Scattering Equations and Amplitudes in String Theory

Amplitudes 2014

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Computation of perturbative amplitudes

Feynman diagrams:
Factorial Growth!

Sum over topological different diagrams

Generic Feynman amplitude

Complex expressions involving e.g.

$(p_i \bullet p_j)$

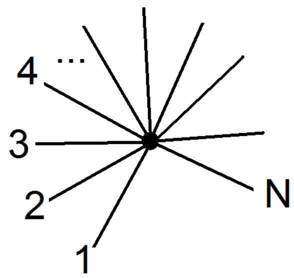
$(p_i \bullet \varepsilon_j)$ $(\varepsilon_l \bullet \varepsilon_j)$

(no manifest symmetry or simplifications)

Amplitudes

Specifying external polarisation tensors $(\varepsilon_i \bullet \varepsilon_j)$

Colour ordering



$Tr(T_1\ T_2\ ..\ T_n)$

Simplifications

Recursion

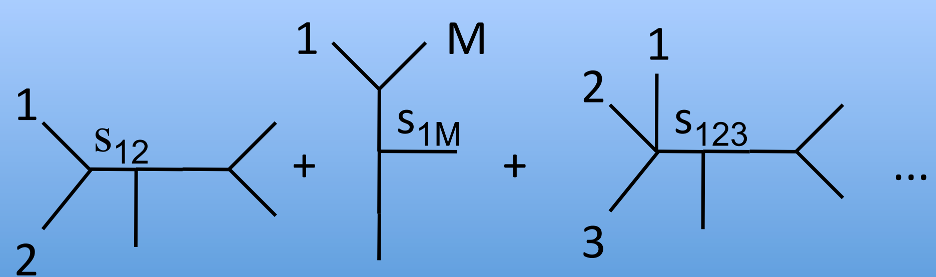
Spinor-helicity formalism

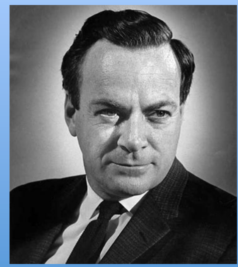
Loop amplitudes: (Unitarity, Symmetries, Supersymmetric decomposition)

Inspiration from String theory

(See various talks)

Amplitude revolution!





Factorial growth ~ billions of terms!!

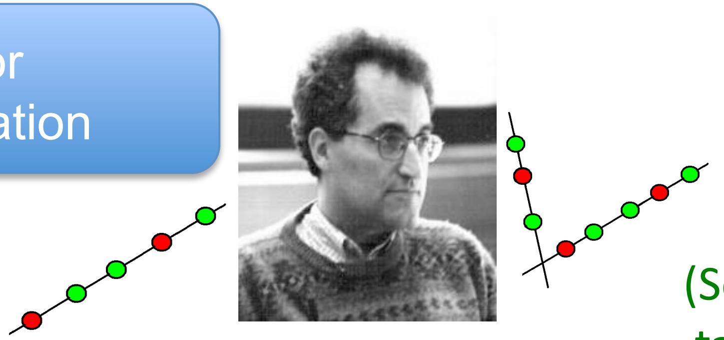
Rich hidden structure

On-shell recursion

MHV one term!

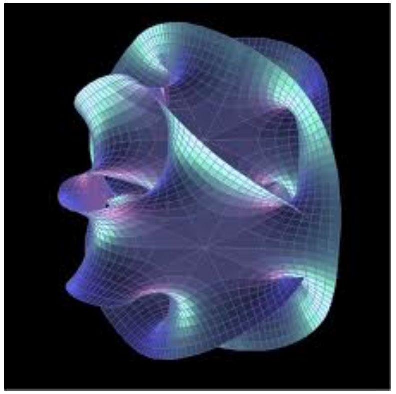
$$\sim \frac{\langle jk\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle M1\rangle}$$

Twistor inspiration

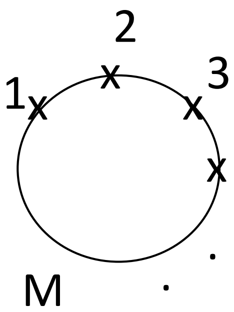


(See Johansson's talk)

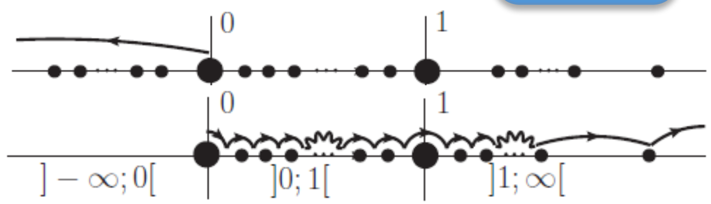
BCJ



String Theory



Monodromy relations



(NEJBB, Damgaard, Vanhove; Stieberger)

