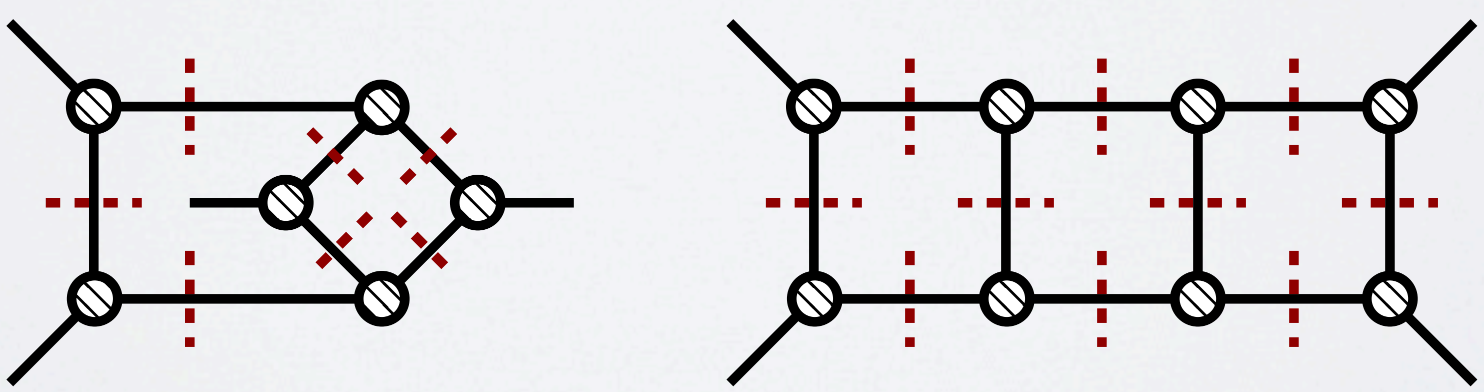
D-dimensional integrand reduction methods for two-loop QCD amplitudes

Simon Badger (CERN)

10th June 2014

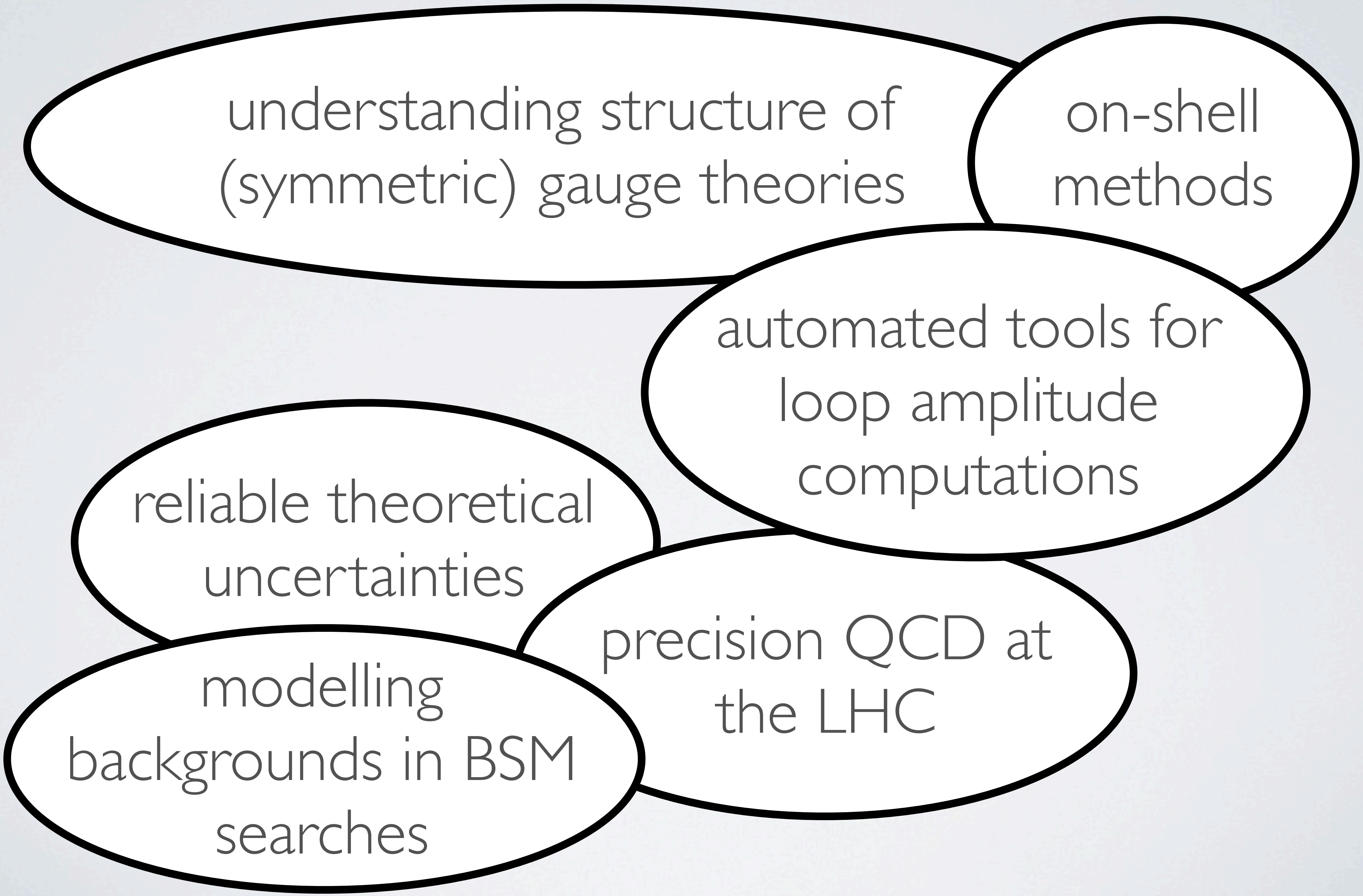
Based on work with Hjalte Frellesvig and Yang Zhang

1202.2019 JHEP 1204 (2012), **1207.2976** JHEP 1208 (2012), **1310.1051** JHEP 1312 (2013)



Amplitudes 2014, IPhT Saclay

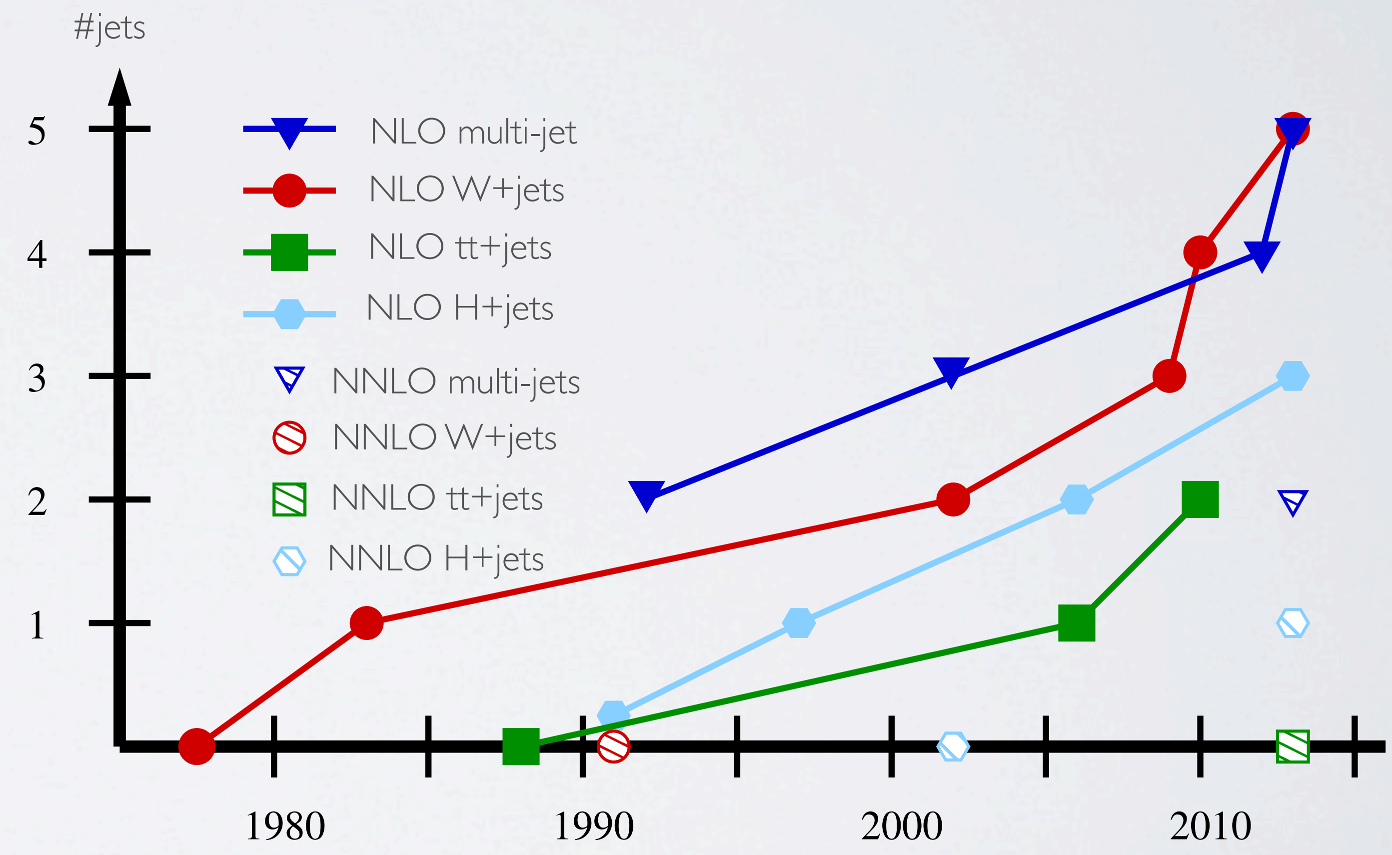
Introduction



Precision QCD for hadron colliders

Modern amplitude methods at one-loop → automated multi-leg NLO

recursion relations
unitarity



lot's more data will come from Run II

access to more exclusive signals

e.g. α_s from $R_{3/2} = \frac{\frac{d\sigma_{pp\to 3j}}{dp_T}}{\frac{d\sigma_{pp\to 2j}}{dp_T}}$

