From Cuts to Coproducts of Feynman Integrals

Ruth Britto

IPhT, CEA Saclay

Amplitudes 2014
(19th Itzykson Conference)
June 10, 2014

Based on 1401.3546 with Samuel Abreu, Claude Duhr, and Einan Gardi

Scattering amplitudes: fundamental interactions

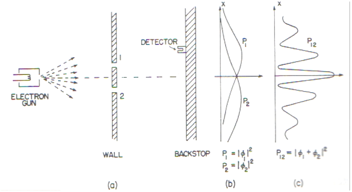


Image: The Feynman Lectures on Physics

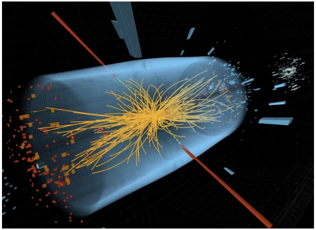


Image: CERN

From double-slit experiments to Higgs discovery at the LHC, and beyond...
scattering amplitudes lie at the heart of quantum theory.

$$\text{Probability} = |A(p_1, p_2, \ldots)|^2$$

Also interesting in formal investigations: structure of gauge theory, integrability, gravity and supergravity, various dualities, ...

Motivation: loops and cuts

- Loop integrals are **necessary**

  ...for high precision at high energy

- Loop integrals are **hard**

- Amplitudes are **simple**

  compared to individual Feynman diagrams.

Amplitude simplicity & the on-shell framework

- Why are amplitudes so simple?
- How can we use the simplicity to calculate?

Key ideas:

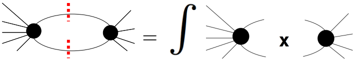
- On-shell framework: recycle amplitudes
- Use singularities to construct integrals

The on-shell framework: replace Feynman rules by constructions from singularities, i.e. complex poles and discontinuities across branch cuts.

On-shell approach at one loop

Unitarity cuts & generalized cuts for 1-loop amplitudes
[Bern, Dixon, Dunbar, Kosower; Anastasiou, RB, Buchbinder, Cachazo, Feng, Kunszt, Mastrolia; ...]

A "cut" takes virtual particles to be physical ("on shell"),
giving the discontinuity across a branch cut of the amplitude.
[Cutkosky; Veltman]



$= \text{Disc}\, A^{1-\text{loop}}.$

Ingredients are complete tree-level amplitudes. Exploit their simplicity.

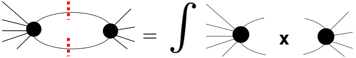
On-shell approach at one loop

Unitarity cuts & generalized cuts for 1-loop amplitudes

[Bern, Dixon, Dunbar, Kosower; Anastasiou, RB, Buchbinder, Cachazo, Feng, Kunszt, Mastrolia; ...]

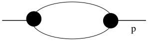
A "cut" takes virtual particles to be physical ("on shell"),
giving the discontinuity across a branch cut of the amplitude.

[Cutkosky; Veltman]

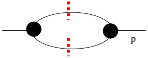


$= \text{Disc}\, A^{1-\text{loop}}.$

Example: bubble integral



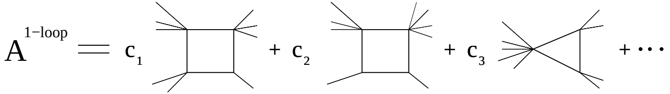
$= \int d^{4-2\epsilon}\ell \,\frac{1}{\ell^2}\,\frac{1}{(\ell+p)^2} = \frac{1}{\epsilon} - \log(-p^2) + \cdots$

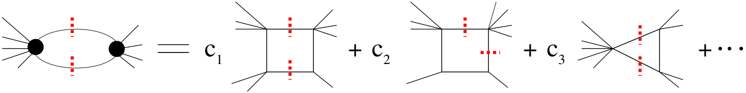


$= \int d^{4-2\epsilon}\ell \;\delta(\ell^2)\delta((\ell+p)^2) = 2\pi i + \cdots$

