Toying with bound states

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(based work with Johannes Henn: 1404.2922 and work to appear)

[Amplitudes 2014, Saclay, 12 juin 2014]

Motivation

`N=4 SYM: the harmonic oscillator of the 21st century?'

- One is generally interested in two types of theories:
  -Theories which precisely describe Nature
  -Approximate models which we can exactly solve
- In four dimensions, there appears to be a unique nontrivial quantum field theory in which we can calculate scattering amplitudes exactly

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Hydrogen atom

`N=4 SYM: the ~~harmonic oscillator~~ of the 21st century!'

- One is generally interested in two types of theories:
  -Theories which precisely describe Nature
  -Approximate models which we can exactly solve
- In four dimensions, there appears to be a unique nontrivial quantum field theory in which we can calculate scattering amplitudes exactly
- I will try to shed a new light on the symmetry which make it possible, by relating it to a more familiar one. I will also present some recent developments which incorporate massive particles.

Plan

1. Symmetries: the hydrogen atom and N=4 SYM
2. A perverse way to compute the spectrum
3. Our original story...

The Kepler problem, 1.

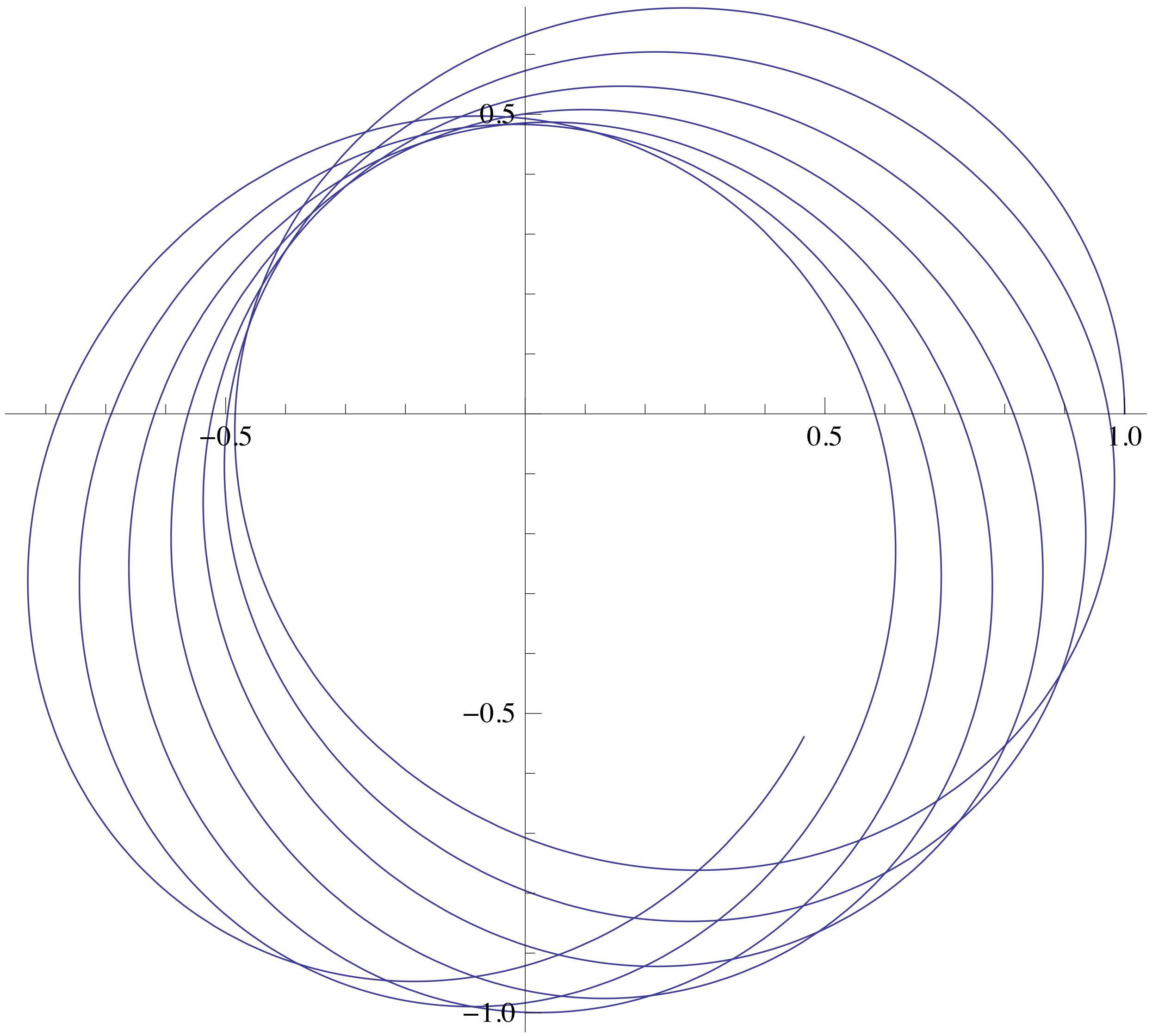
- Consider the classical two-body problem with a 1/r potential
- We can go to a center-of-mass frame;
  four conserved quantities are apparent:
  angular momentum $\vec{J}$ and energy
- These basically fix the dynamics, for example the motion takes place in a plane, ...

The Kepler problem, 2.

- Something special happens when the potential is 1/r: the orbits do not precess

Example:

$V = 1/r^{9/10}$



The Kepler problem, 3.

- For V ∝ -1/r the system possess an additional, *non-obvious* conserved vector:

$$\vec{M} = \frac{\vec{p} \times \vec{J}}{m} - \frac{\vec{x}}{|x|}$$

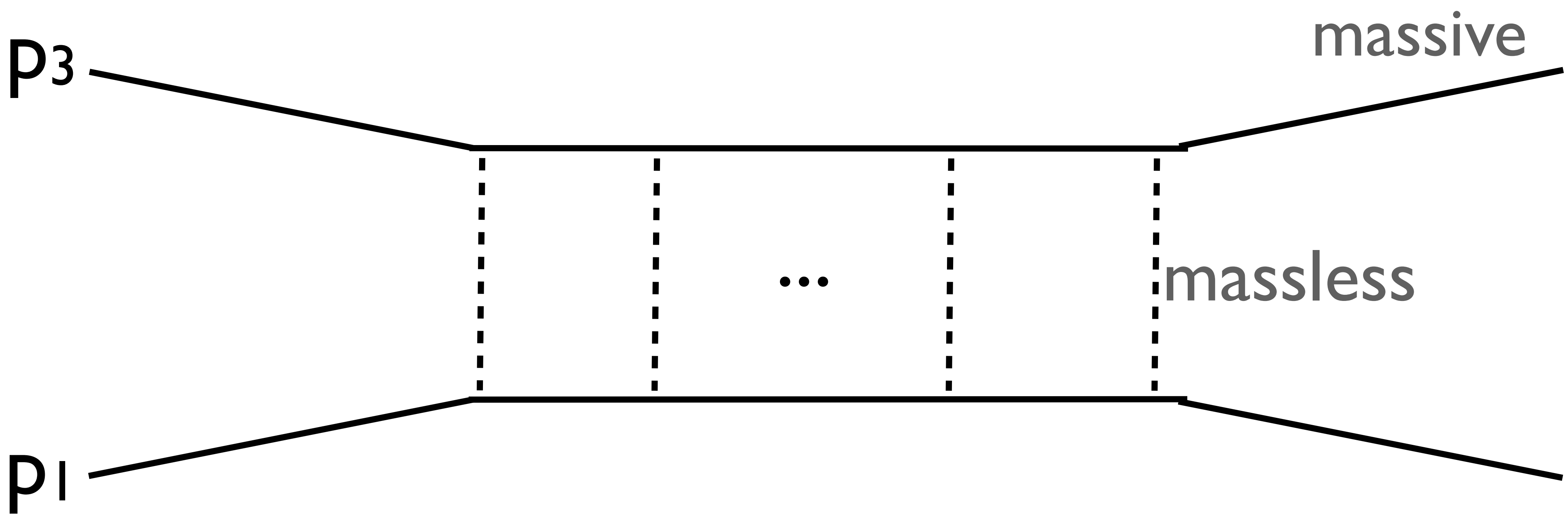
(« Laplace-Runge-Lenz » vector)

- It points in the direction of the eccentricity, preventing it from precessing

- Quantum mechanically, the Laplace-Runge-Lenz vector is *still* conserved
- It explains the well-known degeneracy of the excited states of the Hydrogen atom
  (this was quickly pointed out by Pauli in the early days of the subject)
- In the real world, its conservation is broken by relativistic effects (spin-orbit, …)

Is there a fully consistent, relativistic quantum field theory, in which the Runge-Lenz vector is conserved?

- In the early days of relativistic QFT, Wick and Cutkowski considered the following model:



- This is the ladder approximation to $ep \to ep$, ignoring the spin of the photon.

- In the nonrelativistic limit, for massless exchange, this reduces to the H Hamiltonian

- This model possesses an exact O(4) symmetry, even *away* from the NR limit

- Consider just one rung

$$\cdots \int \frac{d^4\ell_2}{(\ell_2-\ell_1)^2\left[(\ell_2-p_1)^2+m^2\right]\left[(\ell_2+p_2)^2+m^2\right](\ell_2-\ell_3)^2} \cdots$$

- The symmetry is non-obvious in this form, but there is a conformal symmetry in momentum space

- The symmetry can be made evident by using Dirac's embedding formalism

- Rewrite each vector as a 6-vector, with $L^2=0$:

$$L_i^a \equiv \begin{pmatrix} \ell_i^\mu \\ L_i^+ \\ L_i^- \end{pmatrix} = \begin{pmatrix} \ell_i^\mu \\ \ell_i^2 \\ 1 \end{pmatrix}$$

and similarly for the external regions:

$$Y_1^a = \begin{pmatrix} p_1^\mu \\ p_1^2 + m^2 \\ 1 \end{pmatrix}, \qquad Y_3^a = \begin{pmatrix} -p_2^\mu \\ p_2^2 + m^2 \\ 1 \end{pmatrix}$$

