Summary of models analyzed

One-loop logarithmic and rational terms of 2⟶2 amplitudes reproduced:

- ✓ Bosonic relativistic models **up to shifts in the coupling**
- ✓ Supersymmetric relativistic models

- ✓ $AdS_5 \times S^5$
- ✓ $AdS_3 x S^3 x T^4$
- ✓ $AdS_3 x S^3 x S^3 x S^1$ **up to shifts in the coupling**

Shift in the coupling is not seen by integrability (indeed the result respects YB)!
As expected.

A cut-constructibility criterion?

Not a ``uniqueness result'' proving cut constructibility a' la [Bern, Dixon, Dunbar, Kosower 1994]

Power-counting criterion: the m-point integrals have at most m-2 powers of the loop momentum in the numerator of the integrand, and the two-point (bubble) integrals have at most one power of the loop momentum can be completely determined from its cuts.

for one-loop amplitudes in massless supersymmetric gauge theories.

> Here just scalar bubbles in the basis, and no m-point information yet.

> Here no 2d supersymmetry (GS string)

> Up to shift in the coupling: something missing is "expected",
> as we are in a massive theory in d=2 (issues tadpole- and regularization-related).

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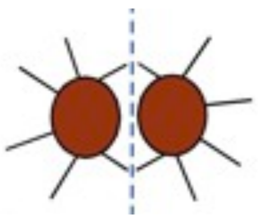
Clear interplay (including external legs corrections) between cut-constructibility and integrability.

Concluding remarks and a wish list

- Enough evidence that for large class of 2-d models (relativistic and not) four-points one-loop amplitudes are cut-constructible

  > Standard unitarity (2-particle cuts) reproduces **all rational terms**, **up to shifts in the coupling**.



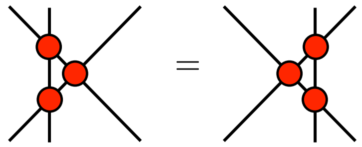
  > It **bypasses** issues of UV divergences emerging in Feynman diagrams

- Efficient way for

  > Checks of quantum integrability aspects (e.g. group factorization).

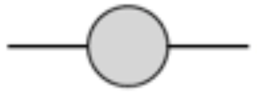
  > Proposing/checking matrix structure and overall phases for models relevant for the AdS/CFT correspondence.

- Cut-constructibility "criterion"



  > Integrability is crucial asset

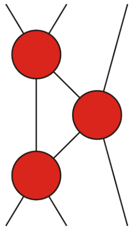
  > External leg corrections required to meet the exact result (up to shifts): **off-shell** continuation of the unitarity method to include form factors on either side of the cut



Wish list

★ Two loops rational terms (all logarithms reproduced in [Engelund McEwan Roiban 2013]

★ Higher points: factorization should emerge $$S_{3\to3} = \left(S_{2\to2}\right)^3$$



★ Extend to other interesting integrable string backgrounds (require a tree-level S-matrix! and massless modes treatment) [Basso 2010]

★ Extend to off-shell objects, including correlation functions.

[Klose McLoughlin 2012/2013]
[Engelund McEwan Roiban 2013]

EXTRA

Other string backgrounds I: “ABJM” worldsheet S-matrix

No direct check of our method (also not in [Roiban Engelung 2013]).

- Same S-matrix as in $AdS_5 \times S^5$ *despite* elementary excitations (BMN modes) differ.

[Gaiotto Giombi Yin 2008][Grignani, Harmark, Orselli 2008][Grignani, Harmark, Orselli Semenoff 2008]
[Astolfi, GM Puletti, Grignani, Harmark, Orselli 2009]

Modes are (4|4) *light* + ~~(4|4) *heavy*~~. → dissolve in the continuum [Zarembo 2009]
[Kalousios Vergu Volovich 2009]

Consistent with Bethe ansatz solution where heavy modes are interpreted as stacks of Bethe roots.

