ON THE COLLINEAR LIMIT OF SCATTERING AMPLITUDES AT STRONG COUPLING

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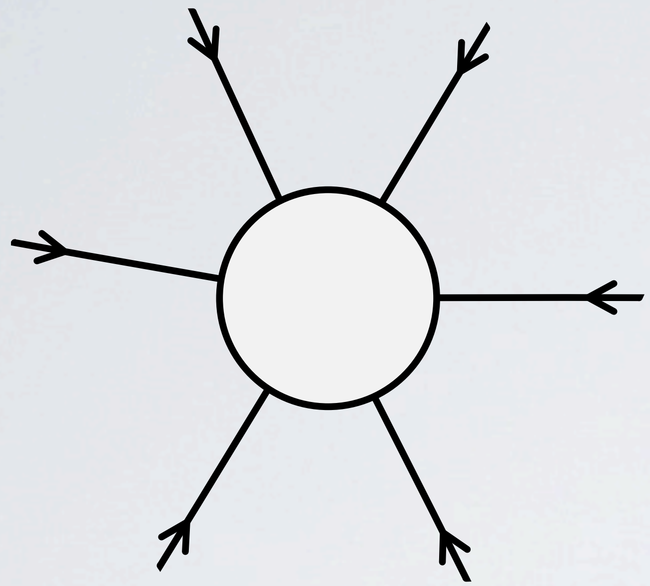
Amplitudes 14 IPhT Saclay

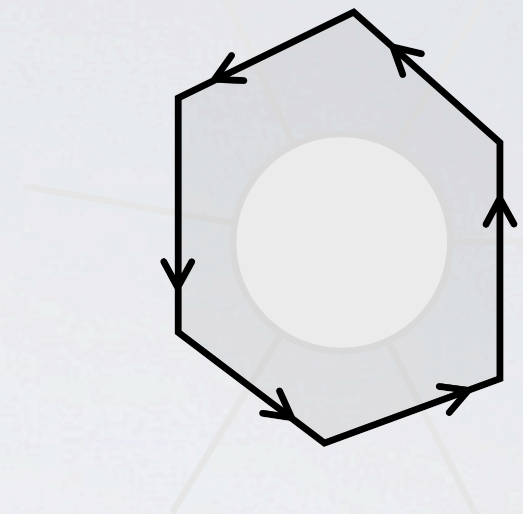
based on work with Amit Sever and Pedro Vieira

SA = WL

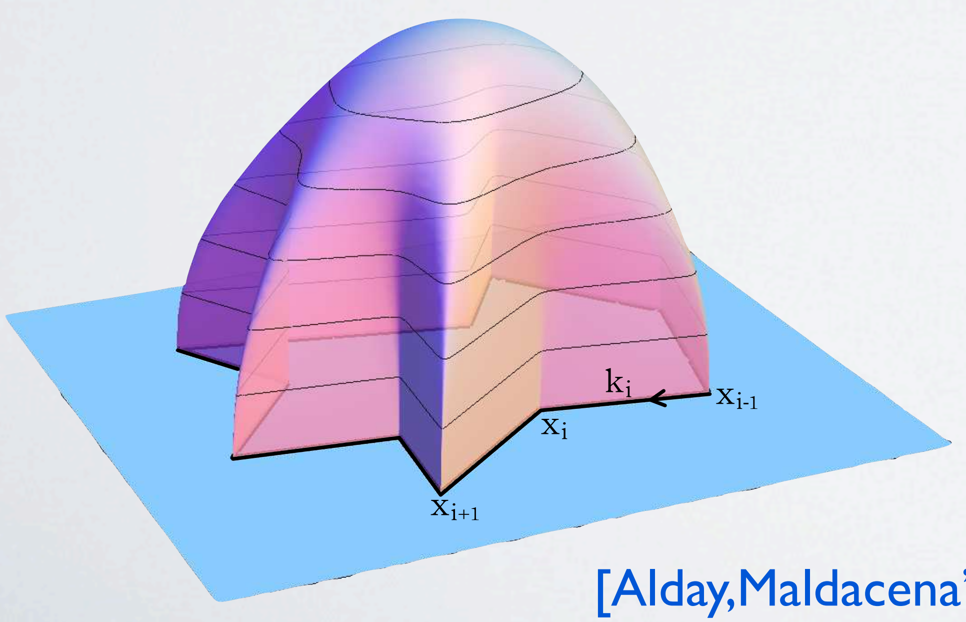
[Alday,Maldacena'07]
[Drummond,Korchemsky,Sokatchev'07]
[Brandhuber,Heslop,Travaglini'07]
[Drummond,Henn,Korchemsky,Sokatchev'07]



$\simeq$



String theory : prediction at strong coupling $\sqrt{\lambda} \gg 1$



[Alday,Maldacena'07]

number of gluons

$$\mathcal{W}_n = e^{-\frac{\sqrt{\lambda}}{2\pi} A_n + \dots}$$

minimal surface area in AdS_5

Goal of this talk

1) Show that there is yet another large contribution

$$\mathcal{W}_n = e^{-\frac{\sqrt{\lambda}}{2\pi} A_n + \frac{\sqrt{\lambda}(n-4)(n-5)}{48n} + o(\sqrt{\lambda})}$$

clash?

Innocent constant? look at collinear limit

$\mathcal{W}_n \to \mathcal{W}_{n-1}$ must hold true

$A_n \to A_{n-1}$ must also be true

Goal of this talk

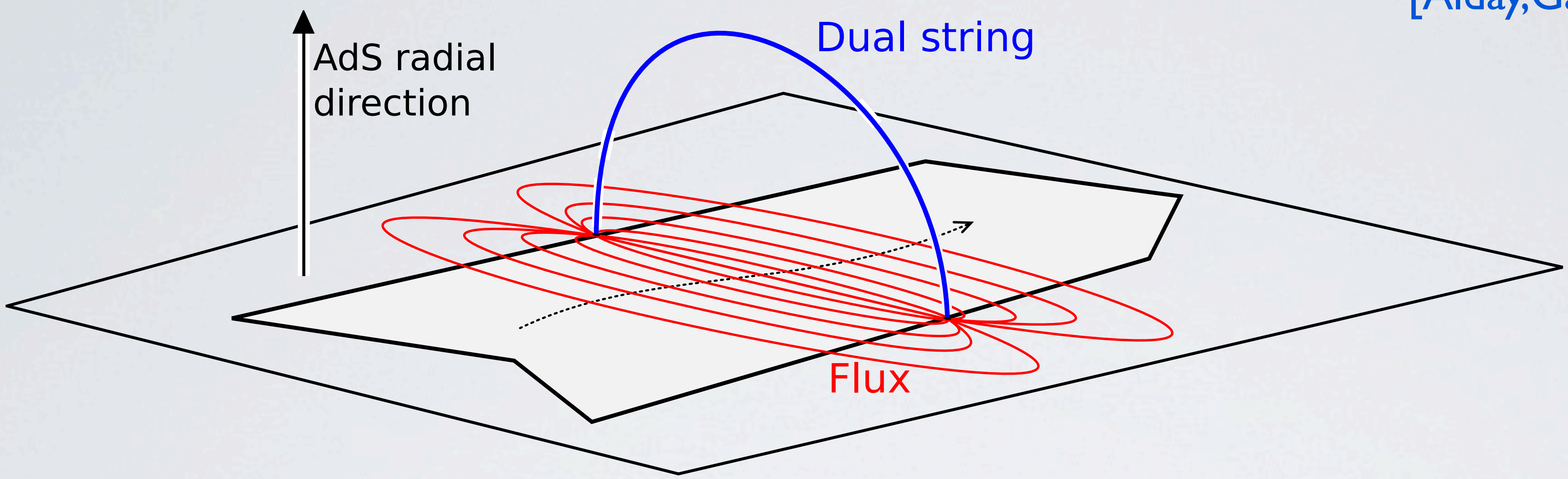
1) Show that there is yet another large contribution

$$\mathcal{W}_n = e^{-\frac{\sqrt{\lambda}}{2\pi} A_n + \frac{\sqrt{\lambda}(n-4)(n-5)}{48n} + o(\sqrt{\lambda})}$$

2) Understand the collinear limit at strong coupling

Our method : the OPE picture

[Alday,Gaiotto,Maldacena,Sever,Vieira'09]



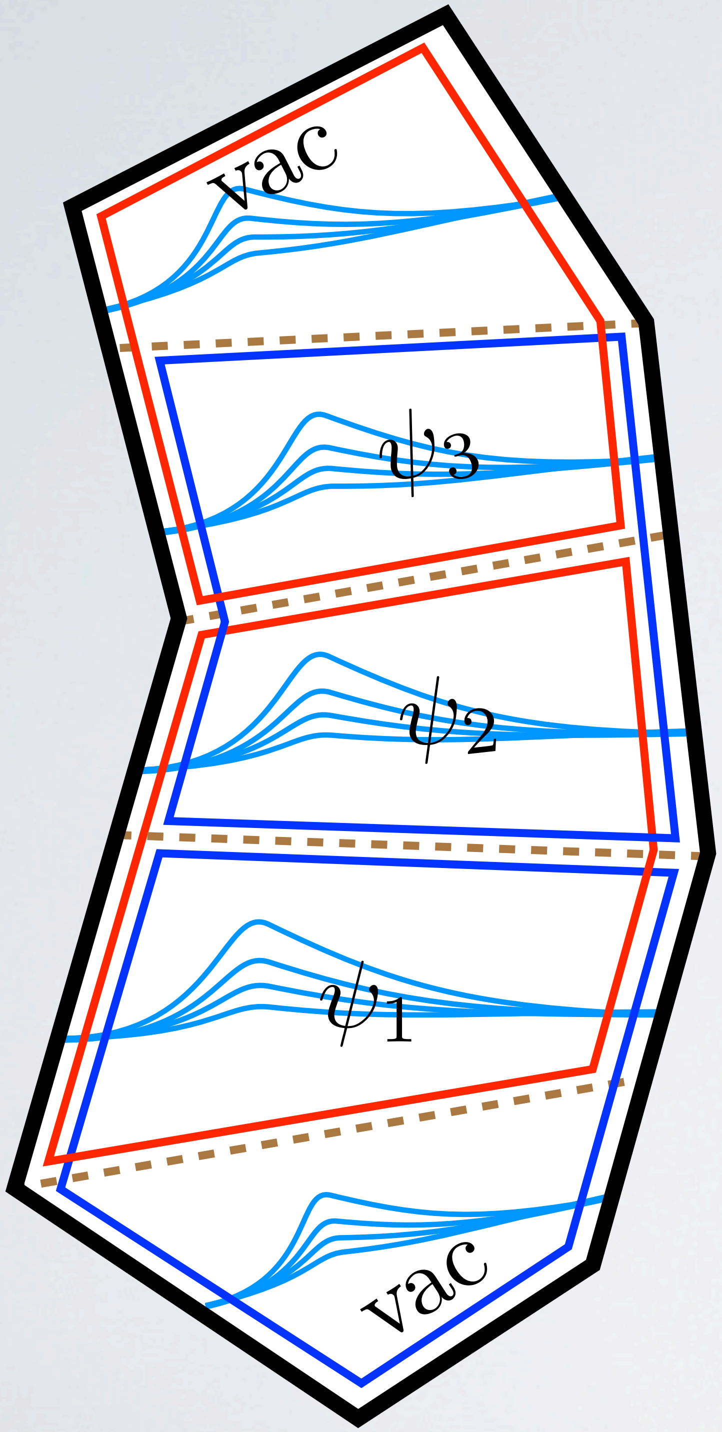
We have a 1+1d background : the flux tube sourced by two parallel null lines