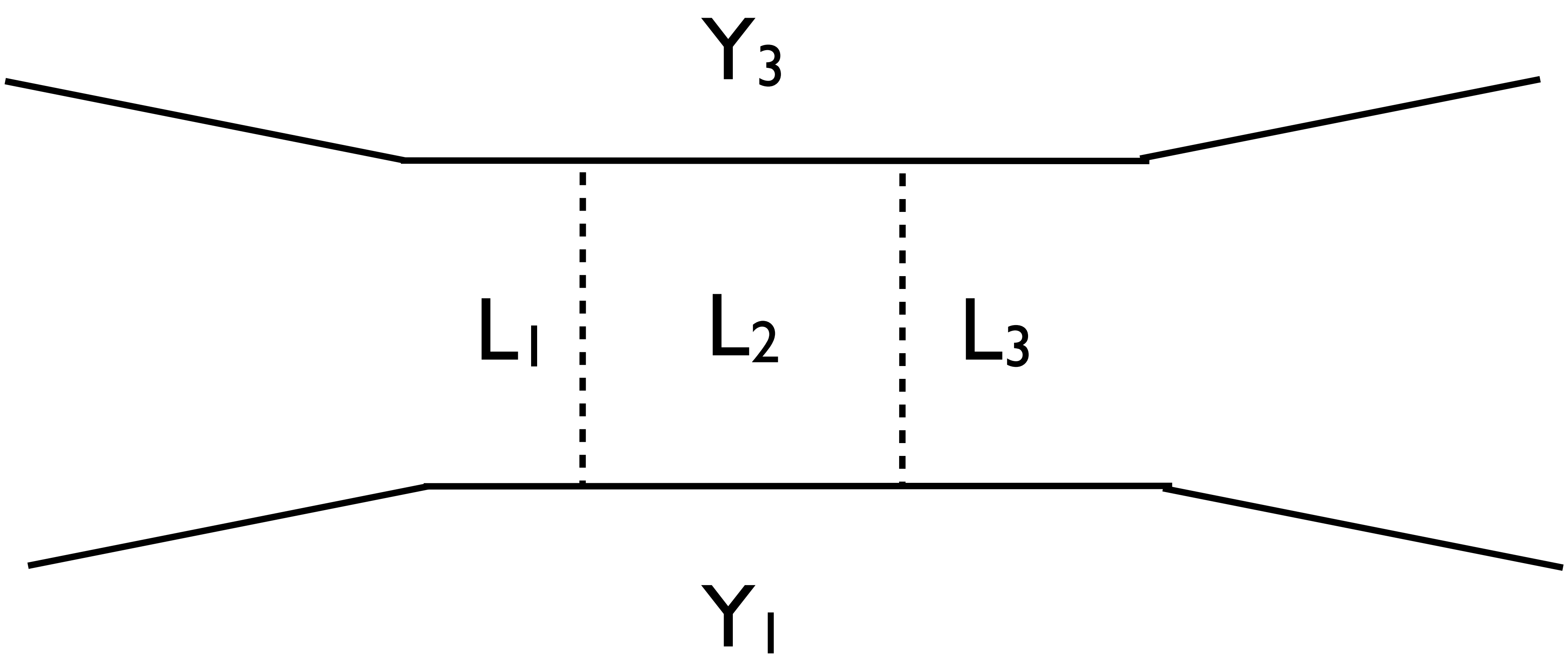
- The 6D vector product gives:

$$L_i \cdot L_j = (\ell_i - \ell_j)^2$$

$$L_i \cdot Y_1 = (\ell_i - p_1)^2 + m^2$$

$$L_i \cdot Y_3 = (\ell_i + p_2)^2 + m^2$$

- The L's and Y's 'live' in regions of the planar graph



$$\cdots \int \text{``}d^4L_2\text{''} \frac{1}{(L_1{\cdot}L_2)(L_2{\cdot}Y_1)(L_2{\cdot}Y_3)(L_2{\cdot}L_3)} \cdots$$

- The integration measure is also important, but let me skip it for now.

$$\cdots \int \text{“}d^4L_2\text{”} \frac{1}{(L_1 \cdot L_2)(L_2 \cdot Y_1)(L_2 \cdot Y_3)(L_2 \cdot L_3)} \cdots$$

- Since everything (incl. measure) depends only on 6-dimensional dot products, there is a natural SO(6) (really SO(4,2)) symmetry
- The point is that the two vectors Y_1, Y_3 reduce the symmetry, but obviously preserve an SO(4).
- This SO(4) contains the usual SO(3) $\vec{J}$ subgroup.
- What are the remaining three generators? The Runge-Lenz vector!

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(In NR regime, loop integral localizes on pole, and can be rotated to an S^4) $L^a \simeq \left(0, 0, \frac{2p_0\vec{p}}{\vec{p}^2 + p_0^2}, \frac{p_0^2 - \vec{p}^2}{\vec{p}^2 + p_0^2}\right)$

- One can also consider the unequal-mass case

$$Y_1^a = \begin{pmatrix} p_1^\mu \\ p_1^2 + m_1^2 \\ 1 \end{pmatrix}, \qquad Y_3^a = \begin{pmatrix} -p_2^\mu \\ p_2^2 + m_3^2 \\ 1 \end{pmatrix}$$

- It is equivalent to previous case: the spectrum depends only on the cross-ratio

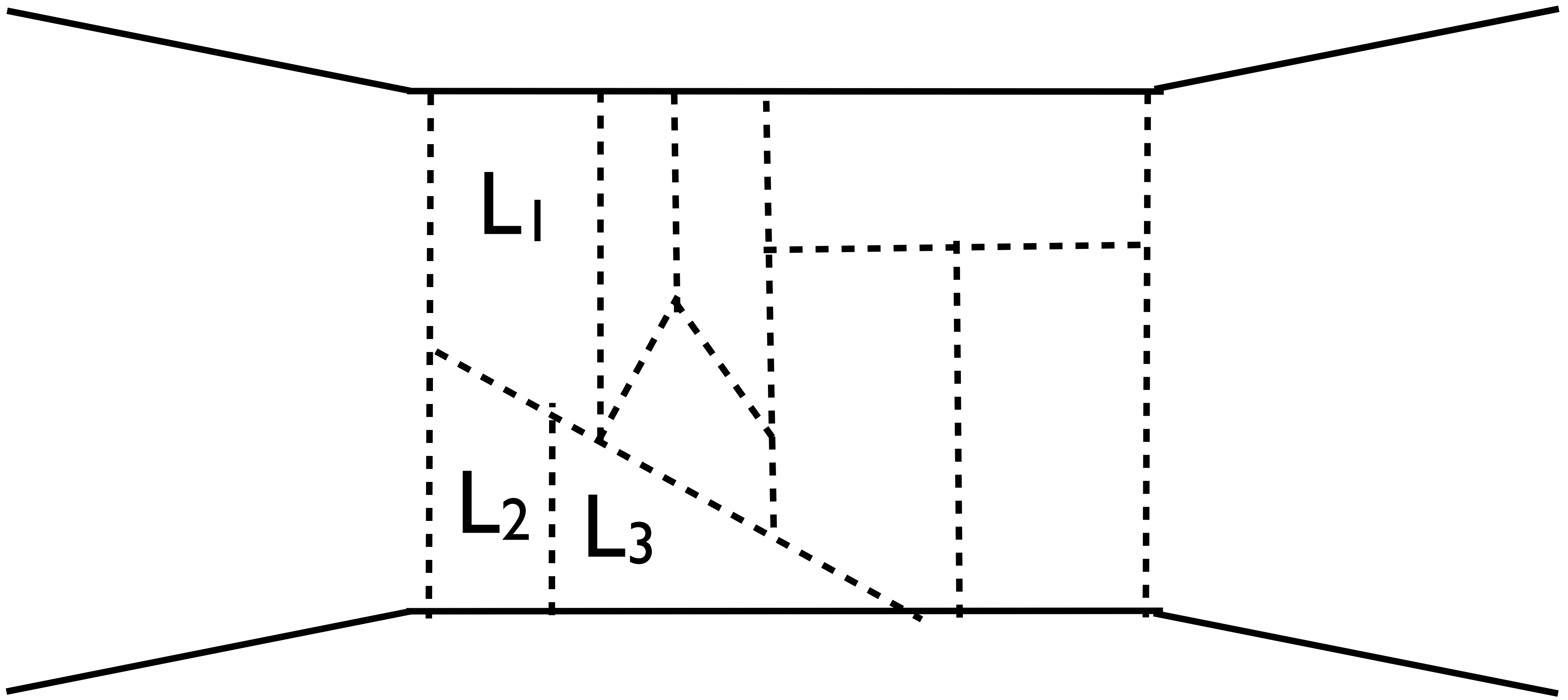
$$u = \frac{4m_1 m_3}{-s + (m_1 - m_3)^2} \qquad \left(= \frac{2\sqrt{Y_1^2 Y_3^2}}{Y_1 \cdot Y_3} \right)$$

[Cutkowski, `54]

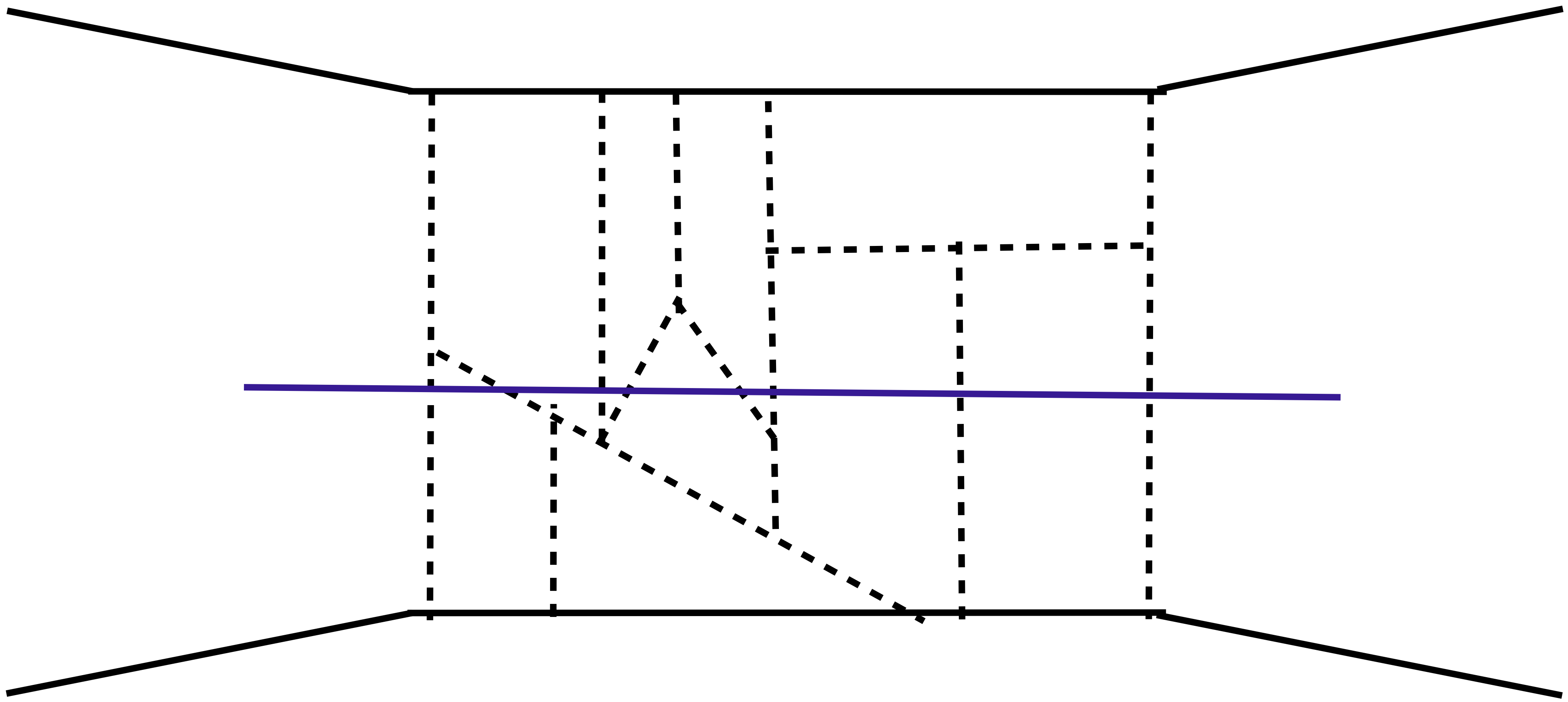
generalizes 'reduced mass'

