# Gravity Scattering Amplitudes

**Amplitudes 2014**

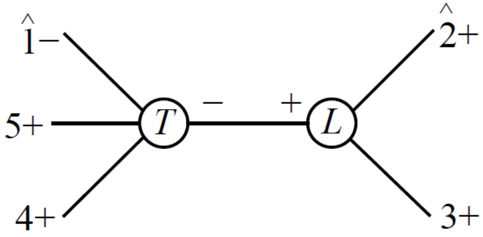
**June 13, 2014**

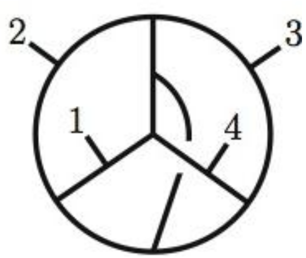
**Zvi Bern, UCLA**

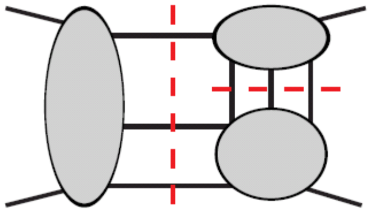
**Review + new work:**

**ZB, Scott Davies, Tristan Dennen, Sasha Smirnov, Volodya Smirnov: arXiv:1309.2498**

**ZB, Scott Davies, Josh Nohle, arXiv:1405.1015**



# Outline

## 1. Review of some basic properties of gravity amplitudes:

- Various new representations of gravity amplitudes.
- Relations between gravity and gauge theory.
- Duality between color and kinematics.
- Tame UV behavior of $N = 8$ supergravity.

## 2. Some recent developments discussed here:

- New surprises in UV properties in $N = 4$ supergravity.
- Subleading soft limits and extended BMS symmetry. Also talk from Huang

## 3. Other talks on gravity amplitudes:

- String theory multiloop amplitudes. talk from Green
- New extensions of color kinematics duality. talks from Johansson, Roiban
- Scattering equations and implications for gravity. talks from Monteiro, Yuan
- Causality and 3 graviton amplitudes talk from Maldacena

# Review of Some Basic Properties

# Various Representations of Gravity Amplitudes

**There are a plethora of representations of gravity tree amplitude:**

1) **Classic KLT relations: gravity tree amplitudes from gauge theory.** Kawai, Lewellen, Tye
2) **Duality between color and kinematics.** ZB, Carrasco and Johansson
3) **Twistor forms.** Hodges; Cachazo and Skinner; Adamo and Mason; Litsey and Stankowicz
4) **Ambitwistor string form.** Mason and Skinner
5) **Grassmannian forms.** Cachazo and Geyer; Cheung
6) **Scattering equations and gravity amplitudes.** Cachazo, He, Yuan; Montiero and O'Connell

**Loop level less well developed, but we have powerful tools and many results in gravity theory.**

1) **Generalized unitarity.** ZB, Dixon, Dunbar, Kosower; ZB, Dixon, Dunbar, Perelstein, Rozowsky
2) **Duality between color and kinematics.** ZB, Carrasco, Johansson; ZB, Dixon, Carrasco, Johansson, Roiban

# KLT Relations

**At tree level, Kawai, Lewellen and Tye presented a relationship between closed and open string amplitudes. In field theory limit, relationship is between gravity and gauge theory.**

$$M_4^{\rm tree}(1,2,3,4) = s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$\begin{aligned} M_5^{\rm tree}(1,2,3,4,5) &= s_{12}s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) \\ &\quad + s_{13}s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5) \end{aligned}$$

**Gravity amplitude**

**Color stripped gauge theory amplitude**

where we have stripped all coupling constants

$$\mathcal{A}_4^{\rm tree} = g^2 \sum\nolimits_{\rm non-cyclic} {\rm Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) A_4^{\rm tree}(1,2,3,4)$$

**Full gauge theory amplitude**


**Holds for any external states.**

**Progress in gauge theory can be imported into gravity theories**

# Gravity and Gauge Theory Amplitudes

$$M_4^{\rm tree}(1_h^-, 2_h^-, 3_h^+, 4_h^+) = \left(\frac{\kappa}{2}\right)^2 s_{12} A_4^{\rm tree}(1_g^-, 2_g^-, 3_g^+, 4_g^+) \times A_4^{\rm tree}(1_g^-, 2_g^-, 4_g^+, 3_g^+)$$

**gravity**

**gauge theory**

$$= \left(\frac{\kappa}{2}\right)^2 s_{12} \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle} \times \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,4\rangle\,\langle 4\,3\rangle\,\langle 3\,1\rangle}$$

$$\langle jl\rangle = \langle k_j^- | k_l^+\rangle = \tfrac{1}{2}\bar{u}(k_j)(1+\gamma_5)u(k_l) = \sqrt{2k_j\cdot k_l}\, e^{i\phi}$$

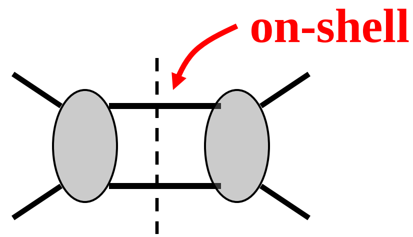
**This can be used to generate *n*-point graviton amplitudes.**

Berends, Giele, Kuijf

Bern, Dixon, Dunbar and Kosower (1998)

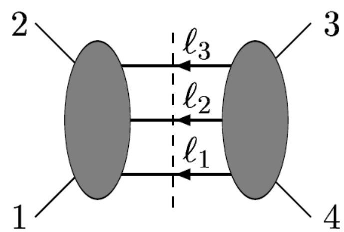
# Unitarity Method

**Two-particle cut:**

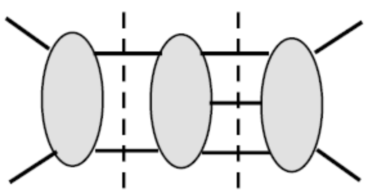


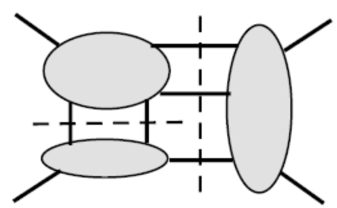

**Systematic assembly of complete amplitudes from cuts for any number of particles or loops.**

**Three-particle cut:**




**Generalized unitarity as a practical tool:**





**Different cuts merged to give an expression with correct cuts in all channels.**

Bern, Dixon and Kosower
Britto, Cachazo and Feng; Forde;
Ossala, Pittau, Papadopolous, and many others

**Standard general purpose tool for turning tree advances into loop advances**

# Duality Between Color and Kinematics

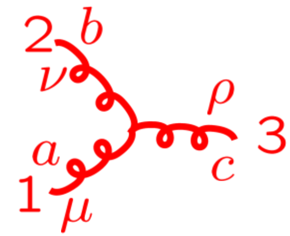
ZB, Carrasco and Johansson

coupling constant

color factor

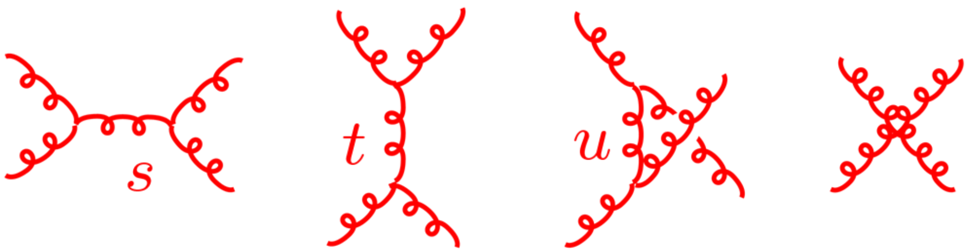
momentum dependent kinematic factor

$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$




**Color factors based on a Lie algebra:** $[T^a, T^b] = i f^{abc} T^c$

**Jacobi Identity** $$f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0$$




**Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.**

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$$s = (k_1 + k_2)^2 \quad u = (k_1 + k_3)^2$$
$$t = (k_1 + k_4)^2$$

