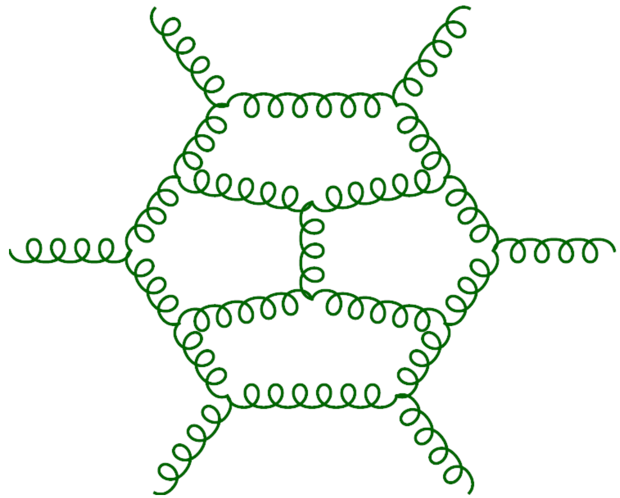
The Hexagon Function Bootstrap

in planar N=4 SYM

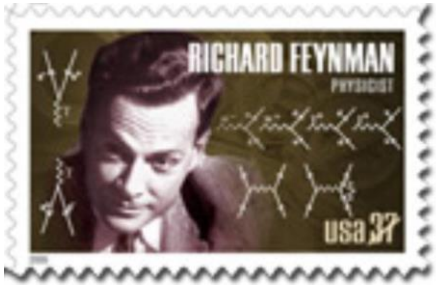


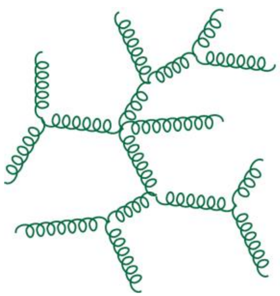
LD, J. Drummond, C. Duhr,
M. von Hippel, J. Pennington

1308.2276, 1402.3300, and to appear

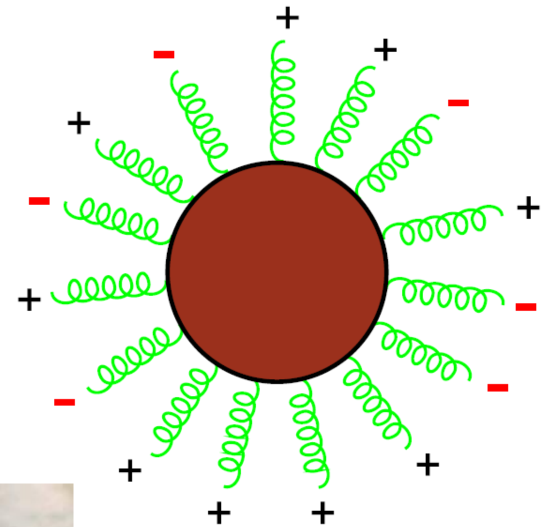
19th Itzykson Conference
Amplitudes 2014 – June 12
Saclay

Granularity vs. Fluidity





+ …

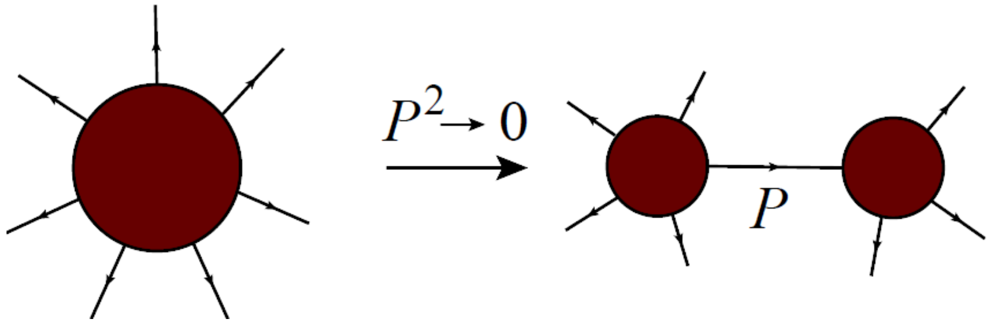


The Analytic *S*-Matrix

1960's: Before even Itzykson & Zuber.
No QCD, no Lagrangian or Feynman rules for strong interactions

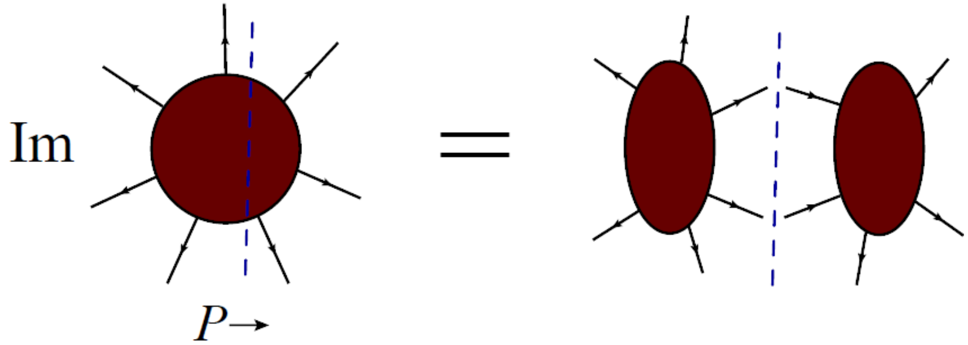
Bootstrap program: Reconstruct scattering amplitudes **directly** from **analytic properties: "on-shell" information**

Poles



Landau; Cutkosky; Chew, Mandelstam; Eden, Landshoff, Olive, Polkinghorne; Veneziano; Virasoro, Shapiro; … (1960s)

Branch cuts



Usually too hard

- Nonperturbative implementation leads to nonlinear integral equations
- Often not enough data to fix ambiguities
- Perturbative implementation is linear, recursive in loops and legs
- Most successful D=4 applications so far: construction of loop integrands (unitarity method, BCFW recursion, …)

  but see talk by Britto; Abreu, Britto, Duhr, Gardi, 1401.3546

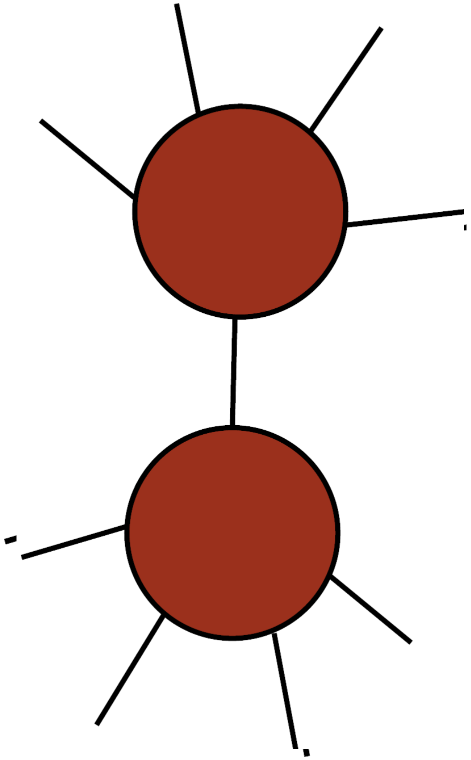
Tree-level fluidity

Amplitudes fall apart into simpler ones in special limits – pole information

Picture leads directly to BCFW (on-shell) recursion relations

Britto, Cachazo, Feng, Witten, hep-th/0501052

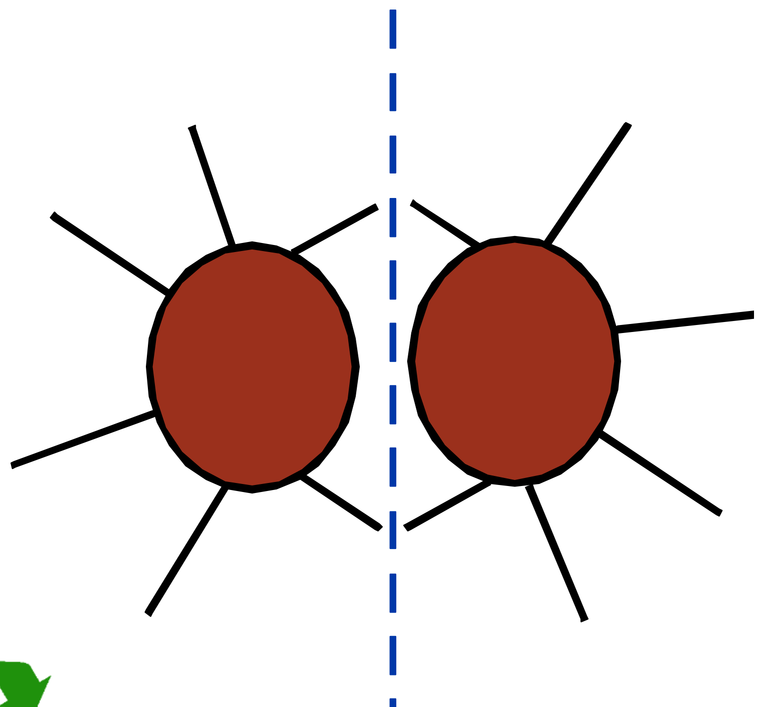
Trees recycled into trees

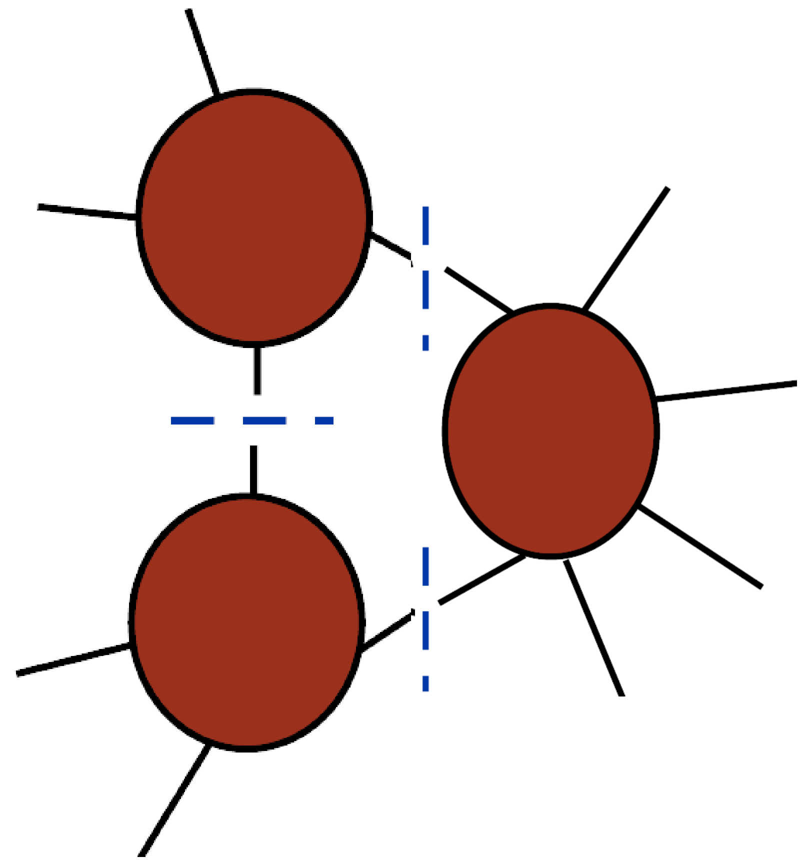


Fluidity of the one-loop integrand

Ordinary unitarity:
put 2 particles on shell

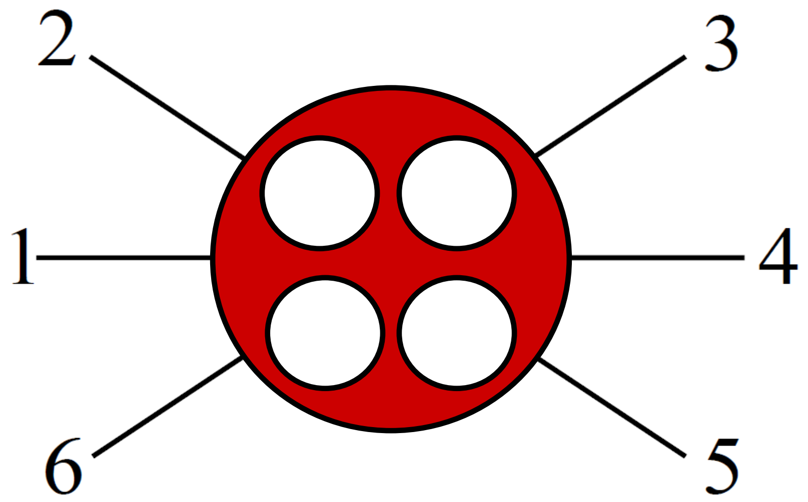
Generalized unitarity:
put 3 or 4 particles on shell





Trees recycled into loops!

Today we will use fluidity, i.e. factorization in kinematical limits, to bootstrap an integrated loop amplitude directly, without ever peeking inside the loops



Similar in spirit to "uplifting" approach for $R^{1,1}$ kinematics

Goddard, Heslop, Khoze, 1205.3448; Caron-Huot, He, 1305.2781

Loop amplitudes without loop integrals

1. Consider N=4 super-Yang-Mills theory in the planar (large N_c) limit. Amplitudes possess many special properties, making integrated bootstrap feasible.
2. Make ansatz for functional form of scattering amplitudes in terms of iterated integrals – hexagon functions
3. Use "boundary value data" to fix constants in ansatz. Linear constraints, leading to rational numbers.
4. Cross check.