- Unfortunately, the ladder approximation is not consistent relativistically.
- (It lacks multi-particle channels and so has deep problems with unitarity)
- For this reason this symmetry appears to have been mostly forgotten, like a curiousity
- Wick and Cutkowski's investigations nonetheless left us the ``Wick rotation"

- The simplest way to imagine a consistent QFT with this symmetry is to take a planar limit:



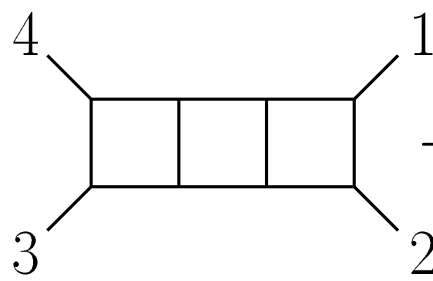
- The Feynman rules would then 'only' need to respect the SO(6) symmetry, which acts in *momentum space*
- Can such a thing exist?



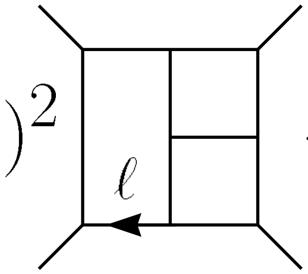
By unitarity, such a theory will contain massless particles. Their self-interactions will then have to respect the dual conformal symmetry.

Fast forward to the 2000's

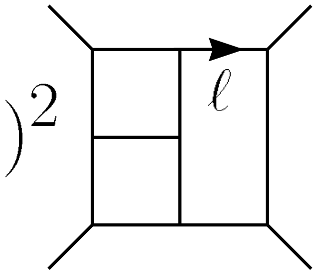
$$\frac{\mathcal{M}_4^{(3-\text{loop})}}{\mathcal{M}_4^{\text{tree}}} = \Big\{ s^2$$



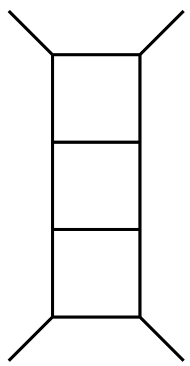
$$+ \; s(\ell + k_2)^2$$



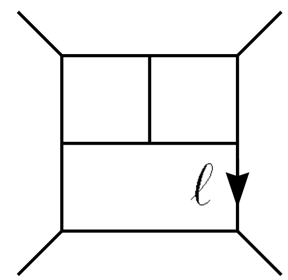
$$+ \; s(\ell + k_4)^2$$



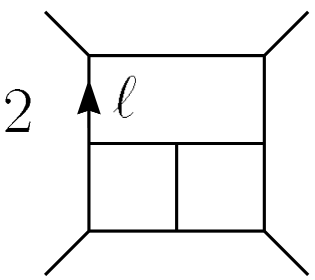
$$+ \; t^2$$



$$+ \; t(\ell + k_1)^2$$



$$+ \; t(\ell + k_3)^2$$



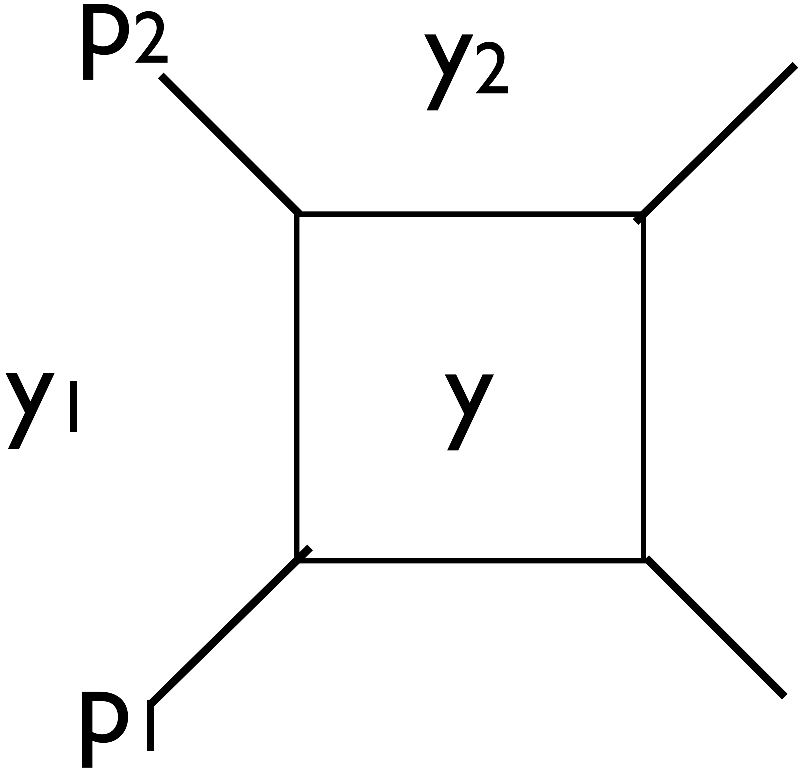
$$\Big\}$$

[Bern,Rozowsky&Yan, `97]

- Bern-Dixon-Smirnov-(Kosower-Anastasiou), and Drummond-Henn-Smirnov-Sokatchev observed:
  -All integrals that contribute are dual-conformal invariant
  -The integrated results exponentiates up to three (four) loops

- Dual conformal symmetry in massless case

(Drummond, Henn, Smirnov& Sokatchev,
Bern, Dixon& Smirnov,
Alday&Maldacena,
Berkovitz&Maldacena
Beisert,Ricci,Tseytlin&Wolf,
...)



$$= \int d^4\ell \frac{st}{\ell^2(\ell-p_1)^2(\ell-p_1-p_2^2)(\ell+p_4)^2}$$

$$= \int d^4y \frac{(y_1-y_3)^2(y_2-y_4)^2}{(y-y_1)^2(y-y_2)^2(y-y_3)^2(y-y_4)}$$

if $p_i = y_i - y_{i-1}$.

Symmetry seen as invariance under inversion: $y_i^\mu \to \frac{y_i^\mu}{y_i^2}$

All integrals in previous slide have this property!

- The SO(2,4) dual conformal symmetry in the massless sector is at the heart of the Wilson loop/amplitude duality, of the integrability of the N=4 theory, and of other recent developments.
- I have just argued that it is a natural QFT extension of the Hydrogen atom's O(4), itself inherited from the classical Kepler problem

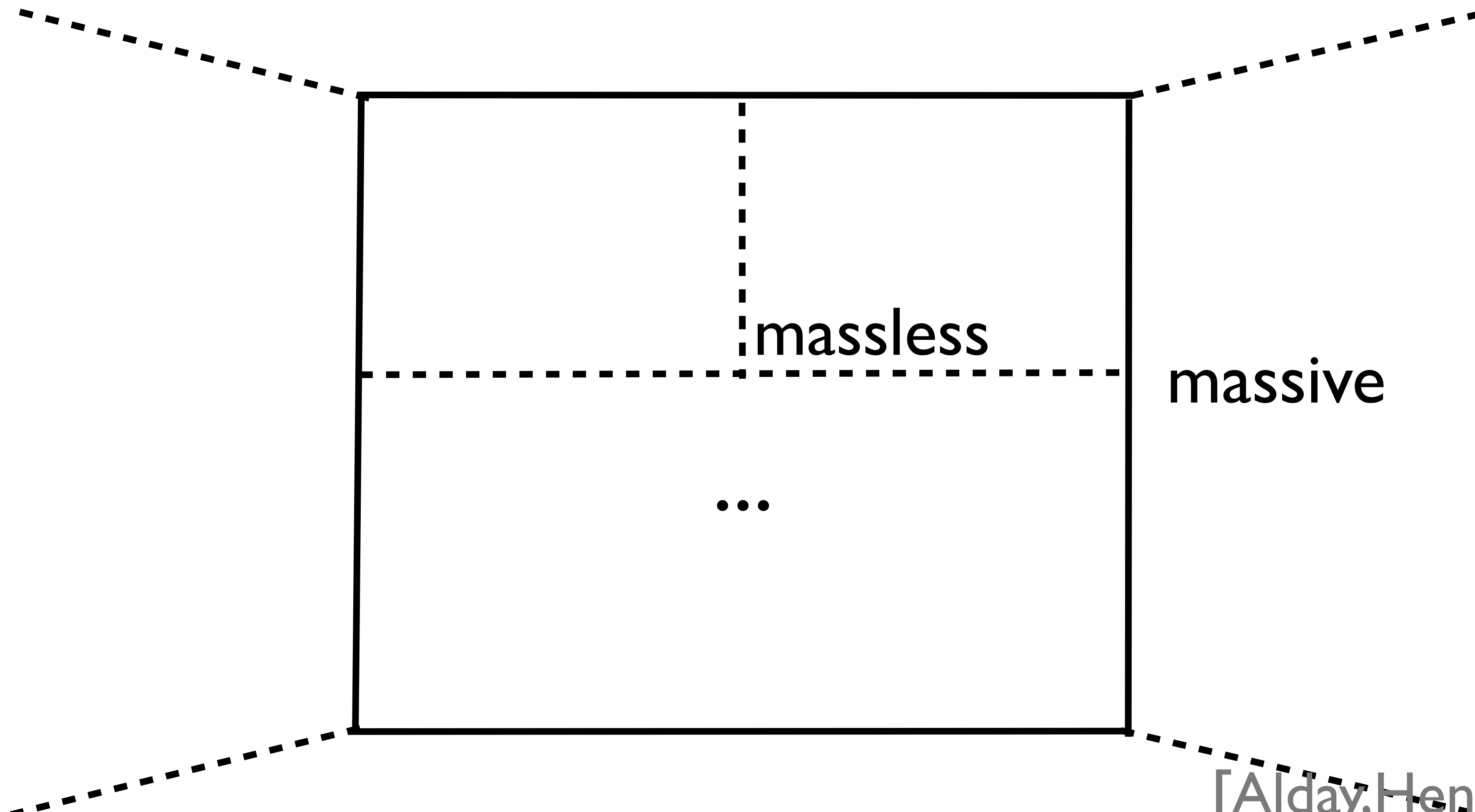
- The SO(2,4) dual conformal symmetry in the massless sector is at the heart of the Wilson loop/amplitude duality, of the integrability of the N=4 theory, and of other recent developments.
- I have just argued that it is a natural QFT extension of the Hydrogen atom's O(4), itself inherited from the classical Kepler problem
- Open question: is N=4 the unique example?
- Let us return to our massive particles!

