Yang-Baxter operators and scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory

Dmitry Chicherin

LAPTH, Annecy, France

Amplitudes 2014, 12 June, Saclay

Based on work in collaboration with
S. Derkachov and R. Kirschner
arXiv/math-ph:1306.0711, hep-th:1309.5748

Yangian symmetry
of scattering amplitudes
$\longleftrightarrow$
Quantum Inverse Scattering Method
(integrable spin chains)

QISM: Complicated nonlocal objects (like Hamiltonian and higher integrals of motion) are constructed from simple local building blocks according to simple local rule $\rightarrow$ dynamic is integrable

BCFW: Construct all superamplitudes out of 3-point superamplitudes  and 

We formulate the **Yangian symmetry** condition of the n-point amplitude M_n in a form of eigenvalue problem for the **quantum monodromy matrix**

$$\mathrm{L}_1(u)\cdots\mathrm{L}_n(u)\,M_n = C(u)\,M_n$$

and we solve it in a spirit of **QISM** as a sequence of R-operators acting on a reference state Ω_n which on its own trivially solves the eigenvalue problem

$$M_n = \mathrm{R}_{ab}\mathrm{R}_{cd}\cdots\mathrm{R}_{ef}\,\Omega_n$$

The bilocal RLL commutation relations allow as to verify that this is indeed solution for an appropriate choice of indices $a, b, c, d, \cdots, e, f = 1, 2, \cdots, n$.

Hidden symmetry of scattering amplitudes

- [Witten '03]

  $psu(2,2|4)$ Superconformal symmetry (**Lagrangian** origin)
  Generators – sums of **local** differential operators acting on the space $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}$

  $$J_a = \sum_{i=1}^{n} J_{i,a}$$

- [Alday, Maldacena '07][Beisert, Ricci, Tseytlin, Wolf '08][Berkovits, Maldacena '08]

  Dual superconformal symmetry (**Dynamical**). Dual coordinates

  $$x_i - x_{i+1} = p_i \qquad p_1 + \cdots + p_n = 0 \leftrightarrow x_1 \equiv x_{n+1}$$

  Covariance of tree amplitudes

  $$x^\mu \to -\frac{x^\mu}{x^2} \qquad M_n \to (x_1^2 \cdots x_n^2)\, M_n$$

  [Drummond,Henn,Korchemsky,Sokatchev '08]

  $psu(2,2|4)$ – infinitesimal form. Generators – sums of **local** differential operators on the space $\{x_i, \theta_i, \lambda_i, \tilde{\lambda}_i, \eta_i\} \longrightarrow$ Sum of **bilocal** differential operators on the space $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}$

  $$J_a^{(1)} = f_a{}^{bc} \sum_{i<k} J_{i,b} J_{k,c}$$

- Closure of two algebras $\longrightarrow$ Yangian algebra $Y(psu(2,2|4))$. Infinite-dimensional

  [Drummond,Henn,Plefka '09]

  $$[\, J_a \,,\, J_b \,] = f_{ab}{}^c J_c \qquad [\, J_a \,,\, J_b^{(1)} \,] = f_{ab}{}^c J_c^{(1)} \quad + \text{ Serre relations } + \text{ Higher levels}$$

$g\ell(4|4)$ integrable spin chain

Spin chain dynamical variables $=$ $p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$, η_A $(A=1,\cdots,4)$
Spinor-helicity variables for asymptotic states of scattering particles

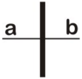
- Heisenberg pairs

$$\{\, x_a \,,\, p_b \,\} = -\delta_{ab} \qquad \mathbf{x} = (\lambda_\alpha, \quad \partial_{\tilde\lambda_{\dot\alpha}}, \quad \partial_{\eta_A}) \;\; , \;\; \mathbf{p} = (\partial_{\lambda_\alpha}, \quad -\tilde\lambda_{\dot\alpha}, \quad -\eta_A)$$

$$a=\alpha,\dot\alpha,A$$

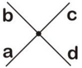
- L-operator for **Jordan-Schwinger** type non-compact representations
u – spectral parameter

$$\big[\mathrm{L}(u)\big]_{ab} = u\,\delta_{ab} + x_a\, p_b =$$



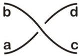
- Yang's $\mathcal{R}$-matrix $(4+4)^2 \times (4+4)^2$

$$\mathcal{R}_{ab,cd}(u) =$$

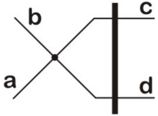


$= u$

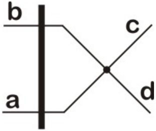
$+$



- Fundamental commutation relation (**FCR**)