A new prescription for perturbative amplitudes

It was suggested recently by Cachazo, He and Yuan that one can compute amplitudes via

(Cachazo, He and Yuan)

$$\mathcal{A}_n = \int \frac{d^n\sigma}{\mathrm{volSL}(2,\mathrm{c})}\prod_a{}' \delta\left(\sum_{a\neq b}\frac{k_a\cdot k_b}{z_a-z_b}\right)$$
$$\left(\frac{\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}\cdots T^{a_n})}{(z_1-z_2)(z_2-z_3)\cdots(z_n-z_1)}+\cdots\right)^{2-s}(\mathrm{Pf}'\Psi)^s$$

Color trace

Algebraic solutions

Pfaffian (dependent on polarisations and momenta)

(See Yuan's talk)

The N-point scalar amplitude

For the N-point scalar amplitude one has

(Cachazo, He and Yuan; Dolan Goddard)

$$A_N = \int \prod_i{}' \delta(S_i) \frac{(z_1 - z_{N-1})(z_1 - z_N)(z_{N-1} - z_N)}{\prod_{i=1}^{N}(z_i - z_{i+1})^2} \prod_{i=2}^{N-1} dz_i$$

Here

$$S_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0$$

Sum over solutions

Generally complicated solutions at higher points. N-roots of Polynomial equations. (can be complex)

are the scattering equations where

$$z_1 = 0, z_{N-1} = 1 \text{ and } z_N = \infty$$

Much like standard Kobe-Nielsen gauge fixing

The scattering equations of Cachazo, He and Yuan

$$S_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0 \quad z_1 = 0, z_{N-1} = 1 \text{ and } z_N = \infty$$

The scattering equations are not new and they in fact are used in early work on Dual Models of (Fairlie and Roberts, 1970-1972)

Some recent results (following Cachazo, He and Yuan's work)

- (Dolan and Goddard) Proof of amplitude formulas via BCFW Recursion
- (Mason and Skinner; Adamo, Casali and Skinner) Calculations of amplitudes from the view point of Ambi-twistor space.
- (Berkovits; Gomez, Yuan) Calculations of amplitudes from the view point of pure spinor formalism.

(See Yuan's talk)

The scattering equations of Cachazo, He and Yuan

$$S_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0 \quad z_1 = 0, z_{N-1} = 1 \text{ and } z_N = \infty$$

View point here is the following:

- Scattering equations and standard Open String theory (NEJBB, Damgaard, Tourkine, Vanhove)
- No assumptions...
- New features as we will see....

The scattering equations

A few comments:

The prescription for amplitudes using the scattering equations appears to be much closer to a String theory framework than field theory (shares a lot of features)

- Picking the $z_1 = 0, z_{N-1} = 1$ and $z_N = \infty$ feels much the like the Kobe-Nielsen gauge fixing encountered In String Theory.

- However no Kobe-Nielsen kinematic factor in the tree amplitude prescription.

Question: Is it supposed to be there?

The scattering equations

- System of solutions to scattering equations is invariant under Mobius transformations (Cachazo, He and Yuan; Dolan and Goddard)

- Interesting feature: Take the e.g. the standard 4 pt (gauge fixed Kobe-Nielsen factor for a string theory (open) amplitude

$$\exp[-\alpha' s \log(x) - \alpha' t \log(1-x)]$$

- One sees that :

$$\partial_x \exp[-\alpha' s \log(x) - \alpha' t \log(1-x)]$$
$$= \alpha' \left(-\frac{s}{x} + \frac{t}{(1-x)} \right) \exp[-\alpha' s \log(x) - \alpha' t \log(1-x)]$$

(Uses of such rewritings can be found in e.g. Polchinski's book)

- I.e. the scattering eq. is brought down by acting with derivatives on the Kobe-Nielsen factor in the string theory integrant (very useful later…)