- Works for 6-gluon amplitude, first "nontrivial" amplitude in planar N=4 SYM, through 4 loops for MHV = (--++++), 3 loops for NMHV = (---+++)

Advantage of bootstrapping the integrated amplitude

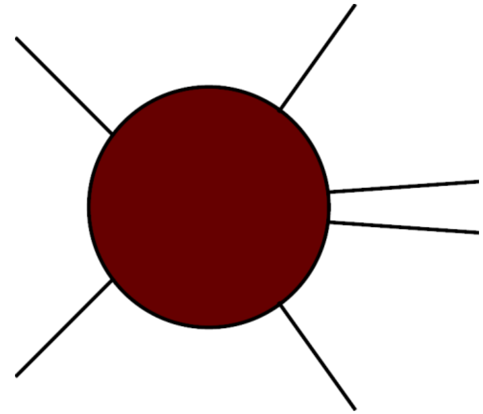
Bypass all difficulties of doing integrals and all subtleties of infrared regularization by working directly with IR finite quantities:

- MHV: Remainder function
- NMHV: Ratio to MHV

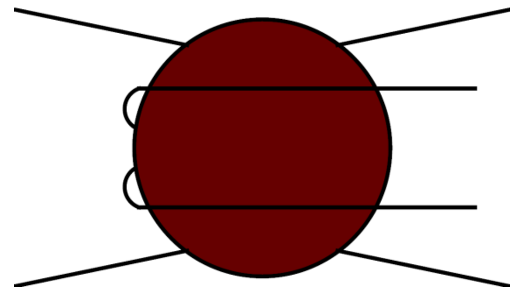
- Just need a good guess for the form of the answer, plus excellent boundary data [Basso, Sever, Vieira]

Three kinematical limits

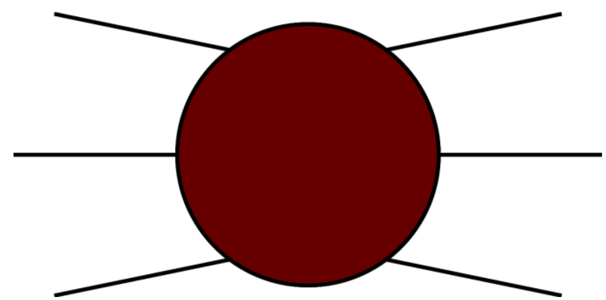
1. (Near) collinear limit



2. High-energy, multi-Regge limit



3. Multi-particle factorization limit
   (NMHV only)



Planar N=4 SYM Scattering Amplitudes

- Uniform transcendental weight: " $\ln^{2L} x$ " at L loops
- Exact exponentiation for *n*=4 or 5 gluons
- Dual (super)conformal invariance for any *n*
- Amplitudes equivalent to Wilson loops
- Strong coupling "soap bubbles" (minimal area)
- Integrability + OPE → exact, nonperturbative predictions for near-collinear limit
- Factorization of amplitude in high-energy, multi-Regge limit
- Finite radius of convergence for pert. theory

Use properties to solve for *n*=6 amplitudes

Exact exponentiation

Bern, LD, Smirnov, hep-th/0505205

BDS Ansatz inspired by IR structure of QCD, Mueller, Collins, Sen, Magnea, Sterman plus evidence collected at 2 and 3 loops for *n=4,5* using **generalized unitarity and collinear limits**:

$$\mathcal{A}_n^{\mathrm{BDS}} = \mathcal{A}_n^{\mathrm{tree}} \times \exp\Big[\sum_{l=1}^{\infty}\big[\frac{\lambda}{8\pi^2}\big]^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon; s_{ij}) + C^{(l)} + \mathcal{O}(\epsilon)\Big)\Big]$$

constants, indep.of kinematics

all kinematic dependence from 1-loop amplitude

n=4 $\Rightarrow$ $$\mathcal{M}_4|_{\mathrm{finite}} = \exp\Big[\frac{1}{8}\gamma_K(\lambda)\ \ln^2\Big(\frac{s}{t}\Big) + \mathrm{const.}\Big]$$

Confirmed at strong coupling using AdS/CFT, for *n=4*,*5*.
Fails for *n = 6,7,...*

Alday, Maldacena (2007)

Dual conformal constraints

Broadhurst (1993); Lipatov (1999); Drummond, Henn, Smirnov, Sokatchev, hep-th/0607160, …

- Amplitude fixed, up to functions of dual conformally invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

- Because $x_{i-1,i}^2 = k_i^2 = 0$ there are no such variables for n = 4,5 (after all loop integrations performed).

For $n = 6$, precisely 3 ratios:

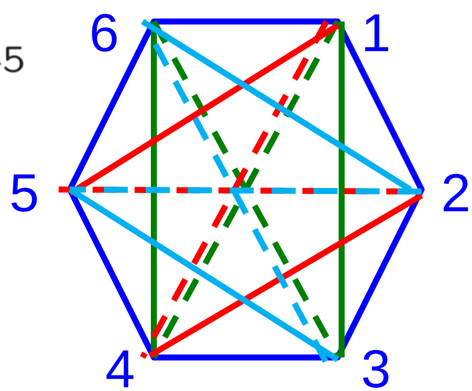
$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

+ 2 cyclic perm's

From 9 variables to just 3:

$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{123}, s_{234}, s_{345}$

$\rightarrow \boxed{u_1, u_2, u_3}$



Six-point remainder function R_6

• $n = 6$ first place BDS Ansatz must be modified, due to dual conformal cross ratios

$$u = u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad v = u_2 = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2} \qquad w = u_3 = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2}$$

$$\mathcal{A}_6^{\rm MHV}(\epsilon; s_{ij}) = \mathcal{A}_6^{\rm BDS}(\epsilon; s_{ij}) \exp[R_6(u_1, u_2, u_3)]$$

Known function, accounts for IR divergences, anomalies in dual conformal symmetry, and tree and 1-loop result

starts at 2 loops

Strong coupling and soap bubbles

Alday, Maldacena, 0705.0303

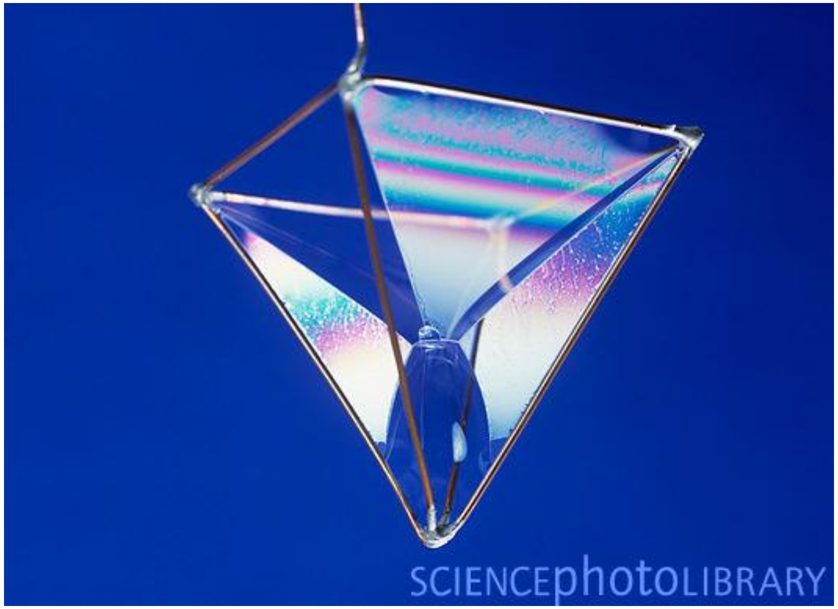
- Use AdS/CFT to compute scattering amplitude
- High energy scattering in string theory semi-classical: two-dimensional string world-sheet stretches long distance, classical solution minimizes area

Gross, Mende (1987,1988)

Classical action imaginary
→ exponentially suppressed tunnelling configuration

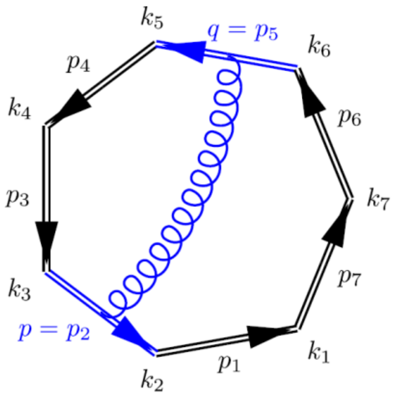
$$A_n \sim \exp[-\sqrt{\lambda} S_{\mathrm{cl}}^{\mathrm{E}}]$$

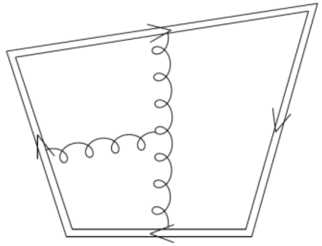
We'll see amazingly similar behavior for strong and weak coupling coefficients – for some kinematics



Wilson loops at weak coupling

Motivated by strong-coupling correspondence, Alday, Maldacena, 0705.0303
use same “soap bubble” boundary conditions as scattering amplitude:





- One loop, *n=4* — Drummond, Korchemsky, Sokatchev, 0707.0243
- One loop, any *n* — Brandhuber, Heslop, Travaglini, 0707.1153
- Two loops, *n=4,5,6* — Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466; Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

- Wilson-loop VEV **always matches** [MHV] scattering amplitude!
- Justifies dual conformal invariance for amplitude DHKS, 0712.1223

Two loop answer: $R_6^{(2)}(u_1,u_2,u_3)$

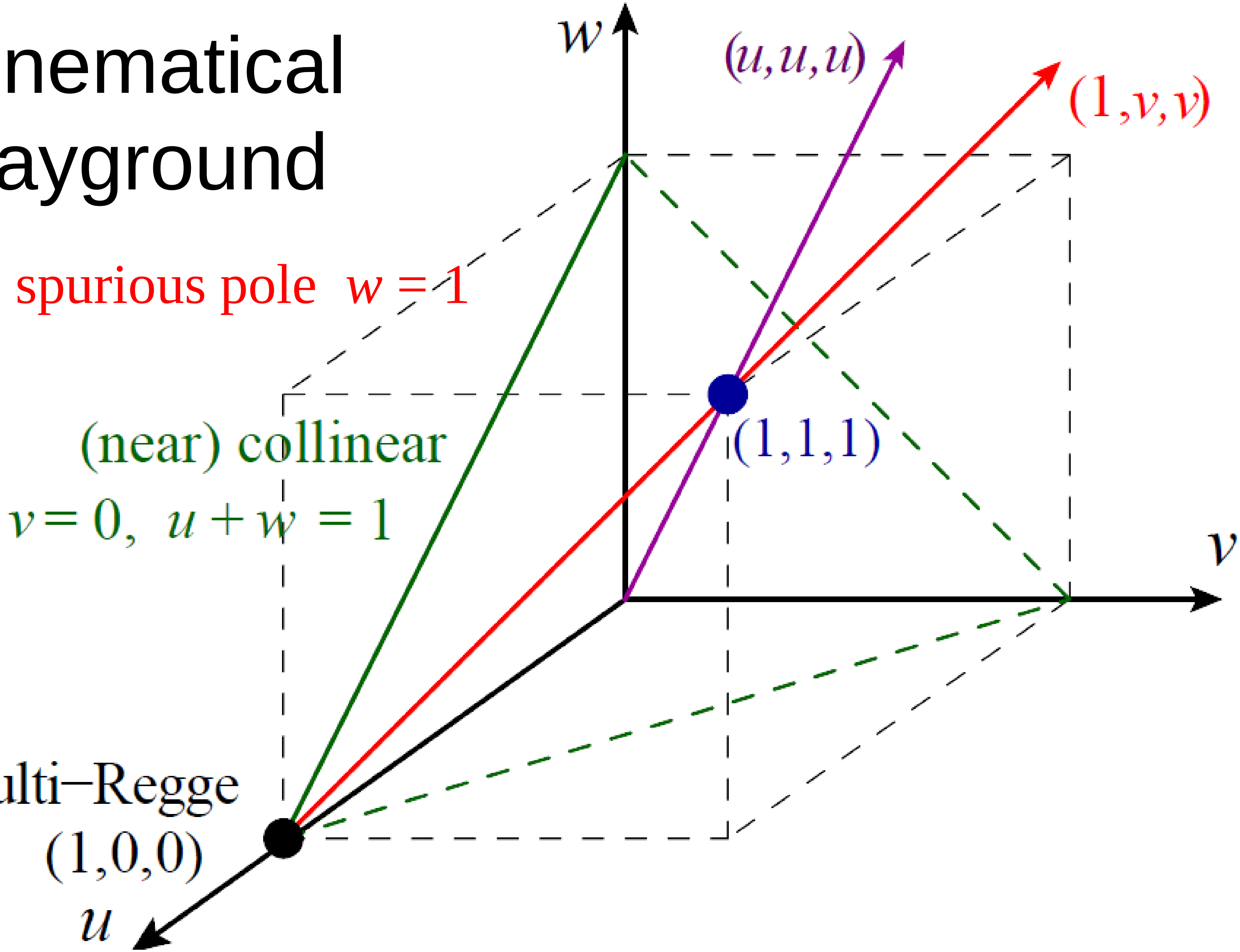
- Wilson loop integrals performed by
Del Duca, Duhr, Smirnov, 0911.5332, 1003.1702
17 pages of multiple polylogarithms $G(\ldots)$.

- Simplified to classical polylogarithms using symbology
Goncharov, Spradlin, Vergu, Volovich, 1006.5703

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

$$x_i^\pm = u_i x^\pm, \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3} \qquad \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$$

Kinematical playground

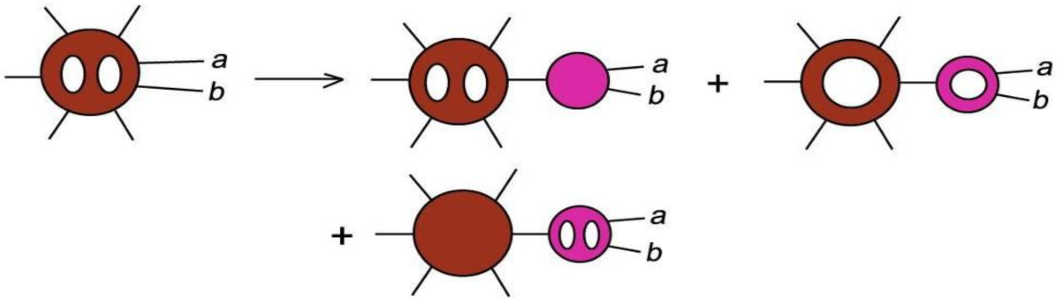


Wilson loop OPEs

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; GMSV, 1010.5009, 1102.0062

- $R_6^{(2)}(u_1,u_2,u_3)$ can be recovered directly from analytic properties, using “near collinear limit”,

$$v \to 0, \quad u + w \to 1$$



- Limit controlled by operator product expansion (OPE)
- **Possible to go to 3 loops, by combining OPE with symbol ansatz** LD, Drummond, Henn, 1108.4461