NNLO beyond 2→2

automated multi-loop computations

QCD amplitudes at work

$$\sigma_n^{NLO} = \int_n (d\sigma^B + d\sigma^V) + \int_{n+1} d\sigma^R$$

QCD amplitudes at work

$$\sigma_n^{NLO} = \int_n (d\sigma^B + d\sigma^V + d\sigma^I) + \int_{n+1} (d\sigma^R - d\sigma^S)$$

QCD amplitudes at work

Automated NLO

$$\sum_{\text{singularities}} \frac{1}{s} f_s |A_n|^2$$

$$\sigma_n^{NLO} = \int_n (d\sigma^B + d\sigma^V + d\sigma^I) + \int_{n+1} (d\sigma^R - d\sigma^S)$$

$$\sum_h \sum_{ij} A_i^{(0),\dagger}(h)\, C_{ij}^{(0,0)}\, A_j^{(0)}(h)$$

$$\sum_h \sum_{ij} A_i^{(0),\dagger}(h)\, C_{ij}^{(0,1)}\, A_j^{(1)}(h)$$

helicity

colour

ordered amplitudes: unitarity cuts, recursion relations, integrand reduction etc.

QCD amplitudes at work

Automa

$\sum \frac{1}{s} f_s |A_n|^2$

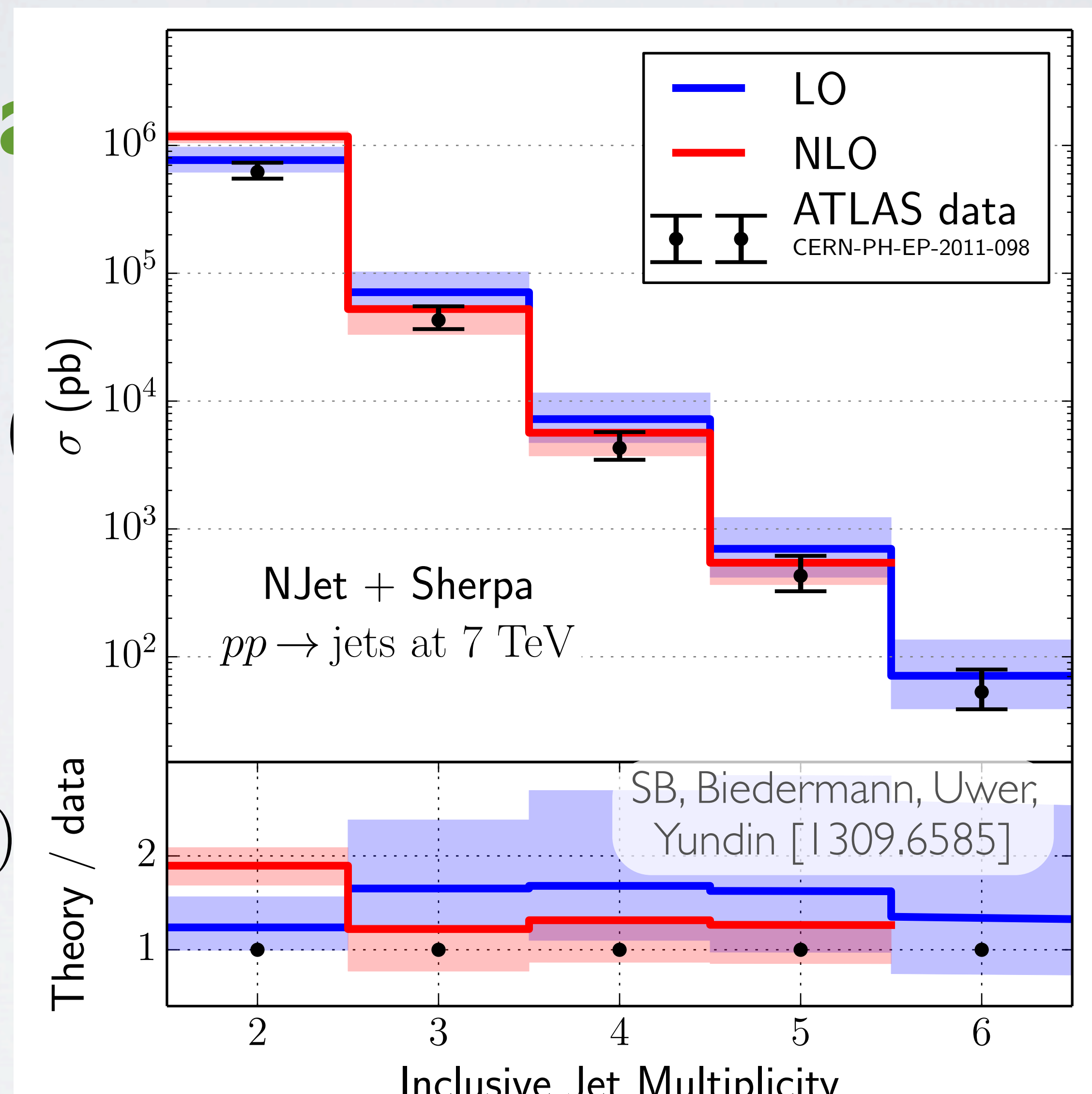
gularities

$\sigma_n^{NLO} = \int_n$

$R - d\sigma^S)$

$\sum_h \sum_{ij} A_i^{(0),\dagger}(h)$

$) C_{ij}^{(0,1)} A_j^{(1)}(h)$



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QCD amplitudes at work

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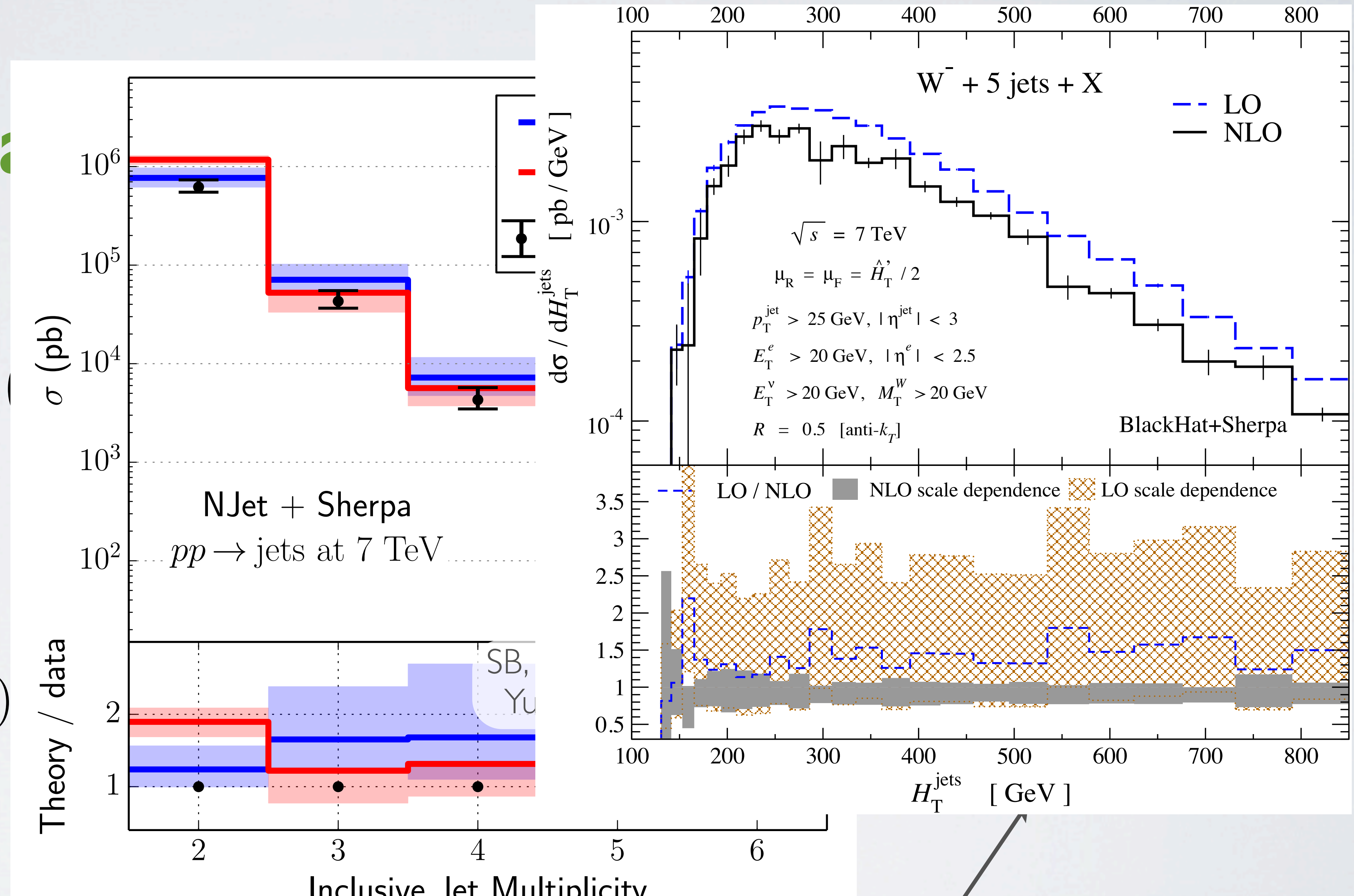
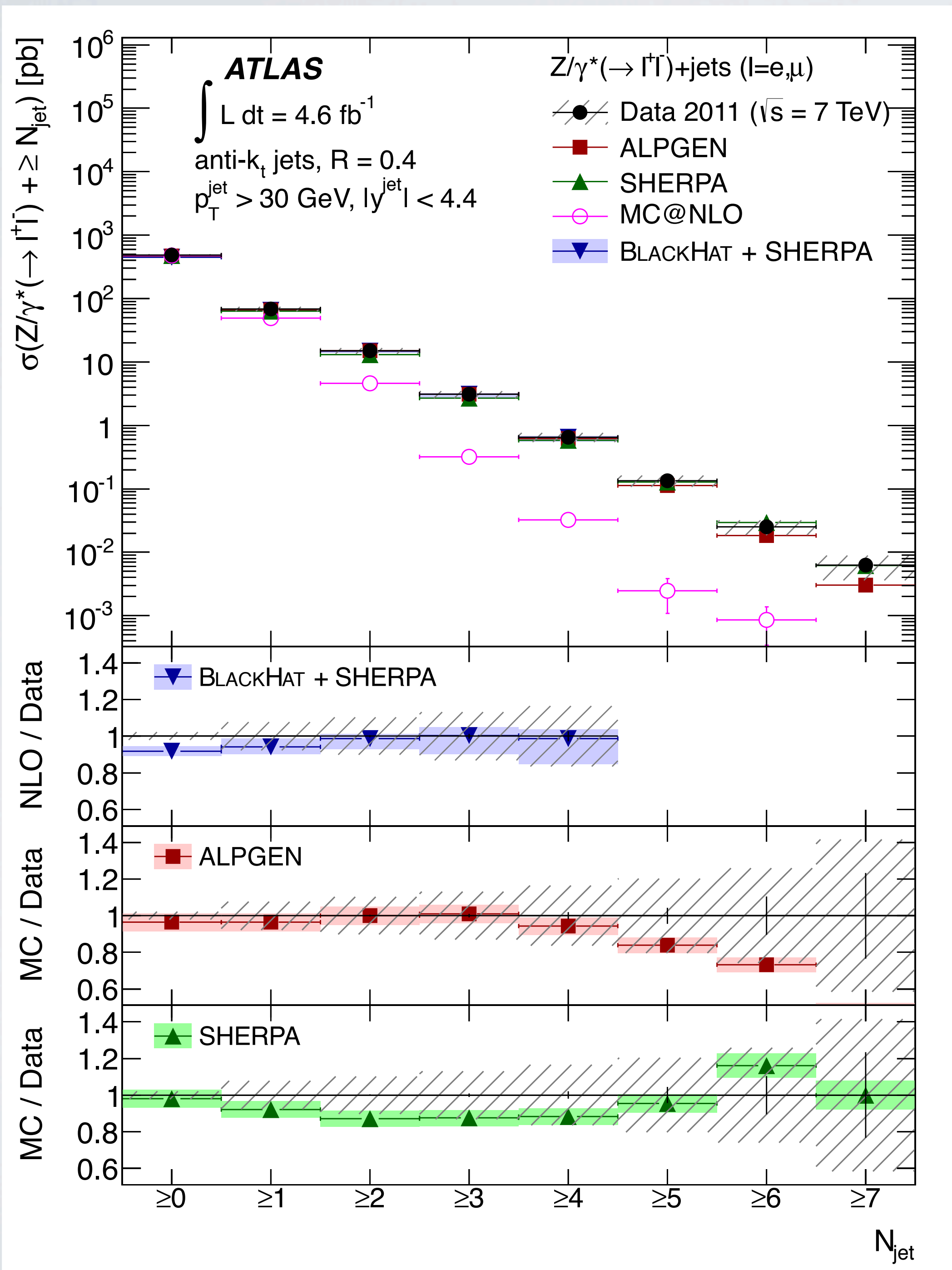
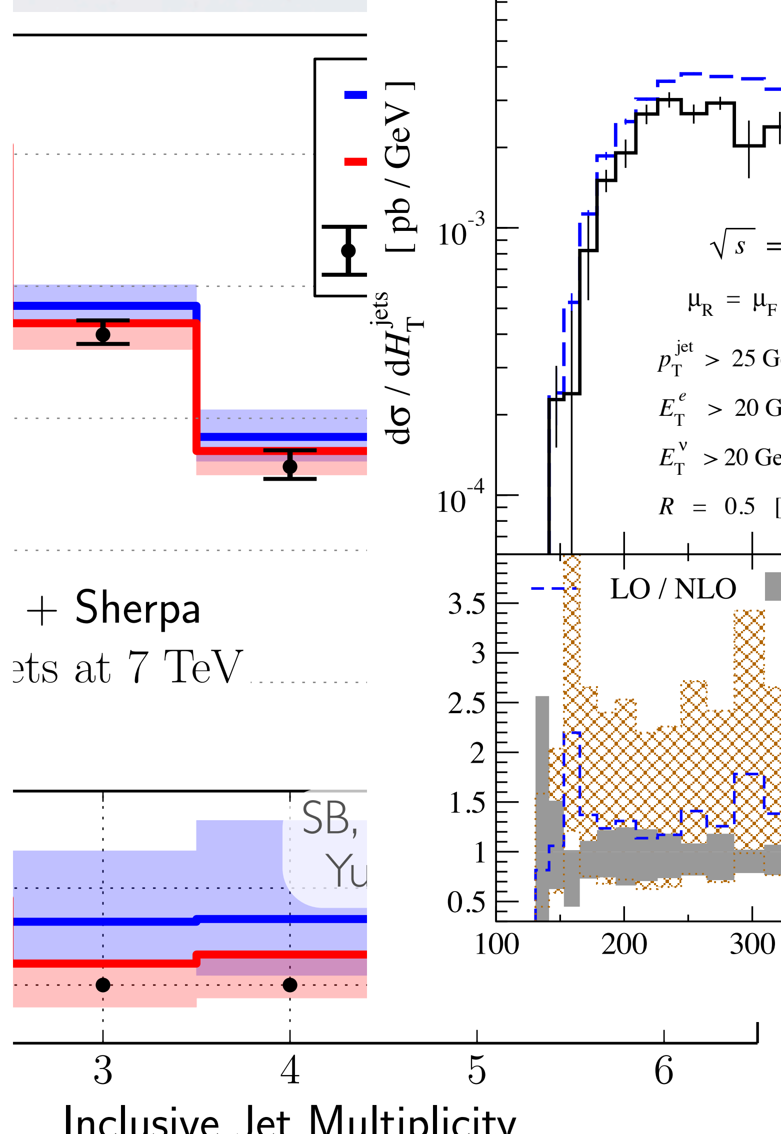
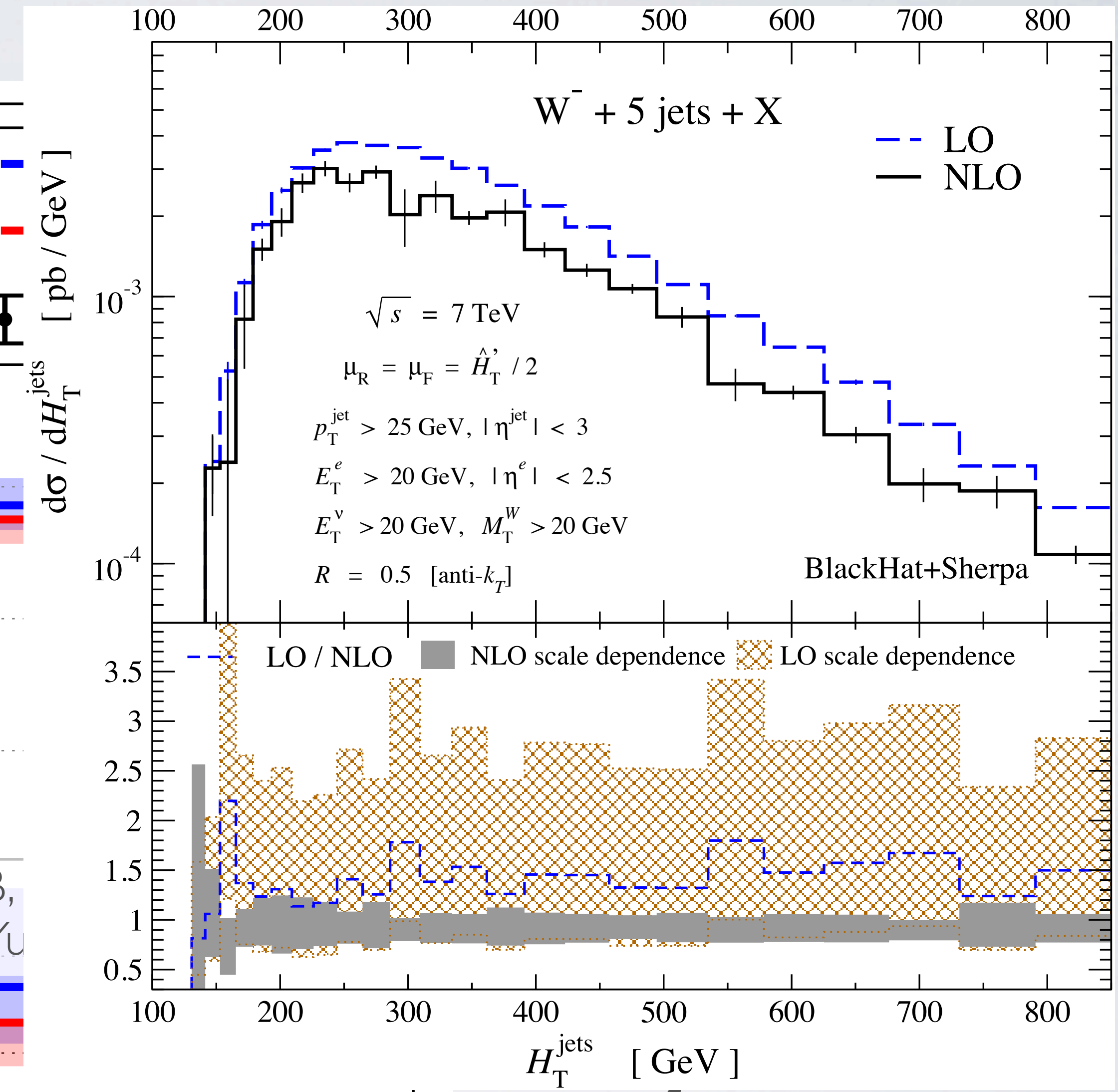