Bottom and top parts of the loop can be thought of as exciting the flux tube out of its ground state

Decomposing over basis of flux tube eigenstates allows us to construct WL as a systematic expansion around the collinear limit

Refinement : the pentagon decomposition

[BB,Sever,Vieira'13]



flux-tube energy

flux-tube momentum

$$= \sum_{\psi_i} \left[\prod_i e^{-E_i\tau_i + ip_i\sigma_i + im_i\phi_i} \right] P(0|\psi_1)P(\psi_1|\psi_2)P(\psi_2|\psi_3)P(\psi_3|0)$$

angular momentum

pentagon transition

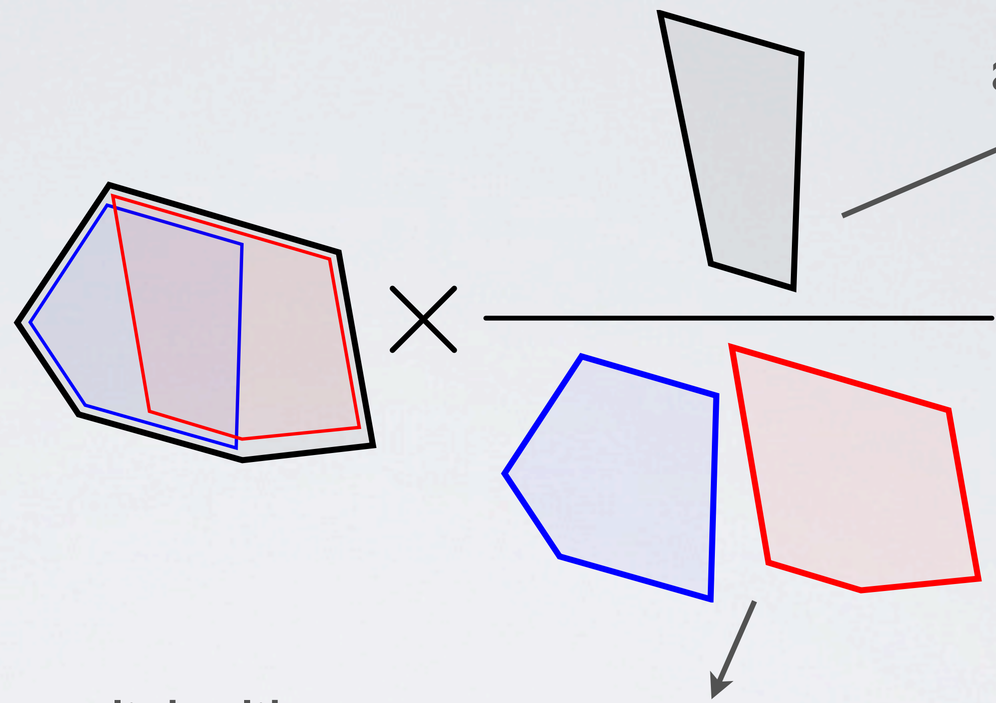
To compute amplitudes we need

- The spectrum of flux-tube states ψ
- All the pentagon transitions $P(\psi_1|\psi_2)$

A bit slower I : the ratio

We want to compute

$$\mathcal{W}_6 \equiv$$



light-like square as overlap of pentagons

light-like pentagons inscribed in the hexagonal WL

$$\log \mathcal{W}_6 = \mathrm{BDS}_6 - \mathrm{BDS}_{5-\text{bottom}} - \mathrm{BDS}_{5-\text{top}} + \mathrm{BDS}_{4-\text{middle}}$$

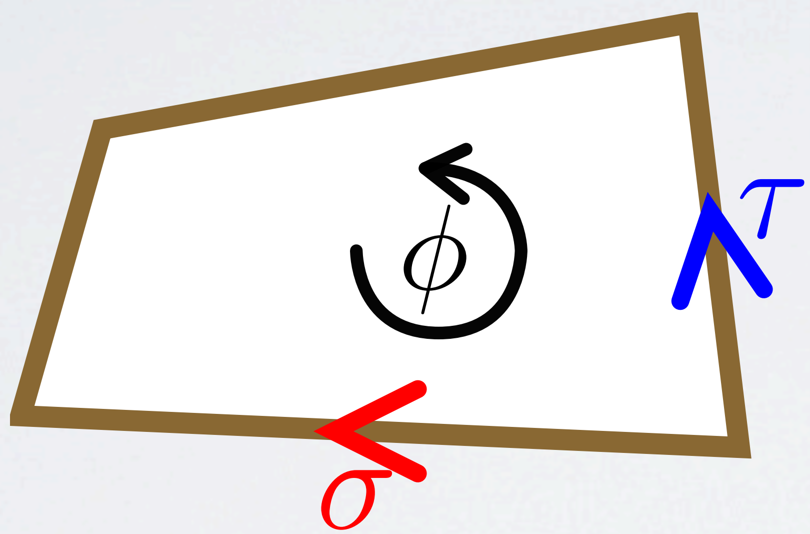
$$+R_6(u_1, u_2, u_3)$$

$$= \tfrac{1}{4}\Gamma_{\text{cusp}}(g)\left[\mathrm{Li}_2(u_2) - \mathrm{Li}_2(1-u_1) - \mathrm{Li}_2(1-u_3) - \ldots\right]$$

Exactly known and UV finite function of cross ratios

No information loss
Free of divergences

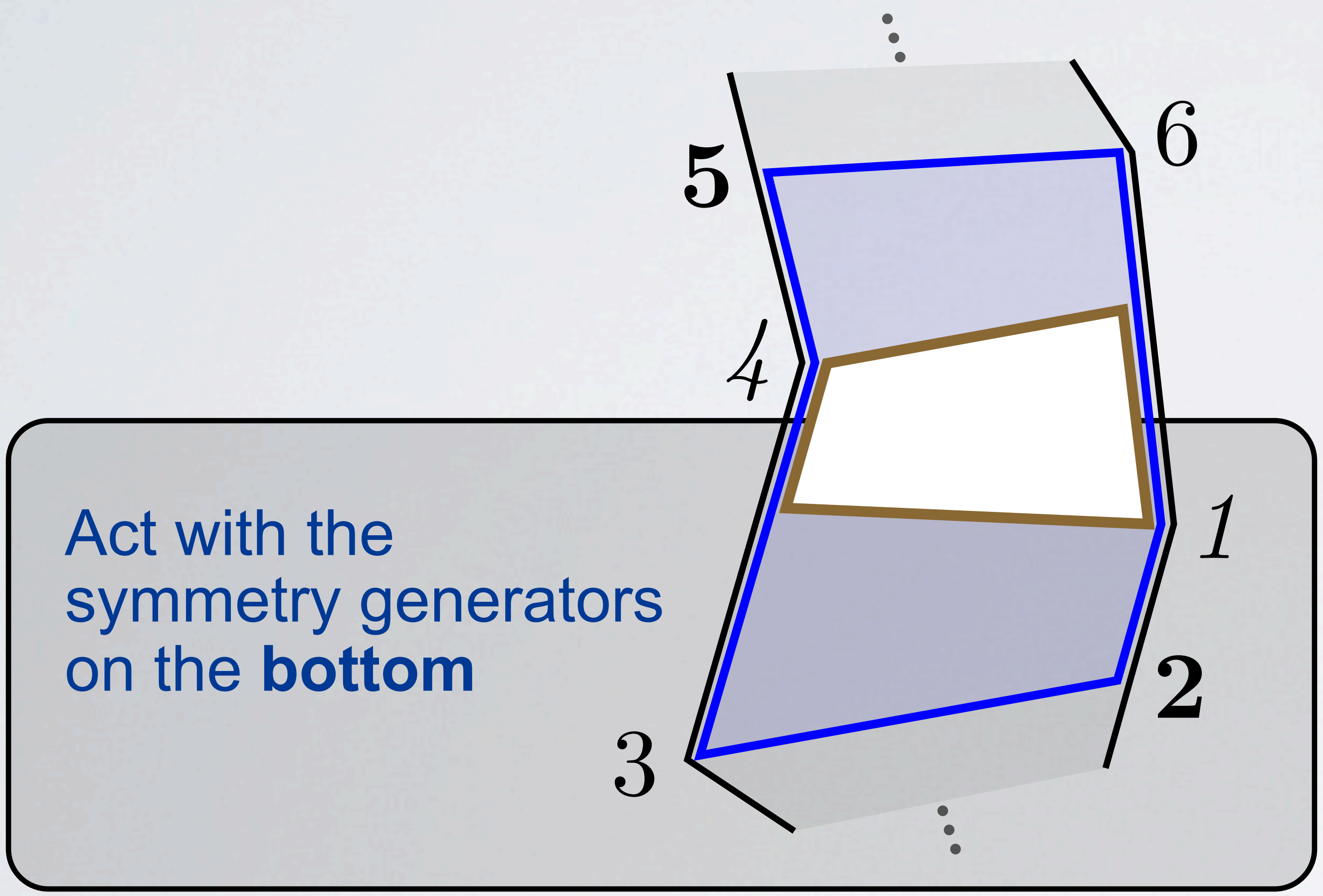
A bit slower II : the OPE parameters



Null square has 3 (abelian) symmetries

[Alday,Gaiotto,Maldacena,Sever,Vieira'09]

A bit slower II : the OPE parameters



cross ratios

$$u_i \equiv \frac{x^2_{i-1,i+1}x^2_{i-2,i+2}}{x^2_{i-1,i+2}x^2_{i+1,i-2}}$$

$$\frac{1}{u_2} = 1 + e^{2\tau}$$

$$\frac{u_1}{u_2 u_3} = e^{2\sigma + 2\tau}$$

$$\frac{1}{u_3} = 1 + (e^{-\tau} + e^{\sigma + i\phi})(e^{-\tau} + e^{\sigma - i\phi})$$