The cut method for one-loop amplitudes

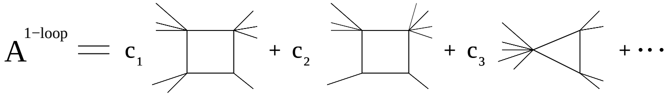
There is a canonical, known set of “master integrals” (with log and Li_2):



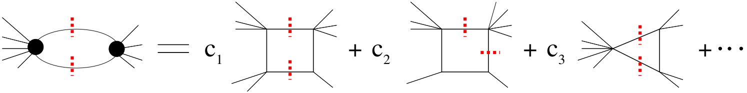


The cut method for one-loop amplitudes

There is a canonical, known set of "master integrals" (with log and Li_2):



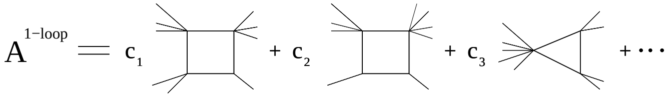
Match cuts of amplitudes with cuts of known master integrals $\rightarrow$ solve for the coefficients.



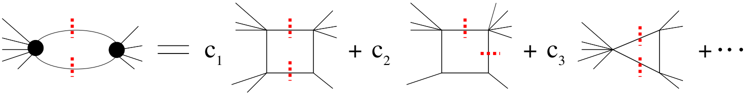
Classic alternative: dispersion relation.

The cut method for one-loop amplitudes

There is a canonical, known set of "master integrals" (with log and Li_2):



Match cuts of amplitudes with cuts of known master integrals $\rightarrow$ solve for the coefficients.



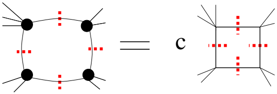
Classic alternative: dispersion relation.

Beyond one loop:

- No standard master integrals
- Many more master integrals
- Few integrals known analytically

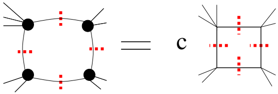
Generalized Cuts

Generalized cuts are the most powerful. The quadruple-cut is extremely effective at 1 loop.

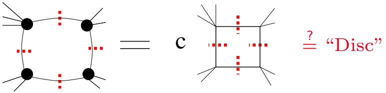


Generalized Cuts

Generalized cuts are the most powerful. The quadruple-cut is extremely effective at 1 loop.



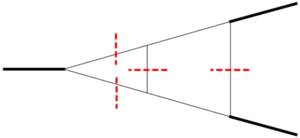
But the deep mathematics has been obscure, making it difficult to extend to more loops.



Hopf algebra structure may be the key!

Cuts are discontinuities

Cutkosky: Cuts are discontinuities across branch cuts



$\overset{?}{=}$ "Disc"

Our claim:
For massless integrals in the class of multiple polylogarithms (MPL),
the discontinuities described by cuts are naturally found within the Hopf algebra of MPL.

3 equivalent definitions of discontinuities: "Cut = Disc = δ"

Known for the first cut; we extend it to sequences of cuts.

In this talk: I explain this claim and give some examples, and then comment on reconstruction of the full integral from its cuts.

Multiple polylogarithms (MPL)

A large class of (massless) integrals are described by multiple polylogarithms:

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) \equiv \int_{a_0}^{a_{n+1}} \frac{dt}{t - a_n} I(a_0; a_1, \ldots, a_{n-1}; t)$$

Examples:

$$I(0;0;z) = \log z, \quad I(0;a;z) = \log\left(1 - \frac{z}{a}\right)$$
$$I(0;\vec{a}_n;z) = \frac{1}{n!}\log^n\left(1 - \frac{z}{a}\right), \quad I(0;\vec{0}_{n-1}, a; z) = -\mathrm{Li}_n\left(\frac{z}{a}\right)$$

Harmonic polylog if all $a_i \in \{-1, 0, 1\}$.

n is the *transcendental weight*.