- The N=4 theory comes with a moduli space of vacua, parametrized by 6 adjoint scalar fields
- We can give them the vev's we want. For example by breaking: $SU(N_c) \to U(1) \times SU(N_c-1)$ we will get U(1) ``photons" coupled to massive W bosons.
- For $2 \to 2$ scattering it is more interesting to break $SU(N_c) \to SU(4) \times U(N_c-4)$:

(Alday,Henn,Plefka&Schuster)

$$\langle \phi_1^a \rangle = \begin{pmatrix} m & 0 & 0 & 0 & 0 & \cdots \\ 0 & m & 0 & 0 & & \\ 0 & 0 & m & 0 & & \\ 0 & 0 & 0 & m & & \\ 0 & \cdots & & & 0 & \cdots \end{pmatrix} \qquad \langle \phi_2^a \rangle, \ldots = 0$$

- The four-point color-ordered amplitude of massless U(4)'s has the following structure:



[Alday,Henn,Plefka&Schuste
Dennen& Huang: 6D,
O'Connell&SCH: 10D]

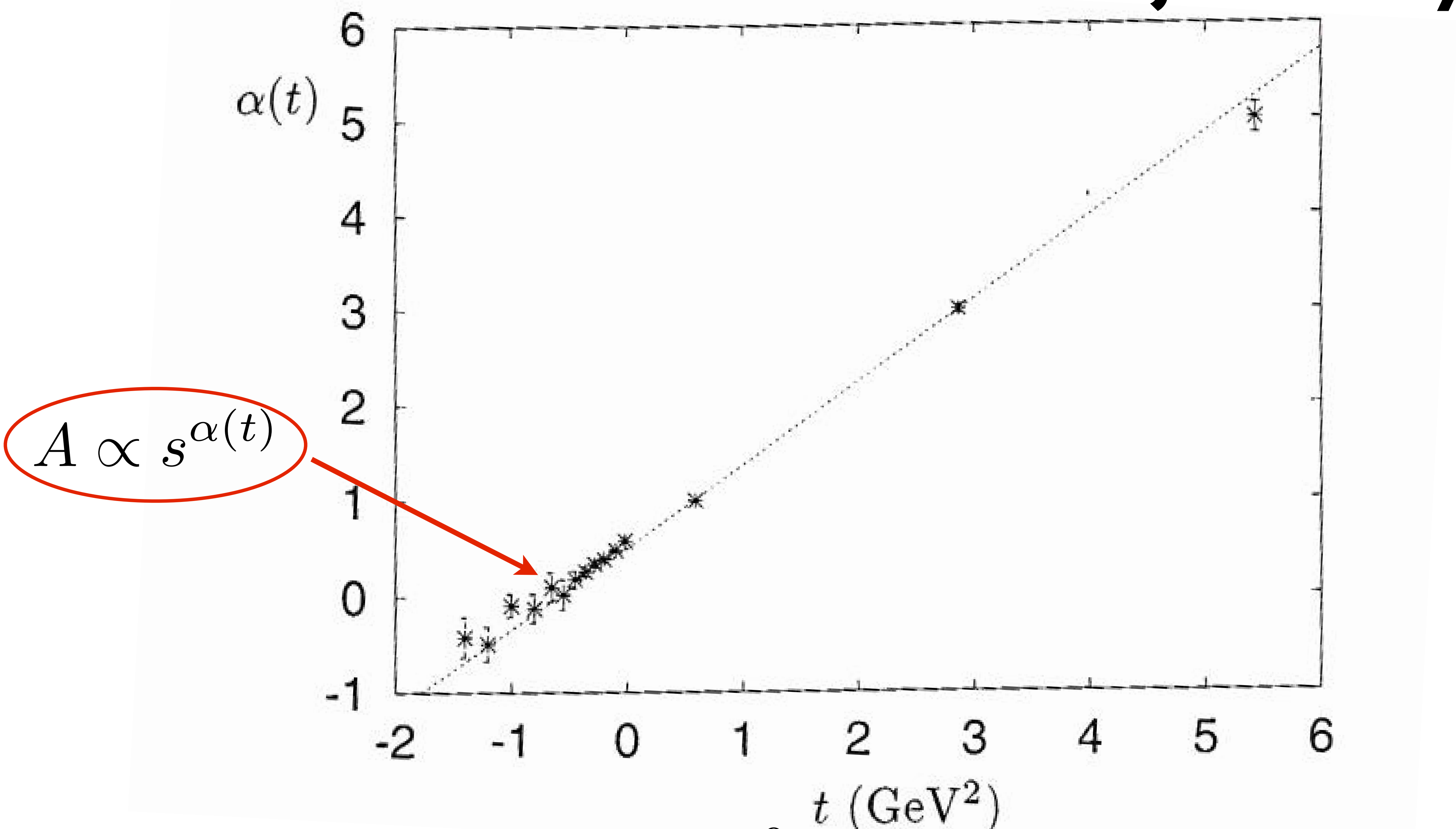
- Exactly what we were looking for! Analogous to light-by-light scattering in QED

- Bound states automatically carry the O(4) 'Runge-Lenz' symmetry at all couplings

Part II

A perverse calculation of the spectrum

connecting t<0 and t>0 in QCD: ex.: the rho meson trajectory



t<0 obtained from $\pi^- p \to \pi^0 n$ data, (@3.6,5.85&13.3GeV/c)

(Donnachie, Dosch,Landshoff&Nachtmann)

- The logic behind this connection is simple:

[Regge; Mandelstam, Gribov, t<1960]

- The spectrum of a theory can be read off from fall-off of correlators at large distances

$$\langle 0|O(0)O(x)|0\rangle \sim \sum_n c_n e^{-m_n|x|}$$

- The same fall-off can be equally well measured by fast particles at large impact parameter:

$$\lim_{\substack{s\to\infty \\ b\to\infty}} A(s,b) \sim \sum_n c_n s^{j_n} e^{-m_n b}$$

[at appropriate rate]

$$\lim_{s\to\infty} A(s,t) \propto s^{j_0(t)+1}$$

(known to three-loops: Correa,Henn,Maldacena&Sever)

- The gluon Regge trajectory for t<0 (equal to -$\Gamma_{\text{cusp}}(\theta)$)

$$j_0(t)+1 = \frac{\lambda}{8\pi^2\sqrt{1-4m^2/t}} \log \frac{\sqrt{1-4m^2/t}-1}{\sqrt{1-4m^2/t}+1} + O(\lambda^2)$$

- For positive t this diverges near threshold:

$$j_0(t)+1 \approx \frac{\lambda}{8\pi\sqrt{4m^2/t-1}}$$

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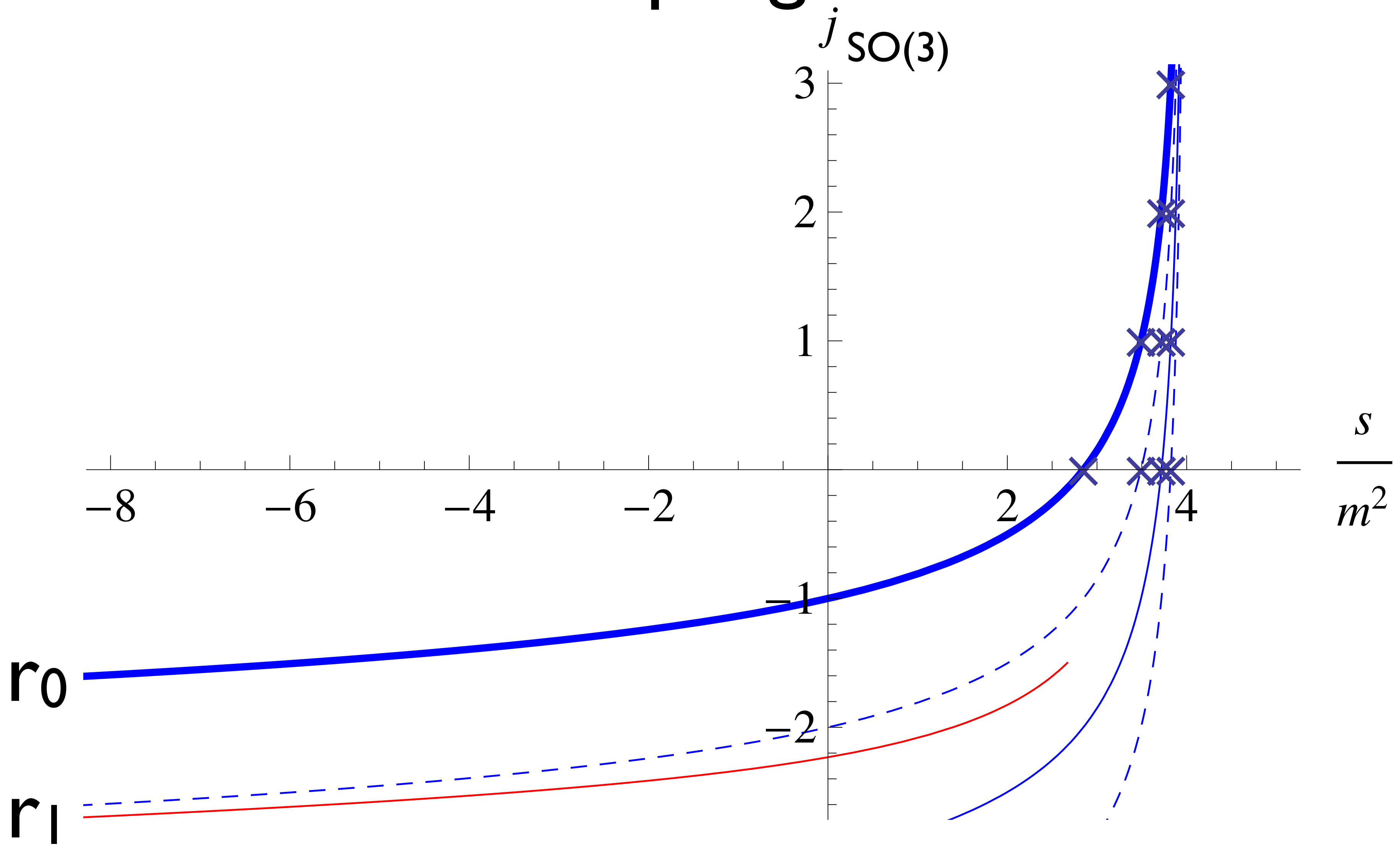
- The condition spin=integer gives the bound states:

$$\epsilon_n = \sqrt{t_n} - 2m = -\frac{m\lambda^2}{64\pi^2 n^2}, \quad n = 1,2,\dots$$