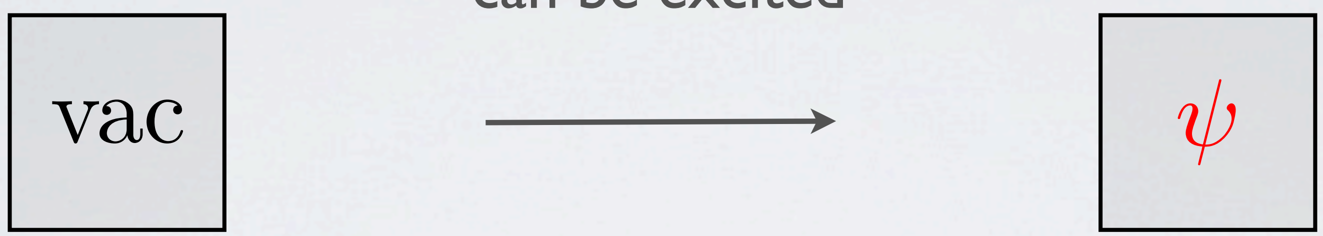
Produce a family of WL parameterized by

$$n - 5 \text{ middle squares } \Rightarrow 3(n-5) \text{ conformal cross ratios}$$

A bit slower III : the flux tube

Square WL as reference state : the flux tube

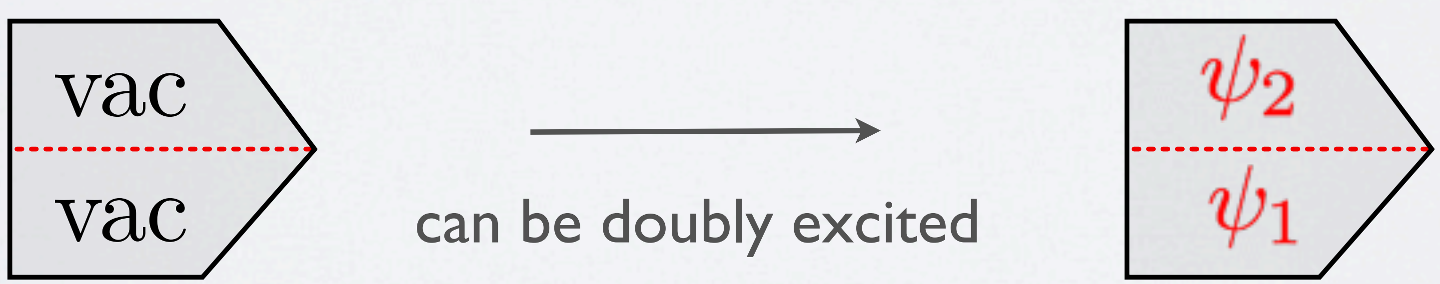
can be excited



flux-tube vacuum

flux-tube excited state ψ

Pentagon WL as flux-tube transition



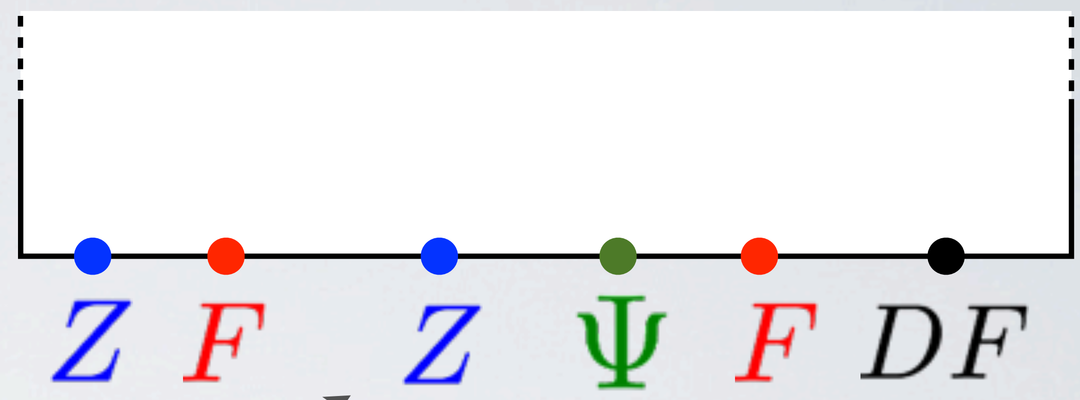
$$= P(\psi_1|\psi_2)$$

vacuum-to-vacuum transition

Pentagon transition from ψ_1 to ψ_2

The flux-tube eigenstates

$\psi = N$ particles state



Field insertions along a light-ray:
create/annihilate state on the flux tube

Adjoint fields of the theory

Spectral data

$$E = E(u_1) + E(u_2) + \ldots + E(u_N) \qquad p = p(u_1) + \cdots + p(u_N) \qquad m = m_1 + \cdots + m_N$$

rapidity

$$E(u) = \text{twist} + g^2 \ldots \qquad p(u) = 2u + g^2 \ldots \qquad m = 0, \pm 1, \ldots$$

= engineering dimension – spin projection along light-ray direction

Square and pentagon transitions

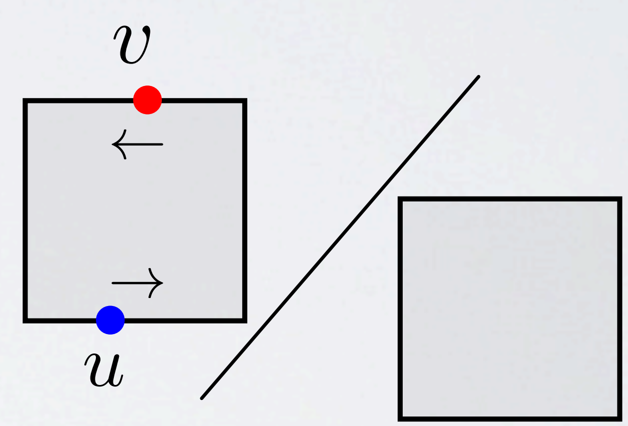
Summing over states

$$\sum_{\psi} = \sum_{\mathbf{a},N} \int du_1 \ldots du_N \, \mu(u_1) \ldots \mu(u_N)$$

$\{Z, \Psi, F, \ldots\}$

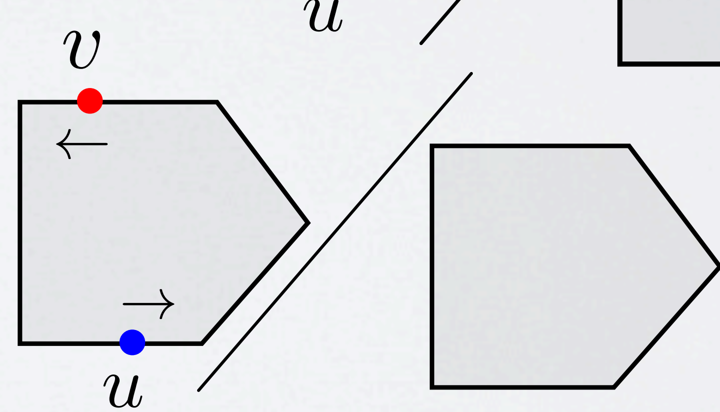
Square measure
or 2-point function
on the square

$$\frac{2\pi}{\mu(u)} \delta(u-v) =$$



Pentagon transition
same but
on a pentagon

$$P(u|v) =$$

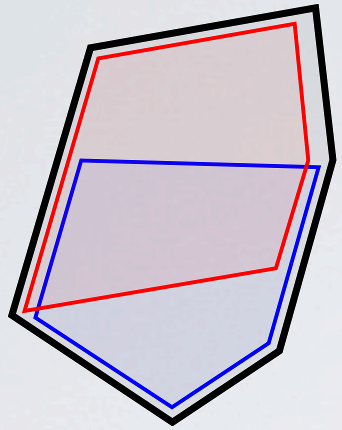


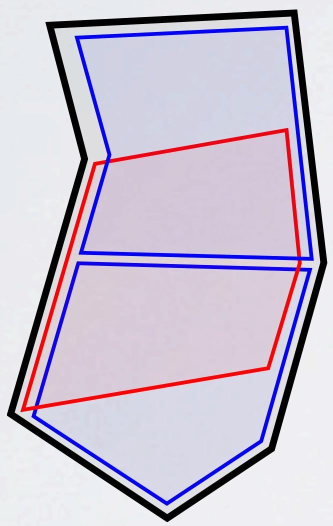
Relation

$$P(u|v) \sim \frac{1}{i(u-v)\mu(u)} \qquad \text{when} \qquad u \sim v$$

Symmetries of the square not
preserved by pentagon
no conservation of flux-tube energy-
momentum

Combining all pieces together

$\mathcal{W}_{\rm hex} =$  $= \sum_{\mathbf{a}} \int d\mathbf{u}\, P_{\mathbf{a}}(0|\mathbf{u})\, e^{-E(\mathbf{u})\tau + ip(\mathbf{u})\sigma + im\phi} P_{\mathbf{a}}(\bar{\mathbf{u}}|0)$

$\mathcal{W}_{\rm hep} =$  $= \sum_{\mathbf{a},\mathbf{b}} \int d\mathbf{u}\, d\mathbf{v}\, P_{\mathbf{a}}(0|\mathbf{u})\, e^{-E(\mathbf{u})\tau_1 + ip(\mathbf{u})\sigma_1 + im_1\phi_1} P_{\mathbf{ab}}(\bar{\mathbf{u}}|\mathbf{v})$