ordered amplitudes: unitarity cuts, recursion relations, integrand reduction etc.

QCD amplitudes at work

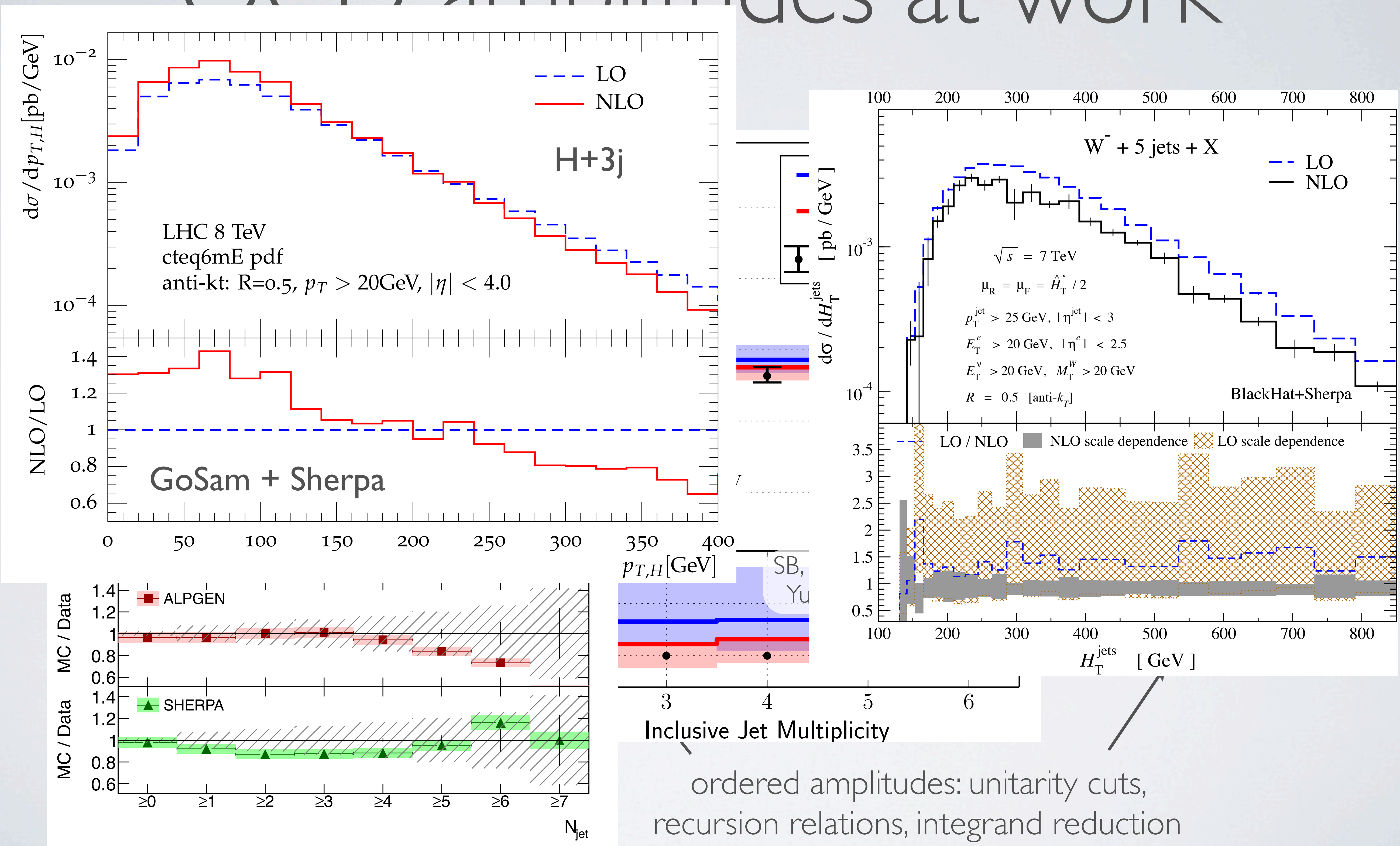






ordered amplitudes: unitarity cuts, recursion relations, integrand reduction etc.

QCD amplitudes at work



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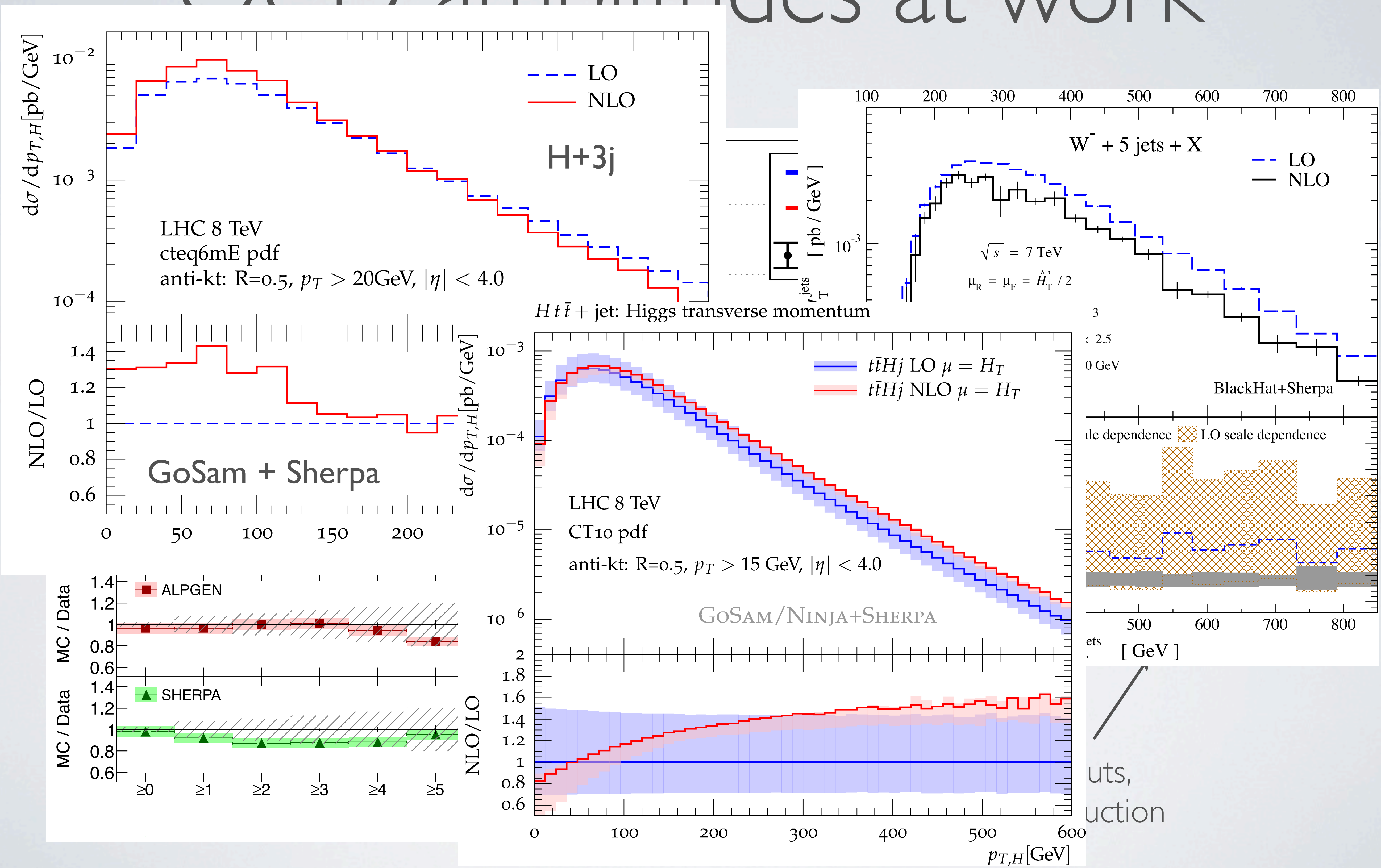