- This is the correct answer! $(H = \frac{p^2}{m} - \frac{\lambda}{4\pi r})$!

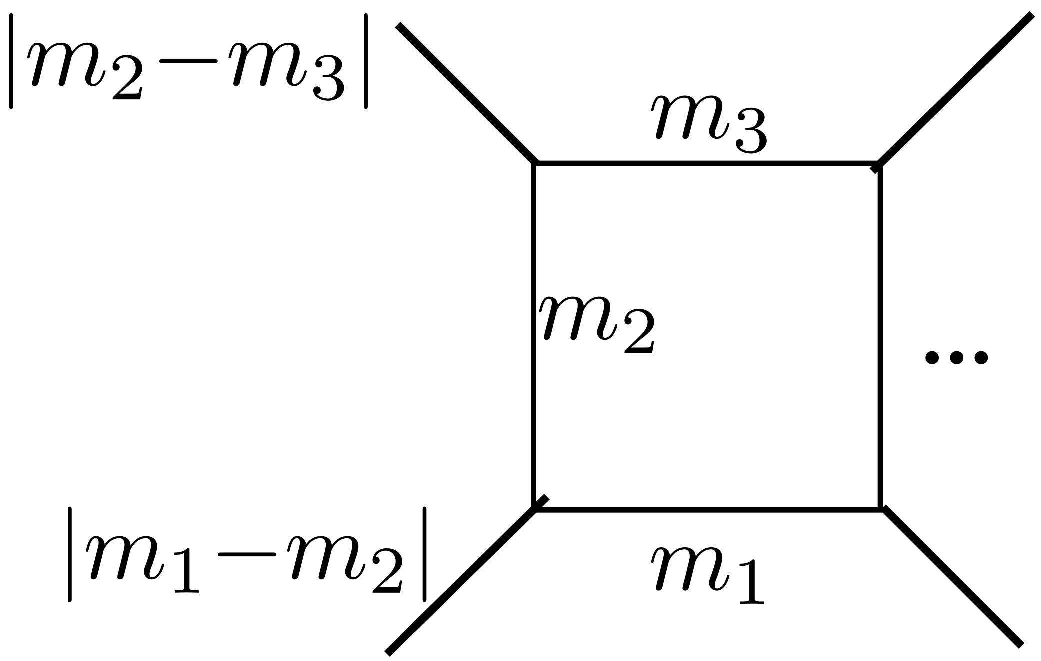
Regge trajectories at weak coupling



All bound states combine into a *single* O(4) Regge trajectory: the next-to-maximal-spin states are *Runge-Lenz* descendents of the leading-spin states

A strange duality, I

- Cutkowski's finding about the unequal-mass case generalizes to the four-mass case:



Amplitude depends on only two cross-ratios!

$$A_4(s, t, m_1, \ldots, m_4) = A_4^{\text{tree}} \times M(u, v)$$

where $u = \dfrac{4 m_1 m_3}{-s + (m_1 - m_3)^2}$ $\quad v = \dfrac{4 m_2 m_4}{-t + (m_2 - m_4)^2}$

[Alday,Henn,Plefka&Schuster]

A strange duality, II

$$u = \frac{4m_1 m_3}{-s + (m_1 - m_3)^2}, \qquad v = \frac{4m^2}{-t}$$

- Keeping v fixed, there are two ways to make u small:

1. Regge limit: s→∞: $M \propto s^{j(t)+1}$
2. Small mass limit: m_3→0: $M \propto m_3^{\Gamma_{\mathrm{cusp}}(v)}$

Conclusion: $j(t) + 1 = -\Gamma_{\mathrm{cusp}}(\theta)$

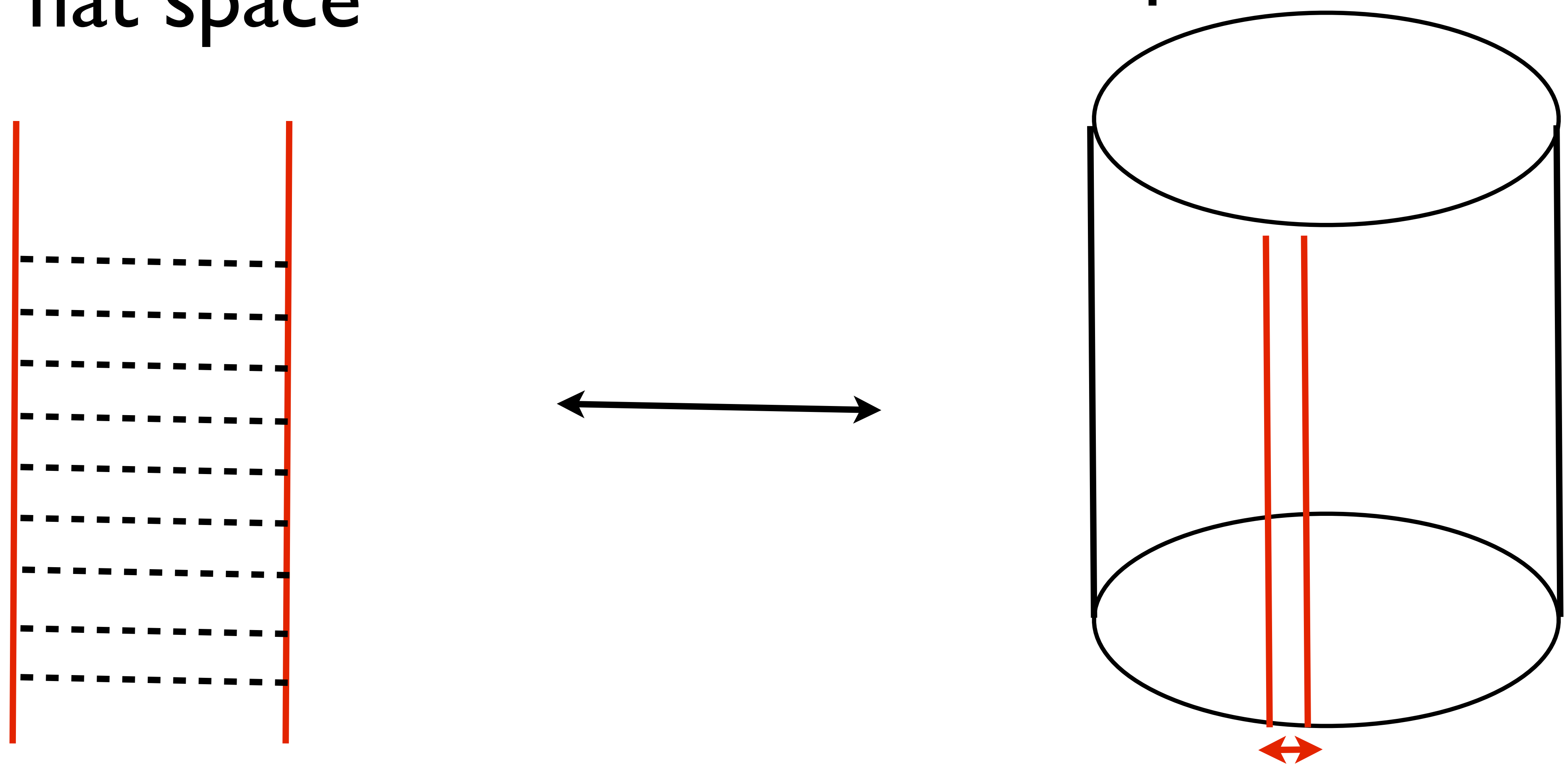
[Correa,Henn,Maldacena&Sever]

where $t = 4m^2 \sin^2 \frac{\theta}{2}$

A strange duality, III

Dynamical quarks in flat space

Static quarks in S^3xR



$\delta = \pi - \theta$

$$E_n = 2m \sin\frac{\theta_n}{2}\,, \quad \text{where} \quad -j-1 = \Gamma_{\text{cusp}}(\theta_n) = -n,$$

'anomalous dimension = minus integer' $(n = 1, 2, \ldots)$

- Checks: NLO calculation of Γ_{cusp}

In the regime dual to bound states, $\delta \sim \lambda$, a nontrivial but understood resummation is necessary ('ultrasoft scalars')

$$\Gamma(\delta) = -\frac{\lambda}{4\pi\delta}(1+\delta\lambda) - \frac{\lambda}{2\pi^2}\int_{\epsilon_{\text{UV}}}^{\infty}\frac{dt}{2(\cosh t - 1)}\left(e^{-t\frac{\lambda}{4\pi\delta}} - 1\right) + \mathcal{O}(\lambda^3)$$

[Pineda]
[Correa,Henn,Maldacena&Sever]

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[Pineda]
[Correa,Henn,Maldacena&Sever]

Equate: $\Gamma(\delta) = -n$

$$-E_n = \frac{m\lambda^2}{64\pi^2 n^2}\left(1 + \frac{\lambda}{\pi^2}\left[S_1(n) + \log\frac{\lambda}{2\pi n} - 1 - \frac{1}{2n}\right] + \mathcal{O}(\lambda^2)\right)$$

→ Perturbative series under uniform control for all n!