**Color factors satisfy Jacobi identity:**
**Numerator factors satisfy similar identity:**

$$c_u = c_s - c_t$$

$$n_u = n_s - n_t$$

**Color and kinematics satisfy the same identity**

Goebel, Halzen ,Leveille Zhu

# Duality Between Color and Kinematics

ZB, Carrasco, Johansson

## Consider five-point tree amplitude:

sum is over diagrams

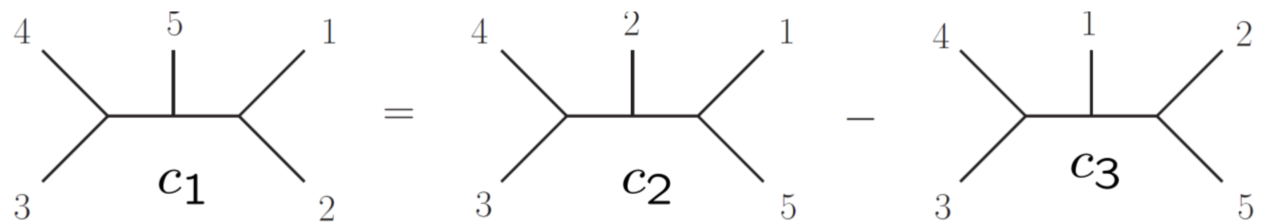
color factor

kinematic numerator factor

**gauge theory**

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i\, n_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

Feynman propagators




$$c_1 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}, \quad c_2 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}, \quad c_3 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$\boxed{c_1 - c_2 + c_3 = 0 \Leftrightarrow n_1 - n_2 + n_3 = 0}$$

**Claim: At n-points we can always find a rearrangement where color and kinematics satisfy the same algebraic constraint equations.**

**Nontrivial constraints on amplitudes in field theory and string theory**

BCJ, Bjerrum-Bohr, Feng,Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer; Cachazo;
Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer

BCJ

# Gravity and Gauge Theory

kinematic numerator

color factor

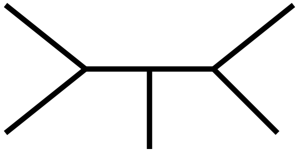
**gauge theory:** $$\frac{1}{g^{n-2}}\mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n) = \sum_i \frac{n_i\, c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

**sum over diagrams with only 3 vertices**

$$c_i \sim f^{a_1 a_2 b_1} f^{b_1 b_2 a_5} f^{b_2 a_4 a_5}$$

***Assume we have:***

$$c_1 + c_2 + c_3 = 0 \quad \Leftrightarrow \quad n_1 + n_2 + n_3 = 0$$



***Then:*** $c_i \Rightarrow \tilde{n}_i$ **kinematic numerator of second gauge theory**

Proof: ZB, Dennen, Huang, Kiermaier

**gravity:** $$\boxed{-i\left(\frac{2}{\kappa}\right)^{(n-2)} \mathcal{M}_n^{\text{tree}}(1,2,\ldots,n) = \sum_i \frac{n_i\, \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}}$$

**Encodes KLT tree relations**

**Gravity numerators are a double copy of gauge-theory ones.**

**This works for ordinary Einstein gravity and susy versions.**

**Cries out for a unified description of the sort given by string theory!**

# Gravity From Gauge Theory

BCJ

$$-i\left(\frac{2}{\kappa}\right)^{(n-2)}\mathcal{M}_n^{\text{tree}}(1,2,\ldots,n)=\sum_i\frac{n_i\,\tilde{n}_i}{\prod_{\alpha_i}p^2_{\alpha_i}}$$

$n$ $\tilde{n}$

**$N$ = 8 sugra: ($N$ = 4 sYM) $\times$ ($N$ = 4 sYM)**

**$N$ = 4 sugra: ($N$ = 4 sYM) $\times$ ($N$ = 0 sYM)**

**Theory controlled by simple tensor product of YM theories.**

**Recent papers show more sophisticated cases.**

**Cases with fewer supersymmetries.**

Carrasco, Chiodaroli, Günaydin and Roiban; ZB, Davies, Dennen, Huang and Nohle

**Bagger, Lambert, Gustavsson (BLG) theory and ABJM theories.**

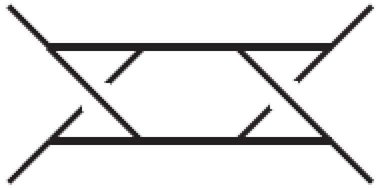
Bargheer, He and McLoughlin; Huang and Johansson; Allic Sivaramakrishnan

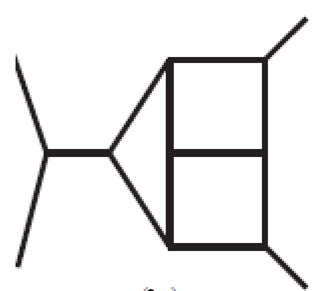
**Fundamental representation matter.** **See Johansson's talk**

**Applications to gauged supergravity.** **See Roiban's talk**

# Loop-Level Conjecture

ZB, Carrasco, Johansson (2010)





$$c_i + c_j + c_k = 0$$
$$n_i + n_j + n_k = 0$$

sum is over diagrams

kinematic numerator

color factor

$$\frac{(-i)^L}{g^{n-2+2L}} \mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

gauge theory

propagators

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} \mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p^2_{\alpha_j}}$$

gravity

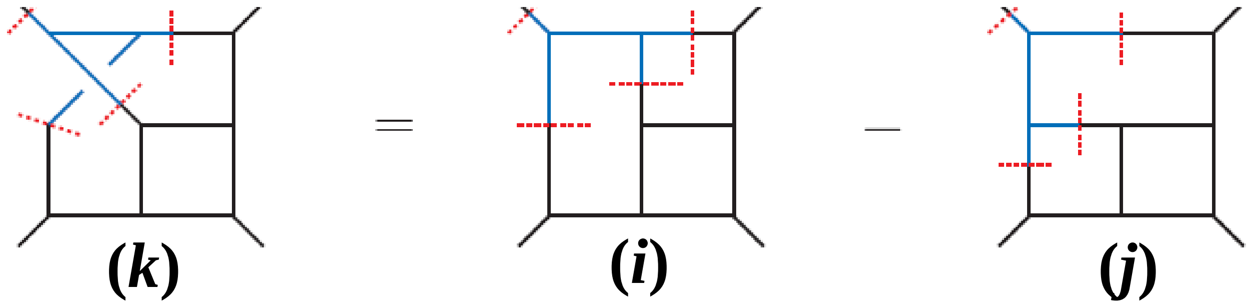
symmetry factor

**Loop-level is identical to tree-level one except for symmetry factors and loop integration.**
**This works if numerator satisfies duality.**