Example the 4-point scalar amplitude

Following the prescription we get : (Cachazo, He and Yuan)

$$A_4 = \int dx \frac{\delta(S_i)}{(z_{12})^2(z_{23})^2}$$

We have the following total (not-independent) scattering equations

$$\left.\begin{aligned} \frac{s_{12}}{z_{12}} + \frac{s_{13}}{z_{13}} = 0 \\ -\frac{s_{12}}{z_{12}} + \frac{s_{23}}{z_{23}} = 0 \\ \frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} = 0 \end{aligned}\right\}$$

$$s_{12} = s, \quad s_{13} = u, \quad s_{23} = t$$

$$z_4 = \infty, \quad z_1 = 0, \quad z_2 = x, \quad z_3 = 1$$

$$z_{12} = -x, \quad z_{23} = x - 1$$

Solution:

$$x = \frac{s}{s+t}$$

Example the 4-point scalar amplitude

Now we can write:

$$A_4 = \int dx \frac{\delta(S_i)}{(z_{12})^2 (z_{23})^2} = \int dx \frac{\delta\left(\frac{s}{x} - \frac{t}{(1-x)}\right)}{x^2(1-x)^2}$$

$$= \frac{st}{(s+t)^3} \frac{(s+t)^2}{s^2} \frac{(s+t)^2}{t^2} = \frac{(s+t)}{st} = \frac{1}{s} + \frac{1}{t}$$

Which is the correct result for the scalar amplitude!

The N-point scalar amplitude

A simple dual model that will also compute the N-point scalar amplitude is (we use unit coupling)

$$\mathcal{A}_N = (\alpha')^{N-3} \int \left(\prod_{i=1}^{N} dz_i \right) / d\omega \prod_{1 \le i < j \le N} |z_i - z_j|^{2\alpha' k_i \cdot k_j} \prod_{i=1}^{N} (z_i - z_{i+1})^{-1}$$

Here

$$d\omega = \frac{dz_1 dz_{N-1} dz_N}{(z_1 - z_{N-1})(z_{N-1} - z_N)(z_N - z_1)}$$

And we again fix

$z_1 = 0, z_{N-1} = 1$ and $z_N = \infty$

Example the 4-point scalar amplitude

For the 4-point we have the following well known result

$$\mathcal{A}_4 = -(\alpha') \int dx \frac{x^{2\alpha' k_1 \cdot k_2}(1-x)^{2\alpha' k_1 \cdot k_4}}{x(1-x)}$$

$$= -(\alpha') \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(-\alpha'(s+t))} = \left(\frac{1}{s} + \frac{1}{t}\right) + O\left(\alpha'^1\right)$$

We arrive at the same result via the normal integration measure. Logic though is completely different!

Lessons learned

In the simple dual model

- Integration in an ordered manner along the real line. Poles comes from pinching regions.

In the delta function prescription

- Integral saturated by delta-function and amplitude becomes localized. (solutions not necessarily on real line [0,1])

However a model that bridges the two approaches is possible.....

A new dual model

It is thus suggestive to consider the following
new dual model

$$\mathcal{A}_N = \int \left(\prod_{i=1}^{N} dz_i \right) \frac{1}{d\omega} (z_1 - z_{N-1})(z_1 - z_N)(z_{N-1} - z_N)$$

$$\prod_{1 \le i < j \le N} |z_i - z_j|^{2\alpha' k_i \cdot k_j} \prod_i{}' \delta(S_i) \prod_{i=1}^{N} (z_i - z_{i+1})^{-1} \prod_{i=1}^{N} (z_i - z_{i+1})^{-1}$$

$$= \int \prod_{i=2}^{N-2} dz_i \prod_{1 \le i < j \le N-2} |z_i - z_j|^{2\alpha' k_i \cdot k_j} \prod_i{}' \delta(S_i) \prod_{i=1}^{N-2} (z_i - z_{i+1})^{-2}$$

$$d\omega = \frac{dz_1 dz_{N-1} dz_N}{(z_1 - z_{N-1})(z_{N-1} - z_N)(z_N - z_1)}$$

KN factor and
New measure

A new dual model

Effectively this is taking the traditional dual model description and sticking in the normalized delta-function constraint

$$(z_1 - z_{N-1})(z_1 - z_N)(z_{N-1} - z_N)\prod_i{}' \delta(S_i) \prod_{i=1}^{N} (z_i - z_{i+1})^{-1}$$

Rather trivial extension

- So what has been achieved? Finite α' no resemblance to ST
- Smooth limits $\alpha' \to 0$ and $\alpha' \to \infty$ similar to ST.

Massive scalar

As a side remark: massive scalars can be dealt with easily through replacing

$$\prod_{i=1}^{N} (z_i - z_{i+1})^{-1} \to \prod_{i=1}^{N} (z_i - z_{i+1})^{-1-\alpha' m^2}$$

By differentiation of the integrand with respect to z_i we obtain the massive scattering equation proposed and proven to be correct by (Dolan and Goddard).