- A unitarity-cut check of recent proposal for $h(\lambda)$ in $\epsilon = \sqrt{1 + h(\lambda)\,\sin^2\frac{p}{2}}$

$$\lambda = \frac{\sinh(2\pi h)}{2\pi}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\sinh^2(2\pi h)\right)$$

$$h(\lambda) = \lambda - \frac{\pi^2\lambda^3}{3} + \frac{5\pi^4\lambda^5}{12} - \frac{893\pi^6\lambda^7}{1260} + \mathcal{O}(\lambda^9),$$

$$h(\lambda) = \sqrt{\frac{1}{2}\left(\lambda - \frac{1}{24}\right)} - \frac{\log 2}{2\pi} + \mathcal{O}\left(e^{-\pi\sqrt{8\lambda}}\right),$$

[Gromov Sizov 2014]

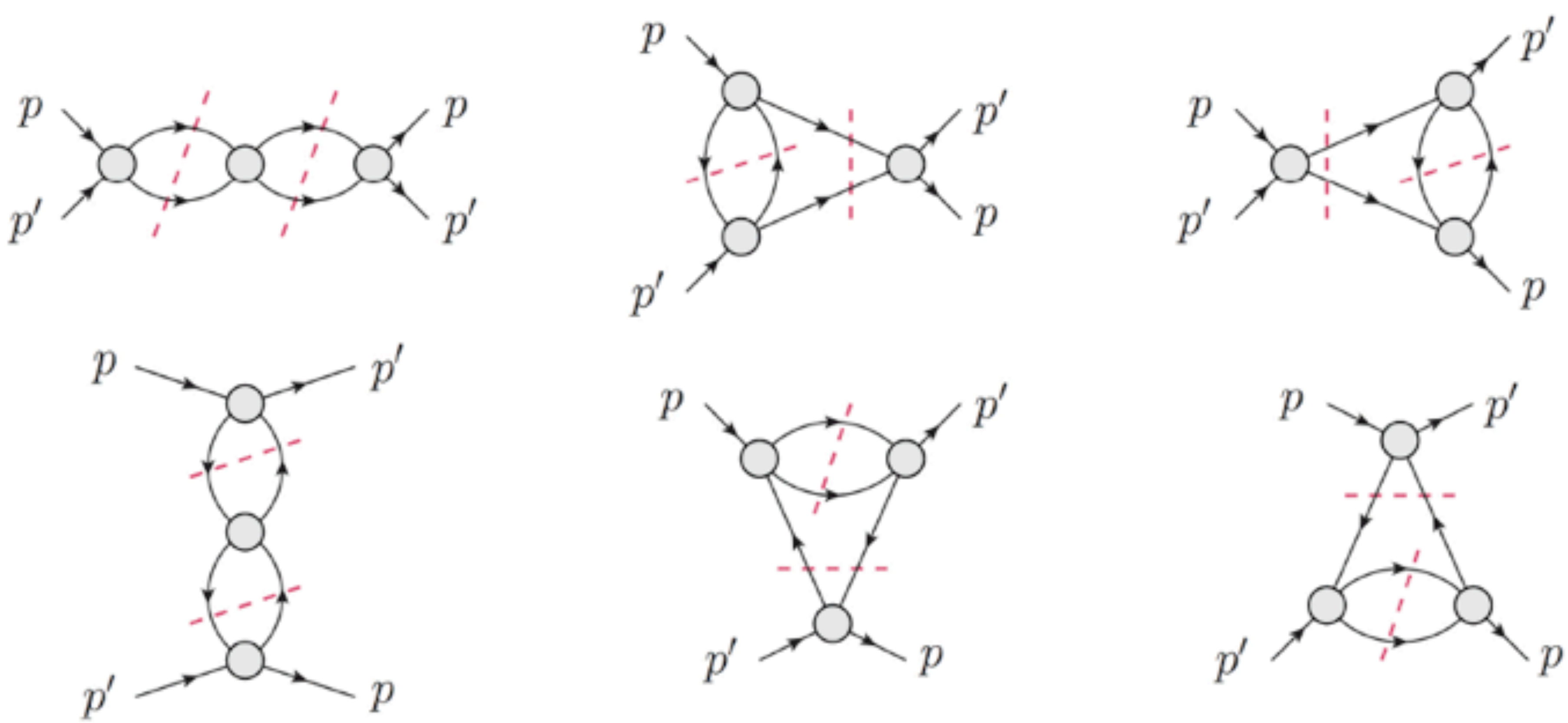
requires treatment of tadpoles.