# Gravity integrands are free!

**BCJ**

**Ideas generalize to loops:**




color factor

$$c_k = c_i - c_j$$

$$n_k = n_i - n_j$$

kinematic numerator

**If you have a set of duality satisfying numerators.**

**To get:**

**gauge theory $\longrightarrow$ gravity theory**

**simply take**

**color factor $\longrightarrow$ kinematic numerator**

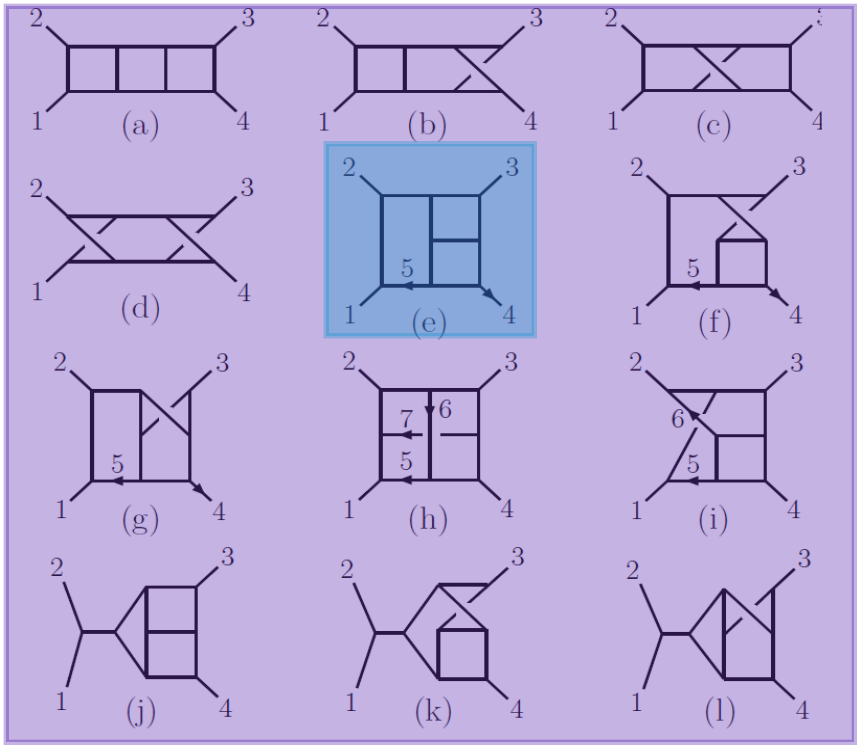
$$c_k \longrightarrow n_k$$

**Gravity loop integrands are trivial to obtain once we have gauge theory in a form where duality works.**

# One diagram numerator to rule them all

ZB, Carrasco, Johansson (2010)

***N* = 4 super-Yang-Mills integrand**




| Integral $I^{(x)}$ | $\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8}$ supergravity) numerator |
|---|---|
| (a)–(d) | $s^2$ |
| (e)–(g) | $\left(s\left(-\tau_{35}+\tau_{45}+t\right)-t\left(\tau_{25}+\tau_{45}\right)+u\left(\tau_{25}+\tau_{35}\right)-s^2\right)/3$ |
| (h) | $\left(s\left(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u\right)\right.$ $\left.+t\left(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}\right)+s^2\right)/3$ |
| (i) | $\left(s\left(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t\right)\right.$ $\left.+t\left(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}\right)+u\,\tau_{25}+s^2\right)/3$ |
| (j)-(l) | $s(t-u)/3$ |

**Diagram (e) is the master diagram.**

**Determine the master integrand in proper form and duality gives all others.**

***N* = 8 sugra given by replacing color with kinematic numerators.**

$$\tau_{ij}=2k_j\cdot l_j$$

## Recent Developments:
## Soft Graviton Behavior

See Yu-tin Huang’s talk

# Subleading Soft Gravity

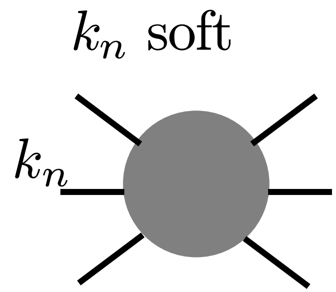
See Yu-tin Huang's talk

**BMS symmetry of asymptotically flat space time: diffeomorphisms leaving null infinity (Scri) invariant.** Bondi, Burg, Metzner; Sachs

**The Weinberg soft graviton theorem is Ward identity of BMS symmetry.** He, Lysov, Pitra and Strominger

**Using extended BMS symmetry Strominger obtained subleading soft limit identity.** Barnich and Troessaert; Strominger; Adamo, Casali and Skinner

**Verified at tree level by Cachazo and Strominger :**




$$M_n^{\text{tree}} \to \left( S_n^{(0)} + S_n^{(1)} + S_n^{(2)} \right) M_{n-1}^{\text{tree}} + \mathcal{O}(k_n^2)$$

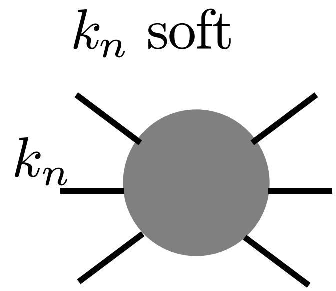
**Weinberg term**

- **Analogous to Low's 1958 soft-photon theorem.**
- **Gross and Jackiw (1968) discussed behavior through 2nd subleading order for a soft graviton coupled to scalars.**
- **Subleading corrections to soft gluons and gravitons had also been studied more recently.** Laenen, Magnea, Stavenga, White; White

# Subleading Soft Graviton Behavior at Tree Level

Cachazo and Strominger

$$M_n^{\text{tree}} \to \left(S_n^{(0)} + S_n^{(1)} + S_n^{(2)}\right) M_{n-1}^{\text{tree}} + \mathcal{O}(k_n^2)$$




$$S_n^{(0)} = \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_n \cdot k_i}\,, \qquad S_n^{(0)} = -\sum_{i=1}^{n-1} \frac{[n\,i]\,\langle x\,i\rangle\,\langle y\,i\rangle}{\langle n\,i\rangle\,\langle x\,n\rangle\,\langle y\,n\rangle}\,,$$

$$S_n^{(1)} = \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^\mu k_{n\rho} J_i^{\rho\nu}}{k_n \cdot k_i}\,, \qquad S_n^{(1)} = -\frac{1}{2}\sum_{i=1}^{n-1} \frac{[n\,i]}{\langle n\,i\rangle}\left(\frac{\langle x\,i\rangle}{\langle x\,n\rangle} + \frac{\langle y\,i\rangle}{\langle y\,n\rangle}\right)\tilde\lambda_n^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}\,,$$

$$S_n^{(2)} = \frac{1}{2}\sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_{n\rho} J_i^{\rho\mu} k_{n\sigma} J_i^{\sigma\nu}}{k_n \cdot k_i}\,, \qquad S_n^{(2)} = -\frac{1}{2}\sum_{i=1}^{n-1} \frac{[n\,i]}{\langle n\,i\rangle}\tilde\lambda_n^{\dot\alpha}\tilde\lambda_n^{\dot\beta}\frac{\partial^2}{\partial\tilde\lambda_i^{\dot\alpha}\partial\tilde\lambda_i^{\dot\beta}}\,,$$

**Holds in *D*-dimensions** Schwab and Volovich, Afkhami-Jeddi

**Similar behavior for gluons:** Casali

$$A_n^{\text{tree}} \to \left(S_{n\,\text{YM}}^{(0)} + S_{n\,\text{YM}}^{(1)}\right) A_{n-1}^{\text{tree}} + \mathcal{O}(k_n)$$

$$S_{n\,\text{YM}}^{(0)} = \frac{\langle (n-1)\,1\rangle}{\langle (n-1)\,n\rangle\langle n\,1\rangle}\,, \qquad S_{n\,\text{YM}}^{(1)} = \frac{1}{\langle (n-1)\,n\rangle}\tilde\lambda_n^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_{n-1}^{\dot\alpha}} - \frac{1}{\langle 1\,n\rangle}\tilde\lambda_n^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_1^{\dot\alpha}}$$

# Loop-Level Subleading Soft Behavior

**Is subleading soft behavior renormalized in gravity? Yes!**

**However, corrections in gravity are fairly tame:**

- **No corrections beyond tree for leading behavior.** ZB, Dixon, Perelstein, Rozowsky
- **No corrections beyond one loop for 1st subleading behavior.**
- **No corrections beyond two loops for 2nd subleading behavior.**

ZB, Davies, Nohle

**Two types of corrections:**

- **Ones connected to IR singularities.**
- **Ones connected to nontrivial complex factorization channels where residues that are not simple products of amplitudes.**