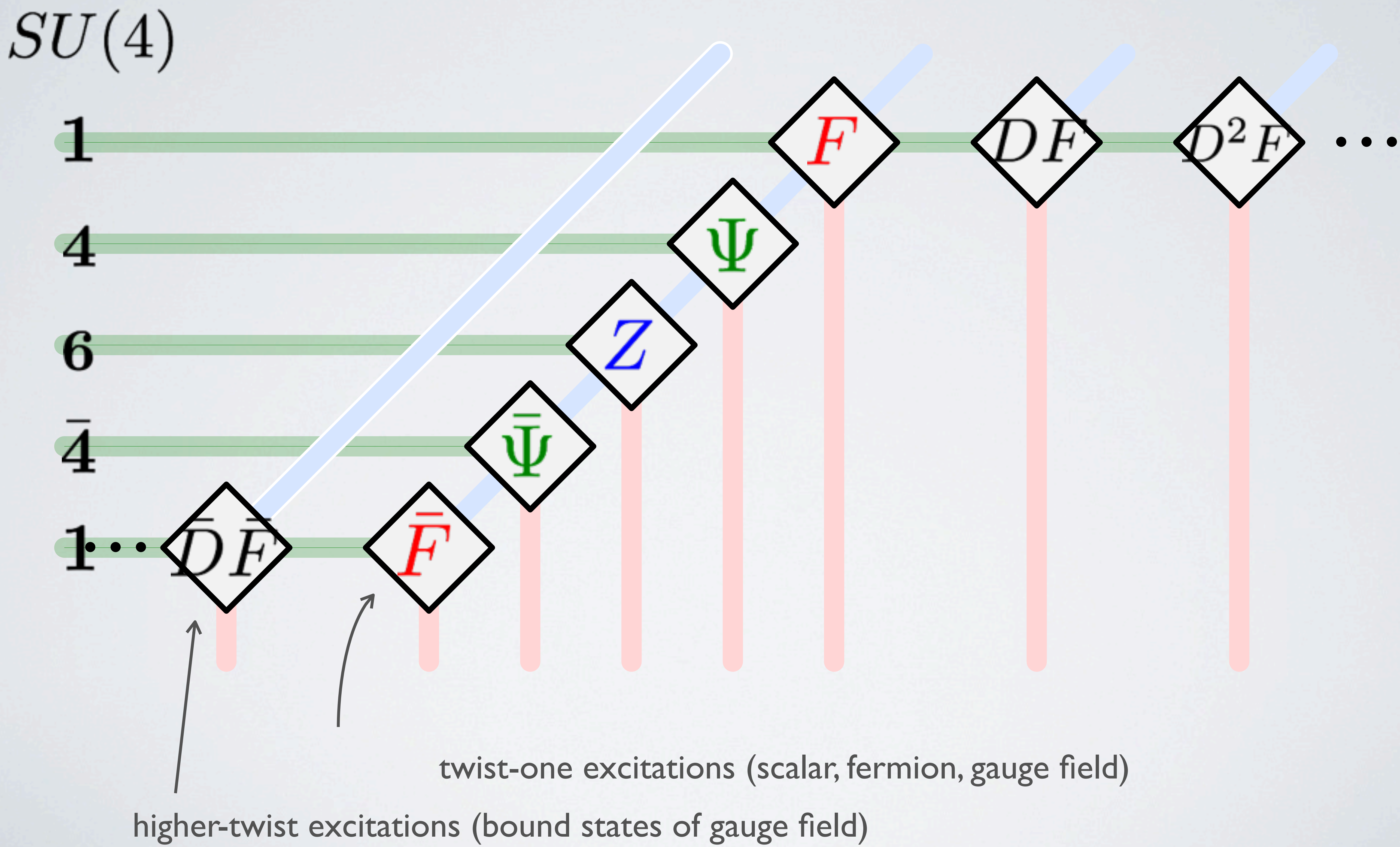
$\times e^{-E(\mathbf{v})\tau_2 + ip(\mathbf{v})\sigma_2 + im_2\phi_2} P_{\mathbf{b}}(\bar{\mathbf{v}}|0)$

i.e. in the collinear limit

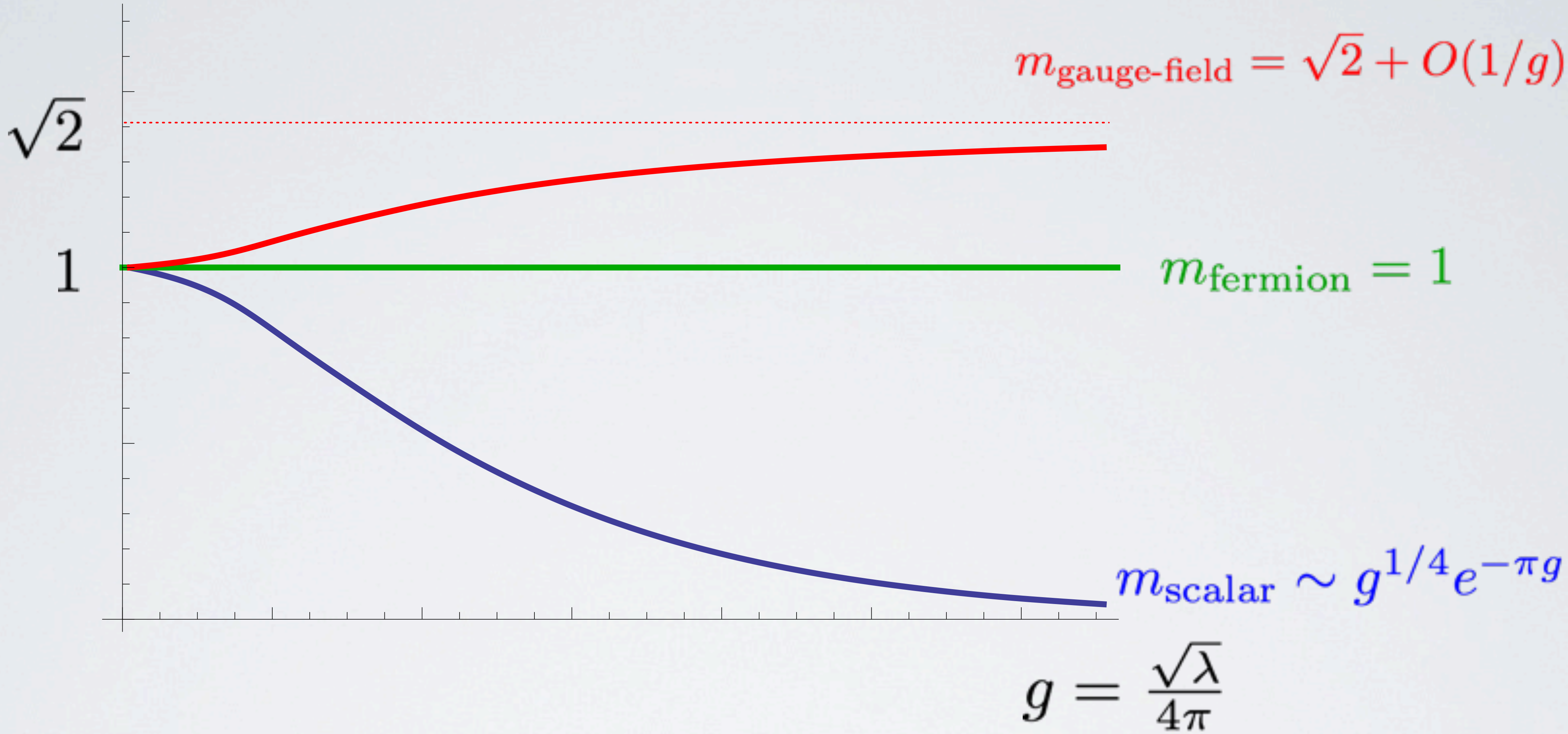
Lightest states dominate at large τ

What are they?

Fundamental flux-tube excitations



Masses at finite coupling



Away from weak coupling : the scalars dominate

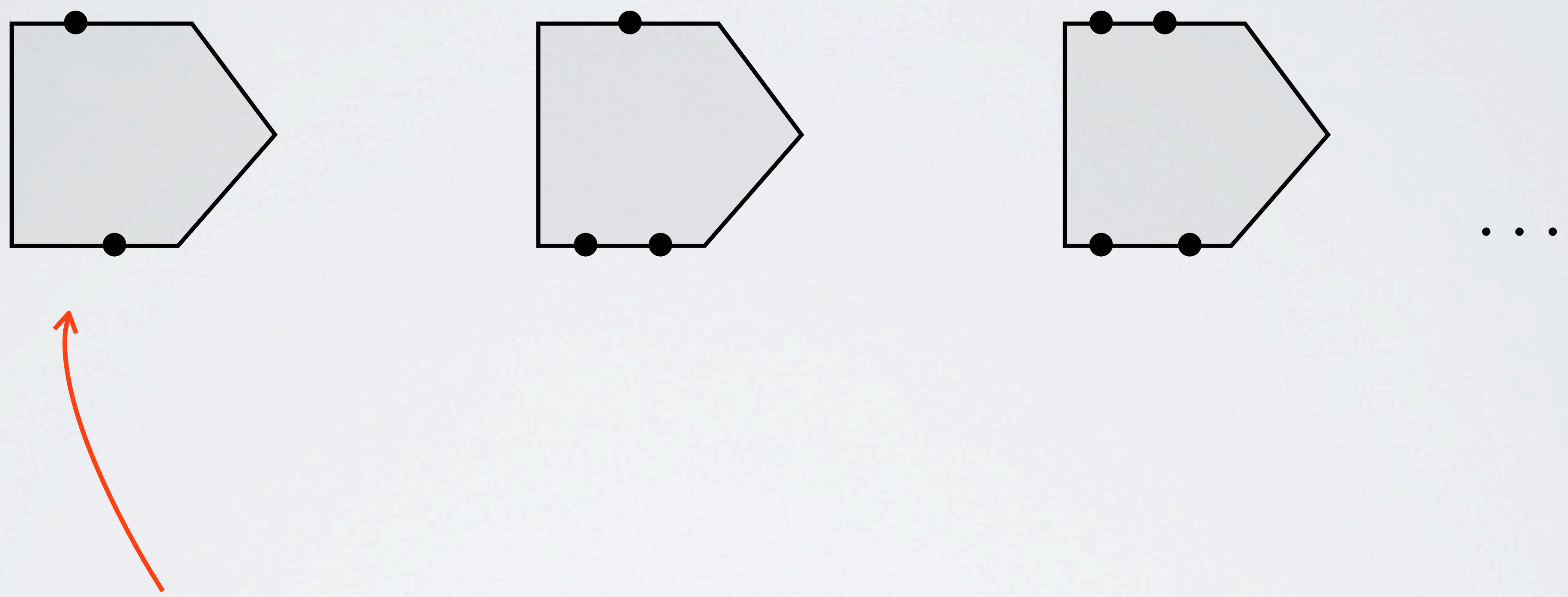
Their mass is exponentially small at strong coupling

and their dynamics is controlled by the O(6) σ model

[Alday,Maldacena'07]

How to get pentagon transitions?