Observation: most known Feynman integrals can be written in terms of classical and harmonic polylogs.

Hopf algebra

Product and coproduct:

$$\mu : \quad \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, \qquad \Delta : \quad \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$$

Compatible:

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)\,,$$

The algebra is graded by transcendental weight:

$$\mathcal{H}_n \xrightarrow{\Delta} \bigoplus_{k=0}^{n} \mathcal{H}_k \otimes \mathcal{H}_{n-k}\,.$$

Coassociative (i.e. $a \otimes b \otimes c$ is unambiguous), and

$$\Delta_{n_1,\ldots,n_k} : \mathcal{H}_n \to \mathcal{H}_{n_1} \otimes \ldots \otimes \mathcal{H}_{n_k}\,.$$

Hopf algebra of MPL

Goncharov's coproduct formula for MPL (modulo π):

$$\Delta I(a_0; a_1, \ldots, a_n; a_{n+1})$$

$$= \sum_{0=i_0<\cdots<i_k<i_{k+1}=n+1} I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{m+1}) \otimes \prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}})$$

Examples:

$$\begin{aligned}
\Delta(a \cdot b) &= \Delta(a) \cdot \Delta(b) \\
\Delta(1) &= 1 \otimes 1 \\
\Delta(\log z) &= 1 \otimes \log z + \log z \otimes 1 \\
\Delta(\log x \log y) &= 1 \otimes (\log x \log y) + \log x \otimes \log y + \log y \otimes \log x + (\log x \log y) \otimes 1 \\
\Delta(\mathrm{Li}_n(z)) &= 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\log^k z}{k!}
\end{aligned}$$

Symbols of MPL

The "symbol" $\mathcal{S}$ is essentially the maximal iteration.

$$\mathcal{S}(F) \equiv \Delta_{1,\ldots,1}(F) \in \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_1 \,.$$

$$\begin{aligned} \mathcal{S}\left(\frac{1}{n!}\log^n z\right) &= \underbrace{z \otimes \cdots \otimes z}_{n \text{ times}} \\ \mathcal{S}(\mathrm{Li}_n(z)) &= -(1-z) \otimes \underbrace{z \otimes \cdots \otimes z}_{(n-1) \text{ times}} \end{aligned}$$

(most familiar from remainder functions [Goncharov, Spradlin, Vergu, Volovich])

Coproducts of Feynman integrals

Observation: first entries are Mandelstam invariants, and

$$\Delta_{1,n-1} F = \sum_i \log(-s_i) \otimes f_{s_i}$$

where f_{s_i} is the discontinuity of F in the channel s_i. [Gaiotto, Maldacena, Sever, Viera]

Thus: the coproduct captures standard cuts.

Is there an extension to generalized cuts?

Cut=Disc=δ for generalized cuts

- Need to define generalized cuts: as a sequence of traditional cuts.
- Need to specify kinematic regions.
- Need to identify the MPL alphabet and explain the correspondence.
- Limited by: number of channels, transcendental weight, and number of independent variables.

Definition of Disc

The discontinuity across the branch cut.

$$\text{Disc}_x\left[F(x \pm i0)\right] = \lim_{\varepsilon \to 0}\left[F(x \pm i\varepsilon) - F(x \mp i\varepsilon)\right],$$

Example:

$$\text{Disc}_x \log(x + i0) = 2\pi i\, \theta(-x).$$

Sequential:

$$\text{Disc}_{x_1,\ldots,x_k} F \quad \equiv \quad \text{Disc}_{x_k}\left(\text{Disc}_{x_1,\ldots,x_{k-1}} F\right).$$

Definition of multiple cuts

$$\mathrm{Cut}_{s_1,\ldots,s_k} F$$

With *real* kinematics.