ZB, Davies, Nohle; He, Huang, Wen

# Loop-Level Subleading Soft Behavior

**Loop corrections linked to IR singularities:** ZB, Davies, Nohle

$$M_n^{\text{1-loop}} \to \left(S_n^{(0)} + S_n^{(1)} + S_n^{(2)}\right) M_{n-1}^{\text{1-loop}} + \left(S_n^{(0)\,\text{1-loop}} + S_n^{(1)\,\text{1-loop}} + S_n^{(2)\,\text{1-loop}}\right) M_{n-1}^{\text{tree}}$$

**Straightforward to work out because the IR divergences are all known.** Weinberg; Naculich and Schnitzer; Akhoury, Saotome and Sterman,

$$M_n^{\text{1-loop}}\Big|_{\text{div.}} = \frac{\sigma_n}{\epsilon} M_n^{\text{tree}} \qquad \sigma_n = -c_\Gamma \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} s_{ij} \log\left(\frac{\mu^2}{-s_{ij}}\right)$$

**IR discontinuities cause loop corrections**

$$\sigma'_n = -c_\Gamma \sum_{i=1}^{n-1} s_{in} \log\left(\frac{\mu^2}{-s_{in}}\right)$$

$$S_n^{(0)\,\text{1-loop}}\Big|_{\text{div.}} = 0\,,$$

$$S_n^{(1)\,\text{1-loop}}\Big|_{\text{div.}} = \frac{1}{\epsilon}\left[\sigma'_n S_n^{(0)} - \left(S_n^{(1)}\sigma_{n-1}\right)\right],$$

$$S_n^{(2)\,\text{1-loop}}\Big|_{\text{div.}} = \frac{1}{\epsilon}\left[\sigma'_n S_n^{(1)} - \left(S_n^{(2)}\sigma_{n-1}\right) + \sum_{i=1}^{n-1} \frac{[n\,i]}{\langle n\,i\rangle}\left(\tilde\lambda_n^{\dot\alpha} \frac{\partial \sigma_{n-1}}{\partial \tilde\lambda_i^{\dot\alpha}}\right) \tilde\lambda_n^{\dot\beta} \frac{\partial}{\partial \tilde\lambda_i^{\dot\beta}}\right]$$

**Good choice of momentum conservation prescription greatly simplifies expressions.**

**There are also IR finite pieces associated with these.**

with P. Di Vecchia

# Loop Corrections from Nontrivial Loop Factorizations

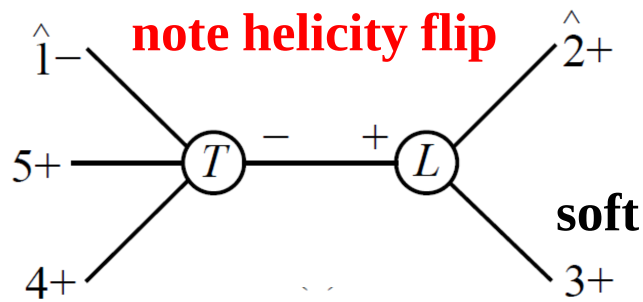
**See Huang's talk** ZB, Davies and Nohle; He, Huang and Wen

**At loop level in general there are factorization channels with no simple factorization properties also causing corrections. These come from careful implementation of dim. reg.** $\epsilon/\epsilon$ $\quad D = 4 - 2\epsilon$

**Most easily seen in IR finite amplitudes of nonsusy Yang-Mills:**

| single pole | double pole | "unreal poles" |
|---|---|---|
| $\frac{1}{\langle 23 \rangle}$ | $\frac{[23]}{\langle 23 \rangle^2}$ | $\frac{[23]}{\langle 23 \rangle}$ |
| standard factorization | | nonstandard factorization |

ZB, Dixon, Kosower; Vaman and Yao; Dunbar, Ettle and Perkins




$A^{\text{1-loop}}_{\text{QCD}}(1^-, 2^+, 3^+, 4^+, 5^+)$

**Unreal poles contaminate the subleading soft behavior, causing nontrivial loop corrections in non-susy theories.**

**Gravity is similar except this happens at 2$^{\text{nd}}$ subleading order.**

**In nonsusy gravity extended BMS symmetry has anomalous 2$^{\text{nd}}$ subleading soft behavior**

# Renormalization of Subleading Soft Behavior

**Very recently, Cachazo and Yuan noted that loop corrections can be avoided by reversing the standard order of limits in dimensional regularization.**

$$D = 4 - 2\epsilon$$

**Recall 1-loop QCD correction to leading soft function:**

$$S_{n\,\mathrm{YM}}^{(0)\,\text{1-loop}} = -S_{n\,\mathrm{YM}}^{(0)} \frac{c_\Gamma}{\epsilon^2} \left( \frac{-\mu^2 s_{(n-1)1}}{s_{(n-1)n} s_{n1}} \right)^{\epsilon} \frac{\pi\epsilon}{\sin(\pi\epsilon)}$$

$s_{ab} = (k_a + k_b)^2$ $\qquad$ $s_{n1} \to 0$ for $k_n \to 0$

**If $\epsilon < 0$ held fixed 1-loop YM soft correction vanishes for $k_n \to 0$**

**However, serious problems:**

- **Soft physics comes out wrong.**
- **In QCD it ruins leading IR cancellations between real emission and virtual contributions.**
- **Keeping $\epsilon$ finite is in general impractical.**

**Keeping standard formulation of soft limits is greatly preferred.**

# Sample Application: UV properties of Gravity

# UV Finiteness of *N* = 8 Supergravity?

**If *N* = 8 supergravity is perturbatively finite it would imply a new symmetry or non-trivial dynamical mechanism. No known symmetry can render a *D* = 4 gravity theory finite.**

**The discovery of such a mechanism would have a fundamental impact on our understanding of gravity.**

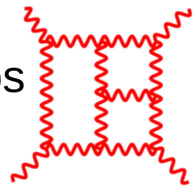
**Of course, perturbative finiteness is not the only issue for consistent gravity: Nonperturbative completions? High-energy behavior of theory? Realistic models?**

**Consensus opinion for the late 1970's and early 1980's: All supergravity theories would diverge by three loops and therefore are not viable as fundamental theories.**

# Feynman Diagrams for Gravity

**SUPPOSE WE WANT TO CHECK IF CONSENSUS OPINION IS TRUE USING FEYNMAN DIAGRAMS.**

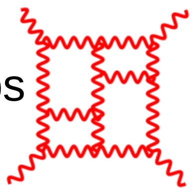
3 loops



$\sim 10^{20}$ TERMS

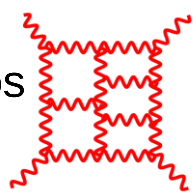
**No surprise it has never been calculated via Feynman diagrams.**

4 loops



$\sim 10^{26}$ TERMS

5 loops



$\sim 10^{31}$ TERMS

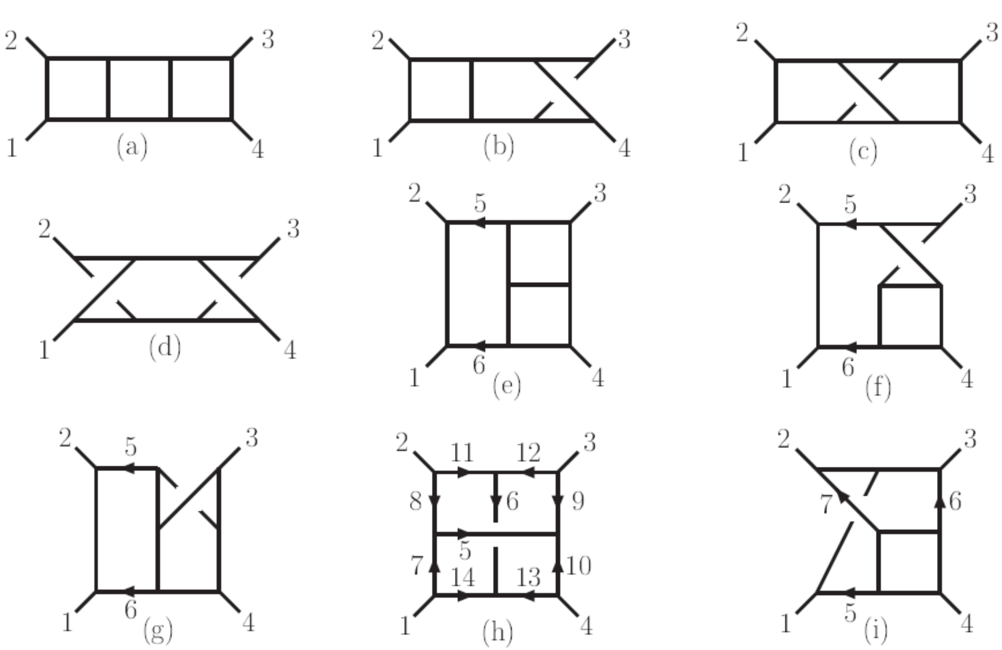
**More terms than atoms in your brain!**