Two loops I [Engelund McEwan Roiban 2013]

- Simply gluing T0 and T1 and uplifting seems to fail (you don't capture all contrs that at the end give you the expected result in terms of logs or double logs)

- [Engelund McEwan Roiban 2013] Since there is no integral basis ->strategy of maximal cuts
  [Bern Carrasco Johansson Kosower 2007]
  [Bern Carrasco Dixon Johansson Roiban 2008/10]

Begin with the generalized cuts imposing D L = 2 L = 4 cut conditions
proceed by releasing the on-shell condition for one propagator at a time (near-maximal cuts).

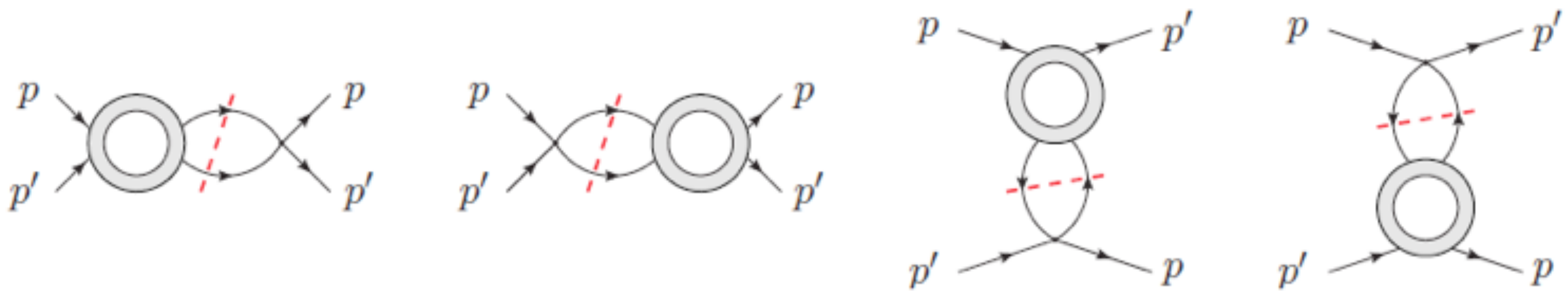


Two loops II [Engelund McEwan Roiban 2013]

$$
\begin{aligned}
i\mathrm{T}^{(2)} &= \frac{1}{4}C_a\tilde{I}_a + \frac{1}{2}C_b\tilde{I}_b + \frac{1}{2}C_c\tilde{I}_c + \frac{1}{4}C_d\tilde{I}_d + \frac{1}{2}C_e\tilde{I}_e + \frac{1}{2}C_f\tilde{I}_f \\
&+ \frac{1}{2}C_{s,\mathrm{extra}}\,\tilde{I}_s + \frac{1}{2}C_{u,\mathrm{extra}}\,\tilde{I}_u \\
&+ \text{rational}
\end{aligned}
$$

Get coefficients by comparing with maximal cuts

Get coefficients by comparing with near-maximal (here 2) cuts



Successfully reproduces expected logarithms behavior

$$
\mathcal{C}_{\ln^2} = \frac{1}{8\pi^2 J^2}\big(-2C_a + C_b + C_c - 2C_d + C_e + C_f\big) = 0
$$

$$
\mathcal{C}_{\ln^1} = \frac{i}{4\pi^2 J^2}(2C_a - C_b - C_c) \propto -\frac{1}{2}\mathrm{T}^{(0)}
$$

Successfully reproduces expected logarithms behavior

To be improved to catch all (+rational) terms.