Defined by: cut propagators + consistent energy flow + corresponding kinematic region

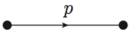
Region is such that cut invariants s_i are positive and all others are negative.

Multiple cuts are taken simultaneously.

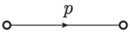
Cutting Rules

Traditional [Veltman]:

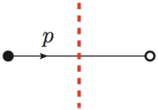
$$\bullet = i \qquad \circ = -i$$



$$= \frac{i}{p^2 + i\varepsilon}$$



$$= \frac{-i}{p^2 - i\varepsilon}$$



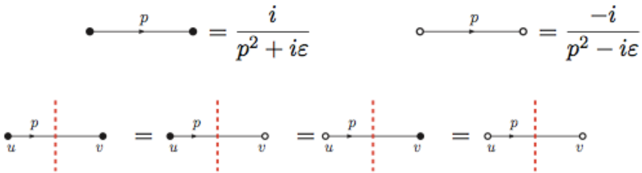
$$= 2\pi\,\delta\left(p^2\right)\theta\left(p_0\right)$$

for massless scalar theory.

Cutting Rules

Generalized:

$$\bullet = i \qquad\qquad \circ = -i$$



$$= 2\pi\,\delta\left(p^2\right) \prod_{i:c_i(u)\neq c_i(v)} \theta\left([c_i(v) - c_i(u)]p_0\right)$$

Colors are $c_i = 0, 1$ for each cut i.

We must allow repeated cuts of same propagator or same loop – but forbid the same channel.

From Mandelstam invariants to MPL variables

From the Largest Time Equation [Veltman]:

$$F + F^* = -\sum_s \mathrm{Cut}_s\, F,$$

Hence:

$$\mathrm{Disc}_s\, F = -\,\mathrm{Cut}_s\, F.$$

Generalize to:

$$\boxed{\mathrm{Cut}_{s_1,\ldots,s_k}\, F = (-1)^k\, \mathrm{Disc}_{s_1,\ldots,s_k}\, F.}$$

Valid in a particular kinematic region: cut invariants s_i positive, others negative.

Strictly real kinematics.

Definition of δ

If

$$\Delta_{\underbrace{1,1,\ldots,1}_{k \text{ times}},n-k} F = \sum_{\{x_1,\ldots,x_k\}} \log x_1 \otimes \cdots \otimes \log x_k \otimes g_{x_1,\ldots,x_k} ,$$

then

$$\delta_{x_1,\ldots,x_k} F \cong g_{x_1,\ldots,x_k} .$$

More precisely: match branch points. The "$\cong$" means modulo π.

Motivated by coproduct identity : $\Delta \, \text{Disc} = (\text{Disc} \otimes \text{id}) \, \Delta$ [Duhr]
and first entry condition.

If $\delta_x F \cong g_x$, then $\text{Disc}_x F \cong (\text{Disc}_x \otimes \text{id})(\log x \otimes g_x) = \pm 2\pi i \, g_x$. Sign determined by $i\varepsilon$ prescription.

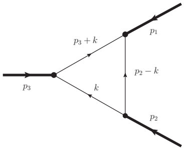
Coproduct and discontinuities for Feynman integrals

$$\mathrm{Disc}_{s_1} F = \theta(s_1)\,(-2\pi i)\,\delta_{s_1} F.$$

$$\boxed{\mathrm{Disc}_{s_1,\ldots,s_k} F \cong \Theta \sum_{x_1,\ldots,x_k} \pm(2\pi i)^k \delta_{x_1,\ldots,x_k} F.}$$

- $\cong$ means mod π
- Θ restricts to the correct kinematic region (corresponding to cuts)
- Assume prior knowledge of alphabet (e.g. from cuts)
- Underlying claim: kinematics put us on the branch cuts, so that it is correct to use our definition of Disc.