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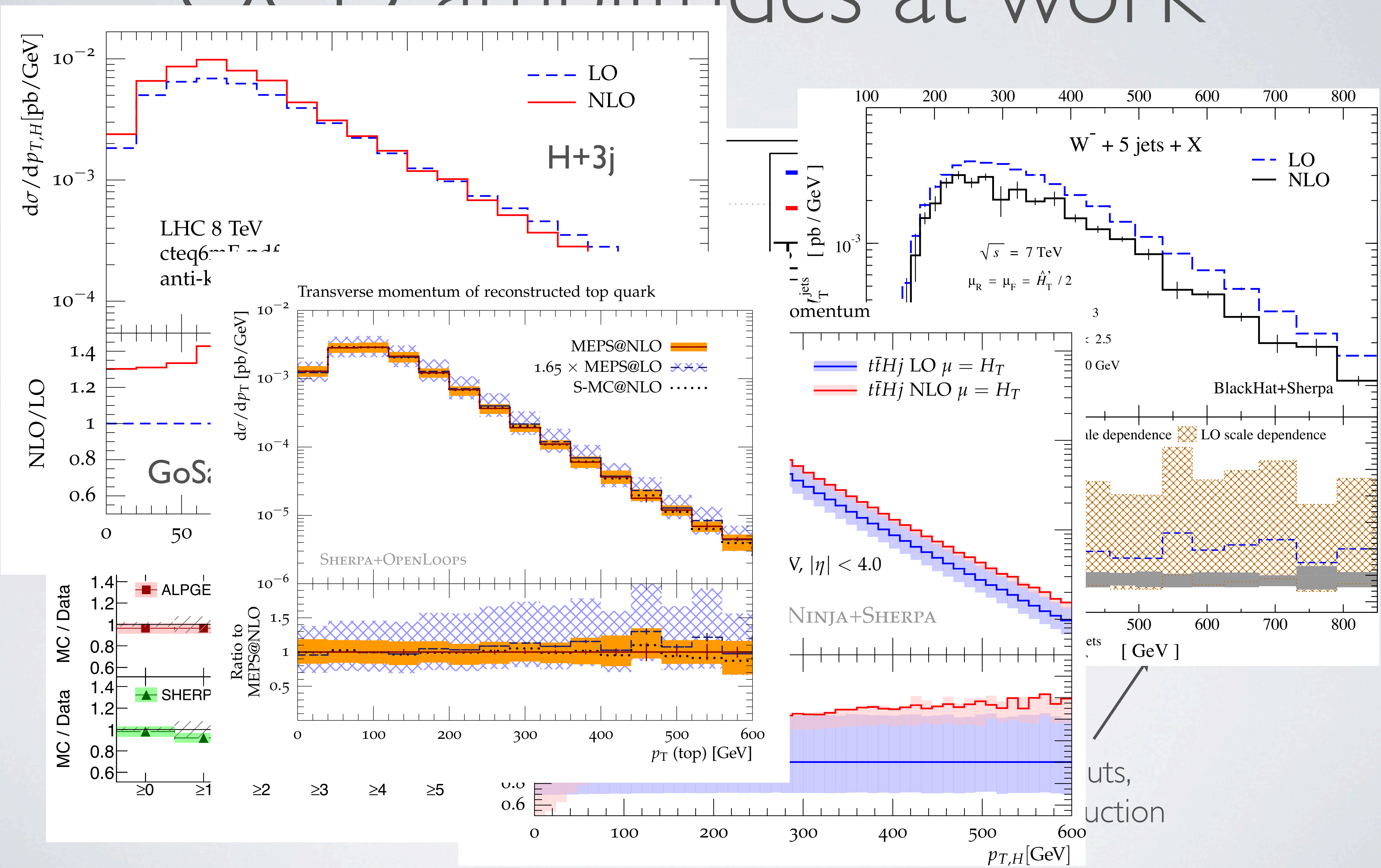
QCD amplitudes at work



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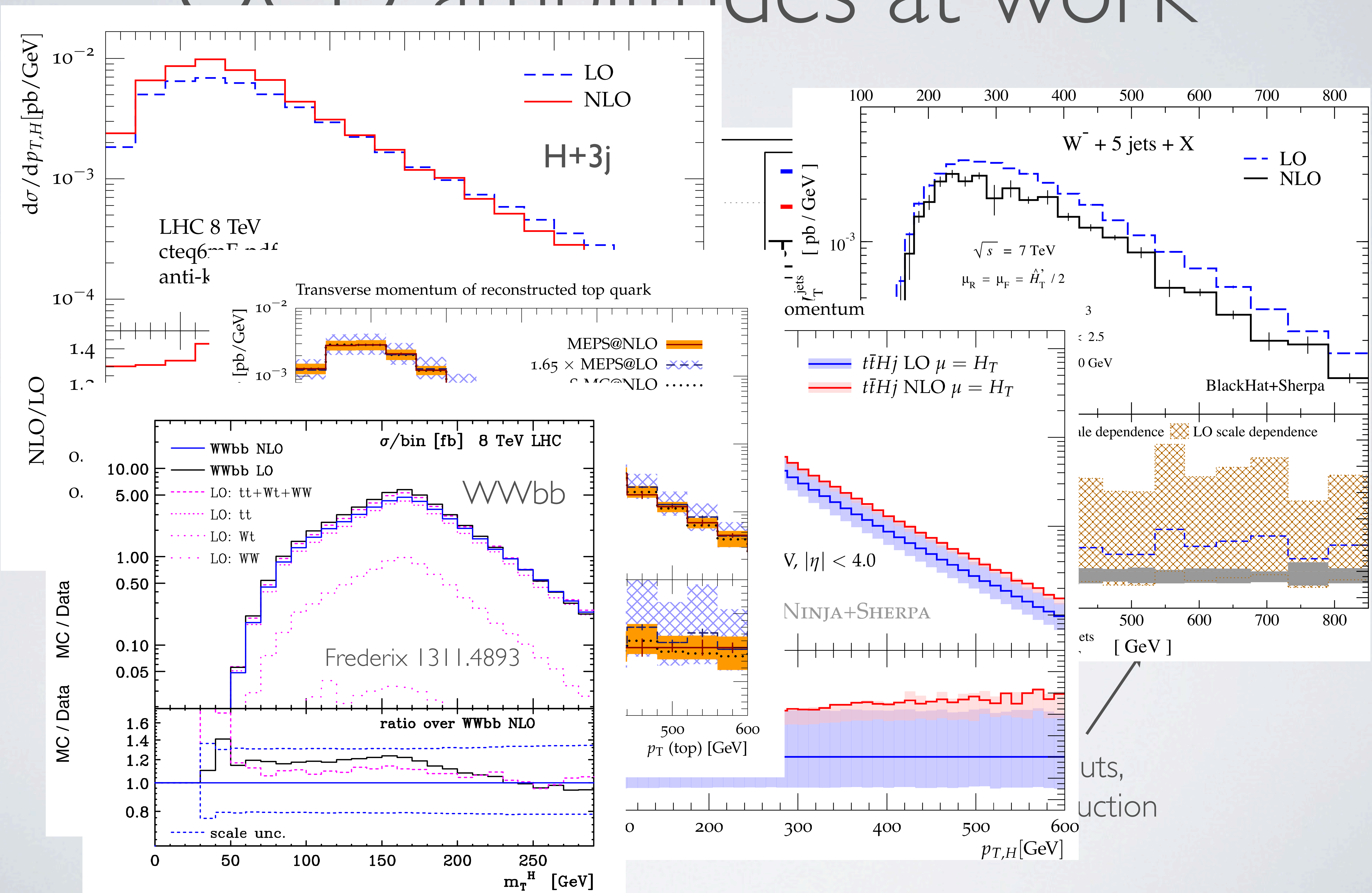