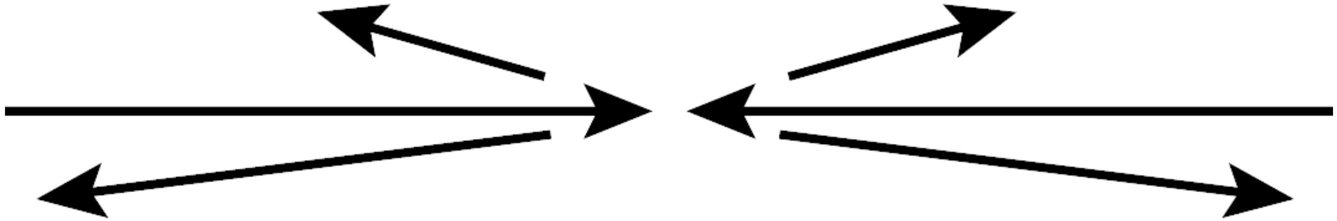
Now symbol → function $R_6^{(3)}(u,v,w)$
LD, Drummond, von Hippel, Pennington, 1308.2276

and 3 loop NMHV, LD, von Hippel, 1406.nnnn

and 4 loops, $R_6^{(4)}$
LD, Duhr, Drummond, Pennington, 1402.3300

Multi-Regge limit

• Minkowski kinematics, large rapidity separations between the 4 final-state gluons:



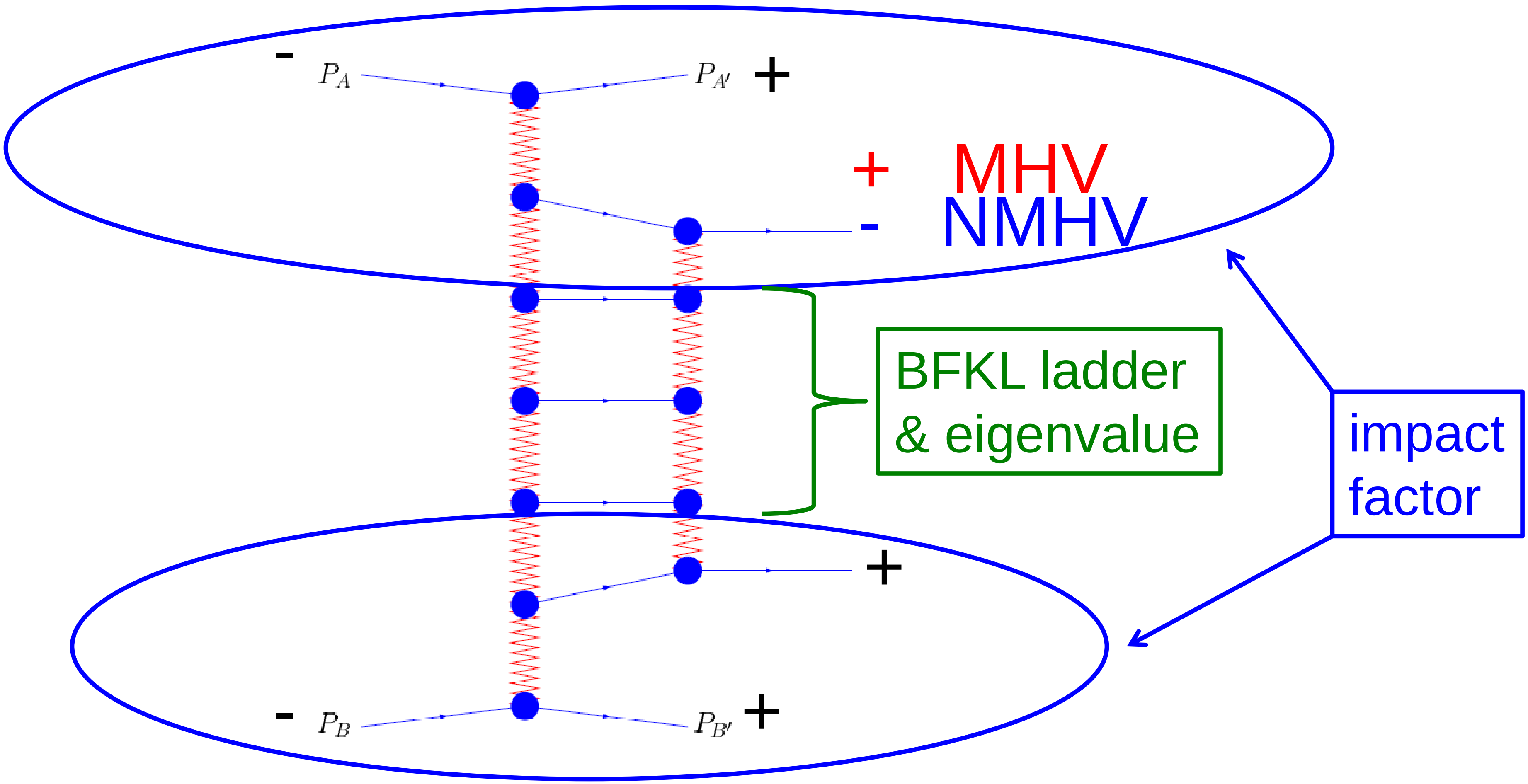
• Properties of planar N=4 SYM amplitude in this limit studied extensively at weak coupling:

Bartels, Lipatov, Sabio Vera, 0802.2065, 0807.0894; Lipatov, 1008.1015; Lipatov, Prygarin, 1008.1016, 1011.2673; Bartels, Lipatov, Prygarin, 1012.3178, 1104.4709; LD, Drummond, Henn, 1108.4461; Fadin, Lipatov, 1111.0782; LD, Duhr, Pennington, 1207.0186

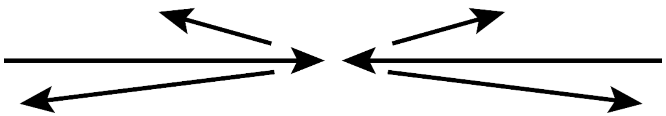
• Factorization and exponentiation in this limit provides additional source of "boundary data" for bootstrapping!

2 → 4 Multi-Regge picture

Bartels, Lipatov, Sabio Vera, 0802.2065



2→4 multi-Regge limit



- Euclidean MRK limit vanishes
- To get nonzero result for physical region, first let

$u_1 \to u_1\, e^{-2\pi i}$, then $u_1 \to 1,\quad u_2, u_3 \to 0$

$$\frac{u_2}{1-u_1} \to \frac{1}{(1+w)(1+w^*)} \qquad \frac{u_3}{1-u_1} \to \frac{w w^*}{(1+w)(1+w^*)}$$

different w, sorry!

$$R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u)\, [g_r^{(L)}(w,w^*) + 2\pi i\, h_r^{(L)}(w,w^*)]$$

$g_{L-1}^{(L)}$ (**LLA**) and $g_{L-2}^{(L)}$ (**NLLA**) well understood

Fadin, Lipatov, 1111.0782;
LD, Duhr, Pennington, 1207.0186;
Pennington, 1209.5357

Put **LLA, NLLA** results into bootstrap;
extract **N^{k}LLA** terms, $k > 1$

MRK Master formulae

- MHV:

$$e^{R+i\pi\delta}|_{\rm MRK} = \cos\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,|w|^{2i\nu}\,\boxed{\Phi_{\rm Reg}(\nu,n)}$$

$$\times\left(-\frac{1}{1-u}\,\frac{|1+w|^2}{|w|}\right)^{\boxed{\omega(\nu,n)}}$$

NLL: Fadin, Lipatov, 1111.0782; Caron-Huot, 1309.6521

- NMHV:

$$\exp(R^{\rm NMHV}+i\pi\delta)|_{\rm MRK} = \boxed{\mathcal{P}}\exp(R^{\rm MHV}+i\pi\delta)$$

$$= \cos\pi\omega_{ab} - i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\boxed{\frac{d\nu}{(i\nu+\frac{n}{2})^2}}|w|^{2i\nu}$$

$$\times\,\Phi_{\rm Reg}^{\boxed{\rm NMHV}}(\nu,n)\left(-\frac{1}{1-u}\,\frac{|1+w|^2}{|w|}\right)^{\omega(\nu,n)}$$

LL: Lipatov, Prygarin, Schnitzer, 1205.0186

Basic bootstrap assumption

- MHV: $R_6^{(L)}(u,v,w)$ is a linear combination of weight $2L$ hexagon functions at any loop order L
- NMHV: Super-amplitude ratio function

$$\mathcal{P}_{\rm NMHV} \equiv \frac{\mathcal{A}_{\rm NMHV}}{\mathcal{A}_{\rm MHV}}$$

Drummond, Henn, Korchemsky, Sokatchev, 0807.1095

(also IR finite) has expansion

$$\mathcal{P}_{\rm NMHV} = \frac{1}{2}\Big[[(1)+(4)]V(u,v,w) + [(2)+(5)]V(v,w,u) + [(3)+(6)]V(w,u,v)$$
$$+[(1)-(4)]\tilde{V}(u,v,w) - [(2)-(5)]\tilde{V}(v,w,u) + [(3)-(6)]\tilde{V}(w,u,v)\Big]$$

Grassmann-containing dual superconformal invariants

$V, \tilde{V}$ = hexagon functions

Functional interlude

Chen; Goncharov; Brown; talks by Vergu, Henn, Duhr, …

- Multiple polylogarithms, or n-fold iterated integrals, or weight n pure transcendental functions f.
- Define by derivatives: $$d\,f = \sum_{s_k \in \mathcal{S}} f^{s_k}\, d\ln s_k$$

$\mathcal{S}$ = finite set of rational expressions, "symbol letters", and

$$f^{s_k} \equiv \{n-1,1\} \text{ coproduct component}$$

Duhr, Gangl, Rhodes, 1110.0458

are also pure functions, weight n-1

- Iterate: $d\,f^{s_k} \;\Rightarrow\; f^{s_j\,s_k} \equiv \{n-2,1,1\} \text{ component}$
- Symbol = {1,1,…,1} component (maximally iterated)

Harmonic Polylogarithms of one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Subsector of hexagon functions
- Define by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \quad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- "Symbol letters": $\mathcal{S} = \{u, 1-u\}$

Hexagon function symbol letters

- Momentum twistors Z_i^A, $i=1,2,\ldots,6$ transform simply under dual conformal transformations. Hodges, 0905.1473
- Construct 4-brackets $\varepsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D \equiv \langle ijkl \rangle$
- 15 projectively invariant combinations of 4-brackets can be factored into 9 basic ones:

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$$

- y_i not independent of u_i

$$y_u \equiv \frac{u - z_+}{u - z_-}, \ldots \quad \text{where} \quad z_\pm = \frac{1}{2}\left[-1 + u + v + w \pm \sqrt{\Delta}\right]$$

$$\Delta = (1-u-v-w)^2 - 4uvw$$

y_i rationalize symbol:

$$u = \frac{y_u(1-y_v)(1-y_w)}{(1-y_u y_v)(1-y_u y_w)}$$

Branch cut condition

- All massless particles → all branch cuts start at origin in

$$s_{i,i+1},\ \ s_{i,i+1,i+2}$$

→ Branch cuts all start from 0 or ∞ in

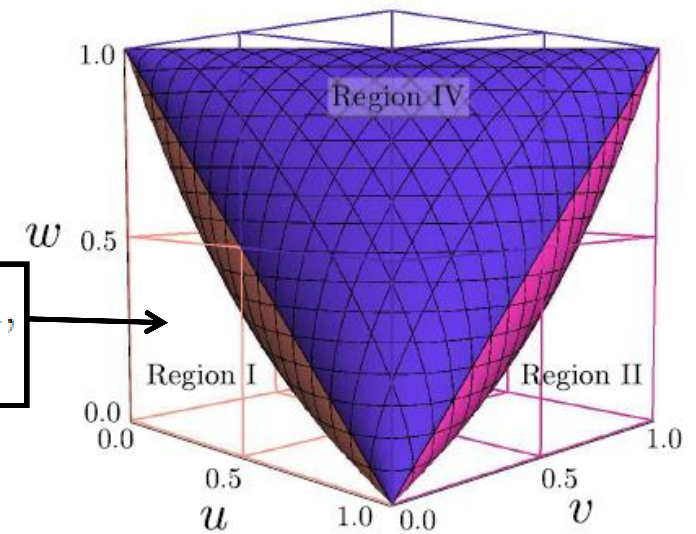
$$u = \frac{s_{12}^2 s_{45}^2}{s_{123}^2 s_{345}^2} \qquad \text{or } v \text{ or } w$$

→ First symbol entry $\in \{u, v, w\}$

GMSV, 1102.0062; talk by Britto

Hexagon functions are multiple polylogarithms in y_i

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)$$



$$\text{Region I}: \quad \begin{cases} \Delta > 0\,, \quad 0 < u_i < 1\,, \quad \text{and} \quad u+v+w < 1, \\ 0 < y_i < 1\,. \end{cases}$$

$$\mathcal{G} = \Big\{ G(\vec{w};y_u) | w_i \in \{0,1\} \Big\} \cup \Big\{ G(\vec{w};y_v) \Big| w_i \in \Big\{0,1,\frac{1}{y_u}\Big\} \Big\} \cup \Big\{ G(\vec{w};y_w) \Big| w_i \in \Big\{0,1,\frac{1}{y_u},\frac{1}{y_v},\frac{1}{y_u y_v}\Big\} \Big\}$$

- Useful for analytics and for numerics for $\Delta > 0$

  GiNaC implementation: Vollinga, Weinzierl, hep-th/0410259

“Coproduct” approach

{n-1,1} coproduct defines coupled linear first-order PDEs

$$\left.\frac{\partial F}{\partial u}\right|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}} F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}} F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}} F^{y_w}$$

$$\sqrt{\Delta}\, y_u \left.\frac{\partial F}{\partial y_u}\right|_{y_v,y_w} = (1-u)(1-v-w)F^u - u(1-v)F^v - u(1-w)F^w - u(1-v-w)F^{1-u} + uv\, F^{1-v} + uw\, F^{1-w} + \sqrt{\Delta}\, F^{y_u}\,.$$

- Integrate numerically.
- Or solve PDEs analytically in special limits, e.g.:

1. Near-collinear limit
2. Multi-regge limit

- Always stay in space of functions with good branch cuts.
- Don’t need $\Delta > 0$

A menagerie of functions

1. HPLs: One variable, symbol letters $\{u, 1-u\}$. Near-collinear limit, lines $(u,u,1)$, $(u,1,1)$
2. Cyclotomic Polylogarithms [Ablinger, Blumlein, Schneider, 1105.6063]: One variable, letters $\{y_u, 1+y_u, 1+y_u+y_u^2\}$. For line (u,u,u).
3. SVHPLs [F. Brown, 2004]: Two variables, letters $\{z, 1-z, \bar{z}, 1-\bar{z}\}$. First entry/single-valuedness constraint (real analytic function in z plane). Multi-Regge limit.
4. Full hexagon functions. Three variables, symbol letters $\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}$, branch-cut condition