Basic example: 3-mass triangle



$$
\begin{aligned}
T &= -\frac{i}{p_1^2}\frac{2}{z-\bar{z}}\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right) \\
&\equiv -\frac{i}{p_1^2}\frac{2}{z-\bar{z}}\ \mathcal{P}_2
\end{aligned}
$$

$$
\mathrm{Disc}_{p_2^2}\ T = -\frac{2\pi}{p_1^2(z-\bar{z})}\log\frac{1-z}{1-\bar{z}}
$$

Coproduct of the 3-mass triangle

$$\begin{aligned}
\Delta \mathcal{P}_2 &= \mathcal{P}_2 \otimes 1 + 1 \otimes \mathcal{P}_2 + \frac{1}{2}\log(z\bar{z}) \otimes \log\frac{1-z}{1-\bar{z}} + \frac{1}{2}\log[(1-z)(1-\bar{z})] \otimes \log\frac{\bar{z}}{z} \\
&= \mathcal{P}_2 \otimes 1 + 1 \otimes \mathcal{P}_2 + \frac{1}{2}\log\left(-p_2^2\right) \otimes \log\frac{1-z}{1-\bar{z}} + \frac{1}{2}\log\left(-p_3^2\right) \otimes \log\frac{\bar{z}}{z} \\
&\quad + \frac{1}{2}\log(-p_1^2) \otimes \log\frac{1-1/\bar{z}}{1-1/z}
\end{aligned}$$

Alphabet: $\{z, \bar{z}, 1-z, 1-\bar{z}\}$.

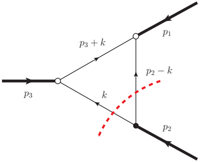
Here

$$z\bar{z} = \frac{p_2^2}{p_1^2}, \qquad (1-z)(1-\bar{z}) = \frac{p_3^2}{p_1^2}$$

$$\sqrt{\lambda} = z - \bar{z}$$

First cut of the 3-mass triangle

Cut in the p_2^2 channel.

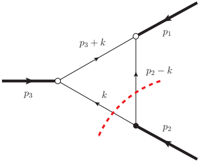


Kinematic region: $p_2^2 > 0; \quad p_1^2, p_3^2 < 0.$

$$\begin{aligned}
\mathrm{Cut}_{p_2^2}\, T &= -2\pi \int_{-1}^{1} d\cos\theta \, \frac{1}{p_3^2 + p_1^2 - p_2^2 - \cos\theta\sqrt{\lambda(p_1^2, p_2^2, p_3^2)}} \\
&= -\frac{2\pi}{p_1^2} \int_0^1 dx \, \frac{1}{1 - \bar{z} - x\sqrt{\lambda}} \\
& \qquad \sqrt{\lambda} = z - \bar{z} \\
&= \frac{2\pi}{p_1^2(z - \bar{z})} \log \frac{1-z}{1-\bar{z}}
\end{aligned}$$

First cut of the 3-mass triangle

Cut in the p_2^2 channel.

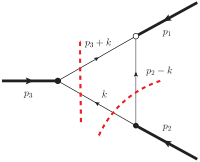


Kinematic region: $p_2^2 > 0; \quad p_1^2, p_3^2 < 0.$

$$\begin{aligned} \mathrm{Cut}_{p_2^2}\, T &= \frac{2\pi}{p_1^2(z-\bar{z})} \log\frac{1-z}{1-\bar{z}} \\ &= -\,\mathrm{Disc}_{p_2^2}\, T \end{aligned}$$

$$\mathrm{Disc}_{p_2^2}\, T \cong (-2\pi i)\delta_{p_2^2} T.$$

Second cut of the 3-mass triangle



$$\text{Cut}_{p_3^2, p_2^2} \; T = \frac{4\pi^2 i}{p_1^2 (z - \bar{z})}$$

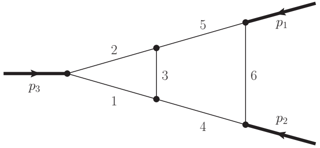
Kinematic region: $p_3^2, p_2^2 > 0; \quad p_1^2 < 0$
Equivalently: $\bar{z} < 0, z > 1$.