There are all kinds of pentagon transitions



These are the most fundamental ones

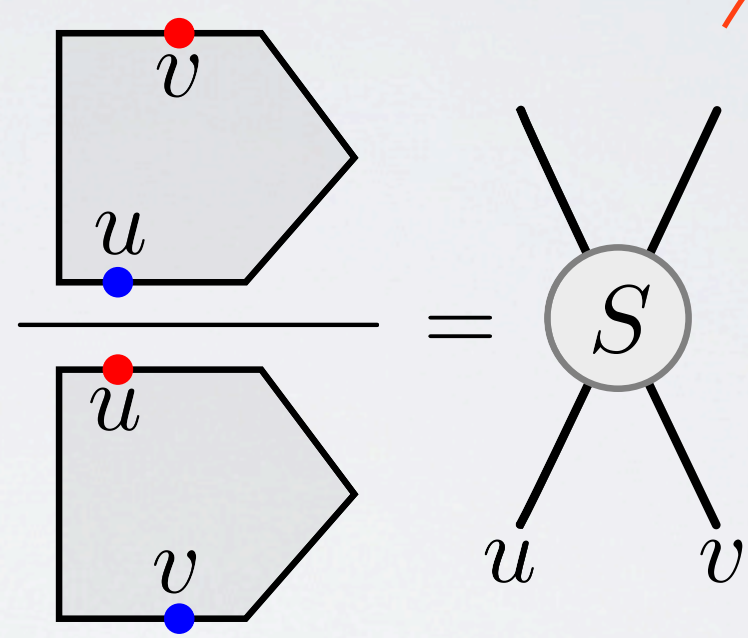
Idea : use single-particle transition to build higher ones

Pentagon bootstrap

Fundamental axiom

flux tube S-matrix

$$\frac{P(u|v)}{P(v|u)} =$$



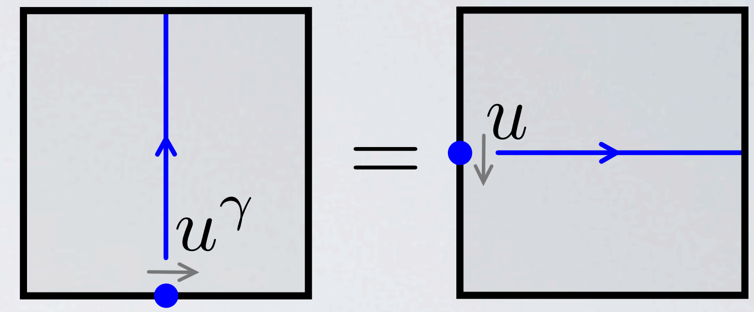
The flux-tube S-matrix determines $P(u|v)$ up to a symmetric function

Further axiom : mirror transformation

Mirror rotation

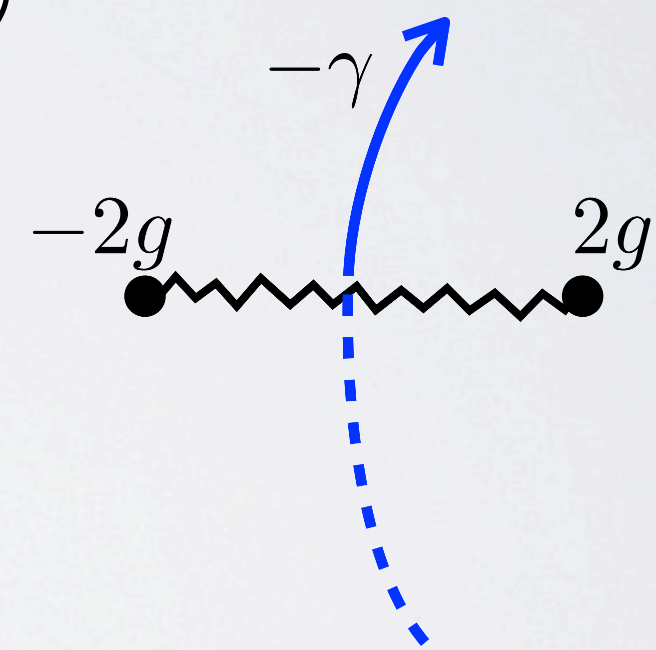
$$E(u^\gamma) = ip(u)$$

$$p(u^\gamma) = iE(u)$$



Non-perturbative!

$$x(u) = \frac{1}{2}\left(u + \sqrt{u^2 - 4g^2}\right)$$

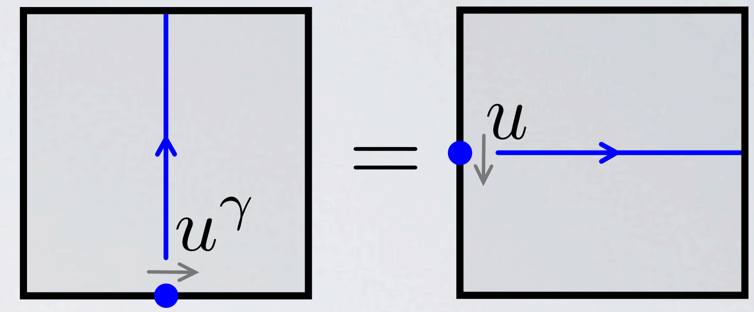


Involve continuation through cut with size controlled by the coupling $g = \frac{\sqrt{\lambda}}{4\pi}$

Further axiom : mirror transformation

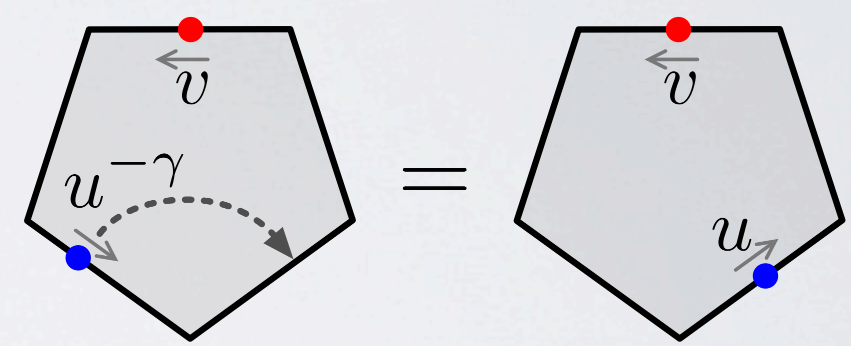
Mirror rotation

$$E(u^\gamma) = ip(u)$$
$$p(u^\gamma) = iE(u)$$

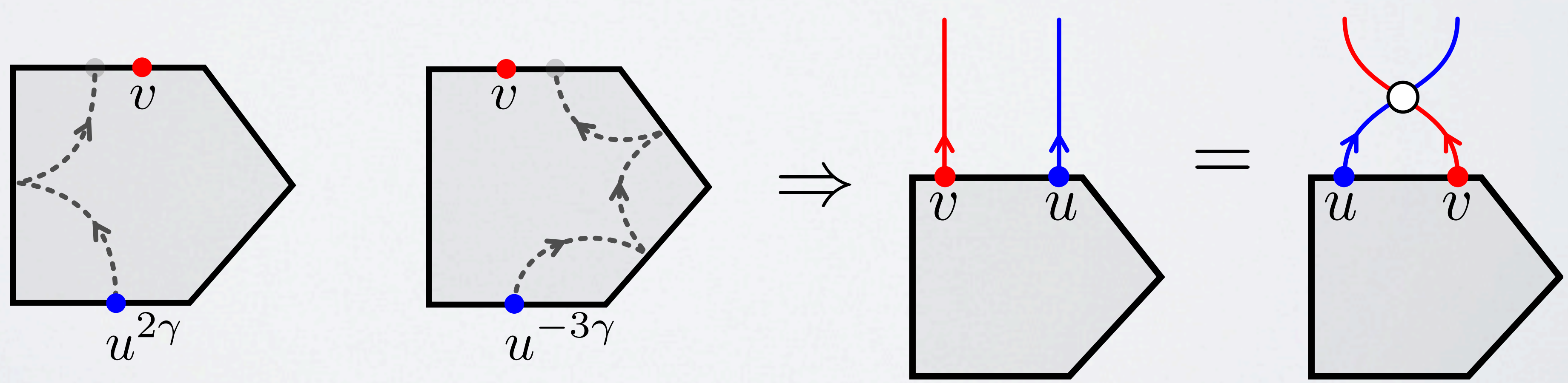


Mirror axiom

$$P(u^{-\gamma}|v) = P(v|u)$$



Consistency check



Watson equation OK

Pentagon transition at any coupling

I. $P(u|v) = P(-v|-u)$

II. $P(u|v) = S(u,v)P(v|u)$

III. $P(u^{-\gamma}|v) = P(v|u)$

Pentagon transition can be expressed in terms of flux tube S-matrix

$$P(u|v)^2 = \frac{S(u,v)}{g^2(u-v)(u-v+i)S(u^\gamma,v)}$$

At strong coupling:

$$P(u|v) \propto \frac{\Gamma(\frac{1}{4}-\frac{i}{2\pi}\theta_{12})\Gamma(\frac{i}{2\pi}\theta_{12})}{\Gamma(\frac{3}{4}-\frac{i}{2\pi}\theta_{12})\Gamma(\frac{1}{2}+\frac{i}{2}\theta_{12})}$$

Function of difference of rapidities $\theta_{12} = \frac{\pi}{2}(u-v)$

(relativistic invariance)

Hexagon in 2pt approximation

$$\mathcal{W}_6 = 1 + \frac{1}{2}\int \frac{d\theta_1 d\theta_2}{(2\pi)^2} |P(0|\theta_1,\theta_2)|^2 e^{-m\tau(\cosh\theta_1+\cosh\theta_2)+im\sigma(\sinh\theta_1+\sinh\theta_2)} + \dots$$

where

$$|P(0|\theta_1,\theta_2)|^2 = |P(\theta_1+i\pi,\theta_2)|^2 = \mu^2 \frac{6}{(\theta_{12}^2+\frac{\pi^2}{4})(\theta_{12}^2+\pi^2)} \times \frac{1}{P(\theta_1|\theta_2)P(\theta_2|\theta_1)}$$

Observation : $\mathcal{W}_6 = \mathcal{W}_6(z) + \dots \qquad z = m\sqrt{\sigma^2+\tau^2}$

as a consequence of relativistic invariance

Decoupling limit

For $\tau \gg 1$ all heavy flux tube excitations decouple

We are left with the scalars

$$\mathcal{W}_6 = \mathcal{W}_{O(6)}(z) + O(e^{-\sqrt{2}\tau})$$

Still the physics remains rich as it is controlled by the O(6) model

$\mathcal{W}_{O(6)}(z)$ is a complicated function of $z = m\sqrt{\sigma^2 + \tau^2}$

Large distance behavior

At large distance $z \gg 1$

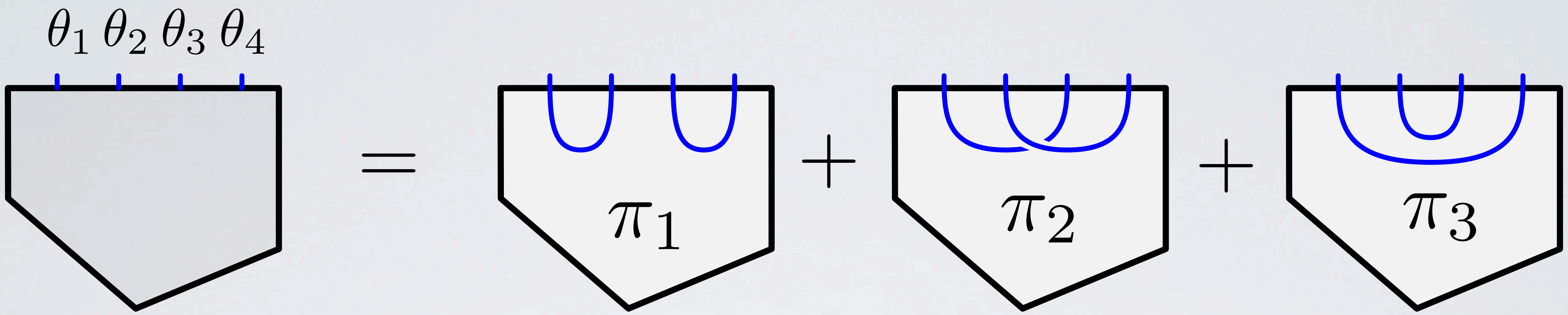
$$\mathcal{W}_6 = 1 + O(e^{-2z})$$

This is the deep (infrared) collinear limit $\tau \gg e^{\sqrt{\lambda}/4}$

It is completely non perturbative

If we want to move away from it we must include states with more than 2 particles