- Compare against standard 'Coulomb resummation'

[pNRQCD; see Beneke,Kiyo&Schuller 1312.4791]

$$\overset{\text{NLO}}{H}\psi^{(s)} = \left[\frac{p^2}{m} - \frac{\lambda}{4\pi r} + \delta V^{(2)}(r)\right]\psi^{(s)} + \phi\psi^{(o)}\,,$$

$$\overset{\text{NLO}}{H}\psi^{(o)} = \frac{p^2}{m}\psi^{(o)} + \phi\psi^{(s)} \qquad (\text{with } \delta V^{(2)} = \frac{\lambda}{2\pi^2\epsilon} + \frac{\lambda^2}{8\pi^3 r}\log\frac{\epsilon}{2r})\,.$$

[Pineda, '08]

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[Pineda, '08]

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Solve perturbatively:

$$G_{\text{us}}^{(\text{NLO})}(\mathbf{p},\mathbf{p}',E) = \frac{\lambda}{2\pi^2}\int\frac{d^3q}{(2\pi)^3}G^{(\text{LO})}(\mathbf{p},\mathbf{q},E)P_{\text{us}}(\mathbf{q},E)G^{(\text{LO})}(\mathbf{q},\mathbf{p}',E)$$

$$\overset{(\text{LO})}{G}(\mathbf{p},\mathbf{p}',E) = -\frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')}{E-\frac{\mathbf{p}^2}{m}} + \frac{1}{E-\frac{\mathbf{p}^2}{m}}\frac{\lambda}{(\mathbf{p}-\mathbf{p}')^2}\frac{1}{E-\frac{\mathbf{p}'^2}{m}} + \frac{1}{E-\frac{\mathbf{p}^2}{m}}\int_0^1 dt\frac{\lambda(j+1)t^{-j-1}}{(\mathbf{p}-\mathbf{p}')^2t-\frac{m}{4E}(E-\frac{\mathbf{p}^2}{m})(E-\frac{\mathbf{p}'^2}{m})(1-t)^2}\frac{1}{E-\frac{\mathbf{p}'^2}{m}}$$

[Schwinger]

- The integrals turn out to be doable, and we get:

$$M^{\mathrm{NLO}} \propto \frac{\lambda}{4\pi}\frac{1}{\kappa - n} + \frac{\lambda^2}{8\pi^3}\frac{nS_1(n) + \log 4\sqrt{\frac{-E}{m}} - n - \frac{1}{2}}{(\kappa - n)^2} + \ldots$$

$$= \frac{\lambda}{4\pi}\frac{1}{\kappa - n - \delta n} \qquad \left(\kappa = \frac{\lambda}{8\pi\sqrt{-Em}}\right)$$

- From the NLO propagator computed within pNRQCD, we deduce the NLO spectrum:

$$-E_n = \frac{m\lambda^2}{64\pi^2 n^2}\left(1 + \frac{\lambda}{\pi^2}\left[S_1(n) + \log\frac{\lambda}{2\pi n} - 1 - \frac{1}{2n}\right] + \mathcal{O}(\lambda^2)\right)$$

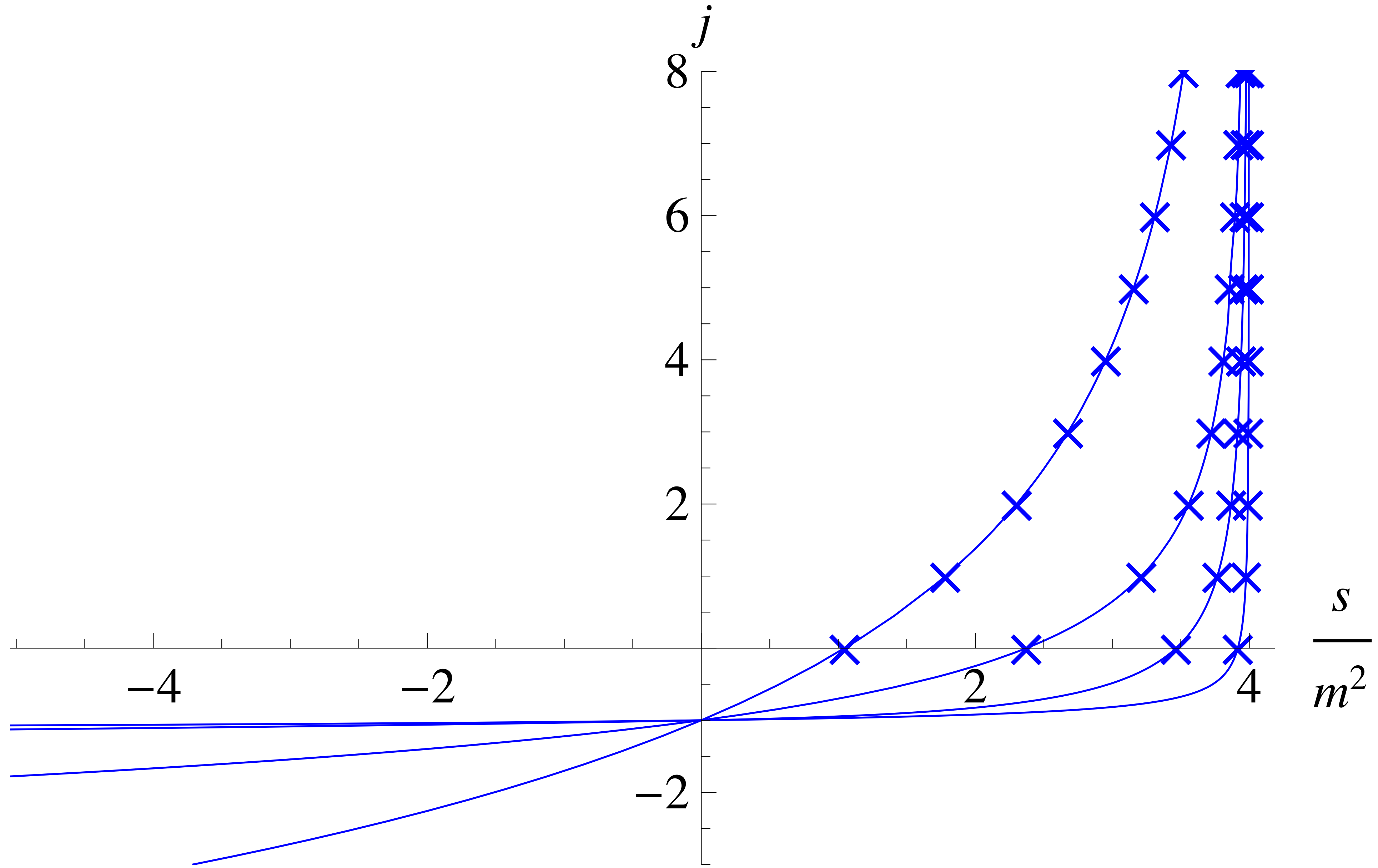
Exactly as predicted by the duality.

- Other check: strong coupling

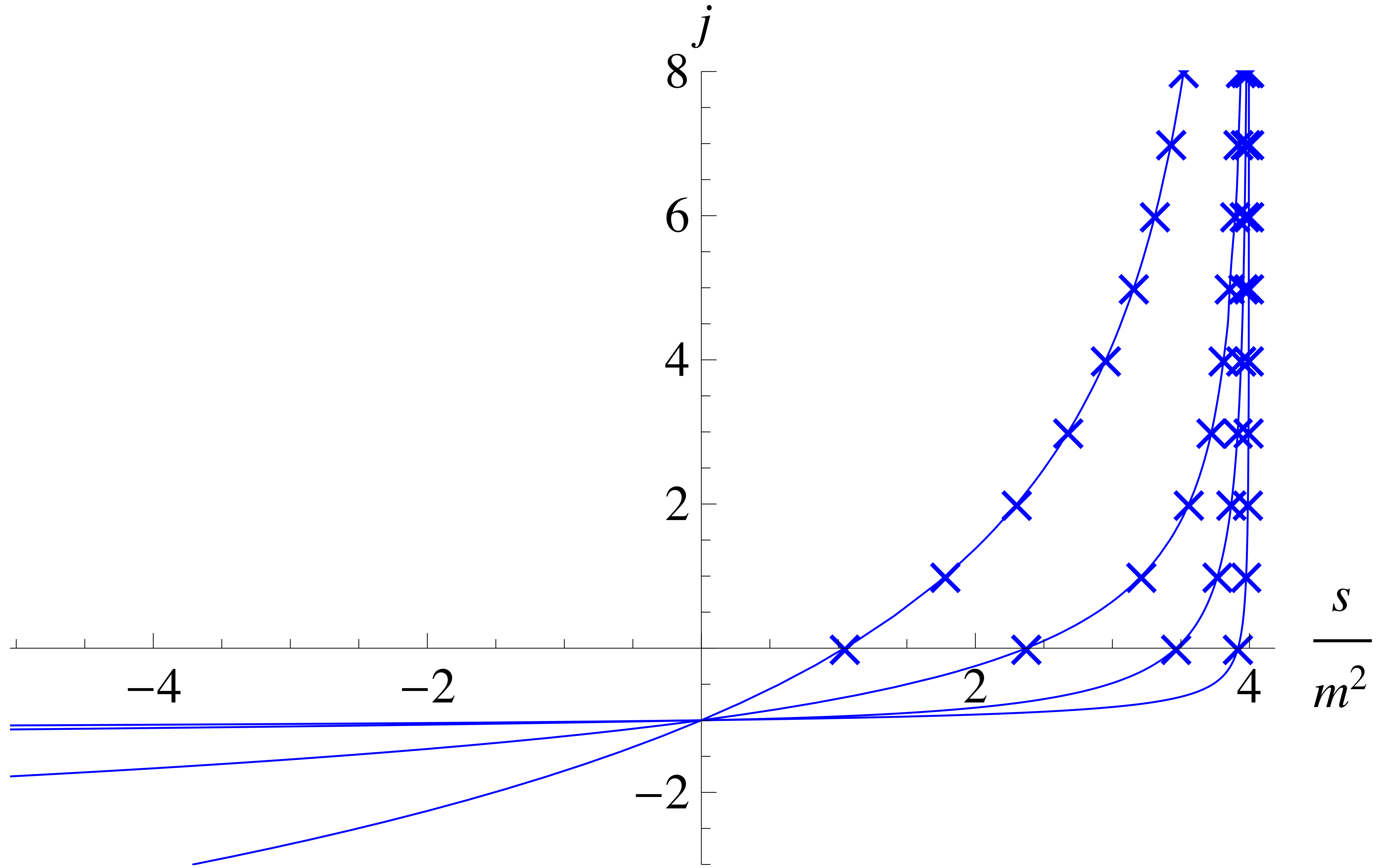
- Cusp anomalous dimension $\Gamma_{\rm cusp}(\theta)$ was computed in 2002 (Kruczensky `02)

- Spectrum (of 'mesons') was computed at strong coupling in 2003 (Kruczensky,Mateos,Myers&Winters `03

- The two curves agree perfectly, once one uses the correct dictionary! $E_n = 2m \sin\frac{\theta_n}{2}$



The trajectories at weak coupling (λ=5,15) and strong coupling (λ=100,1000).