- **Calculations to settle this seemed utterly hopeless!**
- **Seemed destined for dustbin of undecidable questions.**

# Complete Three-Loop Result

**Analysis of unitarity cuts shows highly nontrivial all-loop cancellations.** ZB, Dixon and Roiban (2006); ZB, Carrasco, Forde, Ita, Johansson (2007)

**To test completeness of cancellations, we decided to directly calculate potential three-loop divergence.**




ZB, Carrasco, Dixon, Johansson, Kosower, Roiban (2007)

**Three loops is not only ultraviolet finite it is "superfinite"— finite for $D < 6$.**

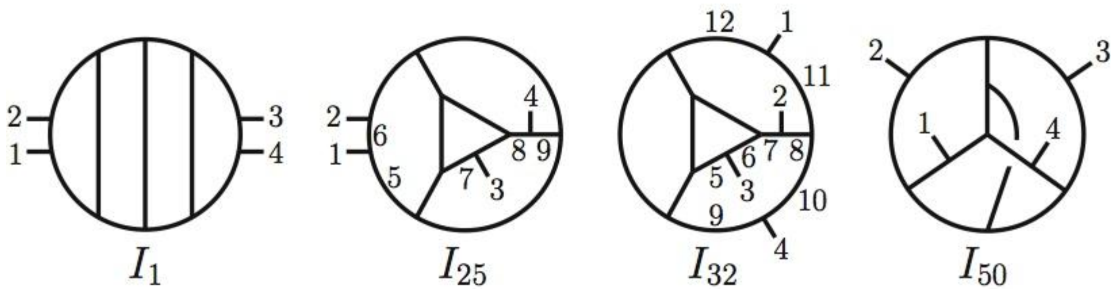
**It is very finite!**

**Obtained via on-shell unitarity method.**

# Four-Loop $N$ = 8 Supergravity Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban (2009)

**Get 85 distinct diagrams or integrals.**




$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

Integral

symmetry factor

leg perms

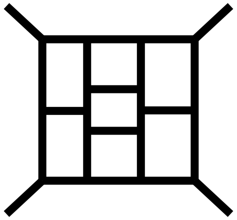
**UV finite for $D <$ 11/2**
**It's very finite!**

**Duality between color and kinematic discovered by doing this calculation.**

# Current Status of $N = 8$ Divergences

**Consensus in $N = 8$ supergravity is that trouble starts at 5 loops and by 7 loops we have valid UV counterterm in $D = 4$ under all known symmetries (suggesting divergences).**

Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Green and Bjornsson ; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger



**For $N = 8$ sugra in $D = 4$:**

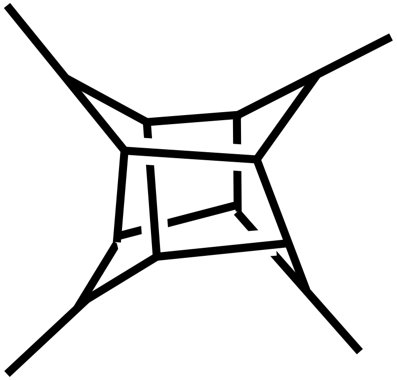
- **All counterterms ruled out until 7 loops.**
- **But $D^8R^4$ available at 7 loops (1/8 BPS) under all known symmetries. No known nonrenormalization theorem.**

Bossard, Howe, Stelle and Vanhove

**Based on this a reasonable person would conclude that $N = 8$ supergravity almost certainly diverges at 7 loops in $D = 4$.**

# $N$ = 8 Sugra 5 Loop Calculation

ZB, Carrasco, Johannson, Roiban



~500 such diagrams with ~1000s terms each

**Being reasonable and being right are not the same.**

**Place your bets:**

- **At 5 loops in $D$ = 24/5 does $N$ = 8 supergravity diverge?**
- **At 7 loops in $D$ = 4 does $N$ = 8 supergravity diverge?**

$D^8 R^4$ **counterterms**

**5 loops**

| | |
|---|---|
| **Kelly Stelle:** | **Zvi Bern:** |
| **English wine** | **California wine** |
| **“It will diverge”** | **“It won’t diverge”** |

# Recent Progress using Half Maximal Supergravity

**While 5 loops $N = 8$ has proven to be formidable even with powerful methods, we have made significant recent progress by reducing the susy.**

# Examples of Magical Cancellations?

**There are a number of curious connections between the divergences of $N$ = 4 sYM and $N$ = 8 supergravity through 4 loops in their critical dimensions (which I won't discuss here).**

ZB, Carrasco, Dixon, Johansson, Roiban

**Do we have any examples where a divergence vanishes but the standard symmetries suggest valid counterterms?**

**Yes!**

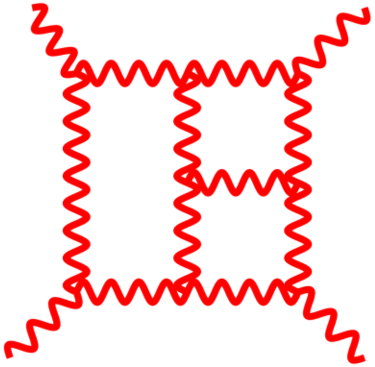
**Two examples in half-maximal supergravity :**

- **$D$ = 5 at 2 loops.**
- **$D$ = 4 at 3 loops.** **$N$ = 4 supergravity of Cremmer, Ferrara and Scherk.**

# $N = 4$ Supergravity in $D = 4$

- $N = 4$ sugra at 3 loops ideal $D = 4$ test case to study.
- Representation where duality between color and kinematics manifest for $N = 4$ sYM 3-loop 4-pt amplitude.