Structure of multi-particle transitions



Factorized ansatz

$$P(0|\theta_1,\ldots,\theta_4)_{i_1,\ldots,i_4} = \ P_{\text{dyn}}(\theta_1,\ldots,\theta_4) \times M_{i_1,\ldots,i_4}(\theta_1,\ldots,\theta_4)$$

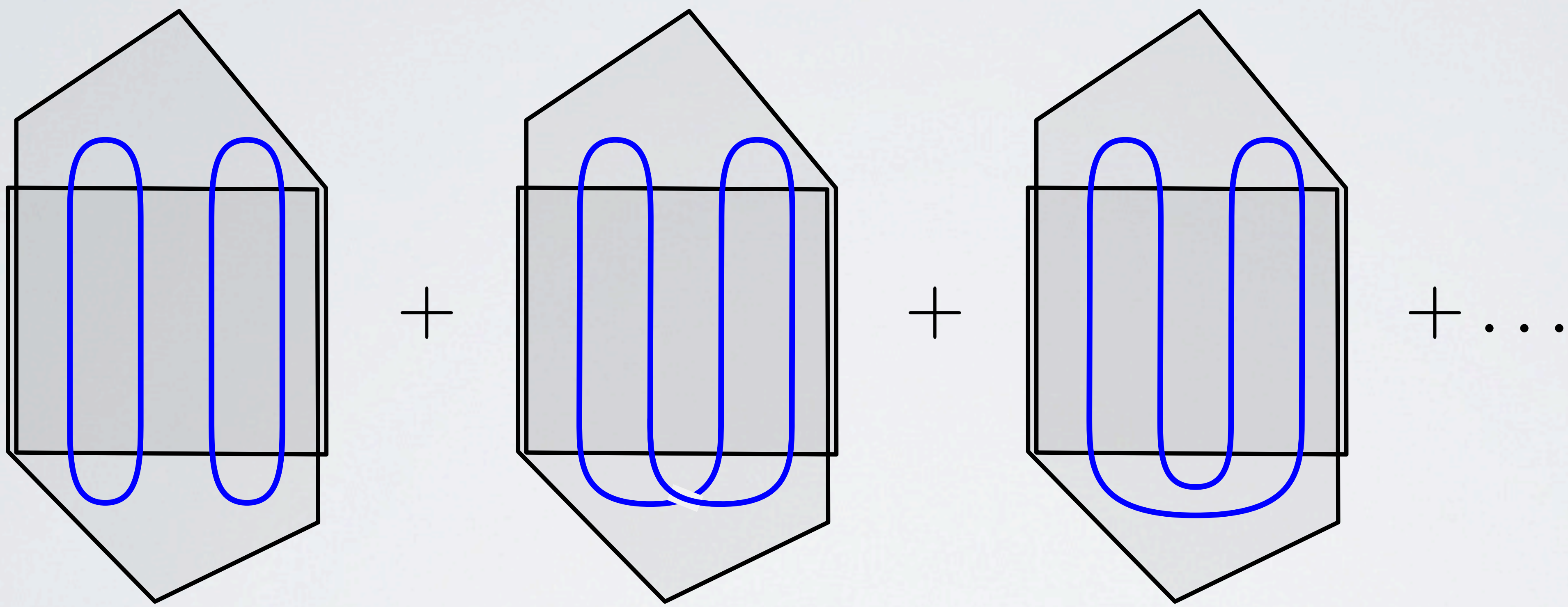
Dynamical (or abelian) part $P_{\text{dyn}} = \prod_{i<j} \frac{1}{P(\theta_i|\theta_j)}$

Matrix part

$$M = \prod_{i>j} \frac{1}{(\theta_i - \theta_j + i\pi)(\theta_i - \theta_j + i\frac{\pi}{2})} \times (\pi_1 \delta_{i_1,i_2}\delta_{i_3,i_4} + \pi_2 \delta_{i_1,i_3}\delta_{i_2,i_4} + \pi_3 \delta_{i_1,i_4}\delta_{i_2,i_3})$$

Application to hexagon

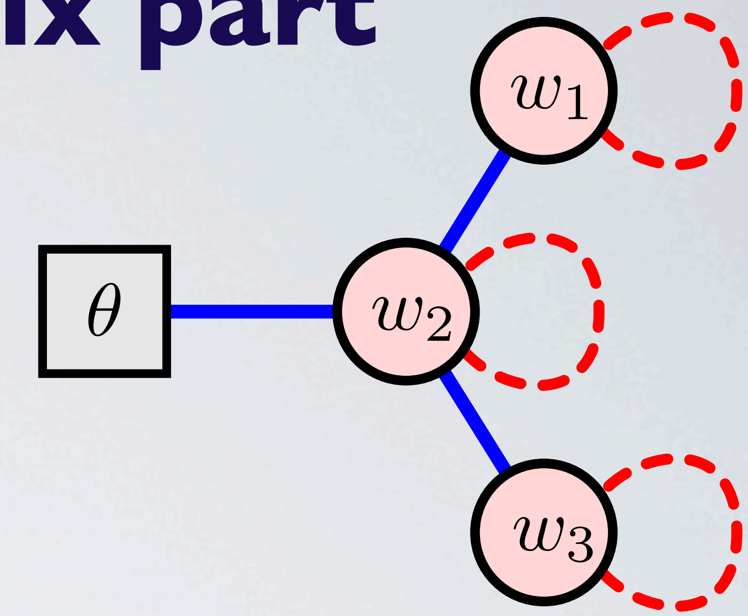
Contract bottom and top pentagon transitions



Get $\quad \text{integrand} = \prod_{i<j} \frac{1}{P(\theta_i|\theta_j)P(\theta_j|\theta_i)} \times \text{rational}$

$$\text{rational} = \prod_{i<j} \frac{1}{((\theta_i - \theta_j)^2 + \pi^2)((\theta_i - \theta_j)^2 + \frac{1}{4}\pi^2)} \times (6^2 \pi_1 \pi_1^* + \ldots)$$

Algebraic structure of matrix part



Integral representation of rational part

$$r = \frac{1}{K_1!K_2!K_3!} \int \prod_i \frac{dw_{1,i}}{2\pi} \prod_i \frac{dw_{2,i}}{2\pi} \prod_i \frac{dw_{3,i}}{2\pi}$$

$$\times \frac{\prod_{i<j} g(w_{1,i} - w_{1,i}) \prod_{i<j} g(w_{2,i} - w_{2,i}) \prod_{i<j} g(w_{3,i} - w_{3,i})}{\prod_{i,j} f(w_{2,i} - \frac{2}{\pi}\theta_j) \prod_{i,j} f(w_{1,i} - w_{2,j}) \prod_{i,j} f(w_{3,i} - w_{2,j})}$$

$$g(x) = x^2(x^2+1) \qquad f(x) = x^2 + \frac{1}{4}$$

corresponding to sum over states with K_θ particles

and $\mathfrak{su}(4)$ weights $= (K_2 - 2K_1, K_\theta - 2K_2 + K_1 + K_3, K_2 - 2K_3)$

In particular for singlet states : $K_2 = K_\theta \qquad K_1 = K_3 = \frac{1}{2} K_\theta$

Short distance analysis

For $z \ll 1$ equivalently $1 \ll \tau \ll e^{\sqrt{\lambda}/4}$

$$\mathcal{W}_{\text{2-pt}} \xrightarrow[z\to 0]{} r \log(1/z) + s \log\log(1/z) + t$$

$$r \simeq 0.031 \quad s \simeq -0.055 \quad t \simeq -0.008$$

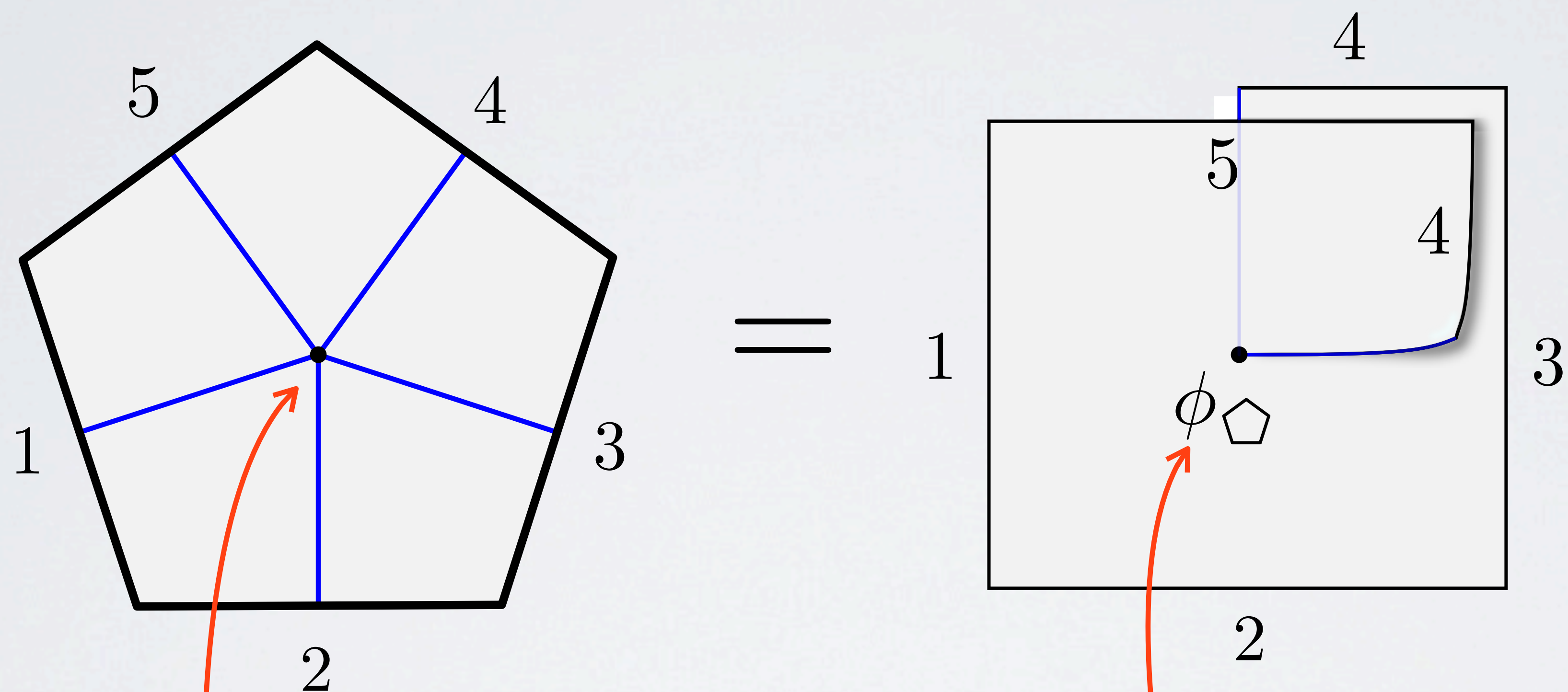
and higher contributions scale stronger

$$\mathcal{W}_{\text{2n-pt}} \sim \log^n (1/z)$$

What is the small z behavior of the full sum?