Back to physics

- enumerate all hexagon functions with weight $\boldsymbol{2L}$
- write most general linear combination with unkown rational-number coefficients
- impose a series of physical constraints until all coefficients uniquely determined
  - sometimes do in two steps: first fix symbol, later the full function (fix $\zeta(k)$ ambiguities)

Simple constraints on R_6

- S_3 permutation **symmetry** in $\{u, v, w\}$

- Even under "**parity**":
every term must have an even number of y_i

$$i\sqrt{\Delta} \leftrightarrow -i\sqrt{\Delta}$$
$$z_+ \leftrightarrow z_-$$
$$y_i \leftrightarrow 1/y_i$$

- Vanishing in **collinear** limit

$$v \to 0 \qquad u + w \to 1$$

Constraint on final entry of symbol or {n-1,1} coproduct

- From super Wilson-loop approach

Caron-Huot, 1105.5606 , Caron-Huot, He, 1112.1060

for remainder function R_6 and for odd part of ratio function $\tilde{V}$, only 6 of 9 possible entries:

$$\left\{\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w\right\}$$

- For even part V, one more entry allowed:

$$\left\{\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, \boxed{\frac{uw}{v}}, y_u, y_v, y_w\right\}$$

OPE Constraints

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; GMSV, 1010.5009; 1102.0062
Basso, Sever, Vieira [BSV], 1303.1396; 1306.2058; 1402.3307

- $R_6^{(L)}(u,v,w)$ vanishes in the collinear limit,

$$v = 1/\cosh^2\tau \to 0 \qquad\qquad \tau \to \infty$$

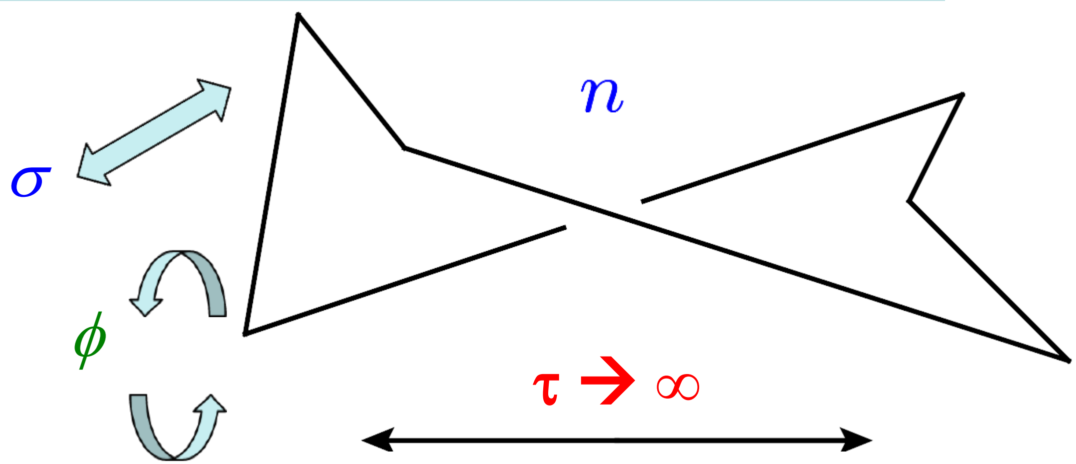
Its **near**-collinear limit is described by an OPE with generic form

$$R_6^{(L)}(u,v,w) = R_6^{(L)}(\tau,\sigma,\phi) \sim \int dn\ C_n(g)\ \exp[-E_n(g)\tau]$$

$$u = \frac{e^{\sigma}\sinh\tau\tanh\tau}{2(\cosh\sigma\cosh\tau + \cos\phi)}$$

$$v = \frac{1}{\cosh^2\tau}$$

$$w = u\, e^{-2\sigma}$$



OPE Constraints (cont.)

- Early OPE constraints fixed "leading discontinuity" terms:

$$\tau^{L-1} \sim [\ln T]^{L-1} \sim [\ln v]^{L-1} \quad \text{where} \quad T \sim \exp(-\tau)$$

- New results of BSV use power of integrability, give all powers of $\ln T$ for leading twist, one flux-tube excitation:

$$T\, e^{\pm i\phi}\ [\ln T]^{k} f_k(\sigma), \qquad k = 0,1,2,\ldots,L\text{-}1$$

and even subleading twist, two flux-tube excitations

$$T^{2}\, \{e^{\pm 2i\phi},1\}\ [\ln T]^{k} f_k(\sigma), \qquad k = 0,1,2,\ldots,L\text{-}1$$

- At **ANY** loop order!

Unknown parameters in $R_6^{(L)}$ symbol

Constraint	$L = 2$ Dim.	$L = 3$ Dim.	$L = 4$ Dim.
1. Integrability	75	643	5897
2. Total S_3 symmetry	20	151	1224
3. Parity invariance	18	120	874
4. Collinear vanishing (T^0)	4	59	622
5. OPE leading discontinuity	0	26	482
6. Final entry	0	2	113
7. Multi-Regge limit	0	2	80
8. Near-collinear OPE (T^1)	0	0	4
9. Near-collinear OPE (T^2)	0	0	0

only need $T^2 \times e^{\pm 2i\phi}$ terms;
$T^2 \times 1$ is pure cross check

Unknown parameters in $V^{(L)}$, $\tilde{V}^{(L)}$ functions

Constraint	One Loop	Two Loops	Three Loops
Symmetry in u and w	7	52	412
Cyclic vanishing of $\tilde{V}$	7	52	402
Final-entry condition	4	25	182
Spurious-pole vanishing	3	15	142
Collinear vanishing	1	8	92
$\mathcal{O}(T^1)$ Operator product expansion	0	0	2
$\mathcal{O}(T^2)$ OPE *or* Multi-Regge kinematics	0	0	0

$[\tilde{\Phi}_6]^2$ and $R_6^{(2)} \times V^{(1)}$

New information in MRK limit: NNLLA BFKL eigenvalue

$$E_{\nu,n} = \psi(\xi^+) + \psi(\xi^-) - 2\psi(1) - \frac{1}{2}\,\mathrm{sgn}(n) N$$

$$E^{(1)}_{\nu,n} = -\frac{1}{4}\Big[\psi^{(2)}(\xi^+) + \psi^{(2)}(\xi^-) - \mathrm{sgn}(n) N\Big(\frac{1}{4}N^2 + V^2\Big)\Big] + \frac{1}{2}V\Big[\psi^{(1)}(\xi^+) - \psi^{(1)}(\xi^-)\Big] - \zeta_2 E_{\nu,n} - 3\zeta_3$$

$$E^{(2)}_{\nu,n} = \frac{1}{8}\Big\{\frac{1}{6}\Big[\psi^{(4)}(\xi^+) + \psi^{(4)}(\xi^-) - 60\,\mathrm{sgn}(n) N\Big(V^4 + \frac{1}{2}V^2N^2 + \frac{1}{80}N^4\Big)\Big] - V\Big[\psi^{(3)}(\xi^+) - \psi^{(3)}(\xi^-) - 3\,\mathrm{sgn}(n) V N (4V^2 + N^2)\Big] + (V^2 + 2\zeta_2)\Big[\psi^{(2)}(\xi^+) + \psi^{(2)}(\xi^-) - \mathrm{sgn}(n) N\Big(3V^2 + \frac{1}{4}N^2\Big)\Big] - V(N^2 + 8\zeta_2)[\psi'(\xi^+) - \psi'(\xi^-) - \mathrm{sgn}(n) V N] + \zeta_3\,(4V^2 + N^2) + 44\,\zeta_4 E_{\nu,n} + 16\,\zeta_2\zeta_3 + 80\,\zeta_5\Big\},$$

$$\xi^{\pm} \equiv 1 \pm i\nu + \frac{|n|}{2}$$

$$V \equiv -\frac{1}{2}\left[\frac{1}{i\nu + \frac{|n|}{2}} - \frac{1}{-i\nu + \frac{|n|}{2}}\right] = \frac{i\nu}{\nu^2 + \frac{|n|^2}{4}} \qquad N \equiv \mathrm{sgn}(n)\left[\frac{1}{i\nu + \frac{|n|}{2}} + \frac{1}{-i\nu + \frac{|n|}{2}}\right] = \frac{n}{\nu^2 + \frac{|n|^2}{4}}$$

Closely related to flux-tube anomalous dimensions

Basso, 1010.5237

New information in MRK limit: NMHV impact factor

- NLL (from two-loop amplitude):

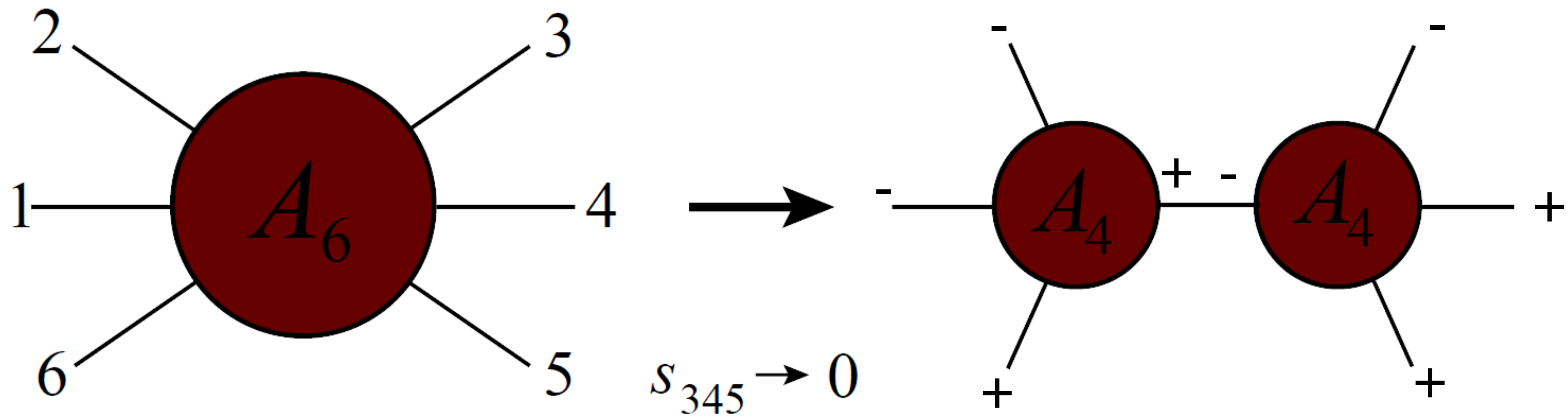
$$\Phi_{\rm Reg}^{\rm NMHV,(1)}(\nu,n) = \Phi_{\rm Reg}^{\rm MHV,(1)}(\nu,n) + \frac{in\nu}{2\left(-\frac{n}{2}+i\nu\right)^2\left(\frac{n}{2}+i\nu\right)^2}$$

- NNLL (from three-loop amplitude):

$$\begin{aligned}\Phi_{\rm Reg}^{\rm NMHV,(2)}(\nu,n) =& \Phi_{\rm Reg}^{\rm MHV,(2)}(\nu,n) \\ &+ \left(\Phi_{\rm Reg}^{\rm MHV,(1)}(\nu,n) + \zeta_2\right)\frac{in\nu}{2\left(-\frac{n}{2}+i\nu\right)^2\left(\frac{n}{2}+i\nu\right)^2} \\ &- \frac{in\nu(n^2 - in\nu - 2\nu^2)}{8(-\frac{n}{2}+i\nu)^4(\frac{n}{2}+i\nu)^4}\end{aligned}$$

- Very suggestive (Basso…)

NMHV Multi-Particle Factorization



$$A_6^{\rm NMHV}(k_i) \overset{s_{345}\to 0}{\longrightarrow} A_4(k_6,k_1,k_2,K)\,\frac{F_6(K^2,s_{i,i+1})}{K^2}\,A_4(-K,k_3,k_4,k_5)$$

Only interesting for NMHV: MHV tree has no pole $\quad \mathcal{A}_{\rm MHV}^{(0)} = i\,\frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \to \infty \qquad w = \frac{s_{61}s_{34}}{s_{345}s_{234}} \to \infty$$

$$u/w \text{ and } v = \frac{s_{23}s_{56}}{s_{234}s_{123}} \text{ fixed}$$

Multi-Particle Factorization (cont.)

$$(1) = (4) \to \infty, \text{ rest finite}$$

→ look at $V(u,v,w)$

- Actually much better to look at $U(u,v,w)$ defined by

$$U = \ln V + R_6 - 1/8\ \gamma_K [\ \mathrm{Li}_2(1-u) + \tfrac{1}{2} \ln^2 u + \mathit{cyclic}]$$

- Don't put MHV amplitude over NMHV tree pole.
- Logs always more instructive.
- Last term cancels part of BDS ansatz

Factorization limit of U

$$U^{(1)}(u,v,w) = -\frac{1}{4}\ln^2(uw/v) - \zeta_2$$

$$U^{(2)}(u,v,w)|_{u,w\to\infty} = \frac{3}{4}\zeta_2\ln^2(uw/v) - \frac{1}{2}\zeta_3\ln(uw/v) + \frac{71}{8}\zeta_4$$

$$U^{(3)}(u,v,w)|_{u,w\to\infty} = \frac{1}{3}\zeta_3\ln^3(uw/v) - \frac{75}{8}\zeta_4\ln^2(uw/v) + (7\zeta_5 + 8\zeta_2\zeta_3)\ln(uw/v) - \frac{721}{8}\zeta_6 - 3(\zeta_3)^2$$

$$\frac{uw}{v} = \frac{s_{12}s_{34}}{s_{56}}\cdot\frac{s_{45}s_{61}}{s_{23}}\cdot\frac{1}{s_{345}^2}$$

Simple polynomial in $\ln(uw/v)$!