ZB, Carrasco, Johansson (2010)



**Consensus had it that a valid $R^4$ counterterm exists for this theory in $D = 4$.**

Bossard, Howe, Stelle; Bossard, Howe, Stelle, Vanhove

$N = 8$ Supergravity $\rightarrow$ $N = 4$ Supergravity

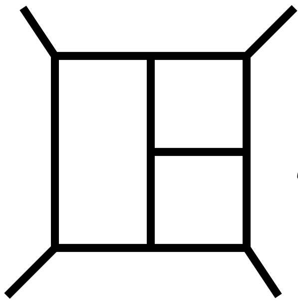
$$D^8 R^4,\quad L = 7 \;\rightarrow\; R^4,\quad L = 3$$
$$D^{10} R^4,\quad L = 8 \;\rightarrow\; D^2 R^4,\quad L = 4$$

# Three-Loop Construction

ZB, Davies, Dennen, Huang

**$N$ = 4 sugra : ($N$ = 4 sYM) x ($N$ = 0 YM)**

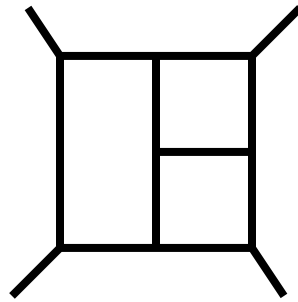
**$N$ = 4 sYM**



$\sim l \cdot k\, s^2 t A_4^{\text{tree}}$

**BCJ representation**

**pure YM**



$\sim (\varepsilon_i \cdot l)^4\; l^4$

**Feynman representation**

$c_i \rightarrow n_i$

**$N$ = 4 sugra linear divergent**

$$\int (d^D l)^3 \frac{k^7 l^9}{l^{20}}$$

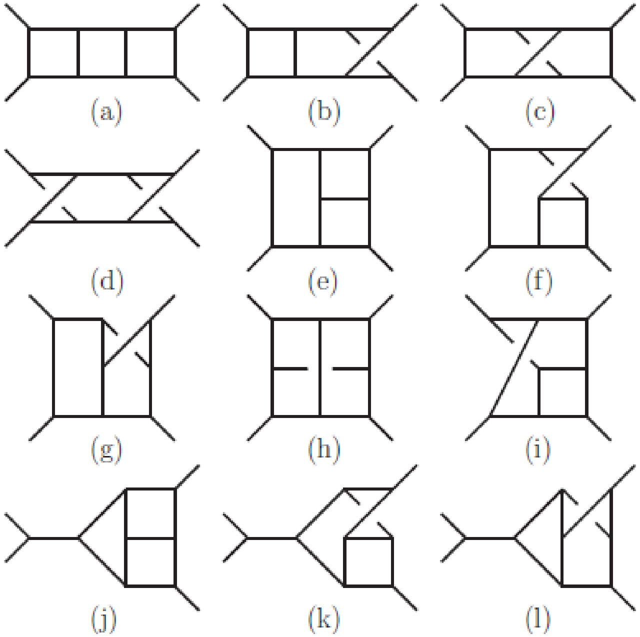
**simple to see finite for $N$=5,6,8 sugra**

- **No covariant formalism exists that can make this manifestly UV finite.**
- **UV divergences are obtained by series expanding small external momentum (or large loop momentum).**
- **Introduce mass regulator for IR divergences.**
- **In general, subdivergences must be subtracted.**

Vladimirov; Marcus and Sagnotti

# The $\mathcal{N}$ = 4 Supergravity UV Cancellation

ZB, Davies, Dennen, Huang




| Graph | (divergence)$/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$ |
|---|---|
| (a)-(d) | $0$ |
| (e) | $\frac{263}{768}\frac{1}{\epsilon^3}+\frac{205}{27648}\frac{1}{\epsilon^2}+\left(-\frac{5551}{768}\zeta_3+\frac{326317}{110592}\right)\frac{1}{\epsilon}$ |
| (f) | $-\frac{175}{2304}\frac{1}{\epsilon^3}-\frac{1}{4}\frac{1}{\epsilon^2}+\left(\frac{593}{288}\zeta_3-\frac{217571}{165888}\right)\frac{1}{\epsilon}$ |
| (g) | $-\frac{11}{36}\frac{1}{\epsilon^3}+\frac{2057}{6912}\frac{1}{\epsilon^2}+\left(\frac{10769}{2304}\zeta_3-\frac{226201}{165888}\right)\frac{1}{\epsilon}$ |
| (h) | $-\frac{3}{32}\frac{1}{\epsilon^3}-\frac{41}{1536}\frac{1}{\epsilon^2}+\left(\frac{3227}{2304}\zeta_3-\frac{3329}{18432}\right)\frac{1}{\epsilon}$ |
| (i) | $\frac{17}{128}\frac{1}{\epsilon^3}-\frac{29}{1024}\frac{1}{\epsilon^2}+\left(-\frac{2087}{2304}\zeta_3-\frac{10495}{110592}\right)\frac{1}{\epsilon}$ |
| (j) | $-\frac{15}{32}\frac{1}{\epsilon^3}+\frac{9}{64}\frac{1}{\epsilon^2}+\left(\frac{101}{12}\zeta_3-\frac{3227}{1152}\right)\frac{1}{\epsilon}$ |
| (k) | $\frac{5}{64}\frac{1}{\epsilon^3}+\frac{89}{1152}\frac{1}{\epsilon^2}+\left(-\frac{377}{144}\zeta_3+\frac{287}{432}\right)\frac{1}{\epsilon}$ |
| (l) | $\frac{25}{64}\frac{1}{\epsilon^3}-\frac{251}{1152}\frac{1}{\epsilon^2}+\left(-\frac{835}{144}\zeta_3+\frac{7385}{3456}\right)\frac{1}{\epsilon}$ |

***All* three-loop divergences cancel completely!**
**All subdivergences cancel amongst themselves with uniform mass regulator.**

**Too bad bet wasn't on *N* = 4 supergravity!**

**Tourkine and Vanhove have understood this result by extrapolating from two-loop heterotic string amplitudes.**

# Explanations?

**Prediction of superspace: If you add $N$ = 4 vector multiplets, amplitude should develop no new 2, 3 loop divergences.**

Bossard, Howe and Stelle (2013)

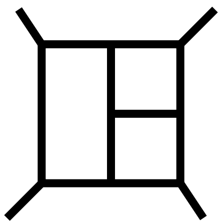
**Note that $N$ = 4 supergravity with matter already diverges at one loop.**

Fischler (1979)

**Prediction motivated us to check cases with vector multiplets.**

ZB, Davies, Dennen (2013)

**Four vector multiplet amplitude diverges at 2, 3 loops!**



$n_V = D_s - 4$

$$\mathcal{M}^{(3)}(1_{\rm H},2_{\rm H},3_{\rm H},4_{\rm H})\big|_{D=4\,\text{div.}} = 0\,,$$

← **external graviton multiplets**

$$\mathcal{M}^{(3)}(1_{\rm H},2_{\rm H},3_{\rm V},4_{\rm V})\big|_{D=4\,\text{div.}} = 0\,,$$

$$\mathcal{M}^{(3)}(1_{\rm V},2_{\rm V},3_{\rm V},4_{\rm V})\big|_{D=4\,\text{div.}} = -\frac{1}{(4\pi)^6}\left(\frac{\kappa}{2}\right)^8 (s^2+t^2+u^2)\,st\,A^{(0)}_{Q=16} \times \frac{(D_s-2)^2}{4}\left(\frac{D_s-2}{2\epsilon^3} - \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\right)$$

**UV divergence**

**matter multiplet**

**Similar story in $D$ = 5**

**Adding vector multiplets causes new divergences both at 2, 3 loops. Conclusion: currently, no viable complete standard symmetry understanding of 3-loop finiteness in pure $N$ = 4 supergravity.**

# Explanations?

**Key Question:**

**Is there an ordinary symmetry explanation for this?**

**Or is something extraordinary happening?**

**Bossard, Howe and Stelle (2013) showed that 3-loop finiteness can be explained by ordinary superspace +duality symmetries, *assuming* a 16 supercharge off-shell superspace exists.**

$$\int d^4x d^{16}\theta \, \frac{1}{\epsilon} \mathcal{L}$$

**More $\theta$s implies more derivatives in operators**

**If true, then there is a perfectly good "ordinary" explanation.**

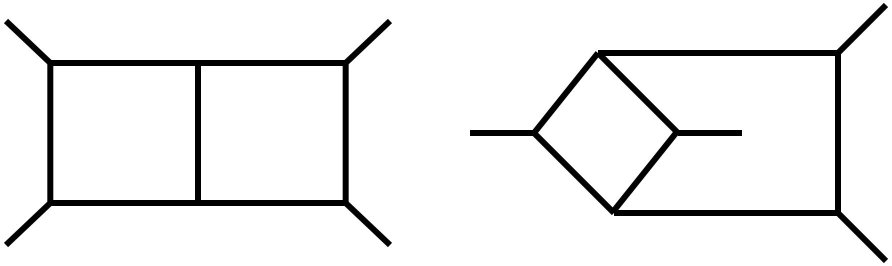
**Does this superspace exist in *D* = 5 or *D* = 4?**