Pentagon as twist operator

Aympotically a pentagon = 5 quadrants glued together

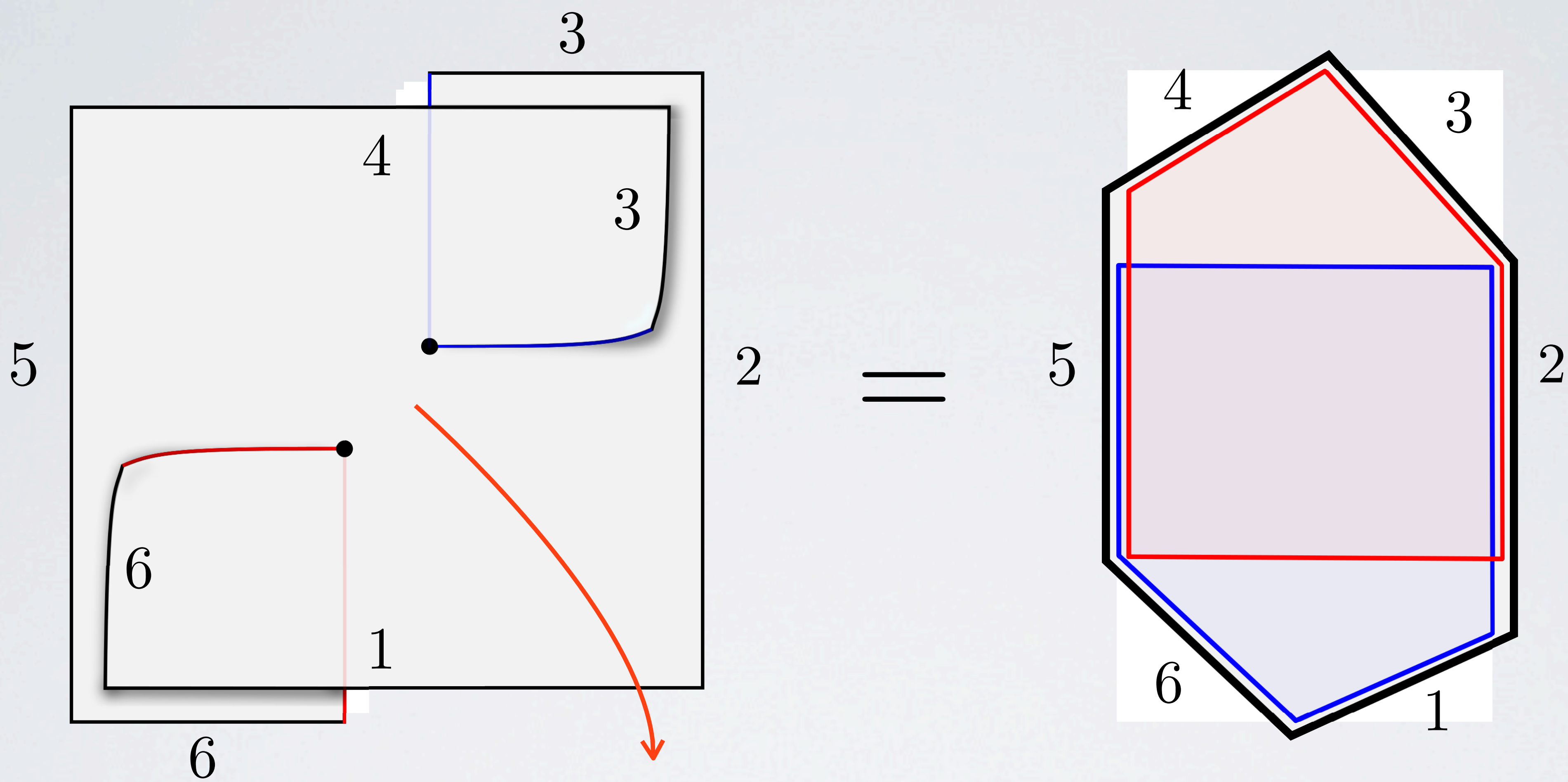


excess angle $= \frac{\pi}{2}$

twist operator

$$P(\psi_{\text{edge }2}|\psi'_{\text{edge }5}) = \langle\psi'|\, \phi_{\pentagon}\, |\psi\rangle$$

Hexagon as a correlator of twist operators



distance = $\sqrt{\sigma^2 + \tau^2}$

$$\mathcal{W}_6 = \langle 0|\, \phi_{\pentagon}(\tau, \sigma)\phi_{\pentagon}(0,0)\, |0\rangle$$

computed in O(6) sigma model

OPE as form factor expansion

Insert complete basis of states

$$\mathcal{W}_6 = \sum_N \frac{1}{N!} \langle 0| \phi_{\pentagon} |\theta_1, \dots, \theta_N\rangle \langle \theta_1, \dots, \theta_N| \phi_{\pentagon} |0\rangle \, e^{-m\tau \sum_i \cosh\theta_i + im\sigma \sum_i \sinh\theta_i}$$

Pentagon transition = form factor of twist operator

$$P(0|\theta_1, \dots, \theta_N) = \langle \theta_1, \dots, \theta_N| \phi_{\pentagon} |0\rangle$$

P goes through all the axioms for form factor of twist operator of [Cardy,Castro-Alvaredo,Doyon'07]

Normalization

$\langle 0| \phi_{\pentagon} |0\rangle = 1$ which enforces that $\mathcal{W}_6 \to 1 \quad z \to \infty$

Short distance analysis revisited I

OPE fusion (valid for $z \ll 1$)

$$\phi_{\text{⬠}}(\tau,\sigma)\phi_{\text{⬠}}(0,0) \sim \frac{\log{(1/z)}^B}{z^A}\phi_{\text{⬡}}(0,0)$$

3-point function

Critical exponent A

$$A = 2\Delta_{\text{⬠}} - \Delta_{\text{⬡}} = 2\Delta_{5/4} - \Delta_{3/2}$$

with Δ_k the scaling dimension of the twist operator ϕ_k

$$\Delta_k = \frac{c}{12}(k - \frac{1}{k})$$

[Knizhnik'87]

$$\begin{cases} c = & \text{central charge} \\ 2\pi(k-1) = & \text{excess angle for } \phi_k \end{cases}$$

Short distance analysis revisited I

OPE fusion (valid for $z \ll 1$)

$$\phi_{\pentagon}(\tau,\sigma)\phi_{\pentagon}(0,0) \sim \frac{\log{(1/z)}^{B}}{z^{A}}\phi_{\hexagon}(0,0)$$

3-point function

Critical exponent A

$$A = \frac{1}{36} \qquad \text{since in our case } c = 5$$

Critical exponent B from one-loop anomalous dimensions

$$B = -\frac{3}{2}A = -\frac{1}{24}$$

Short distance analysis revisited II

For $z \ll 1$

include subleading RG logs

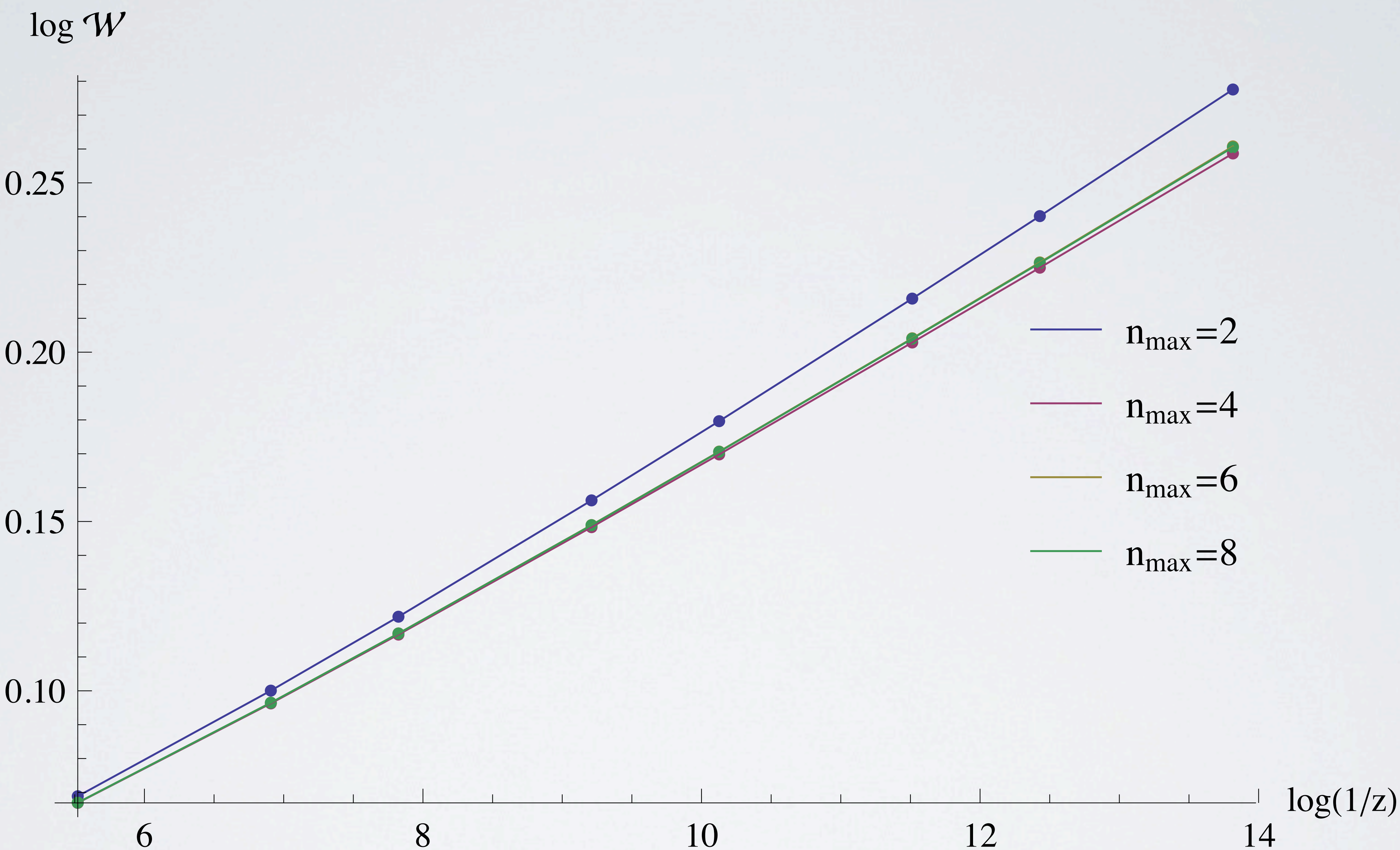
$$\mathcal{W}_6 = \frac{C}{z^{1/36} \log\left(1/z\right)^{1/24}} + \dots$$