A new dual model

In order to understand this model better we now turn to gluon amplitudes

We have the following prescription from Cachazo, He and Yuan

$$A_N = \int \mathrm{Pf}'\Psi_N(z_i) \prod_i{}' \delta(S_i) \prod_{i=1}^{N} \frac{1}{(z_i - z_{i+1})} \prod_{i=2}^{N-2} dz_i \,,$$

$$S_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0$$

A new dual model

One has

$$\mathrm{Pf}'\Psi = \frac{(-1)^{i+j}}{z_i - z_j}\mathrm{Pf}(\Psi^{ij}_{ij})$$

$$\Psi_N(z_i) = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

Legs singled out

Dependence on momenta and polarizations

$$A_{i,j} = \frac{k_i \cdot k_j}{z_i - z_j} \quad i \neq j \qquad C_{i,j} = \frac{\epsilon_i \cdot k_j}{z_i - z_j} \quad i \neq j$$

$$B_{i,j} = \frac{\epsilon_i \cdot \epsilon_j}{z_i - z_j} \quad i \neq j \qquad C_{i,i} = -\sum_{l \neq i} \frac{\epsilon_i \cdot k_l}{z_i - z_l}$$

Question: Does ST reproduce this?

Pfaffian structure in open string theory

One has for the N gluon amplitude in the open bosonic string

$$\mathcal{A}^{open}(\sigma(1),\cdots,\sigma(n)) = \int F_n \prod_{1\le i<j\le n} (x_i - x_j)^{-\alpha' k_i\cdot k_j} \prod_{i=2}^{n-2} dx_i$$

where

$$F_n^{bos} = \int \prod_{i=1}^{n} d\eta_i \, \exp\left(-\sqrt{\alpha'} \sum_{i\neq j} \frac{\eta_i(\epsilon_i\cdot k_j)}{x_i - x_j} - \frac{\eta_i\eta_j(\epsilon_i\cdot\epsilon_j)}{(x_i - x_j)^2} \right)$$

$$\int d\eta_i\eta_j = \delta_{ij} \qquad \int d\eta_i = 0$$

Pfaffian structure basically comes out from the fermionic Integration over the η_i

This is easiest seen in the Picture Changing formalism.

Pfaffian structure in open string theory

The (-1)-picture of the unintegrated vertex operator for the emission of a gauge boson is then given by

$$U^{(-1)} = g_o T^a : e^{-\varphi} \epsilon \cdot \psi \, e^{ik \cdot X} :$$

While in the (0) picture one has

$$U^{(0)} = g_o \sqrt{\frac{2}{\alpha'}} T^a : (i\partial X^\mu + 2\alpha'(k \cdot \psi)(\epsilon \cdot \psi)) e^{ik \cdot X} :$$

$$X^\mu(z) X^\nu(0) \simeq -\alpha' \log |z|^2$$

$$\psi^\mu(z)\psi^\nu(0) \simeq \frac{\eta^{\mu\nu}}{z} \qquad e^{q_1\varphi(z)} e^{q_2\varphi(0)} \simeq \frac{1}{z^{q_1 q_2}}$$

Pfaffian structure in open string theory

The *N* point gluon amplitude can then be written as in RNS formalism

$$\mathcal{A}_N = \langle cU^{(-1)}(z_1)cU^{(0)}(z_{N-1})cU^{(0)}(z_N)$$
$$\int \prod_{i=2}^{N-2} dz_i V^{(-1)}(z_2) \cdots V^{(0)}(z_{N-2})\rangle$$

A Pfaffian comes out of this integral simply because of the Grassmann integral over a product of fermionic fields.

Pfaffian structure in open string theory

Focusing first on the purely fermionic part of the correlator, it has 2*N*-2 fermionic fields, among which *N*-3 are bilinears:

$$\langle(\epsilon_1 \cdot \psi(z_1))(\epsilon_2 \cdot \psi(z_2)) \prod_{i=3}^{N} :(k_i \cdot \psi(z_i))(\epsilon_i \cdot \psi):\rangle$$

Giving

$$\int [d\psi](\epsilon_1 \cdot \psi(z_1))(\epsilon_2 \cdot \psi(z_2)) \prod_{i=3}^{N} :(k_i \cdot \psi(z_i))(\epsilon_i \cdot \psi): \exp\left(-1/2 \int \psi\bar{\partial}\psi\right)$$

We see from this that we will get various fermionic contractions out (dot products of momenta, momenta and polarizations and polarizations) (+ signs i.e. Pfaffian)