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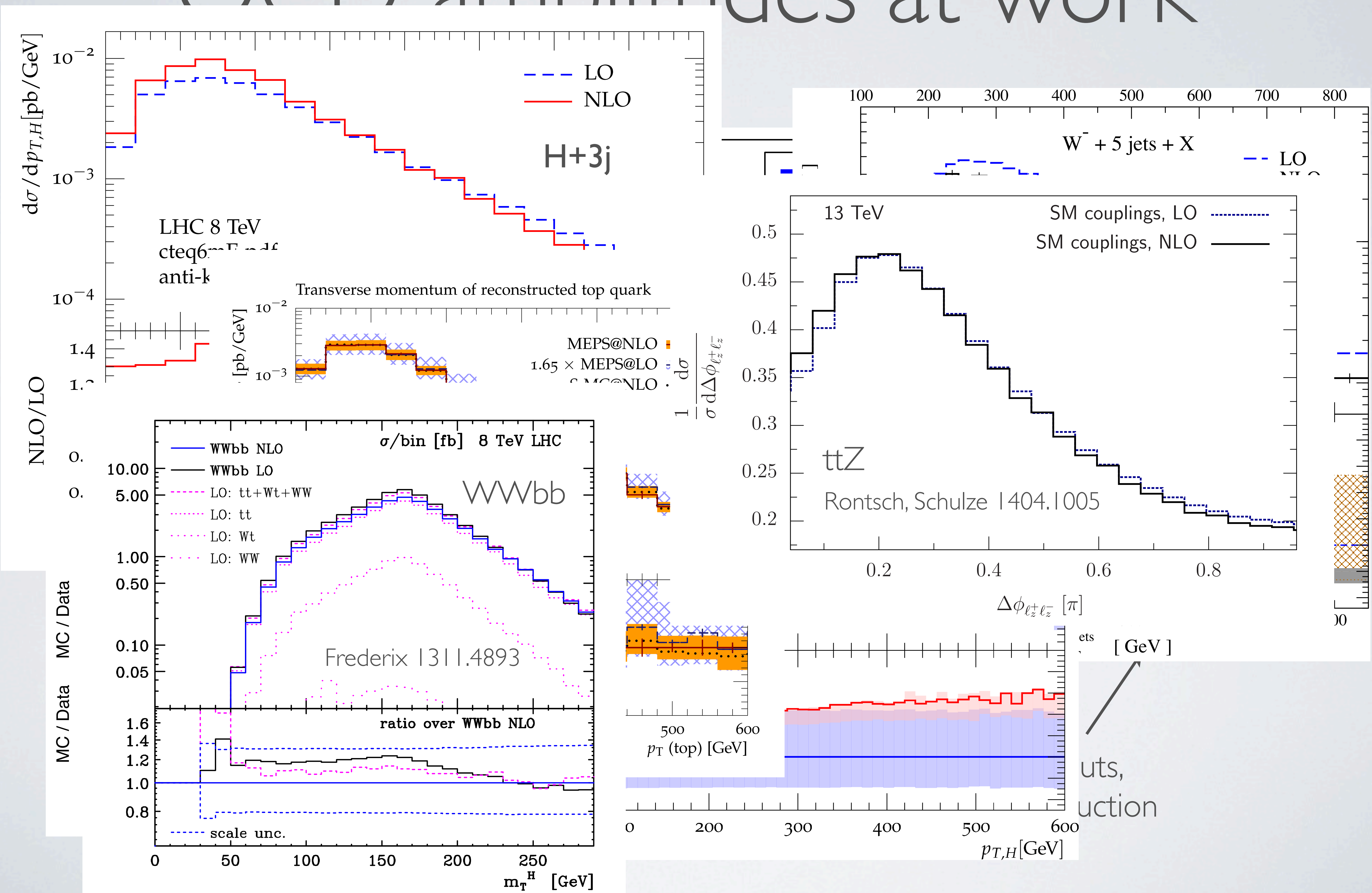
QCD amplitudes at work



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Amplitudes for NNLO

Can we achieve the same level of automation at higher precision?

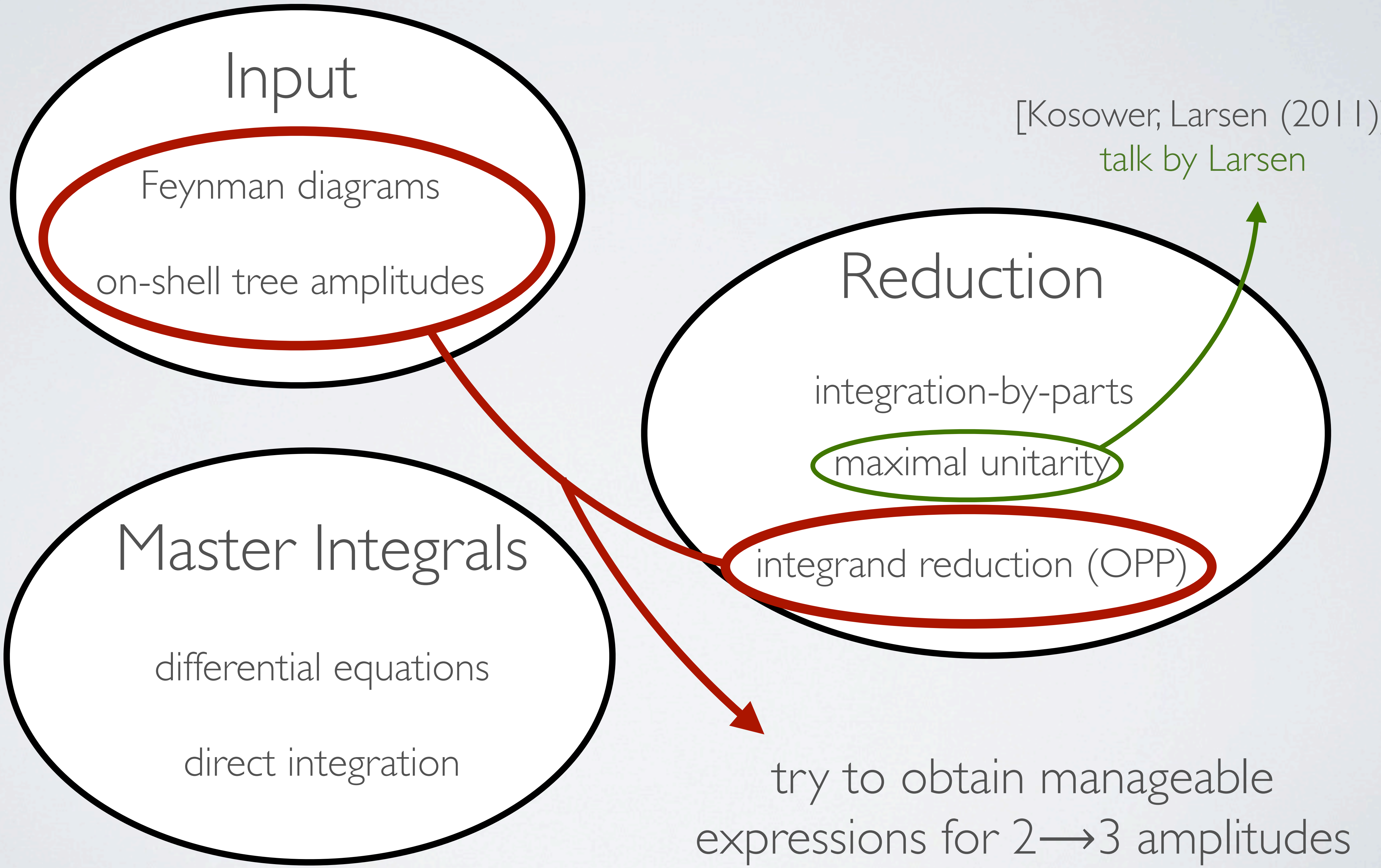
$$\sigma_n^{NNLO} = \int_n (d\sigma^B + d\sigma^V + d\sigma^{VV}) + \int_{n+1} d\sigma^R + d\sigma^{RV} + \int_{n+2} d\sigma^{RR}$$

Traditional approach: Feynman diagrams + integration-by-parts

suitable for 2→2 processes

complexity grows **very** fast with additional legs

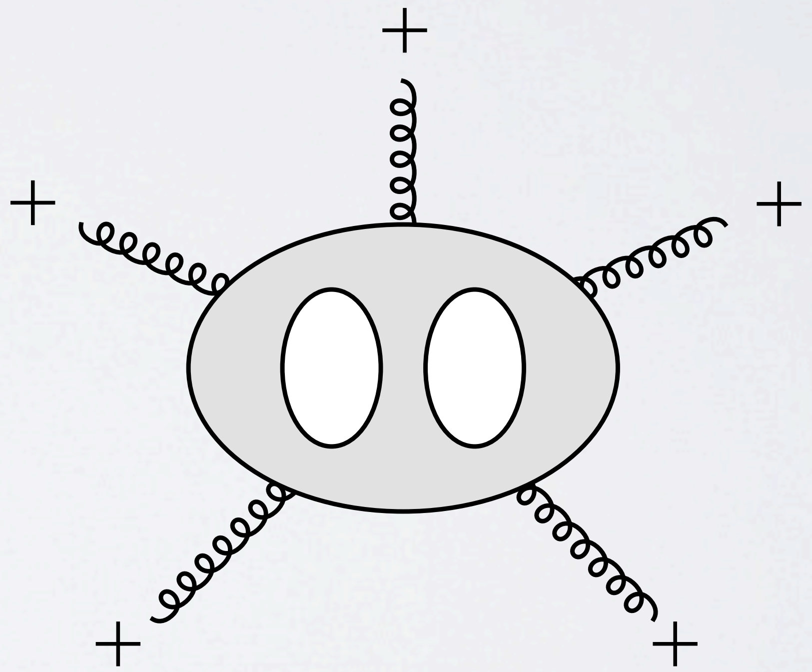
Automation for multi-leg NNLO



Outline

- Multi-loop integrand reduction using computational algebraic geometry
- Extension to dimensionally regulated cuts
- Application to $A_5^{(2)}(1^+, 2^+, 3^+, 4^+, 5^+)$