Now we have to match the alphabet with Mandelstam invariants:

$$\text{Disc}_{p_2^2, p_3^2} \; T = \text{Cut}_{p_2^2, p_3^2} \; T.$$

$$\text{Disc}_{p_2^2, p_3^2} \; T \cong 4\pi^2 \, \Theta \;\; \delta_{p_2^2, 1-z} \, T$$

2-loop example: 3-point ladder

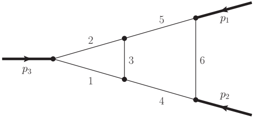


$$L = i\left(p_1^2\right)^{-2}\frac{1}{(1-z)(1-\bar{z})(z-\bar{z})}F$$

$$\begin{aligned} F &= 6\left[\mathrm{Li}_4(z)-\mathrm{Li}_4(\bar{z})\right]-3\log(z\bar{z})\left[\mathrm{Li}_3(z)-\mathrm{Li}_3(\bar{z})\right] \\ &+\frac{1}{2}\log^2(z\bar{z})\left[\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar{z})\right]. \end{aligned}$$

$$z\bar{z}=\frac{p_2^2}{p_1^2}, \quad (1-z)(1-\bar{z})=\frac{p_3^2}{p_1^2}$$

Coproduct of the ladder

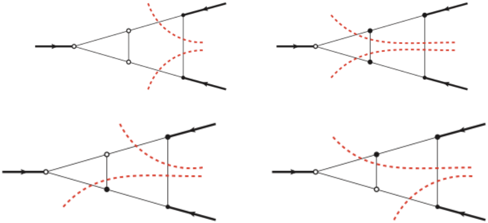


$$
\begin{aligned}
\Delta_{1,1,2}F \quad = \quad & \log\frac{p_3^2}{p_1^2} \otimes \log z \otimes \left(\log z \log \bar{z} - \frac{1}{2}\log^2 \bar{z}\right) \\
& - \log\frac{p_3^2}{p_1^2} \otimes \log \bar{z} \otimes \left(\log z \log \bar{z} - \frac{1}{2}\log^2 z\right) \\
& - \log\frac{p_2^2}{p_1^2} \otimes \log(1-z) \otimes \left(\log z \log \bar{z} - \frac{1}{2}\log^2 z\right) \\
& + \log\frac{p_2^2}{p_1^2} \otimes \log(1-\bar{z}) \otimes \left(\log z \log \bar{z} - \frac{1}{2}\log^2 \bar{z}\right) \\
& + \log\frac{p_2^2}{p_1^2} \otimes \log(z\bar{z}) \otimes [\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})] \ ,
\end{aligned}
$$

Individual cuts diverge, but sum is finite.

Same alphabet as triangle. $\frac{p_2^2}{p_1^2} = z\bar{z}, \ \frac{p_3^2}{p_1^2} = (1-z)(1-\bar{z})$

Two cuts of the ladder

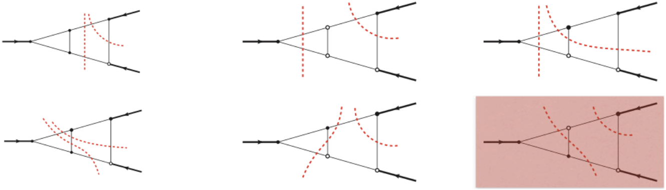


$$\mathrm{Cut}_{p_1^2,p_2^2} F = \mathrm{Disc}_{p_1^2,p_2^2}\, F \cong -(2\pi i)^2\, \Theta\; \delta_{p_1^2,\bar{z}} F\,,$$

$$\mathrm{Cut}_{p_2^2,p_1^2} F = \mathrm{Disc}_{p_2^2,p_1^2}\, F \cong (2\pi i)^2\, \Theta\, [\delta_{p_2^2,z} + \delta_{p_2^2,1-z}] F\,,$$

Variables matched within the correct kinematic region:
$p_1^2, p_2^2 > 0$ and $p_3^2 < 0$, or equivalently $0 < \bar{z} < 1 < z$.
Follow $i\varepsilon$ for signs.