Pfaffian structure in open string theory

Hence we can write the integral in terms of the following (2N-2) x (2N-2) matrix

$$M' = \begin{pmatrix} A & -C'^{\mathrm{T}} \\ C' & B \end{pmatrix}$$

$$A_{i,j} = \frac{k_i \cdot k_j}{z_i - z_j} \qquad i,j = 3,4,\ldots,N$$

$$B_{i,j} = \frac{\epsilon_i \cdot \epsilon_j}{z_i - z_j} \qquad i,j = 1,2,\ldots,N$$

$$C'_{i,i} = 0\,, \qquad C'_{ij} = \frac{\epsilon_i \cdot k_j}{z_i - z_j} \quad i = 1,2,\ldots,N\,, \qquad j = 3,4,\ldots,N$$

Almost like the Cachazo, He, Yuan Pfaffian except…

Pfaffian structure in open string theory

An additional factor comes from the boson contraction:

$$: (\epsilon_i \cdot \partial X(z_i)) e^{i\sum_l k_l X(z_l)} : \sim \left(-2\alpha' \sum_l \frac{\epsilon_i \cdot k_l}{z_i - z_l} \right) : e^{i\sum_{l \neq i} k_l X(z_l)} :$$

$$+ O(z_i - z_l)$$

That is equal to the additional factor:

$$C_{i,i} = -\sum_l \frac{\epsilon_i \cdot k_l}{z_i - z_l}\,, \qquad C_{ij} = C'_{ij}\,, \qquad j \neq i$$

Also seen to come out in the formalism of (Mason, Skinner)

Pfaffian structure in open string theory

Important different interpretation:

In (Mason and Skinner's) formalism one has fixed

$$\partial X(z_i) \sim \sum_j \frac{k_i}{z_i - z_j}$$

I.e. no more contractions than than just considered on previous slide... no α'

However in actual string theory this comes dynamically from the contraction of

$$\partial X(z_i) X(z_k) \sim \alpha' \sum_j \frac{k_i}{z_i - z_j}$$

And there are also other contractions such as e.g.

$$\partial X(z_i) \partial X(z_k) \sim \frac{1}{(z_i - z_k)^2} \cdots$$

Such contractions have the important function in String Theory that they prevent tachyon poles from appearing.

(Mafra, Schlotterer and Stieberger)

Pfaffian structure in open string theory

$$\mathcal{A}_N = \int \prod_{i=2}^{N-2} dz_i \prod_{1<i<j<N-1} |z_{ij}|^{2\alpha' k_i\cdot k_j} \times (z_{1,N-1} z_{N-1,n} z_{N1}) \times$$

$$\left(\mathrm{Pf}'(\Psi) + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{(2\alpha')^k} \sum_{\substack{\text{distinct pairs} \\ (i_3,i_4),\dots,(i_{2k-1},i_{2k})}} \prod_{p=3}^{2k-1} \frac{(\epsilon_{i_p}\cdot \epsilon_{i_{p+1}})}{(z_{i_p i_{p+1}})^2} \mathrm{Pf}'(\Psi^{i_3 i_4 \dots i_{2k}}_{i_3 i_4 \dots i_{2k}}) \right)$$

$$\mathrm{Pf}'(\Psi^{i_3 i_4 \dots i_{2k}}_{i_3 i_4 \dots i_{2k}}) \rightarrow \frac{1}{z_{12}} \mathrm{Pf}(\Psi^{12 i_3 i_4 \dots i_{2k}}_{12 i_3 i_4 \dots i_{2k}})$$

How does tree-level come out right?

So we see an extended structure than just the Pfaffian of (Cachazo, He and Yuan), and a series in α'^{-1}

It is clear that in the limit $\alpha' \to \infty$ we get (Gross, Mende)..

Four point gluon scattering: example

We have:

$$A_4(1,2,3,4) = \int_0^1 \left(\mathrm{Pf}'(\Psi) + \frac{(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4)}{2\alpha' x^2} \right) x^{2\alpha' k_1 \cdot k_2} (1-x)^{2\alpha' k_2 \cdot k_3}\, dx$$

The second term is crucial for removing the tachyon contribution.

It comes from the

$$\partial X(z_i) \partial X(z_k) \sim \frac{1}{(z_i - z_k)^2} \cdots$$

type contractions.

Four point gluon scattering: example

Looking exclusively at the terms we now observe:

$$\delta A_4 = \int_0^1 dz_2 \, \frac{1}{z_{12}^2} \exp\Big(2\alpha' k_1 \cdot k_2 \log(-z_{12}) + 2\alpha' k_2 \cdot k_3 \log(-z_{23})\Big)$$

can be integrated by parts to

$$\delta A_4 = -\int_0^1 dx \, \partial_x \left(-\frac{1}{x}\right)$$
$$\exp\Big(2\alpha' k_1 \cdot k_2 \log(x) + 2\alpha' k_2 \cdot k_3 \log(1-x)\Big) =$$

$$\int_0^1 dx \, -\frac{1}{x} \partial_x \Big(\exp\Big(2\alpha' k_1 \cdot k_2 \log(x) + 2\alpha' k_2 \cdot k_3 \alpha' \log(1-x)\Big)\Big)$$