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Should be computable exactly using TBA for Γ_{cusp}

[Drukker; Correa,Maldacena&Sever]

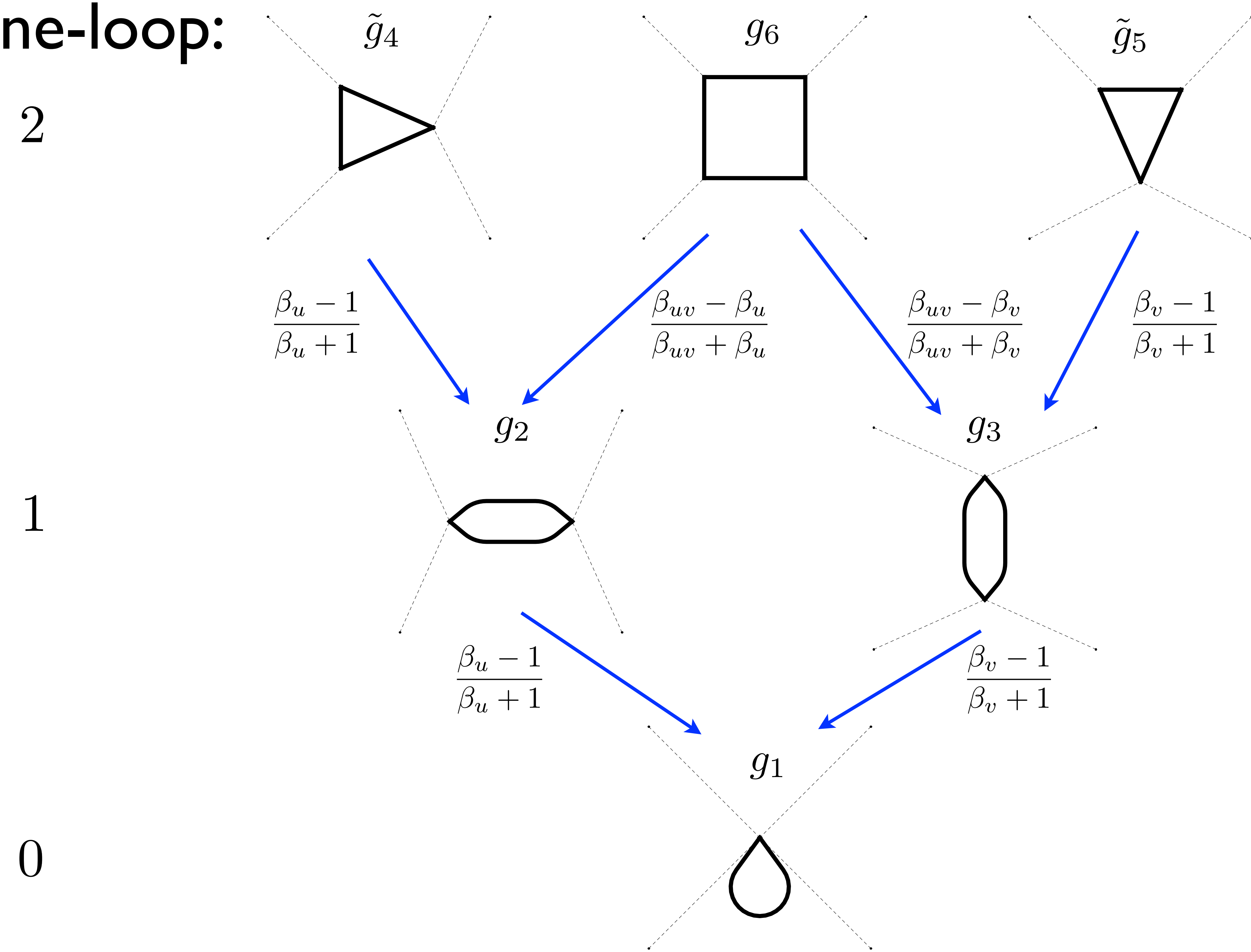
Part III

Our original story...

- Our original goal was to compute the simplest amplitude involving massive particles in N=4, (2→2), to see if the 'simplicity' of N=4 survived finite masses.
- Previously the mass had been used as a regulator
- Long after 'guessing' the 2-loop result, we found a simple way to derive it, using IBPs and differential equations, restricted to (convergent) DCI integrals defined in D=4.

[Henn & SCH]

Ex. One-loop:



In D=4, the non-DCI triangles decouple from the box.

- Simple differential equation for box, bubbles and tadpole:

$$d\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_6 \end{pmatrix} = d\begin{pmatrix} 0 & 0 & 0 & 0 \\ \log\left(\frac{\beta_u-1}{\beta_u+1}\right) & 0 & 0 & 0 \\ \log\left(\frac{\beta_v-1}{\beta_v+1}\right) & 0 & 0 & 0 \\ 0 & \log\left(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}\right) & \log\left(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}\right) & 0 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_6 \end{pmatrix}$$

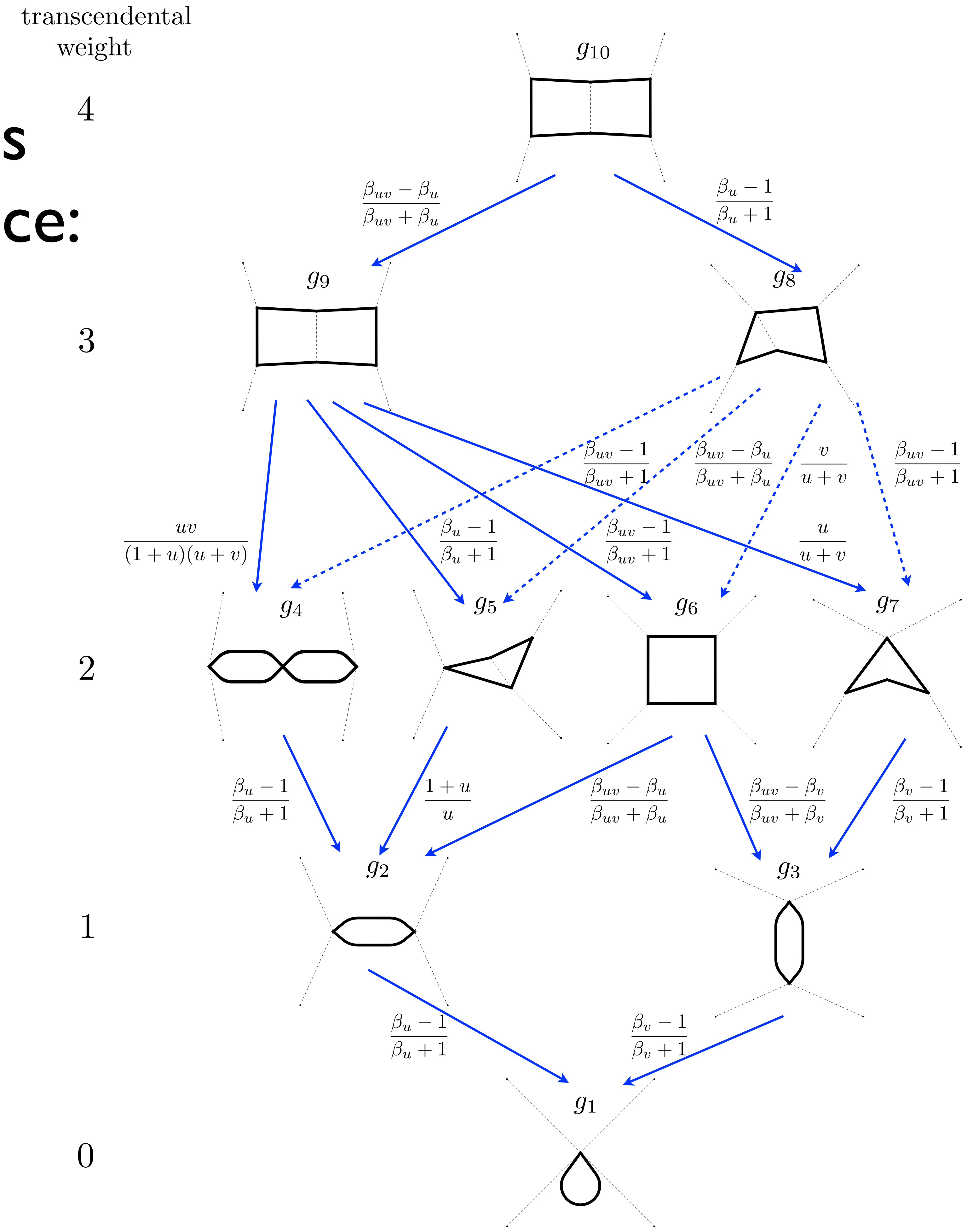
(where: $u = \frac{4m^2}{-s} \qquad v = \frac{4m^2}{-t}$