Two cuts of the ladder

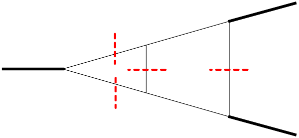


$$\mathrm{Cut}_{p_1^2,p_3^2} F = \mathrm{Disc}_{p_1^2,p_3^2} F \cong -(2\pi i)^2 \,\Theta\, \delta_{p_1^2,1-z} F\,,$$
$$\mathrm{Cut}_{p_3^2,p_1^2} F = \mathrm{Disc}_{p_3^2,p_1^2} F \cong (2\pi i)^2 \,\Theta\, [\delta_{p_3^2,\bar{z}} + \delta_{p_3^2,1-\bar{z}}] F\,,$$

Kinematic region: $p_1^2, p_3^2 > 0$ and $p_2^2 < 0$, or equivalently $\bar{z} < 0 < z < 1$.

Cuts live strictly within their own regions

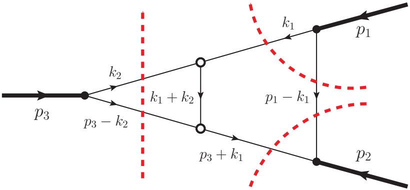
Notice: the "1236" cut diagram is common to both double-cuts, but gives different values!



$\stackrel{?}{=}$ "Disc"

Must place individual cut diagrams in kinematic contexts.

Third cut of the ladder?



Cut $= 0$ in the kinematic region $p_1^2, p_2^2, p_3^2 > 0$.

Disc $= 0$: no region detects the three cuts simultaneously.

It's consistent, but we must continue to more complicated examples. (in progress)

Reconstructing the full coproduct from cuts

Integrability condition on symbols: for each k,

$$\sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n}\, d\log a_{i_k} \wedge d\log a_{i_{k+1}}\; a_{i_1} \otimes \cdots \otimes a_{k-1} \otimes a_{k+2} \otimes \cdots \otimes a_{i_n} = 0\,.$$

Combine with first entry condition (=Mandelstam invariant) and known cut(s).

Remark: integrability may sometimes account for exchanging order of cuts.

Reconstructing the full coproduct from cuts

Integrability condition on symbols: for each k,

$$\sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n}\, d\log a_{i_k} \wedge d\log a_{i_{k+1}}\; a_{i_1} \otimes \cdots \otimes a_{k-1} \otimes a_{k+2} \otimes \cdots \otimes a_{i_n} = 0\,.$$

Combine with first entry condition (=Mandelstam invariant) and known cut(s).
Example: p_2^2 cut of triangle.

$$\frac{1}{2}\,(z\bar{z}) \otimes \frac{1-z}{1-\bar{z}}\,.$$

- For integrability, add

$$\frac{1}{2}\,(1-z) \otimes \bar{z} - \frac{1}{2}\,(1-\bar{z}) \otimes z\,.$$

- For the first entry condition, add

$$\frac{1}{2}\,(1-\bar{z}) \otimes \bar{z} - \frac{1}{2}\,(1-z) \otimes z\,.$$

Both conditions are satisfied.

Result:

$$\mathcal{S}(\mathcal{T}) = \frac{1}{2}\, z\bar{z} \otimes \frac{1-z}{1-\bar{z}} + \frac{1}{2}\,(1-z)(1-\bar{z}) \otimes \frac{\bar{z}}{z},$$

Reconstructing the full coproduct from cuts

Integrability condition on symbols: for each k,

$$\sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n}\, d\log a_{i_k} \wedge d\log a_{i_{k+1}}\; a_{i_1} \otimes \cdots \otimes a_{k-1} \otimes a_{k+2} \otimes \cdots \otimes a_{i_n} = 0\,.$$

Combine with first entry condition (=Mandelstam invariant) and known cut(s).