Four point gluon scattering: example

Looking exclusively at that terms we now observe:

$$\delta A_4 = \int_0^1 dz_2 \frac{1}{z_{12}^2} \exp\left(2\alpha' k_1 \cdot k_2 \log(-z_{12}) + 2\alpha' k_2 \cdot k_3 \log(-z_{23})\right)$$

or rewriting

$$\delta A_4 = \int_0^1 dx \frac{1}{x}\left(\frac{k_1 \cdot k_2}{x} - \frac{k_2 \cdot k_3}{1-x}\right)\Bigg(\exp\Big(2\alpha' k_1 \cdot k_2 \log(x) +$$
$$2\alpha' k_2 \cdot k_3 \alpha' \log(1-x)\Big)\Bigg)$$

Where we of course recognize the scattering equation

$$S_2 = \frac{k_1 \cdot k_2}{x} - \frac{k_2 \cdot k_3}{1-x}$$

Tachyon term removal by scattering equations

We see that in string theory we can trade the explicit tachyon term by an integration over a term proportional to the scattering equation.

The same phenomenon occurs for amplitudes with higher N.

It gets increasingly tedious to carry out the sequence of partial integrations

Integration by parts at 5 pt (gluons)

$$\sim \ldots - \frac{\epsilon_1 \cdot k_3 \epsilon_2 \cdot \epsilon_3 \epsilon_4 \cdot \epsilon_5}{\alpha' z_{12} z_{13} z_{23} z_{45}^2} + \frac{\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot k_3 \epsilon_4 \cdot \epsilon_5}{\alpha' z_{12} z_{13} z_{23} z_{45}^2} - \frac{\epsilon_1 \cdot k_4 \epsilon_2 \cdot \epsilon_4 \epsilon_3 \cdot \epsilon_5}{\alpha' z_{12} z_{14} z_{24} z_{35}^2} + \frac{\epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot k_4 \epsilon_3 \cdot \epsilon_5}{\alpha' z_{12} z_{14} z_{24} z_{35}^2} - \frac{\epsilon_1 \cdot k_5 \epsilon_2 \cdot \epsilon_5 \epsilon_3 \cdot \epsilon_4}{\alpha' z_{12} z_{15} z_{25} z_{34}^2} + \frac{\epsilon_1 \cdot \epsilon_5 \epsilon_2 \cdot k_5 \epsilon_3 \cdot \epsilon_4}{\alpha' z_{12} z_{15} z_{25} z_{34}^2}$$

$$- \frac{\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 \left(\frac{\epsilon_5 \cdot k_1}{z_{15}} + \frac{\epsilon_5 \cdot k_2}{z_{25}} + \frac{\epsilon_5 \cdot k_3}{z_{35}} + \frac{\epsilon_5 \cdot k_4}{z_{45}} \right)}{\alpha' z_{12}^2 z_{34}^2} - \frac{\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_5 \left(\frac{\epsilon_4 \cdot k_1}{z_{14}} + \frac{\epsilon_4 \cdot k_2}{z_{24}} + \frac{\epsilon_4 \cdot k_3}{z_{34}} - \frac{\epsilon_4 \cdot k_5}{z_{45}} \right)}{\alpha' z_{12}^2 z_{35}^2} + \frac{\epsilon_1 \cdot \epsilon_2 \epsilon_4 \cdot \epsilon_5 \left(-\frac{\epsilon_3 \cdot k_1}{z_{13}} - \frac{\epsilon_3 \cdot k_2}{z_{23}} + \frac{\epsilon_3 \cdot k_4}{z_{34}} + \frac{\epsilon_3 \cdot k_5}{z_{35}} \right)}{\alpha z_{12}^2 z_{45}^2}$$

IBP on z_4

IBP on z_3

IBP on z_1

Tachyon term removal by scattering equations

Let us summarize the main point:

We use integration by parts to rewrite the full string theory integrand.

After having done these partial integrations, the new integrand now has the property that it:

- obviously reproduces the string theory amplitude.

- reproduces the field theory answer at tree level using the scattering equation prescription for the measure.

Amplitudes with fermions and mixed matter

So far we have looked at pure scalar amplitudes as well as pure gluon amplitudes.

To demonstrate the generality of the new integration measure prescription as well as the removal prescription for tachyon type terms.

We will here consider some examples of

- Fermion and mixed fermion amplitudes
- Mixed scalar / gluon amplitude example.