**Not easy to construct: A non-Lorentz covariant harmonic superspace .**

# What is the new “magic”?

**To shed light on the source of magic we need a simpler example:**

## Half maximal supergravity at 2 loop in $D = 5$



**Susy + duality symmetry does not appear to protect against divergence in $D = 5$.**

Bossard, Howe and Stelle
ZB, Davies, Dennen

**Story is similar to $D = 4$, $N = 4$ sugra at 3 loops except that it is much simpler to work with.**

# One-Loop Warmup in Half-Maximal Sugra

ZB, Boucher-Veronneau ,Johansson
ZB, Davies, Dennen, Huang

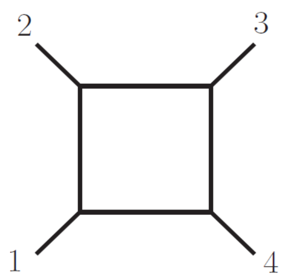
## Generic color decomposition:

Dixon, Del Duca, Maltoni

**color factor**

$$\mathcal{A}_Q^{(1)} = ig^4 \left[ c_{1234}^{(1)} A_Q^{(1)}(1,2,3,4) + c_{1342}^{(1)} A_Q^{(1)}(1,3,4,2) + c_{1423}^{(1)} A_Q^{(1)}(1,4,2,3) \right]$$

**Q = # supercharges** **Q = 0 is pure non-susy YM**




$c_{1234}^{(1)}$ **is color factor of this box diagram**

$$s = (k_1 + k_2)^2$$
$$t = (k_2 + k_3)^2$$

**To get Q + 16 supercharge supergravity take 2nd copy *N* = 4 sYM**

***N* = 4 sYM numerators very simple: independent of loop momentum**

$$n_{1234} = n_{1342} = n_{1423} = st A_{Q=16}^{\text{tree}}(1,2,3,4) \qquad c_{1234}^{(1)} \to n_{1234}$$

$$\mathcal{M}_{Q+16}^{(1)} = i \left(\frac{\kappa}{2}\right)^4 st A_{Q=16}^{\text{tree}}(1,2,3,4) \left[ A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3) \right]$$

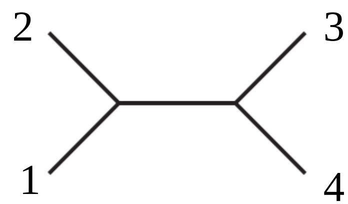
# One-loop divergences in pure YM

ZB, Davies, Dennen, Huang

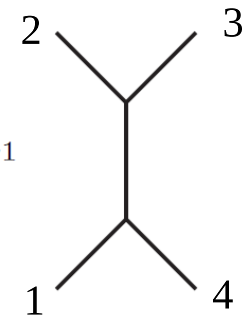
## Go to a basis of color factors

## Three independent one-loop color tensors

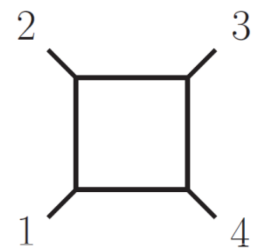
$$b_1^{(0)} = \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}$$




$$b_2^{(0)} = \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}$$




$$b_1^{(1)} \equiv c_{1234}^{(1)} = \tilde{f}^{a_1 b_2 b_1}\tilde{f}^{a_2 b_3 b_2}\tilde{f}^{a_3 b_4 b_3}\tilde{f}^{a_4 b_1 b_4}$$




## All other color factors expressible in terms of these three:

one-loop color tensor

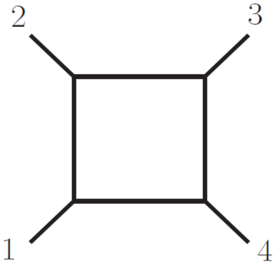
$$\mathcal{A}_Q^{(1)} = i g^4 \Big[ b_1^{(1)} \Big( A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3) \Big) - \frac{1}{2} C_A b_1^{(0)} A_Q^{(1)}(1,3,4,2) - \frac{1}{2} C_A b_2^{(0)} A_Q^{(1)}(1,4,2,3) \Big]$$

tree color tensor

$C_A = 2\, N_c$ for $SU(N_c)$

# One-loop divergences in pure YM

## In a basis of color factors:




one-loop color tensor

$$\mathcal{A}_Q^{(1)} = ig^4\Big[b_1^{(1)}\Big(A_Q^{(1)}(1,2,3,4)+A_Q^{(1)}(1,3,4,2)+A_Q^{(1)}(1,4,2,3)\Big)$$
$$-\frac{1}{2}C_A b_1^{(0)} A_Q^{(1)}(1,3,4,2)-\frac{1}{2}C_A b_2^{(0)} A_Q^{(1)}(1,4,2,3)\Big]$$

tree color tensor

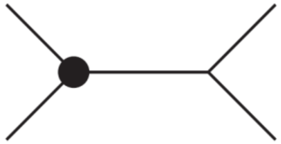
**$Q$ supercharges (mainly interested in $Q$ = 0)**

**$D$ = 4: $F^2$ is only allowed counterterm by renormalizability**

**1-loop color tensor *not* allowed.**

**$D$ = 6: $F^3$ counterterm: 1-loop color tensor again *not* allowed.**

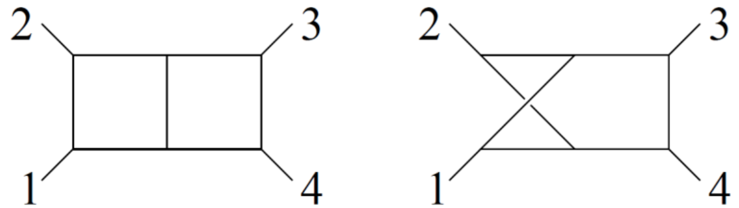
$$F^3 = f^{abc}F_\nu^{a\mu}F_\sigma^{b\nu}F_\mu^{c\sigma}$$



$$A_Q^{(1)}(1,2,3,4)+A_Q^{(2)}(1,3,4,2)+A_Q^{(1)}(1,4,2,3)\Big|_{D=4,6\ \text{div.}}=0$$

$$\boxed{M_{Q+16}^{(1)}(1,2,3,4)\Big|_{D=4,6\ \text{div.}}=0}$$

# Two Loop Half Maximal Sugra in $D = 5$




ZB, Davies, Dennen, Huang

$$\mathcal{A}_Q^{(2)} = -g^6\Big[c^{\rm P}_{1234}A^{\rm P}_Q(1,2,3,4) + c^{\rm P}_{3421}A^{\rm P}_Q(3,4,2,1) + c^{\rm NP}_{1234}A^{\rm NP}_Q(1,2,3,4) + c^{\rm NP}_{3421}A^{\rm NP}_Q(3,4,2,1) + \text{ cyclic}\Big]$$

**$D = 5$ $F^3$ counterterm: 1,2-loop color tensors forbidden!**

1) **Go to color basis.**
2) **Demand no forbidden color tensors in pure YM divergence.**
3) **Replace color factors with kinematic numerators.**

**gravity** $$\mathcal{M}^{(2)}_{16+Q}(1,2,3,4)\Big|_{D=5\,\text{div.}} = 0$$

**Half-maximal supergravity four-point divergence vanishes because forbidden color tensor cancels in pure YM theory.**
**Note: this cancellation is mysterious from standard symmetries.**

# Two Loop $D = 5$ UV Magic

ZB, Davies, Dennen, Huang

**At least for 2 loops in $D = 5$ we have identified the source of unexpected UV cancellations in half-maximal supergravity:**