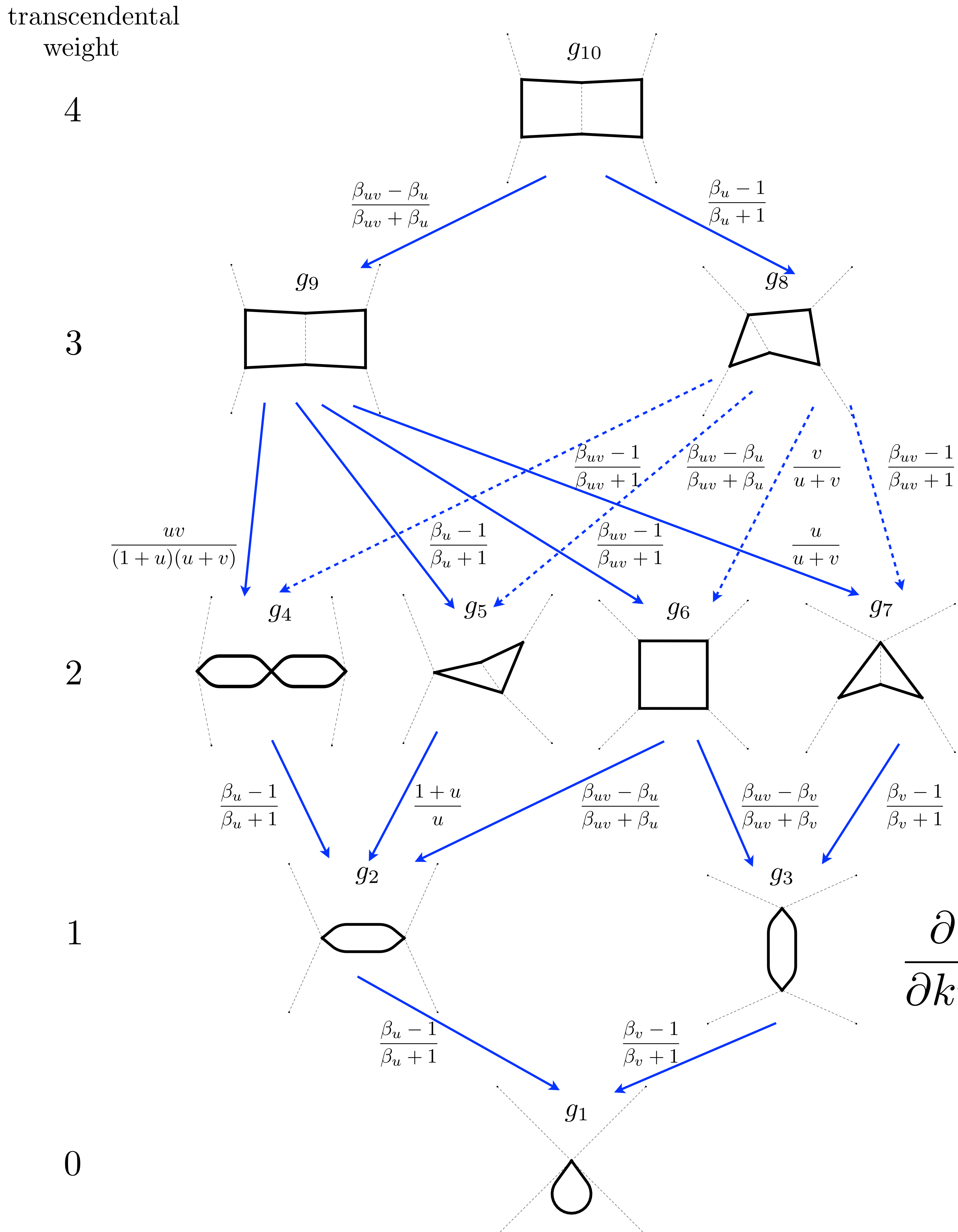
$$\beta_u = \sqrt{1+u}\,, \quad \beta_v = \sqrt{1+v}\,, \quad \beta_{uv} = \sqrt{1+u+v}$$)

- Boundary condition is extremely simple for this problem:

$$\lim_{s,t\to 0} g_i(s,t) = \delta_{i,1}$$

Two-loops also very nice:





Note mixing of loop orders!

$$\frac{\partial}{\partial k_1^\mu}\frac{(k_1-k_2)^\mu}{(k_1-k_2)^4}=2\pi^2 i\delta^4(k_1-k_2)$$

- The two-loop calculation, correctly formulated, turned out to be fully automated and to require little CPU time (~min).
- This prompted us to do three-loops.
- The result is recorded in [1404.2922]; I want to discuss one feature of it.

- The subleading 1/s term in the Regge limit, at leading-log, turns out to be a sum of two exponentials:

$$M(s,t) \sim s^{g^2 j_0^{(1)}(t)} \left(1 + \frac{c_1(t)}{s}\right) + \frac{c_2(t)}{s} s^{g^2 c_3(t)} + O(1/s^2)$$

- Note that testing this hypothesis *required* 3-loops

- Furthermore there seemed to be very nice structure in c_1

- Trying to explain this 'fine detail' in terms of the symmetries of the t-channel led us to Itzykson&Zuber's treatment of the ladders, and to Bander&Itzykson's review of O(4) symmetry, ...

- The result is that using O(4) instead of O(3) partial waves completely decouples the two powers:

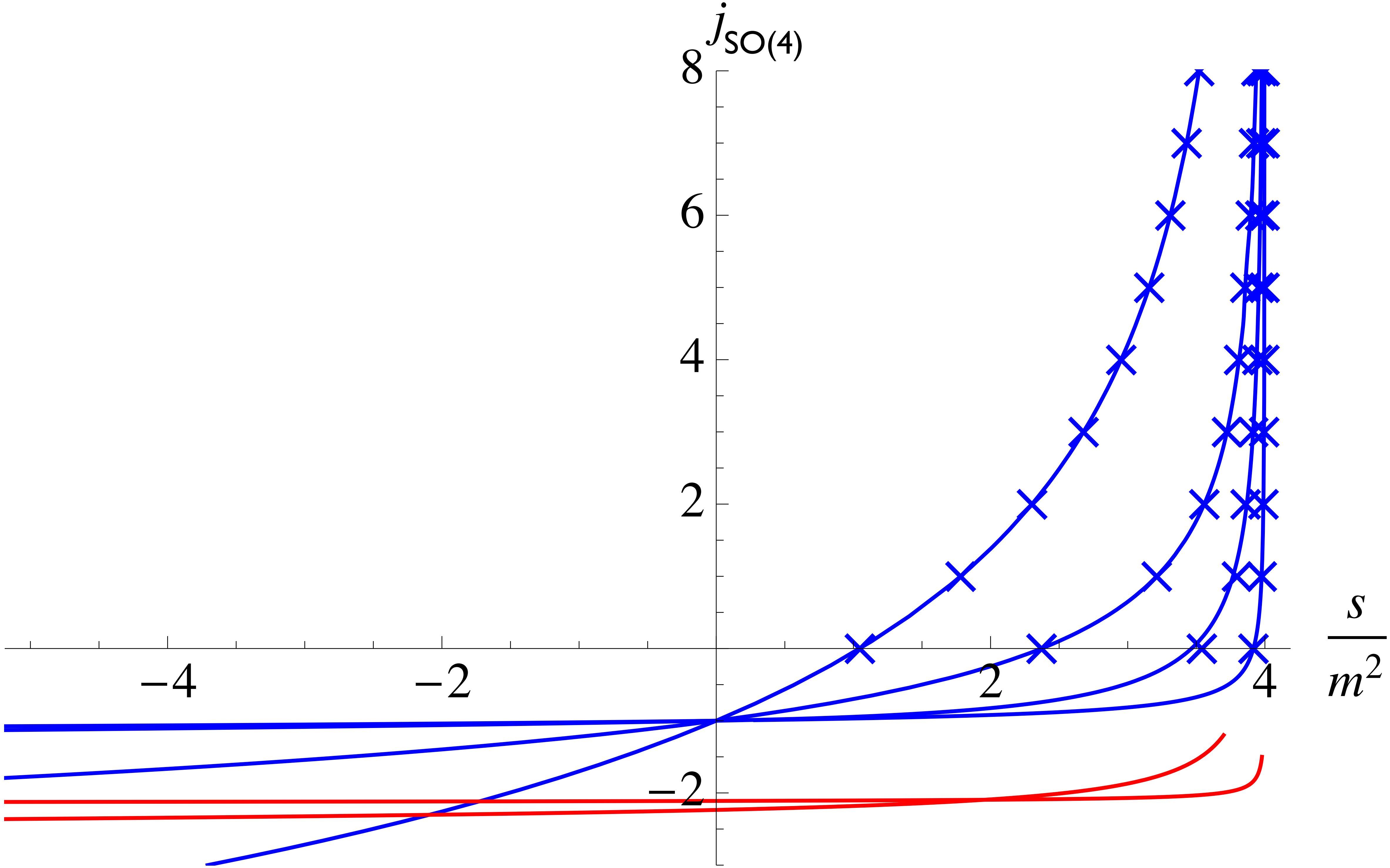
$$\lim_{s\to\infty} \frac{1+e^{-\rho}}{1-e^{-\rho}} M\left(\frac{4m^2}{-s}, \frac{4m^2}{-t}\right) = r_0(t) e^{(j_0(t)+1)\rho} + r_1(t) e^{(j_1(t)+1)\rho} + \mathcal{O}(e^{-2\rho}), \quad \left(\cosh\rho = 1 + \frac{2s}{t} - \frac{s}{2m^2}\right)$$

- We thus obtain the subleading trajectory to 3-loops,

$$\begin{aligned} j_1 =& -2 - 4g^2 + g^4\left(16 - \frac{4}{3\xi}\varphi^3 + 8(\varphi - 2\xi)(\varphi - \frac{1}{\xi}\zeta_2)\right) \\ &+ g^6\left[\frac{24}{\xi}\mathrm{Li}_4(e^{-2\varphi}) + \left(64 + \frac{16\varphi}{\xi}\right)\mathrm{Li}_3(e^{-2\varphi}) + 64(\varphi+\xi)\mathrm{Li}_2(e^{-2\varphi}) - 128\varphi\xi\log(1-e^{-2\varphi})\right. \\ &\quad + \frac{8}{5\xi}\varphi^5 - \frac{8}{3}\varphi^4\left(5 + \frac{1}{\xi}\right) + \frac{16}{3}\varphi^3\left(4 + 7\xi + \frac{1+4\zeta_2}{\xi}\right) - 16\varphi^2\left(3 + 6\zeta_2 + 4\xi + 2\xi^2 + \frac{\zeta_2}{\xi}\right) \\ &\quad \left. + 16\varphi\left(4\zeta_2 + 6\xi(2+\zeta_2) + \frac{11\zeta_4 - \zeta_3 + 2\zeta_2}{\xi}\right) - 24\zeta_4\left(10 + \frac{1}{\xi}\right) + 32\zeta_3 - 64\zeta_2(1+\xi) - 128\right] \end{aligned}$$

- This should be dual to a dimension-one ‘decoration’ of the cusped Wilson line.

Q: Identify and reproduce it from TBA?



The leading O(4) trajectory (blue), for λ=1000,100,15,5 (extrapolating weak/strong coupling); first subleading trajectory(red)

Conclusions

- The planar N=4 SYM model is part of a natural series of integrable systems:

  classical Kepler problem → H atom
  → planar N=4 SYM → … ?

- Although conformal, the model incorporates massive particles in a natural way
- Leading Regge trajectory, aka Γ_{cusp} here, gives bound state; how to interpret subleading ones?
- Analytic computations with three different mass scales possible! Apply to other processes?