Reconstruction of the symbol of the ladder is unique from any of its single *or double* cuts.

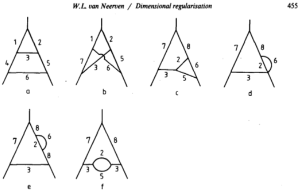
Knowledge of alphabet is crucial.

Reconstructing the full function–from dispersion relations

From the imaginary part, reconstruct the integral:

$$A(K^2) = \frac{1}{\pi} \int_0^\infty ds \; \frac{\mathrm{Im}\; A(s)}{s - K^2}$$

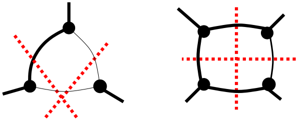
Classic example: On-shell vertex function, 2 loops. [Van Neerven, 1986]



Integration is still hard work.

Double dispersion relations

Previously computed at one loop, with strictly real momenta. [Mandelstam; Ball, Braun, Dosch]. From iterated cuts.



Dispersion relations and MPL

Classic dispersion relation for triangle is complicated:

$$T = -\frac{1}{2\pi i}\int_0^\infty \frac{ds}{s-(p_2^2+i\varepsilon)}\frac{2\pi}{\sqrt{\lambda(p_1^2,s,p_3^2)}}\log\frac{p_1^2-s+p_3^2-\sqrt{\lambda(p_1^2,s,p_3^2)}}{p_1^2-s+p_3^2+\sqrt{\lambda(p_1^2,s,p_3^2)}}.$$

With MPL alphabet:

$$T = \frac{-i}{p_1^2}\frac{1}{z-\bar{z}}\int_0^1 dw\left(\frac{1}{w-\bar{z}}-\frac{1}{w-z}\right)\left[2\log(1-w)-\log(1-z)(1-\bar{z})\right]$$

Dispersion relations and MPL

Classic dispersion relation for triangle is complicated:

$$T = -\frac{1}{2\pi i}\int_0^\infty \frac{ds}{s-(p_2^2+i\varepsilon)}\frac{2\pi}{\sqrt{\lambda(p_1^2,s,p_3^2)}}\log\frac{p_1^2-s+p_3^2-\sqrt{\lambda(p_1^2,s,p_3^2)}}{p_1^2-s+p_3^2+\sqrt{\lambda(p_1^2,s,p_3^2)}}\,.$$

With MPL alphabet:

$$T = \frac{-i}{p_1^2}\frac{1}{z-\bar{z}}\int_0^1 dw\left(\frac{1}{w-\bar{z}}-\frac{1}{w-z}\right)\left[2\log(1-w)-\log(1-z)(1-\bar{z})\right]$$

Double dispersion relation:

$$T = \frac{-i}{p_1^2}\int_1^\infty dw\int_{-\infty}^0 d\bar{w}\,\frac{1}{w\bar{w}-z\bar{z}}\,\frac{1}{(1-w)(1-\bar{w})-(1-z)(1-\bar{z})}\,.$$

Similar for ladder.

Summary & Outlook

- We propose relations of the form Cut $=$ Disc $= \delta$ for sequential cuts and explain how they can be made precise.
  - Important to posit correct alphabet – and cuts gives clues!
  - MPL formalism helps for performing cut integrals.
- We propose avenues for reconstructing the coproduct/symbol/full integral making use of the integrability of the symbol.
- Need to study a wide range of examples.
- What about masses? Hopf algebra not yet formulated for that case.
- Make contact with maximal cuts/ complex residues of multiloop integrals?
- Should cutting rules be expanded to capture more of the coproduct? e.g. repeated cuts in the same channel, crossed cuts, ...