$=$



$$\mathcal{R}_{ab,ef}(u-v)\,\big[\mathrm{L}(u)\big]_{ec}\,\big[\mathrm{L}(v)\big]_{fd} = \big[\mathrm{L}(v)\big]_{bf}\,\big[\mathrm{L}(u)\big]_{ae}\,\mathcal{R}_{ef,cd}(u-v)$$

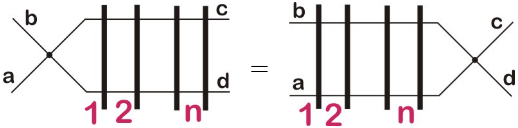
- The spin chain – quantum-mechanical system with many degrees of freedom
  # sites $= n$

$$(\mathbf{x}_1, \cdots, \mathbf{x}_n) \quad \text{and} \quad (\mathbf{p}_1, \cdots, \mathbf{p}_n)$$

- Homogeneous monodromy matrix

$$\left[\mathrm{T}(u)\right]_{ac} = \left[\mathrm{L}_1(u)\right]_{ab_1} \left[\mathrm{L}_2(u)\right]_{b_1 b_2} \cdots \left[\mathrm{L}_n(u)\right]_{b_{n-1} c} =$$

- Co-multiplication property



- $\mathrm{T}(u)$ – "generating function" polynomial in spectral parameter u

$$\left[\mathrm{T}(u)\right]_{ab} = \sum_{m=-1}^{n-1} u^{n-m-1} J_{ab}^{(m)}$$

FCR is equivalent to a set of commutation relations for generators $J_{ab}^{(m)}$

$$J_{ab}^{(0)} = \sum_{1 \le i \le n} x_{a,i}\, p_{b,i} \quad , \quad J_{ab}^{(1)} = \sum_{1 \le i < j \le n} x_{a,i}\, p_{c,i}\, x_{c,j}\, p_{b,j}$$

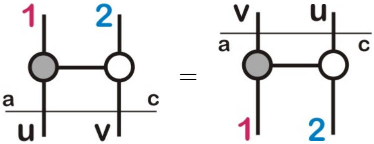
- **Yangian symmetry condition**
  M – Yangian invariant

$$\left[\mathrm{T}(u)\right]_{ab} M = C\, \delta_{ab}\, M \,, \quad M = M(\lambda_1, \tilde{\lambda}_1, \eta_1, \cdots, \lambda_n, \tilde{\lambda}_n, \eta_n)$$

The R-operator

We use the R-operator as the main building block in the construction of Yangian invariants

The R-operator is defined by the intertwining RLL-relation

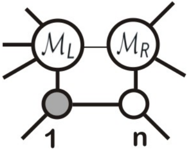


$$\mathrm{R}_{12}(u-v)\left[\mathrm{L}_1(u)\right]_{ab}\left[\mathrm{L}_2(v)\right]_{bc}=\left[\mathrm{L}_1(v)\right]_{ab}\left[\mathrm{L}_2(u)\right]_{bc}\mathrm{R}_{12}(u-v)$$

The R-operator reproduces BCFW-shift: $\mathrm{R}_{12}(u)=(\mathbf{p}_1\cdot\mathbf{x}_2)^{-u}$

$$\left[\mathrm{R}_{12}(u)F\right](\lambda_1,\tilde\lambda_1,\eta_1|\lambda_2,\tilde\lambda_2,\eta_2)=\int\frac{\mathrm{d}\mathbf{z}}{\mathbf{z}^{1-u}}F(\lambda_1-\mathbf{z}\lambda_2,\tilde\lambda_1,\eta_1|\lambda_2,\tilde\lambda_2+\mathbf{z}\tilde\lambda_1,\eta_2+\mathbf{z}\eta_1)$$