Constant C is fixed in the IR by

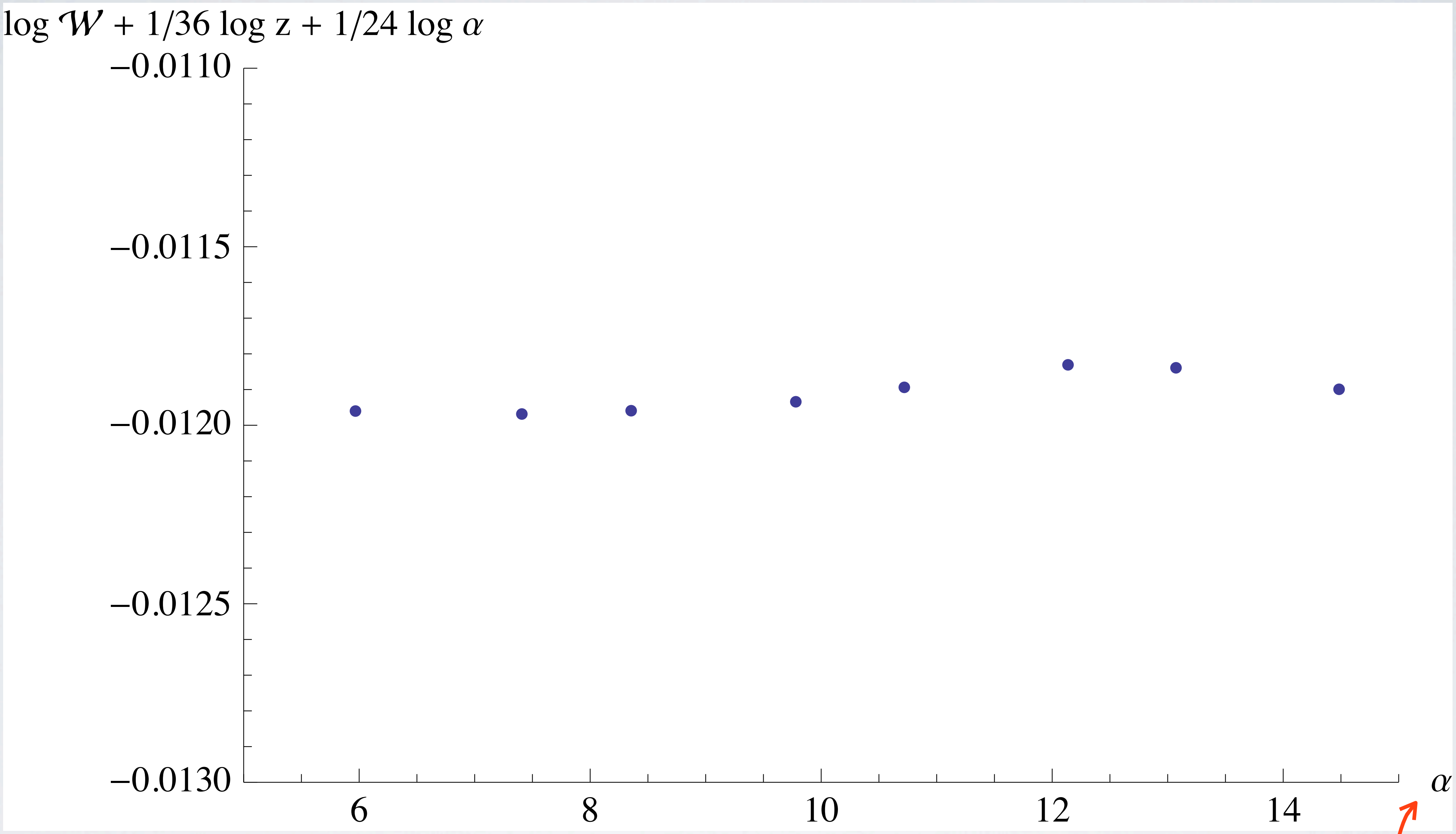
$$\mathcal{W}_6 \to 1 \qquad \text{when} \qquad z \to \infty$$

and thus non-perturbative

Numerical analysis I



Numerical analysis II

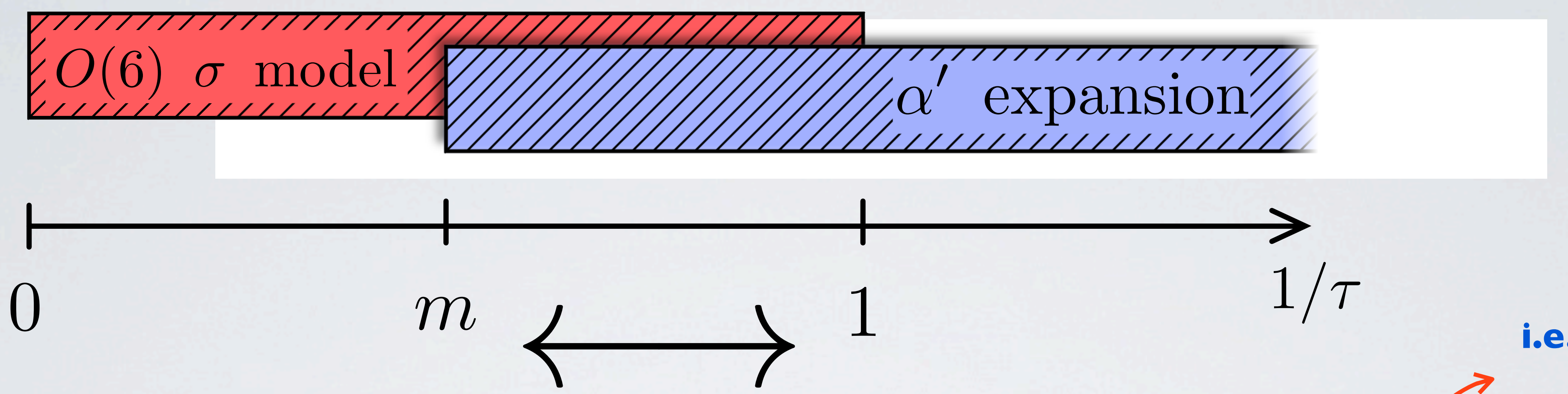


$$\log C = -0.01$$

running coupling

$$\alpha = \log(1/z) + \ldots$$

Cross over I



here
we can match short-distance O(6) analysis
with
string perturbative expansion

i.e. expression for

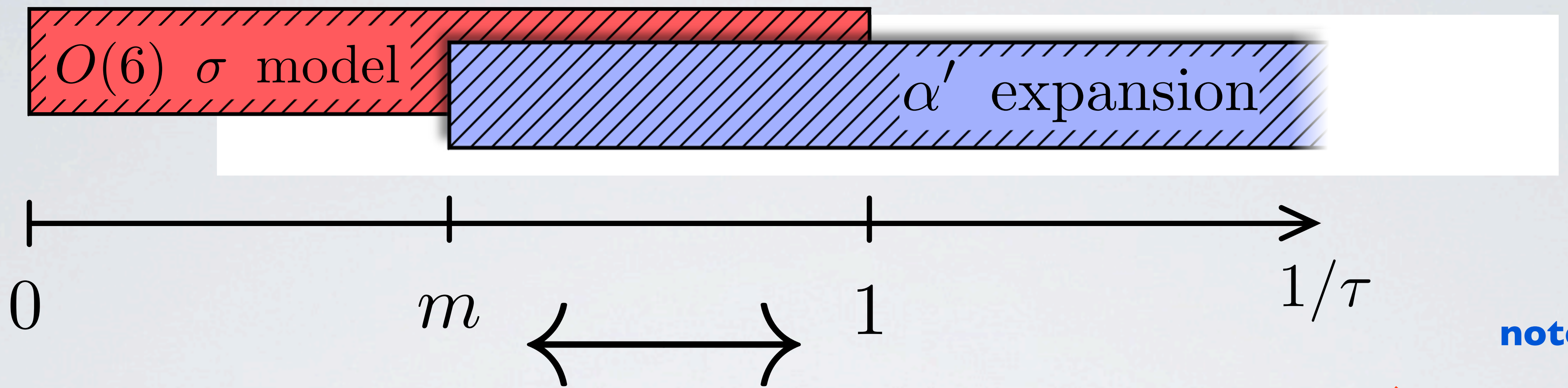
$$z \ll 1$$

Recall that

$$m = \frac{2^{1/4}}{\Gamma(5/4)} \lambda^{1/8} e^{-\frac{\sqrt{\lambda}}{4}} (1 + O(1/\sqrt{\lambda})) \ll 1$$

[Alday,Maldacena'07]

Cross over II



note that

$A_6 = O(e^{-\sqrt{2}\tau})$

in collinear limit

Write string expansion as

$$\mathcal{W}_6 = f_6\, \lambda^{-\frac{7}{288}}\, e^{\frac{\sqrt{\lambda}}{144} - \frac{\sqrt{\lambda}}{2\pi} A_6}\left(1 + O(1/\sqrt{\lambda})\right)$$

and compare with O(6) result for $z \ll 1$

$$f_6 = \frac{1.04}{(\sigma^2 + \tau^2)^{1/72}} + \dots$$

include terms $O(e^{-\sqrt{2}\tau})$ in collinear limit

Generalization

Higher points amplitudes correspond to higher points correlators

$$\mathcal{W}_n = \langle 0 | \, \phi_{\pentagon}(\tau_{n-4}, \sigma_{n-4}) \dots \phi_{\pentagon}(\tau_1, \sigma_1) \, | 0 \rangle$$

Overall short-distance scaling is controlled by OPE fusion

$$\underbrace{\phi_{\pentagon} \dots \phi_{\pentagon}}_{n-4} \sim m^{-(n-4)\Delta(\frac{5}{4})+\Delta(\frac{n}{4})} \phi_{\varphi}$$

with final excess angle $\varphi = 2\pi \times \dfrac{n-4}{4}$

This leads to the constant reported earlier

Conclusion

At strong coupling SA develop a non-perturbative regime in the near collinear limit

The string α' expansion breaks down for extremely large values of $\tau \sim -\log u_2 \sim e^{\sqrt{\lambda}/4}$

This follows from the fact that the flux tube mass gap m becomes extremely small

To properly understand this regime one should think in terms of correlators of twist operators

This way one can fix the collinear limit of SA at strong coupling