[SB, Frellesvig, Zhang arXiv:1310.1051]



Notation

$$\bar{k}_i \cdot p_j,\ \bar{k}_i \cdot \varepsilon_j,\ \bar{k}_i \cdot \bar{k}_j,\ \mu_{ij} = -k_i^{[-2\epsilon]} \cdot k_j^{[-2\epsilon]}\ \ldots$$

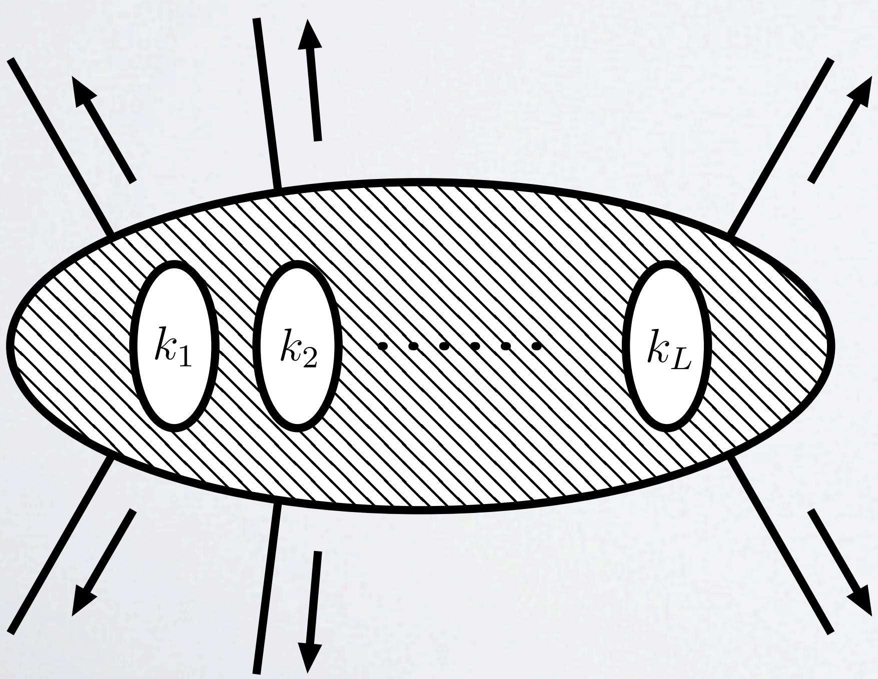
$$A_n^{(L),[D]}(\{p\}) = \int \prod_{i=1}^{L} \frac{d^D k_i}{(2\pi)^D} \frac{N(\{k\},\{p\})}{\prod_{l=1}^{L(L+9)/2} D_l(\{k\},\{p\})}$$

$$= \int \prod_{i=1}^{L} \frac{d^D k_i}{(2\pi)^D} \sum_{c=1}^{L(L+9)/2} \sum_{T \in P_c} \frac{\Delta_{c;T}(\{\bar{k}\cdot v, \mu_{ij}\})}{\prod_{l \in T} D_l(\{k\},\{p\})}$$

$$= \sum_{i \in MI} c_i^{[D]}(\{p\}) I_i(\{k\},\{p\})$$

basis of irreducible scalar products

master integral basis



$$k_i = \bar{k}_i + k^{[-2\epsilon]}$$

Integrand reduction strategy

[Mastrolia, Ossola arXiv:1107.6041]

[SB, Frellesvig, Zhang arXiv:1202.2019]

[Zhang arXiv:1205.5707]

[Mastrolia, Mirabella, Ossola, Peraro arXiv:1205.7087]

- top down: start with maximal number of propagators
- identify basis of irreducible scalar products (ISPs)

  spanning basis e.g. Van Neerven-Vermaseren $x_{ij} = k_i \cdot v_j$

- parametrize integrand using propagators

  $\Delta = \sum c_i m_i(x_{ij}, \mu_{ij})$

  Gröbner basis and polynomial division

- parametrize on-shell solutions and solve for

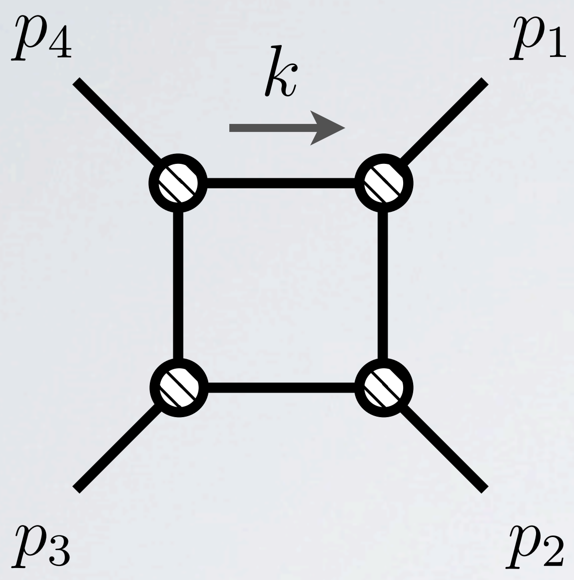
  $N(k^{(s)}(\tau_j)) = \Delta(k^{(s)}(\tau_j)) \Rightarrow c_i$

  primary decompostion

- continue to lower propagator topologies subtracting known singularities

One-loop box example

(four-dimensional case)



$$P = \langle k^2, (k-p_1)^2, (k-p_2)^2, (k+p_4)^2 \rangle$$

$$k_i \cdot k_j = (k_i \cdot v)^T . G_4^{-1} . (k_j \cdot \cdot v) \qquad G_4 = \frac{1}{2}\begin{pmatrix} 0 & s & t & 0 \\ s & 0 & u & 0 \\ 0 & u & t & 0 \\ 0 & 0 & 0 & stu \end{pmatrix}$$

$$v^\mu = \{p_1^\mu, p_2^\mu, p_4^\mu, \omega^\mu = \epsilon^{\mu 124}\}$$

$$P = \langle x_{14}^2 - stu, x_{11}, x_{12}, x_{13} \rangle$$

quadratic part: **ISP** $= \{x_{14}\}$

RSPs in linear part

constraint from renormalizability

$$\Delta_4^{\text{ansatz}} = \sum_{i=0}^{4} c_i x_{14}^i$$

$$\Delta_4^{\text{ansatz}} / \mathrm{Gr}(\langle x_{14}^2 - stu \rangle) = \Delta_4 = c_0 + c_1 x_{14}$$

One-loop box example

two (complex) on-shell solutions

$$\langle x_{14}^2 - stu \rangle = \langle x_{14} - \sqrt{stu} \rangle \cap \langle x_{14} + \sqrt{stu} \rangle$$

primary decomposition (Lasker-Noether theorem)

original numerator

$$\Delta_4(k^{(\pm)}) = c_0 \pm c_1 \sqrt{stu} = d_\pm = N(k^{(\pm)}, p_1, p_2, p_3, p_4) = \prod_{i=1}^{4} A^{(0)}(k^{(\pm)}, p_i)$$

or tree amplitudes

$$\begin{pmatrix} 1 & \sqrt{stu} \\ 1 & -\sqrt{stu} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} d_+ \\ d_- \end{pmatrix}$$

invert linear system

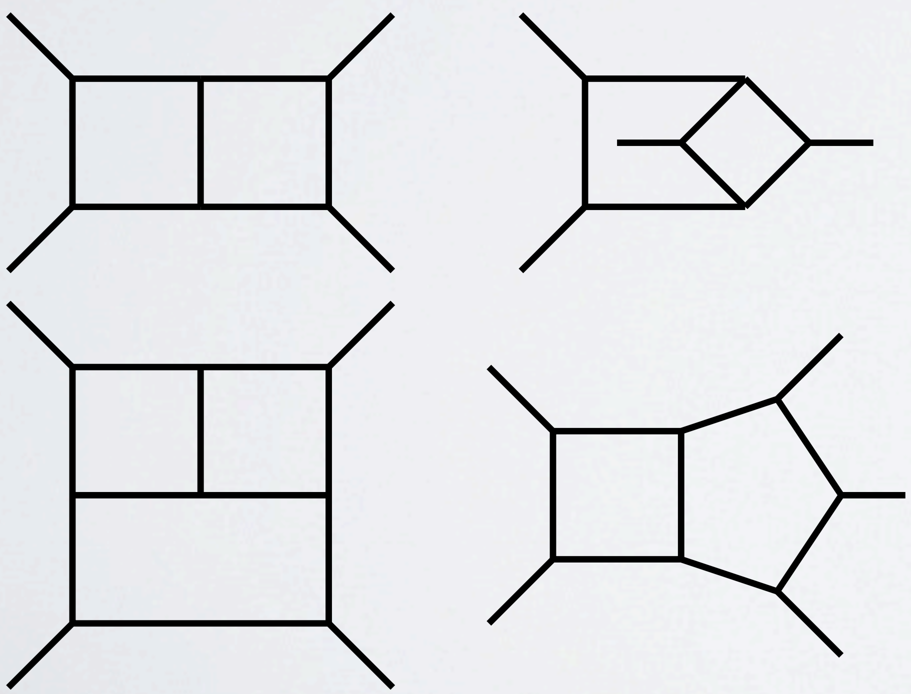
- spurious integrals important at integrand level
- not all linear systems will be square