Full NMHV factorization function in terms of U :

$$[\ln F_6]^{(L)} = \frac{\gamma_K^{(L)}}{8\epsilon^2L^2}\left(1 + 2\,\epsilon\,L\,\frac{\mathcal{G}_0^{(L)}}{\gamma_K^{(L)}}\right)\left[\left(\frac{(-s_{12})(-s_{34})}{(-s_{56})}\right)^{-L\epsilon} + \left(\frac{(-s_{45})(-s_{61})}{(-s_{23})}\right)^{-L\epsilon}\right] - \frac{\gamma_K^{(L)}}{8}\left[\frac{1}{2}\ln^2\left(\frac{(-s_{12})(-s_{34})}{(-s_{56})}\Big/\frac{(-s_{45})(-s_{61})}{(-s_{23})}\right) + 6\,\zeta_2\right] + U^{(L)}(u,v,w)|_{u,w\to\infty} + \frac{f_2^{(L)}}{L^2} + C^{(L)}\,.$$

Global simplicity of U

$$
\begin{aligned}
U^u + U^{1-u} &= U^w + U^{1-w} = -(U^v + U^{1-v}) \\
U^{1-v} &= 0 \\
U^{y_u} &= U^{y_w}
\end{aligned}
$$

- These {n-1,1} coproduct relations hold globally in (u,v,w) through 3 loops
- First relation was imposed (7th final entry allowed for V)
- Next two are quite surprising
- They imply that U has only 5 final entries: $\left\{ \frac{u}{1-u}, \frac{w}{1-w}, y_u y_w, y_v, \frac{uw}{v} \right\}$
- And that one derivative of U is very simple:

$$\sqrt{\Delta}\frac{\partial U}{\partial \ln(y_u/y_w)} = (1-v)(U^u - U^w)$$

Numerical results

- Plot perturbative coefficients on various lines and surfaces
- Instructive to take ratios of successive loop orders $R_6^{(L)}/R_6^{(L-1)} = \bar{R}_6^{(L)}$
  - Planar N=4 SYM has no instantons and no renormalons.
  - Its perturbative expansion has a finite radius of convergence, 1/8
  - For "asymptotically large orders", $R_6^{(L)}/R_6^{(L-1)}$ should approach -8

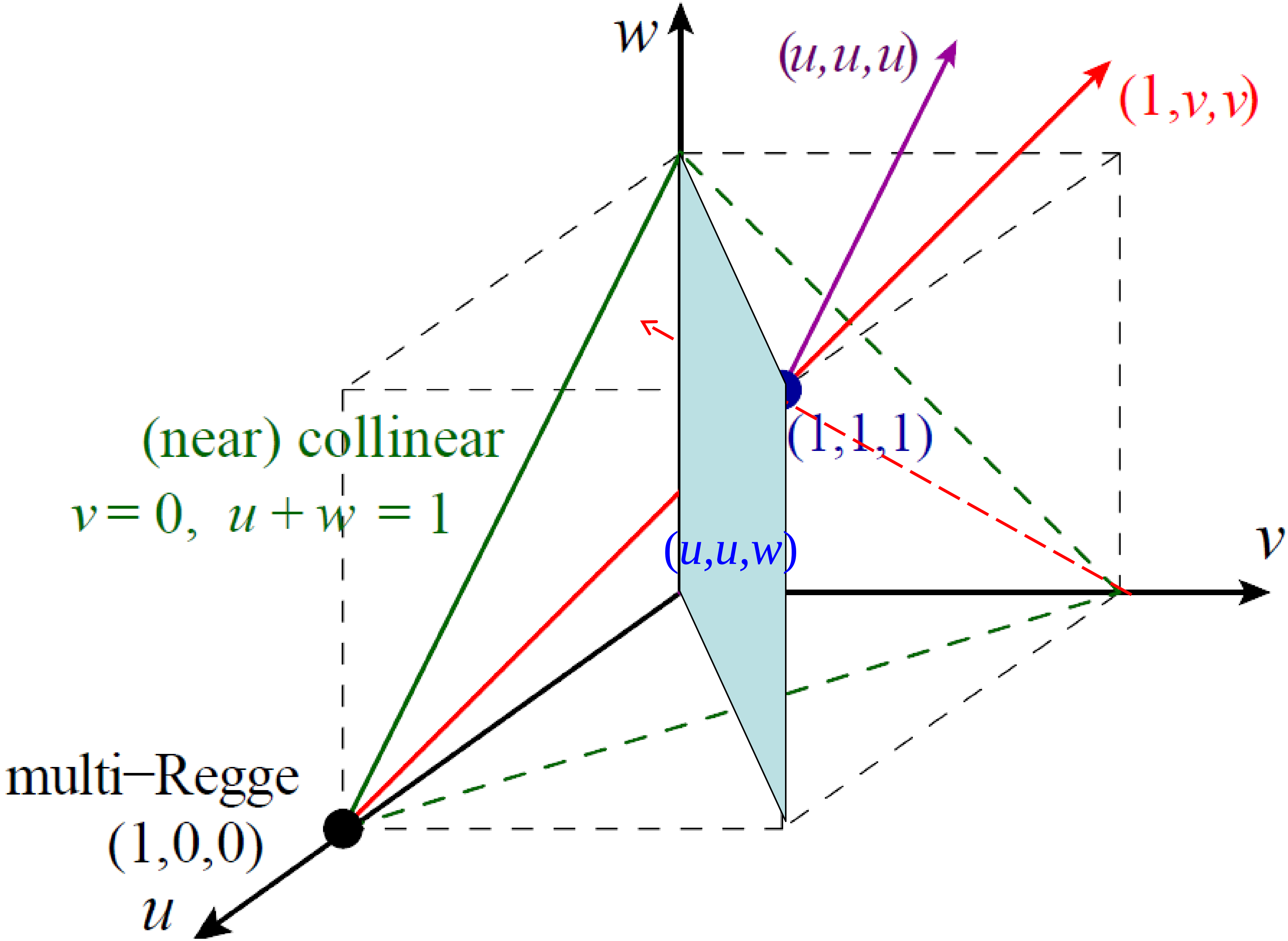
Cusp anomalous dimension $\gamma_K(\lambda)$

- Known to all orders, Beisert, Eden, Staudacher [hep-th/0610251]

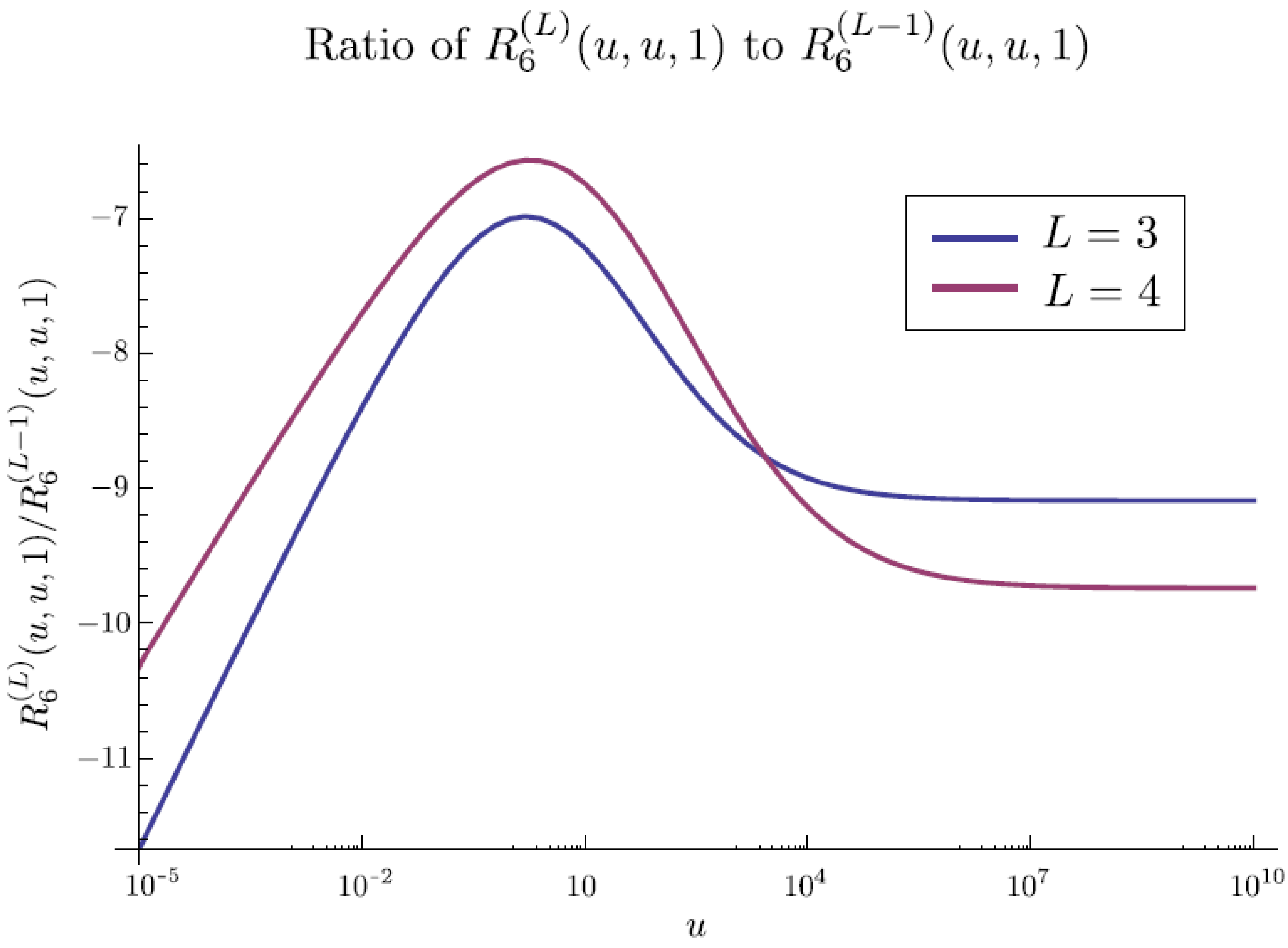
closely related to amplitude/Wilson loop, use as benchmark for approach to large orders:

L	$\gamma_K^{(L)}/\gamma_K^{(L-1)}$	$\bar{R}_6^{(L)}(1,1,1)$	$\overline{\ln \mathcal{W}}_{\text{hex}}^{(L)}(\frac{3}{4},\frac{3}{4},\frac{3}{4})$	$\overline{\ln \mathcal{W}}_{\text{hex}}^{(L)}(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
2	-1.6449340	∞	-2.7697175	-2.8015275
3	-3.6188549	-7.0040885	-5.0036164	-5.1380714
4	-4.9211827	-6.5880519	-5.8860842	-6.0359857
5	-5.6547494	–	–	–
6	-6.0801089	–	–	–
7	-6.3589220	–	–	–
8	-6.5608621	–	–	–

↓ -8

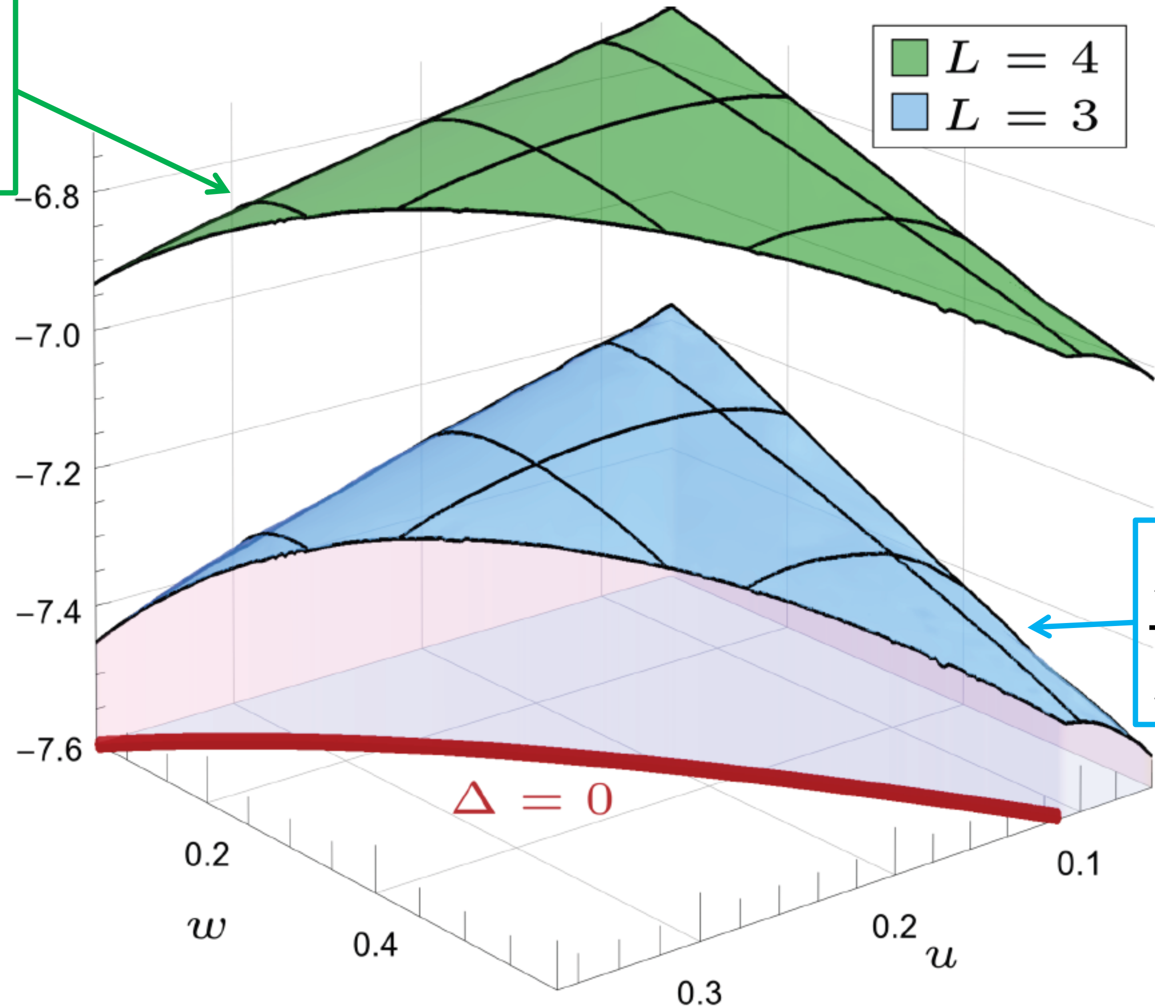

w
(u,u,u)
(1,v,v)
(1,1,1)
(near) collinear
v = 0, u + w = 1
(u,u,w)
v
multi−Regge
(1,0,0)
u

On $(u,u,1)$, everything collapses to HPLs of u



Ratio of successive loop orders extremely flat on (u,u,w)