BCFW-recursion and BCFW-bridge



$$M_n=\mathrm{R}_{1n}\int \mathrm{d}^4\eta_0\,\mathrm{d}^4P_0\,\delta(P_0{}^2)\,M_L(\eta_1\,,\,\lambda_1\,,\,\tilde\lambda_1\,;\,\eta_0\,,\,-P_0)\,M_R(\eta_n\,,\,\lambda_n\,,\,\tilde\lambda_n\,;\,\eta_0\,,\,P_0)$$

In order to construct Yangian invariants we start with the simplest solution of the eigenvalue problem – **Basic state** (Not interacting degrees of freedom)

$$\Omega_{k,n} = \cdots \delta^2(\lambda_i)\cdots\delta^2(\tilde\lambda_j)\delta^4(\eta_j)\cdots\delta^2(\tilde\lambda_l)\delta^4(\eta_l)\cdots \implies \mathrm{T}(u)\,\Omega_{k,n} = u^k(u-1)^{n-k}\Omega_{k,n}$$

n scattering particles $\equiv$ n-site monodromy. Grassmann degree $4k$

- 3-particle anti-MHV

$$\Omega_{1,3} = \delta^2(\lambda_1)\,\delta^2(\lambda_2)\,\delta^2(\tilde\lambda_3)\delta^4(\eta_3) \implies \mathrm{L}_1(u)\mathrm{L}_2(u)\mathrm{L}_3(u)\,\Omega_{1,3} = u(u-1)^2\,\Omega_{1,3}$$

Introduce interaction in an integrable way (entangling degrees of freedom)

$\mathrm{R}_{12}\,\mathrm{R}_{23}\,\Omega_{1,3} =$ anti-MHV$_3$ $=$  $\implies$ $\mathrm{T}(u)$  $= u(u-1)^2$ 

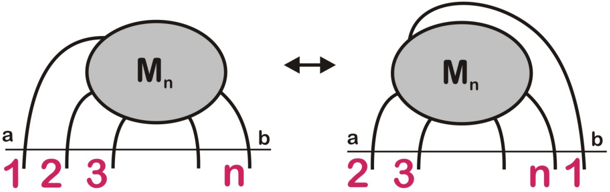
- 3-particle MHV. $\Omega_{2,3} = \delta^2(\lambda_1)\,\delta^2(\tilde\lambda_2)\delta^4(\eta_2)\,\delta^2(\tilde\lambda_3)\delta^4(\eta_3)$

$\mathrm{R}_{23}\,\mathrm{R}_{12}\,\Omega_{2,3} =$ MHV$_3$ $=$  $\implies$ $\mathrm{T}(u)$  $= u^2(u-1)$ 

- The general tree superamplitude of the type $\mathrm{N}^{\mathrm{k}-2}\mathrm{MHV}_n$

$$\mathrm{T}(u)\,M_{k,n} = u^k(u-1)^{n-k}\,M_{k,n}$$

- **Cyclicity** of superamplitudes = cyclicity of the spin chain



$$\mathrm{L}_1(u)\cdots\mathrm{L}_n(u)\,M = C\,M \quad\Leftrightarrow\quad \mathrm{L}_{\sigma_1}(u)\cdots\mathrm{L}_{\sigma_n}(u)\,M = C\,M$$

where $\sigma_1,\cdots,\sigma_n$ is a cyclic permutation of $1,2,\cdots,n$

- **Reflection** of the particle ordering = reflection of the spin chain sites

$$\mathrm{L}_1(u)\cdots\mathrm{L}_n(u)\,M = C\,M \quad\Leftrightarrow\quad \mathrm{L}_n(u')\cdots\mathrm{L}_1(u')\,M = C'\,M$$

- The eigenvalue relation for the monodromy matrix T is compatible with BCFW:

We act by the monodromy matrix on the BCFW term $\mathrm{L}_{n-1}\cdots\mathrm{L}_1\mathrm{L}_n\,\mathrm{R}_{1n}\int_0 M_L(1,2,\cdots,i,0)M_R(0,i+1,\cdots,n) =$

$$= \mathrm{R}_{1n}\int_0 \mathrm{L}_{n-1}\cdots\mathrm{L}_{i+1}\underbrace{\mathrm{L}_i\cdots\mathrm{L}_1\,M_L}_{C_L\cdot\mathrm{L}_0^{-1}M_L}\mathrm{L}_n\,M_R = \frac{C_L}{u(u-1)}\mathrm{R}_{1n}\int_0 M_L\underbrace{\mathrm{L}_{n-1}\cdots\mathrm{L}_{i+1}\mathrm{L}_0\mathrm{L}_n M_R}_{C_R\,M_R} =$$

Inversion and transposition of L-operators

$$\mathrm{L}(u)\,\mathrm{L}(1-u-\mathrm{c}) = u(1-u-c)\ ,\ \mathrm{L}^T(u) = -\mathrm{L}(1-u)$$

$$= \frac{C_L C_R}{u(u-1)}\mathrm{R}_{1n}\int_0 M_L M_R$$

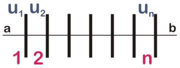
On the space of amplitudes the central charge $\mathrm{c} = 0$

Conclusions and Outlook

- **Deformed** amplitudes and Yangian invariants

[Ferro, Lukowski, Meneghelli, Plefka, Staudacher '12, '13] [Frassek, Kanning, Ko, Staudacher '13]
[Beisert, Broedel, Rosso '14] [Kanning, Lukowski, Staudacher '14] [Broedel, De Leeuw, Rosso '14]

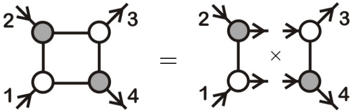
Inhomogeneous monodromy matrix $\left[\mathrm{L}_1(u_1)\mathrm{L}_2(u_2)\cdots\mathrm{L}_n(u_n)\right]_{ab} =$



4-point Yangian invariant (due to RLL-relation):

$$\mathrm{R}_{14}(u_{32})\mathrm{R}_{12}(u_{21})\mathrm{R}_{34}(u_{43})\mathrm{R}_{23}(u_{41})\,\Omega_{2,4} = \frac{\delta^4(p)\,\delta^8(q)}{\langle 12\rangle^{1+u_{23}}\,\langle 23\rangle^{1+u_{34}}\,\langle 34\rangle^{1+u_{41}}\,\langle 41\rangle^{1+u_{12}}}$$

- Factorization of R-operators and kernels of integral operator



Yang-Baxter type relation $\quad \mathrm{T}(u_4,u_1,u_2,u_3)\,M = C\,M \;\leftrightarrow\; \mathrm{L}_2(u_2)\,\mathrm{L}_3(u_3)\,\widehat{M} = C\,\widehat{M}\,\mathrm{L}_2(u_1')\,\mathrm{L}_3(u_4')$

- Momentum super-twistor variables $\quad \mathbf{x} = \mathcal{Z} = (\underbrace{\lambda,\mu}_{\text{twistor } Z}, \chi)\,, \quad \mathbf{p} = \partial_{\mathcal{Z}} = (\partial_Z, -\partial_\chi)$

R-operator $\quad \left[\mathrm{R}_{ij}(u)F\right](\mathcal{Z}_i|\mathcal{Z}_j) = \int \mathrm{d}\mathbf{z}\,\mathbf{z}^{u-1}\,F(\mathcal{Z}_i - \mathbf{z}\mathcal{Z}_j|\,\mathcal{Z}_j)$

R-invariant $\quad [1,2,3,4,5] = \mathrm{R}_{45}\mathrm{R}_{34}\mathrm{R}_{23}\mathrm{R}_{12}\,\delta^{4|4}(\mathcal{Z}_1)$

[Drummond,Henn,Korchemsky,Sokatchev '08][Mason, Skinner '09]