Fermion amplitudes

The four fermion amplitude in open string theory reads

$$\mathcal{A}_4 = \frac{g^2}{2}\alpha' \int dz z^{-2\alpha' t-1}(1-z)^{-2\alpha' s-1}$$

$$[(1-z)(\gamma^\mu)_{\alpha_1\alpha_2}(\gamma_\mu)_{\alpha_3\alpha_4} - z(\gamma^\mu)_{\alpha_1\alpha_4}(\gamma_\mu)_{\alpha_2\alpha_3}]$$

(Friedan, Martinec, Shenker; Cohn, Friedan, Qiu, Shenker)

which is to be sandwiched between the external spinors

$$\bar{v}_{\alpha_1} u_{\alpha_2} \bar{v}_{\alpha_3} u_{\alpha_4}$$

Now we see that substituting:

$$(z_1 - z_{N-1})(z_1 - z_N)(z_{N-1} - z_N)\prod_i{}' \delta(S_i) \prod_{i=1}^{N}(z_i - z_{i+1})^{-1}$$

$$\rightarrow \frac{1}{z}\frac{1}{(1-z)}\delta\left(\frac{s}{z} - \frac{t}{(1-z)}\right)$$

with $z = \dfrac{s}{(s+t)}$

Fermion amplitudes

Gives

$$\mathcal{A}_4 = \frac{g^2}{2}\alpha' \int dz z^{-2\alpha' t-1}(1-z)^{-2\alpha' s-1}$$

$$[(1-z)(\gamma^\mu)_{\alpha_1\alpha_2}(\gamma_\mu)_{\alpha_3\alpha_4} - z(\gamma^\mu)_{\alpha_1\alpha_4}(\gamma_\mu)_{\alpha_2\alpha_3}]$$

$$\frac{1}{z^2(1-z)} \to \frac{(s+t)^3}{s^2 t}\frac{st}{(s+t)^3} \to \frac{1}{s}$$

$$\frac{1}{z(1-z)^2} \to \frac{(s+t)^3}{t^2 s}\frac{st}{(s+t)^3} \to \frac{1}{t}$$

$$\mathcal{A}_4 \sim \left[\frac{1}{s}(\gamma^\mu)_{\alpha_1\alpha_2}(\gamma_\mu)_{\alpha_3\alpha_4} - \frac{1}{t}(\gamma^\mu)_{\alpha_1\alpha_4}(\gamma_\mu)_{\alpha_2\alpha_3}\right]$$

Fermion amplitudes

Similar story for the 2 gluons and 2 fermions amplitude

Prescription exactly again gives the right tree level amplitude.

No tachyon contribution in the case of 2 gluons and 2 fermions.

General prescription at all n expressions:

- Take superstring integrands for amplitudes including fermions and integrate over a measure that localizes exactly on the scattering equations.

Mixed scalar gluon example

Now we will turn to the first mixed example featuring:

Four external scalar states and one gluon state.

String theory integrand of this amplitude
contains two tachyonic terms cancelling each other in the integral.

Cancellation manifest by means of a single partial integration

Mixed scalar gluon example

Amplitude reads:

$$\mathcal{A}_5(\phi_1,\phi_2,\phi_3,\phi_4,g_5) = K_a \int \Big(\prod_{k=1}^{5} dz_k\Big)\Big(\prod_{i<j} |z_{ij}|^{\alpha' s_{ij}}\Big)$$
$$\Big(\frac{1}{z_{35}}\Big(\frac{(\zeta_5\cdot k_4)}{z_{45}}\frac{\alpha' s_{12} z_{34}}{z_{24}z_{13}z_{14}z_{23}}\Big)$$
$$+\frac{(\zeta_5\cdot k_1)}{z_{15}z_{24}}\Big(\frac{(1-\alpha' s_{24})}{z_{24}z_{13}} + \frac{\alpha' s_{24}}{z_{14}z_{23}}\Big)$$
$$+\frac{(\zeta_5\cdot k_2)}{z_{14}z_{25}}\Big(\frac{(1-\alpha' s_{14})}{z_{14}z_{23}} + \frac{\alpha' s_{14}}{z_{13}z_{24}}\Big)\Big)$$

(Stieberger, Taylor)