**It is the *same* magic found by 't Hooft and Veltman 40 years ago preventing forbidden divergences appearing in ordinary non-susy gauge theory!**

**Completely explains the $D = 5$ two-loop half-maximal sugra case, which still remains mysterious from standard supergravity symmetry viewpoint.**

**Half maximal supergravity at $L = 2$, $D = 5$ or $L = 3$, $D = 4$ are borderline cases.**

**We need to go beyond these and also develop a general understanding of the structure.**

# Four-loop $\mathcal{N} = 4$ Supergravity Divergences

ZB, Davies, Dennen, Smirnov, Smirnov

**Four loops done same way as three loops.**

**Similar to three loops except industrial level: C++ FIRE 4**

***N* = 4 sugra: (*N* = 4 sYM) x (*N* = 0 YM)**

***N* = 4 sYM**

$$\sim (l \cdot k)^2 \, s^2 t A_4^{\text{tree}}$$

**BCJ representation**

**pure YM**

$$\sim (\varepsilon_i \cdot l)^4 \, l^6$$

**Feynman representation**

***N* = 4 sugra diagrams quadratically divergent**

$$\int (d^D l)^4 \, \frac{k^8 l^{12}}{l^{26}}$$

***D*² *R*⁴ counterterm**

**82 nonvanishing diagram types using *N* = 4 sYM BCJ form.**

**Are there cancellations between the pieces?**

# The 4 loop Divergence of $N = 4$ Supergravity

ZB, Davies, Dennen, Smirnov, Smirnov

**Pure $N = 4$ supergravity is divergent at 4 loops with divergence**

**dim. reg. UV pole**

**Result is for Siegel dimensional reduction.**

$$\boxed{\mathcal{M}^{4\text{-loop}}\Big|_{\text{div.}} = \frac{1}{(4\pi)^8} \frac{1}{\epsilon} \left(\frac{\kappa}{2}\right)^{10} \frac{1}{144} (1 - 264\zeta_3)\, \mathcal{T}}$$

$$\mathcal{T} = st A^{\text{tree}}_{\mathcal{N}=4} \left(\mathcal{O}_1 - 28\mathcal{O}_2 - 6\mathcal{O}_3\right)$$

$$\mathcal{O}_1 = \sum_{\mathcal{S}_4} (D_\alpha F_{1\mu\nu})\,(D^\alpha F_2^{\mu\nu})\, F_{3\rho\sigma} F_4^{\rho\sigma}$$

$$\mathcal{O}_2 = \sum_{\mathcal{S}_4} (D_\alpha F_{1\mu\nu})\,(D^\alpha F_2^{\nu\sigma})\, F_{3\sigma\rho} F_4^{\rho\mu}$$

$$\mathcal{O}_3 = \sum_{\mathcal{S}_4} (D_\alpha F_{1\mu\nu})\,(D_\beta F_2^{\mu\nu})\, F_{3\sigma}{}^{\alpha} F_4^{\sigma\beta}$$

$$s = (k_1 + k_2)^2$$
$$t = (k_2 + k_3)^2$$

$$F_j^{\mu\nu} \equiv i(k_j^\mu \varepsilon_j^\nu - k_j^\nu \varepsilon_j^\mu)\,,$$
$$D^\alpha F_j^{\mu\nu} \equiv -k_j^\alpha (k_j^\mu \varepsilon_j^\nu - k_j^\nu \varepsilon_j^\mu)$$

**Valid for all nonvanishing 4-point amplitudes of pure $N = 4$ sugra**

# Some Peculiar Properties

ZB, Davies, Dennen, Smirnov, Smirnov

**Linear combinations to expose *D* = 4 helicity structure**

**Refers to helicities of pure YM component**

$$\mathcal{O}^{--++} = \mathcal{O}_1 - 4\mathcal{O}_2$$

$$\mathcal{O}^{-+++} = \mathcal{O}_1 - 4\mathcal{O}_3$$

$$\mathcal{O}^{++++} = \mathcal{O}_2$$

$$\mathcal{O}^{--++} = 4s^2t\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle},$$

$$\mathcal{O}^{-+++} = -12s^2t^2\frac{[2\,4]^2}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]},$$

$$\mathcal{O}^{++++} = 3st(s+t)\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle},$$

**The latter two configurations would vanish if a *U*(1) symmetry were not anomalous.**

See Carrasco, Kallosh, Tseytlin and Roiban

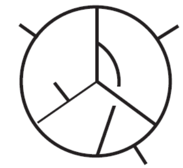
**All three independent configurations have similar divergence.**

**Very peculiar because the nonanomalous sector should have a very different analytic structure. Not related by any supersymmetry Ward identities.**

**For anomalous sectors:**

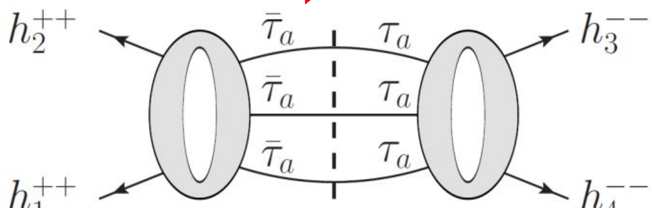
- ***D* = 4 generalized cuts decomposing into tree amplitudes vanish.**
- **Anomaly is $\epsilon/\epsilon$ (UV divergence suppressed by $\epsilon$ ).** $D = 4 - 2\epsilon$

# Relation to $U(1)$ Anomaly



**Anomalous sector feeds poor UV behavior into non-anomalous sector**

unitarity cut




Anomalous 1-loop amplitudes

Figure from arXiv:1303.6219
Carrasco, Kallosh, Tseytlin and Roiban

- As pointed out by Carrasco, Kallosh Roiban,Tseytlin the anomalous amplitudes are poorly behaved and contribute to a 4-loop UV divergence (unless somehow canceled as they are at 3 loops).
- Via the anomaly it is easy to understand why all three sectors can have similar divergence structure.
- The dependence of the divergence on vector multiplets matches anomaly.

$$\mathcal{M}^{4\text{-loop}}_{n_V}\Big|_{\text{div.}} = \frac{1}{(4\pi)^8}\left(\frac{\kappa}{2}\right)^{10}\frac{\boxed{n_V+2}}{2304}\Big[\frac{6(n_V+2)n_V}{\epsilon^2} + \frac{(n_V+2)(3n_V+4)-96(22-n_V)\zeta_3}{\epsilon}\Big]\mathcal{T}$$

**anomaly has exactly this factor**

***$n_V$* is number vector multiplets**

**Bottom line: The 4 loop divergence looks specific to *N* = 4 sugra and likely due to anomaly. Won't be present in *N* $\geqslant$ 5 sugra.**

# $\mathcal{N}$ = 5 Supergravity

ZB, Davies, Dennen, Smirnov, Smirnov

**An obvious test is to check $\mathcal{N}$ = 5 supergravity at four loops. No anomalies and it is therefore expected to be finite.**

**$\mathcal{N}$ = 4 sYM**

**BCJ representation**

**$\mathcal{N}$ = 1 sYM**

**Feynman representation**

**Straightforward to obtain by adding a single adjoint fermion to the pure Yang-Mills side of double copy.**

**Almost finished, but didn't quite make it in time for conference.**

# Summary

- Gravity amplitudes are currently under intense study with many new representations and uncovered structures.
- Soft graviton behavior of amplitudes are corrected by loops.
- Surprisingly good UV behavior of supergravity uncovered. In $N = 4$ supergravity as yet no standard symmetry explanation for observed cancellations. Finiteness of $N \geqslant 5$ supergravity still an open problem.
- In the case of half maximal sugra at 2 loops in $D = 5$, where we could analyze in detail, "magical cancellations" follow from "standard magic" of Yang-Mills preventing forbidden divergences from appearing.

**We can expect many more structures and surprises as we probe (super)gravity theories using modern tools and ideas.**