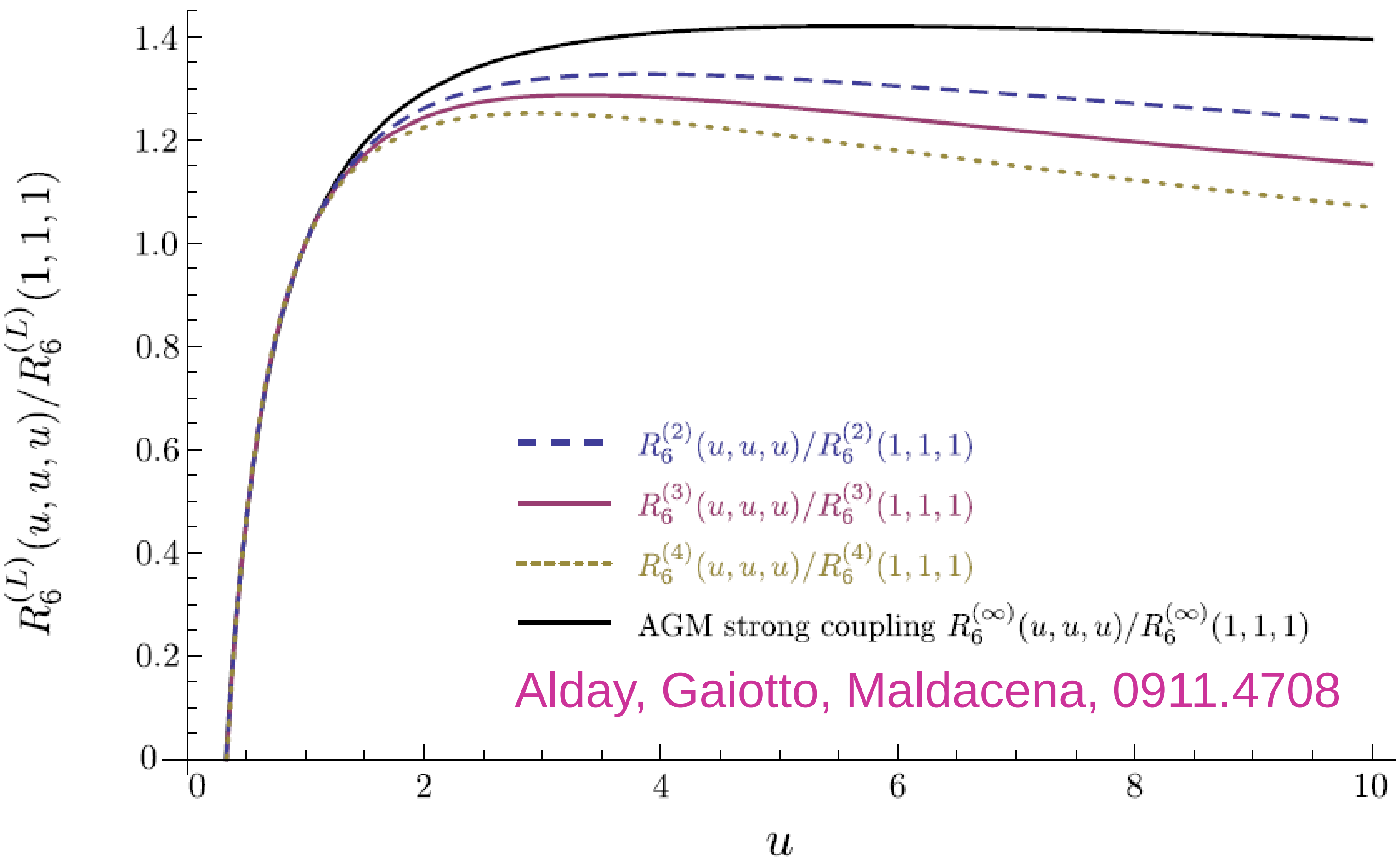
$$\frac{R_6^{(4)}(u,u,w)}{R_6^{(3)}(u,u,w)}$$



$$\frac{R_6^{(3)}(u,u,w)}{R_6^{(2)}(u,u,w)}$$

Uniform negative value in Region I consistent with conjecture of Arkani-Hamed, Trnka based on positive Grassmannian

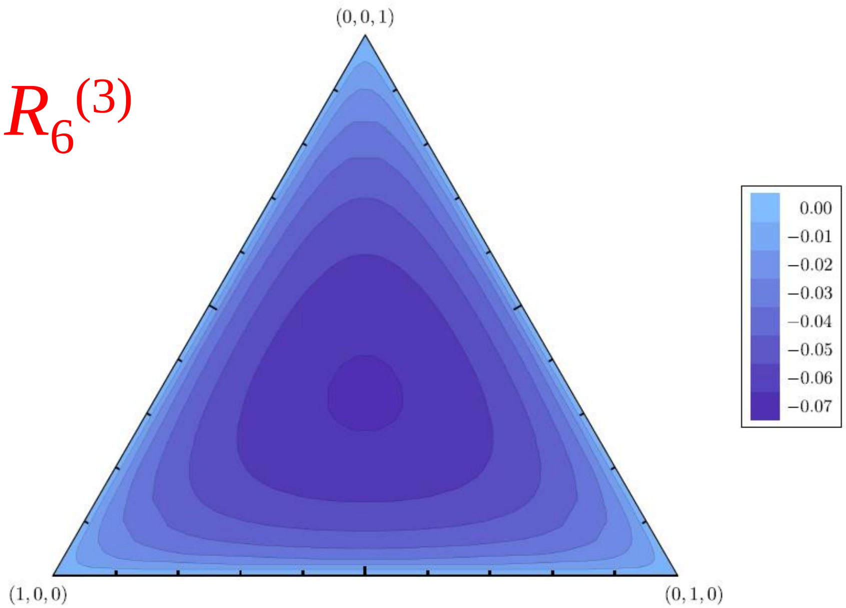
Rescaled $R_6^{(L)}(u,u,u)$ and strong coupling



(u,u,u) → cyclotomic polylogs (weak coupling)
$\arccos^2(1/4/u)$ (strong coupling)

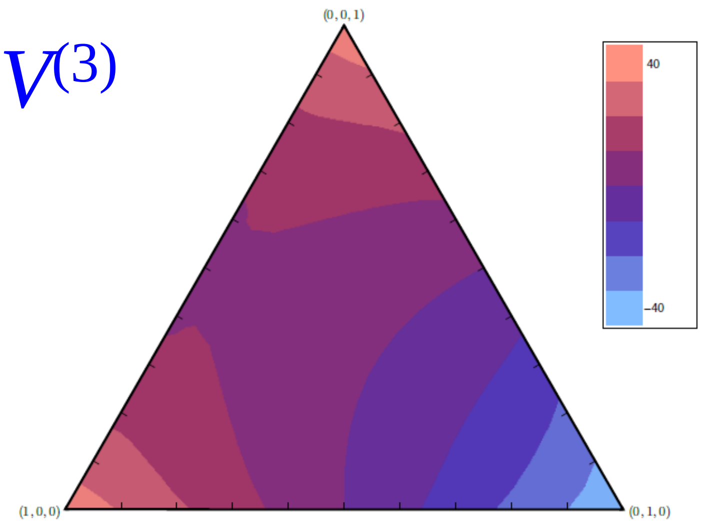
$R_6^{(3)}$, $V^{(3)}$, $\tilde{V}^{(3)}$ on $u + v + w = 1$

$R_6^{(3)}$

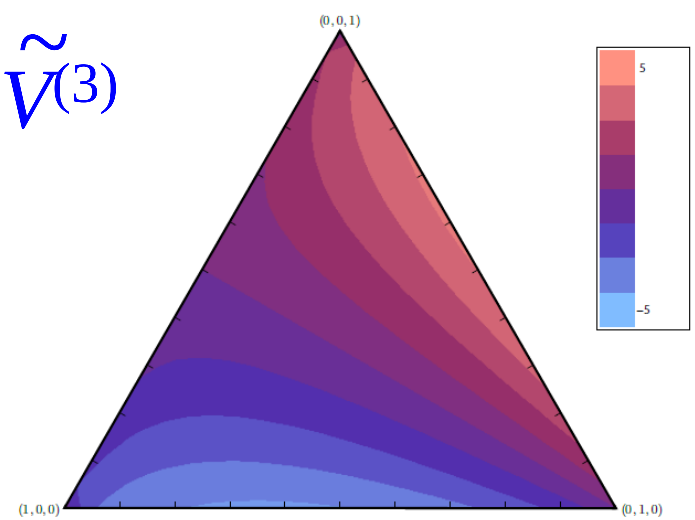


almost vanishes

$V^{(3)}$

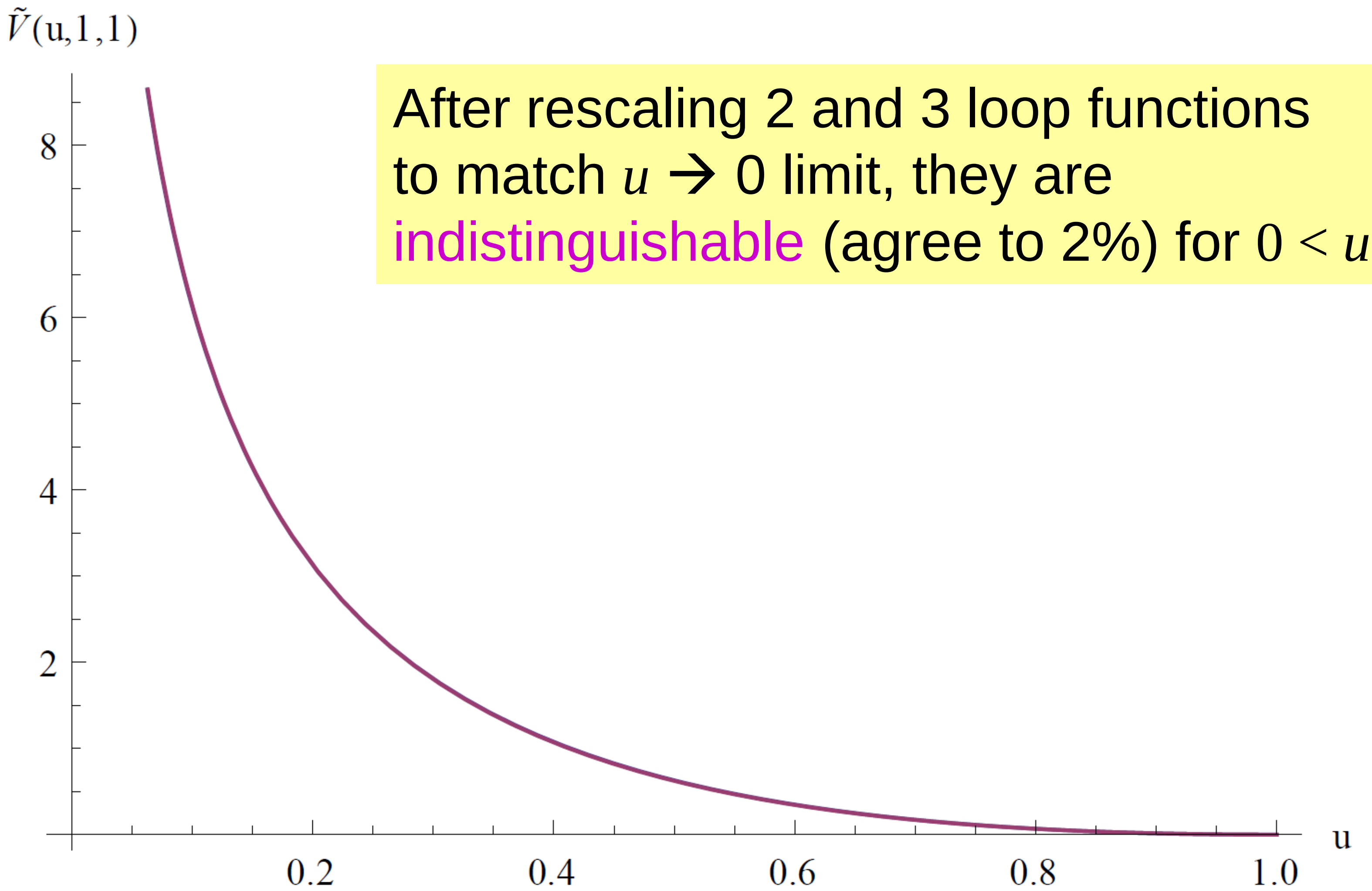


$\tilde{V}^{(3)}$



Ratio function odd part $\tilde{V}(u,1,1)$

After rescaling 2 and 3 loop functions to match $u \rightarrow 0$ limit, they are indistinguishable (agree to 2%) for $0 < u < 1$.



Recent progress in 7 point MHV too

$R_7^{(2)}$ just computed in terms of

$\mathrm{Li}_{2,2}(x,y)$, $\mathrm{Li}_4(x)$, $\mathrm{Li}_4(x)$, $\mathrm{Li}_4(x)$, $\ln(x)$

Golden, Spradlin, 1306.0833, 0406.2055

based a form for the

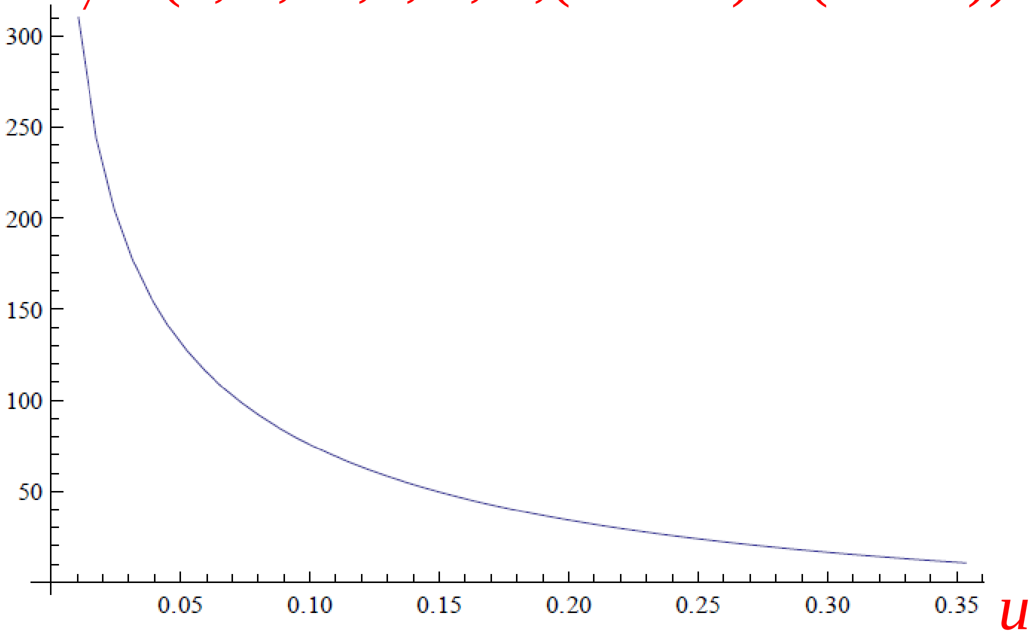
differential $dR_7^{(2)}$

Caron-Huot, 1105.5606

+ cluster coord's

Golden, et al. 1305.1617, 1306.1833

$R_7^{(2)}(u, u, u ,u, u, u,(1-u-u^2)^2/ (1-2u^2))$



Conclusions & Outlook

- Hexagon function ansatz → integrated planar N=4 SYM amplitudes over full kinematical phase space, for both MHV and NMHV for 6 gluons
- No need to know any integrands at all
- Important additional inputs from boundary data: near-collinear and/or multi-Regge limits
- Numerical and analytical results intriguing!
- Can one go to all orders?
- Extensions to other theories?