We first rewrite (using integration by parts (IBP)):

$$(1-\alpha' s_{14}) \to (\mathrm{IBP}(S)_4 z_{14} - \alpha' s_{14})$$

Mixed scalar gluon example

Amplitude now reads:

$$\mathcal{A}_5(\phi_1,\phi_2,\phi_3,\phi_4,g_5) = K_a \int \Big(\prod_{k=1}^{5} dz_k\Big)\Big(\prod_{i<j} |z_{ij}|^{\alpha' s_{ij}}\Big)$$

$$\delta(S_4)\,\delta(S_5)\,\frac{1}{z_{23}z_{34}z_{45}}$$

$$\Bigg(\frac{1}{z_{35}}\Big(\frac{(\zeta_5\cdot k_4)}{z_{45}}\frac{\alpha' s_{12} z_{34}}{z_{24}z_{13}z_{14}z_{23}}\Big)$$

$$+\frac{(\zeta_5\cdot k_1)}{z_{15}z_{24}}\Big(\frac{\alpha' z_{24}S_4 - \alpha' s_{24}}{z_{24}z_{13}} + \frac{\alpha' s_{24}}{z_{14}z_{23}}\Big)$$

$$+\frac{(\zeta_5\cdot k_2)}{z_{14}z_{25}}\Big(\frac{\alpha' z_{14}S_4 - \alpha' s_{14}}{z_{14}z_{23}} + \frac{\alpha' s_{14}}{z_{13}z_{24}}\Big)\Bigg)$$

Where we see that the terms featuring S_4 drops out yielding…

Mixed scalar gluon example

Amplitude after imposing the measure yields:

$$\mathcal{A}_5(\phi_1,\phi_2,\phi_3,\phi_4,g_5) = K_{ft}\Bigg((\zeta_5\cdot k_1)\left(\frac{1}{s_{23}}-\frac{s_{34}}{s_{23}s_{15}}\right) + (\zeta_5\cdot k_2)\left(\frac{1}{s_{23}}\right) + (\zeta_5\cdot k_4)\left(\frac{s_{12}}{s_{23}s_{45}}\right)\Bigg)$$

Which is the correct answer. It should be noted that this result is far easier achieved through the delta function prescription than via traditional integration techniques.

Mixed scalar gluon example

Mixed amplitudes.

- Presents no additional complications
- Generic amplitudes of fermions, scalars, gluons in any combination can be computed in the same manner.

I.e.

- Imposing the same delta function prescription
- Manifestly cancel all tachyon poles (if present) through integrations by parts.

Conclusions

Shown how to identify a unifying framework for string theory and scattering equations.

Naturally such a framework leads to a new kind of dual model, string theory localized on the surface of the solutions to the scattering equations.

Connection to the formalism of string theory, not tied to 4D, inherits many of the same symmetries and capabilities.

- Numerous other examples: mixed amplitudes gluons + fermions, scalars + fermions, scalars + fermions, and so on.
- We have provided some examples, and argued that the general prescription manifestly free of tachyonic terms.

Conclusions

Clear that a dual model for gravity exist as well through KLT or directly in string theory framework?

- No additional complications should occur
- Should be a full ST understanding of the orthogonally of scattering equations. (Cachazo, He, Yuan)
    - Here the connection to the S kernel for KLT appear to be crucial!

    (NEJBB, Damgaard, Feng Sondergaard; NEJBB, Damgaard, Sondergaard and Vanhove)

Conclusions

The modifications of amplitudes at finite α' may have no significance – however potentially they may find an interpretation.

Rigid minimal area constraint?? (Fairlie et al's original idea) (precursor of strong coupling formalism of Alday and Maldacena)

α' look most of all like a regularization factor that deforms the amplitude.

What happens at the genuine quantum level, i.e. at loop order??

Correlation functions on higher genus surfaces??

Conclusions

Open questions:

- Better mathematical understanding of the two different measures at tree level?
- Can this map be made more precise? (Like a mathematical identity in the limit? But very different properties…)
- Appears connected to a fundamental meaning to scattering equations?
- How important is twistor space/pure spinors to prescription? (Inspiration yes -- but prescription in ST seems to be independent)

Closed string expression

$$\mathcal{A}_N^{II} = \int \prod_{i=2}^{N-2} d^2 z_i \prod_{1<i<j<N-1} |z_{ij}|^{\alpha' k_i \cdot k_j} \times |z_{1,N-1} z_{N-1,N} z_{N1}|^2 \times$$

$$\left| \mathrm{Pf}'(\Psi) + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{(\alpha')^k} \sum_{\substack{\text{distinct pairs} \\ (i_3,i_4),\ldots,(i_{2k-1},i_{2k})}} \prod_{p=3}^{2k-1} \frac{(\epsilon_{i_p} \cdot \epsilon_{i_{p+1}})}{(z_{i_p i_{p+1}})^2} \mathrm{Pf}\,'(\Psi^{i_3 i_4 \ldots i_{2k}}_{i_3 i_4 \ldots i_{2k}}) \right|^2$$