Extra Slides

T^1 OPE for NMHV: 1111 component

- Evaluate (i) prefactors →

$$\mathcal{P}^{(1111)}|_{T^1} = \frac{1}{2}\{V(u,v,w) + V(w,u,v) - \tilde{V}(u,v,w) + \tilde{V}(w,u,v)$$
$$+FT[\frac{1+S^4}{S(1+S^2)}V(v,w,u) - \frac{1-S^2}{S}V(u,v,w)]\}$$

$T = e^{-\tau}$
$S = e^{\sigma}$
$F = e^{i\phi}$

- BSV:

$$\mathcal{P}^{(1111)} = 1 + e^{i\phi-\tau}\int \frac{du}{2\pi}\mu(u)(h(u)-1)e^{ip(u)\sigma-\gamma(u)\tau} + e^{-i\phi-\tau}\int \frac{du}{2\pi}\mu(u)(\bar{h}(u)-1)e^{ip(u)\sigma-\gamma(u)\tau} + \dots$$

$$h(u) = \frac{x^+(u)x^-(u)}{g^2}, \qquad \bar{h}(u) = \frac{g^2}{x^+(u)x^-(u)} \qquad x^{\pm}(u) = x(u \pm \tfrac{i}{2}) \qquad x(u) = \tfrac{1}{2}(u + \sqrt{u^2-4g^2})$$

- Quantities μ, p, γ meromorphic in rapidity u
- Evaluate u integral as (truncated) residue sum

See also Papathanasiou, 1310.5735

NMHV MRK limit

Like g, h for R_6:
Extract p, q from V, $\tilde{V}$
→ linear combinations of SVHPLs [Brown, 2004]

$$R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \left[g_r^{(L)}(w,w^*) + 2\pi i \, h_r^{(L)}(w,w^*) \right]$$

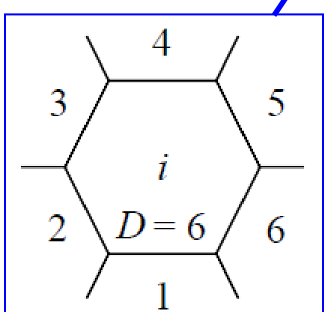
$$\mathcal{P}_{\text{MRK}}^{(L)} = (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \Big[\frac{1}{1+w^*} \big(p_r^{(L)}(w,w^*) + 2\pi i \, q_r^{(L)}(w,w^*) \big) + \frac{w^*}{1+w^*} \big(p_r^{(L)}(w,w^*) + 2\pi i \, q_r^{(L)}(w,w^*) \big) \Big|_{(w,w^*) \to (\frac{1}{w}, \frac{1}{w^*})} \Big] + \mathcal{O}(1-u)$$

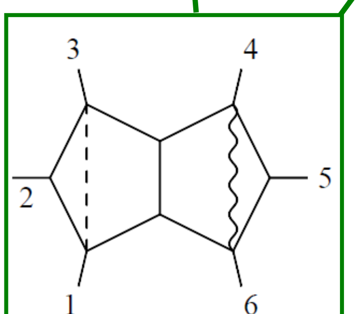
- Then match p, q to master formula for factorization in Fourier-Mellin space

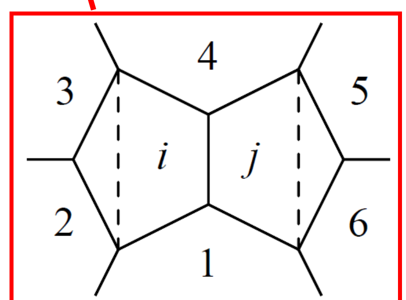
How many hexagon functions?

Irreducible (non-product) ones:

Weight	y^0	y^1	y^2	y^3	y^4
1	3 HPLs	-	-	-	-
2	3 HPLs	-	-	-	-
3	6 HPLs	$\tilde{\Phi}_6$	-	-	-
4	9 HPLs	$3 \times F_1$	$3 \times \Omega^{(2)}$	-	-
5	18 HPLs	$G,\ 3 \times K_1$	$5 \times M_1,\ N,\ O,\ 6 \times Q_{\rm ep}$	$3 \times H_1,\ 3 \times J_1$	-
6	27 HPLs	4	27	29	$3 \times R_{\rm ep} + 15$







$R_6^{(3)}(u,v,w)$ {5,1} coproduct

Many related

$$R_6^{(3),1-u} = -R_6^{(3),u}\,, \qquad R_6^{(3),1-v} = -R_6^{(3),v}\,, \qquad R_6^{(3),1-w} = -R_6^{(3),w}$$

$$R_6^{(3),v}(u,v,w) = R_6^{(3),u}(v,w,u)\,, \qquad R_6^{(3),w}(u,v,w) = R_6^{(3),u}(w,u,v)\,,$$

$$R_6^{(3),y_v}(u,v,w) = R_6^{(3),y_u}(v,w,u)\,, \qquad R_6^{(3),y_w}(u,v,w) = R_6^{(3),y_u}(w,u,v)\,.$$

→ Only 2 independent components to list, y_u and u

$$\begin{aligned} R_6^{(3),y_u} = \frac{1}{32}\Big\{ & -4\Big(H_1(u,v,w) + H_1(u,w,v)\Big) - 2\,H_1(v,u,w) \\ & + \frac{3}{2}\Big(J_1(u,v,w) + J_1(v,w,u) + J_1(w,u,v)\Big) \\ & - 4\Big[H_2^u + H_2^v + H_2^w + \frac{1}{2}\Big(\ln^2 u + \ln^2 v + \ln^2 w\Big) - 9\,\zeta_2\Big]\,\tilde{\Phi}_6(u,v,w)\Big\} \end{aligned}$$

$R_6^{(3)}(u,v,w)$ {5,1} coproduct (cont.)

$$R_6^{(3),u} = \frac{1}{32}\Big[A(u,v,w) + A(u,w,v)\Big]$$

$$A = M_1(u,v,w) - M_1(w,u,v) + \frac{32}{3}\Big(Q_{\rm ep}(v,w,u) - Q_{\rm ep}(v,u,w)\Big)$$
$$+ (4\ln u - \ln v + \ln w)\,\Omega^{(2)}(u,v,w) + (\ln u + \ln v)\,\Omega^{(2)}(v,w,u)$$
$$+ 24H_5^u - 14H_{4,1}^u + \frac{5}{2}H_{3,2}^u + 42H_{3,1,1}^u + \frac{13}{2}H_{2,2,1}^u - 36H_{2,1,1,1}^u + H_2^u\Big[-5H_3^u + \frac{1}{2}H_{2,1}^u + 7\zeta_3\Big]$$

+ 12 more lines of HPLs

Multiple zeta values at $(u,v,w) = (1,1,1)$

$$R_6^{(2)}(1,1,1) = -(\zeta_2)^2 = -\frac{5}{2}\zeta_4$$

$$R_6^{(3)}(1,1,1) = \frac{413}{24}\zeta_6 + (\zeta_3)^2$$

$$R_6^{(4)}(1,1,1) = -\frac{3}{2}\zeta_2(\zeta_3)^2 - \frac{5}{2}\zeta_3\zeta_5 - \frac{471}{4}\zeta_8 + \frac{3}{2}\zeta_{5,3}$$

First irreducible MZV

On the line $(u,u,1)$, everything collapses to HPLs of u.
In a linear representation, and a very compressed notation,

$$H_1^u H_{2,1}^u = H_1^u H_{0,1,1}^u = 3H_{0,1,1,1}^u + H_{1,0,1,1}^u \to 3h_7^{[4]} + h_{11}^{[4]}$$

The 2 and 3 loop answers are:

$$\begin{aligned}
R_6^{(2)}(u,u,1) &= h_1^{[4]} - h_3^{[4]} + h_9^{[4]} - h_{11}^{[4]} - \frac{5}{2}\zeta_4\,, \\
R_6^{(3)}(u,u,1) &= -3h_1^{[6]} + 5h_3^{[6]} + \frac{3}{2}h_5^{[6]} - \frac{9}{2}h_7^{[6]} - \frac{1}{2}h_9^{[6]} - \frac{3}{2}h_{11}^{[6]} - h_{13}^{[6]} - \frac{3}{2}h_{17}^{[6]} \\
&\quad + \frac{3}{2}h_{19}^{[6]} - \frac{1}{2}h_{21}^{[6]} - \frac{3}{2}h_{23}^{[6]} - 3h_{33}^{[6]} + 5h_{35}^{[6]} + \frac{3}{2}h_{37}^{[6]} - \frac{9}{2}h_{39}^{[6]} \\
&\quad - \frac{1}{2}h_{41}^{[6]} - \frac{3}{2}h_{43}^{[6]} - h_{45}^{[6]} - \frac{3}{2}h_{49}^{[6]} + \frac{3}{2}h_{51}^{[6]} - \frac{1}{2}h_{53}^{[6]} - \frac{3}{2}h_{55}^{[6]} \\
&\quad + \zeta_2\left[-h_1^{[4]} + 3h_3^{[4]} + 2h_5^{[4]} - h_9^{[4]} + 3h_{11}^{[4]} + 2h_{13}^{[4]}\right] \\
&\quad + \zeta_4\left[-2h_1^{[2]} - 2h_3^{[2]}\right] + \zeta_3^2 + \frac{413}{24}\zeta_6\,,
\end{aligned}$$

And the 4 loop answer is:

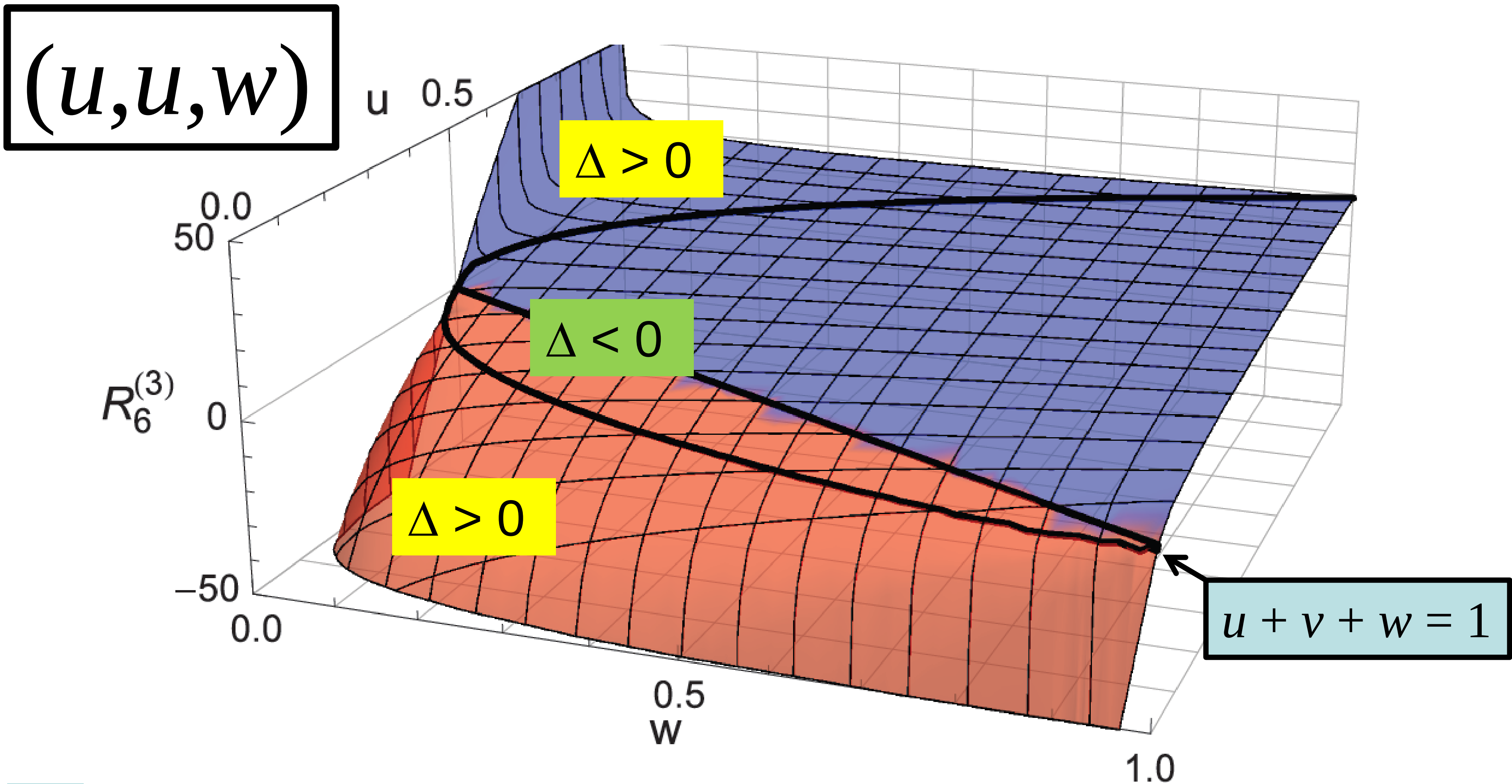
$$\begin{aligned}
R_6^{(4)}(u,u,1) = {} & 15h_1^{[8]} - 41h_3^{[8]} - \frac{31}{2}h_5^{[8]} + \frac{105}{2}h_7^{[8]} - \frac{7}{2}h_9^{[8]} + \frac{53}{2}h_{11}^{[8]} + 12h_{13}^{[8]} - 42h_{15}^{[8]} \\
& + \frac{5}{2}h_{17}^{[8]} + \frac{11}{2}h_{19}^{[8]} + \frac{9}{2}h_{21}^{[8]} - \frac{41}{2}h_{23}^{[8]} + h_{25}^{[8]} - 13h_{27}^{[8]} - 7h_{29}^{[8]} - 5h_{31}^{[8]} \\
& + 6h_{33}^{[8]} - 11h_{35}^{[8]} - 3h_{37}^{[8]} + 3h_{39}^{[8]} - 4h_{43}^{[8]} - 4h_{45}^{[8]} - 11h_{47}^{[8]} + \frac{3}{2}h_{49}^{[8]} - \frac{3}{2}h_{51}^{[8]} \\
& - 3h_{53}^{[8]} - 5h_{55}^{[8]} + \frac{3}{2}h_{57}^{[8]} - \frac{3}{2}h_{59}^{[8]} + 9h_{65}^{[8]} - 25h_{67}^{[8]} - 9h_{69}^{[8]} + 27h_{71}^{[8]} - 2h_{73}^{[8]} \\
& + 9h_{75}^{[8]} + 2h_{77}^{[8]} - 23h_{79}^{[8]} + 2h_{81}^{[8]} - h_{85}^{[8]} - 8h_{87}^{[8]} + 2h_{89}^{[8]} - 3h_{91}^{[8]} + \frac{5}{2}h_{97}^{[8]} \\
& - \frac{7}{2}h_{99}^{[8]} - \frac{1}{2}h_{101}^{[8]} + \frac{5}{2}h_{103}^{[8]} + \frac{1}{2}h_{105}^{[8]} + \frac{1}{2}h_{107}^{[8]} + \frac{1}{2}h_{109}^{[8]} - \frac{5}{2}h_{111}^{[8]} + 15h_{129}^{[8]} \\
& - 41h_{131}^{[8]} - \frac{31}{2}h_{133}^{[8]} + \frac{105}{2}h_{135}^{[8]} - \frac{7}{2}h_{137}^{[8]} + \frac{53}{2}h_{139}^{[8]} + 12h_{141}^{[8]} - 42h_{143}^{[8]} \\
& + \frac{5}{2}h_{145}^{[8]} + \frac{11}{2}h_{147}^{[8]} + \frac{9}{2}h_{149}^{[8]} - \frac{41}{2}h_{151}^{[8]} + h_{153}^{[8]} - 13h_{155}^{[8]} - 7h_{157}^{[8]} \\
& - 5h_{159}^{[8]} + 6h_{161}^{[8]} - 11h_{163}^{[8]} - 3h_{165}^{[8]} + 3h_{167}^{[8]} - 4h_{171}^{[8]} - 4h_{173}^{[8]} \\
& - 11h_{175}^{[8]} + \frac{3}{2}h_{177}^{[8]} - \frac{3}{2}h_{179}^{[8]} - 3h_{181}^{[8]} - 5h_{183}^{[8]} + \frac{3}{2}h_{185}^{[8]} - \frac{3}{2}h_{187}^{[8]} \\
& + 9h_{193}^{[8]} - 25h_{195}^{[8]} - 9h_{197}^{[8]} + 27h_{199}^{[8]} - 2h_{201}^{[8]} + 9h_{203}^{[8]} + 2h_{205}^{[8]} - 23h_{207}^{[8]} \\
& + 2h_{209}^{[8]} - h_{213}^{[8]} - 8h_{215}^{[8]} + 2h_{217}^{[8]} - 3h_{219}^{[8]} + \frac{5}{2}h_{225}^{[8]} - \frac{7}{2}h_{227}^{[8]} - \frac{1}{2}h_{229}^{[8]} \\
& + \frac{5}{2}h_{231}^{[8]} + \frac{1}{2}h_{233}^{[8]} + \frac{1}{2}h_{235}^{[8]} + \frac{1}{2}h_{237}^{[8]} - \frac{5}{2}h_{239}^{[8]} \\
& + \zeta_2\Big[2h_1^{[6]} - 14h_3^{[6]} - \frac{15}{2}h_5^{[6]} + \frac{37}{2}h_7^{[6]} - \frac{5}{2}h_9^{[6]} + \frac{25}{2}h_{11}^{[6]} + 7h_{13}^{[6]} - \frac{1}{2}h_{17}^{[6]} \\
& \quad + \frac{5}{2}h_{19}^{[6]} + \frac{7}{2}h_{21}^{[6]} + \frac{9}{2}h_{23}^{[6]} - 3h_{25}^{[6]} + 3h_{27}^{[6]} + 2h_{33}^{[6]} - 14h_{35}^{[6]} - \frac{15}{2}h_{37}^{[6]} \\
& \quad + \frac{37}{2}h_{39}^{[6]} - \frac{5}{2}h_{41}^{[6]} + \frac{25}{2}h_{43}^{[6]} + 7h_{45}^{[6]} - \frac{1}{2}h_{49}^{[6]} + \frac{5}{2}h_{51}^{[6]} + \frac{7}{2}h_{53}^{[6]} \\
& \quad + \frac{9}{2}h_{55}^{[6]} - 3h_{57}^{[6]} + 3h_{59}^{[6]}\Big] \\
& + \zeta_4\left[\frac{15}{2}h_1^{[4]} - \frac{55}{2}h_3^{[4]} - \frac{41}{2}h_5^{[4]} + \frac{15}{2}h_9^{[4]} - \frac{55}{2}h_{11}^{[4]} - \frac{41}{2}h_{13}^{[4]}\right] \\
& + \left(\zeta_2\zeta_3 - \frac{5}{2}\zeta_5\right)\left[h_3^{[3]} + h_7^{[3]}\right] - \left(\zeta_3^2 - \frac{73}{4}\zeta_6\right)\left[h_1^{[2]} + h_3^{[2]}\right] \\
& - \frac{3}{2}\zeta_2\zeta_3^2 - \frac{5}{2}\zeta_3\zeta_5 - \frac{471}{4}\zeta_8 + \frac{3}{2}\zeta_{5,3}\,.
\end{aligned}$$

"beyond-the-symbol" parameters for $R_6^{(4)}$

k	MZVs of weight k	Functions of weight $8-k$	Total parameters
2	ζ_2	38	38
3	ζ_3	14	14
4	ζ_4	6	6
5	$\zeta_2\zeta_3,\ \zeta_5$	2	4
6	$\zeta_3^2,\ \zeta_6$	1	2
7	$\zeta_2\zeta_5,\ \zeta_3\zeta_4,\ \zeta_7$	0	0
8	$\zeta_2\zeta_3^2,\ \zeta_3\zeta_5,\ \zeta_8,\ \zeta_{5,3}$	1	4
			68

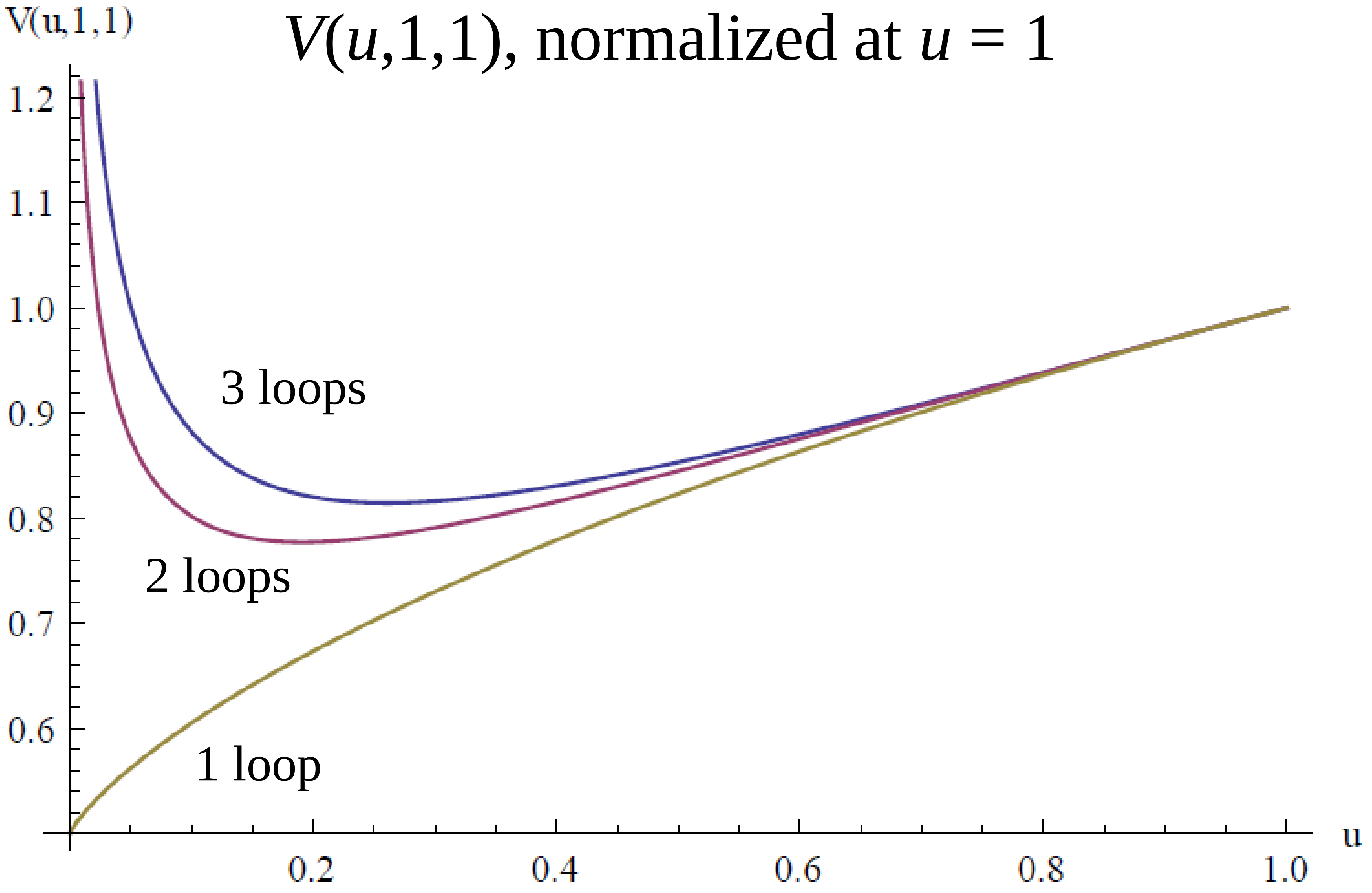
- Collinear limit fixes all but 10
- Near-collinear limit at order T^1 fixes all but 1
- Near-collinear limit at order T^2 fixes the last 1

$R_6^{(3)}$ sign stable within $\Delta > 0$ regions

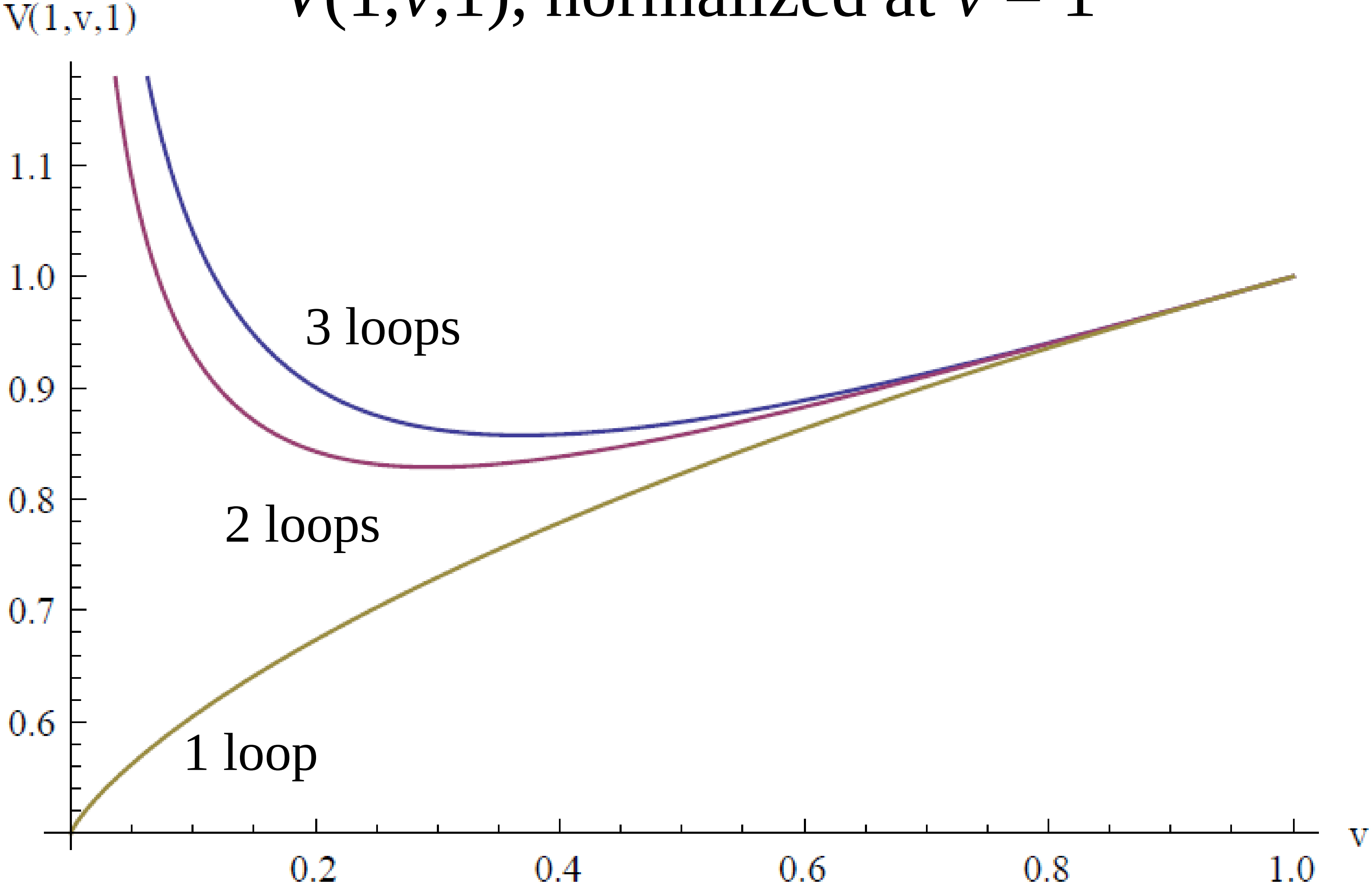


relation to positive Grassmannian? Arkani-Hamed, Trnka conjecture

$V(u,1,1)$, normalized at $u = 1$



$V(1,v,1)$, normalized at $v = 1$



Integration contours in (*u*,*v*,*w*)

$$F(u,v,w) \ = -\sqrt{\Delta}\int_1^u \frac{du_t}{v_t[u(1-w)+(w-u)u_t]}\frac{\partial F}{\partial \ln y_v}(u_t,v_t,w_t)$$

base point $(u,v,w) = (1,1,1)$

$y_u\, y_v\, y_w = 1$

$$F(u,v,w) = F(1,0,0) + \sqrt{\Delta}\int_1^u \frac{\quad}{(1 - \cdots}$$

base point $(u,v,w) = (1,0,0)$

$y_u = 1$