Multi-loop integrand parametrization

[Zhang arXiv:1205.5707]

automated computation of integrand basis for each topology

BasisDet Mathematica package
http://www.nbi.dk/~zhang/BasisDet.html



determination of all on-shell branches using primary decomposition

Macaulay2: http://www.math.uiuc.edu/Macaulay2/

complex multi-loop structures investigated in [Huang, Zhang arXiv:1302.1023]

D-dimensional reduction

Is the integrand system well defined? will there linear system always have a solution?

complications 4-d

an ISP monomial vanishes on all on-shell solutions

i.e. ideal is not *radical*

different on-shell solutions have different dimensions

i.e. integrand systems with different numbers of propagators may need to be solved simultaneously

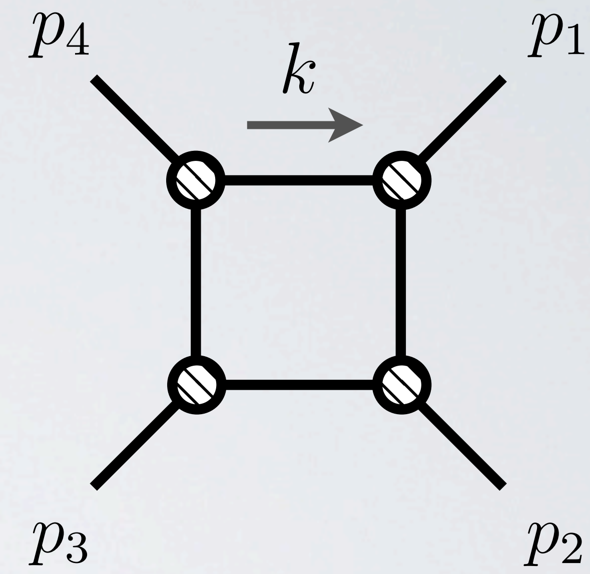
in D-d

all propagator ideals are *radical*

all integrands have exactly one on-shell branch

One-loop box example

$$P = \langle x_{14}^2 - \mu_{11} - stu, x_{11}, x_{12}, x_{13} \rangle$$



$$\Delta_4 = c_0 + c_1 x_{14} + c_2 \mu_{11} + c_2 x_{14} + c_3 x_{14} \mu_{11} + c_4 \mu_{11}^2$$

$$\bar{k}^\mu = \frac{s(1+\tau)}{4\langle 4|2|1]} \langle 4|\gamma^\mu|1] + \frac{s(1-\tau)}{4\langle 1|2|4]} \langle 1|\gamma^\mu|4]$$

$$x_{14} = \frac{st}{2}\tau \qquad \mu_{11} = -\frac{st}{4u}(1-\tau^2)$$

$$\Delta_4(k(\tau)) = \sum_{i=0}^{4} d_i \tau^i$$

$$\begin{pmatrix} 1 & -\frac{t}{2} & 0 & 0 & 0 \\ 0 & t & -\frac{st}{u} & \frac{st^2}{2u} & 0 \\ 0 & 0 & \frac{st}{u} & -\frac{3st^2}{2u} & \frac{s^2t^2}{u^2} \\ 0 & 0 & 0 & \frac{st^2}{u} & -\frac{2s^2t^2}{u^2} \\ 0 & 0 & 0 & 0 & \frac{s^2t^2}{u^2} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

continue reduction
with subtractions

$$\Delta_{3;123}(k(\tau_1,\tau_2)) = N(k(\tau_1,\tau_2), p_1, p_2, p_3, p_4) - \frac{\Delta_4(k(\tau_1,\tau_2))}{(k(\tau_1,\tau_2)+p_4)^2}$$

More examples

Both the order of the polynomial division and choice of spanning basis affect the simplicity of the representation

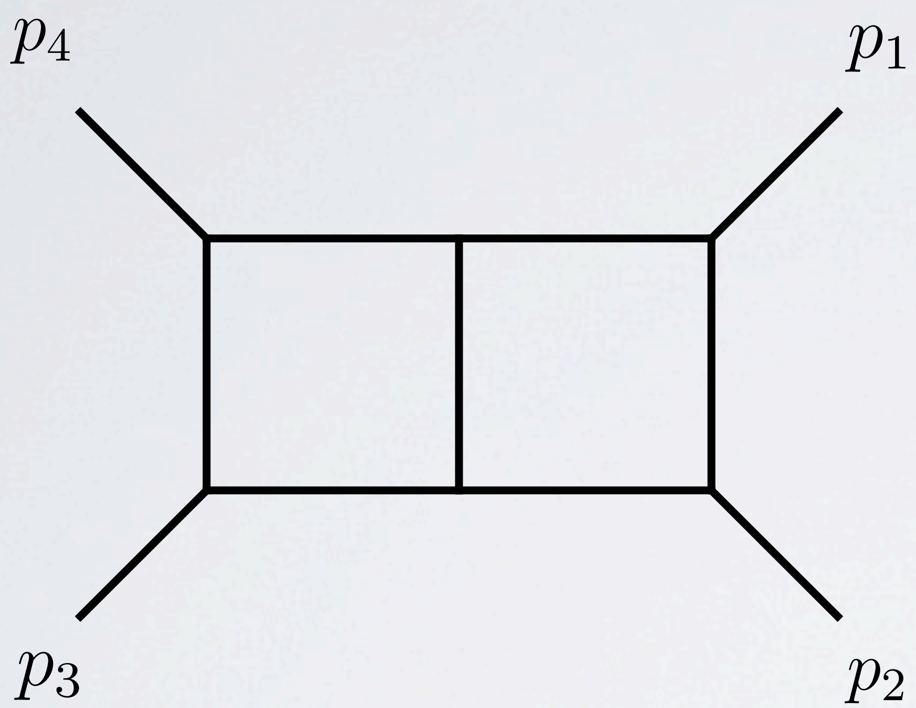
Dimension shifted integrals e.g. one-loop pentagon

$$I = \langle \mu_{11} - \text{ const} \rangle \Rightarrow \Delta_5 = c_0 \text{ or } \Delta_5 = c_0 \mu_{11} \qquad I_5[\mu_{11}] = \mathcal{O}(\epsilon)$$

Vanishing integrals: e.g. one-loop triangles

$$I = \langle \mu_{11} + (k_1 \cdot \omega_1)^2 + (k_1 \cdot \omega_1)^2 - \text{ const} \rangle \Rightarrow I_3[(k_1 \cdot \omega_1)^2 - (k_1 \cdot \omega_2)^2] = 0$$

Two-loop example



$$v = \{p_1, p_2, p_4, \omega_{124}\}$$
$$\text{ISP} = \{x_{13}, x_{21}, x_{14}, x_{24}, \mu_{11}, \mu_{12}, \mu_{22}\}$$

32 spurious terms

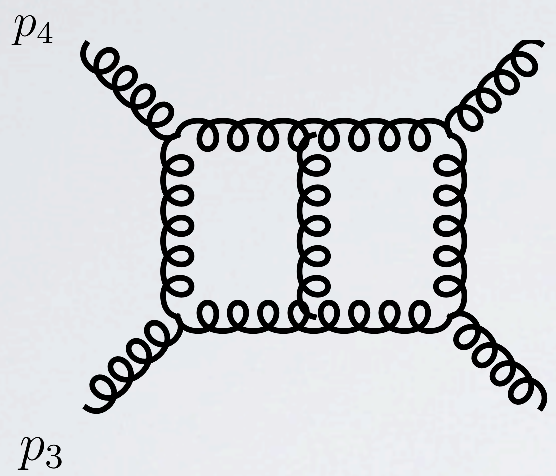
38 non-spurious terms

However: $k_1 \leftrightarrow k_2$ symmetry leaves 22 independent integrals

only 17 remain as $D \to 4$

$\mathcal{O}(\epsilon^{-4})$	8
$\mathcal{O}(\epsilon^{-2})$	4
$\mathcal{O}(\epsilon^{-1})$	4
$\mathcal{O}(1)$	1
$\mathcal{O}(\epsilon)$	5

Two-loop four gluon amplitudes



complete amplitudes FD+IBPs:

Glover, Oleari, Tejeda-Yeomans (2001)

Bern, Dixon, De Freitas (2002)

$$\int \Delta^{++++}_{12*34*} = \frac{2(D_s-2)st^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} I_{12*34*}[\mu_{11}^2] + \mathcal{O}(\epsilon)$$

$$\int \Delta^{--++}_{12*34*} = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(2(D_s-2)t\, I_{12*34*}[\mu_{11}^2] + 4st\, I_{12*34*}[\mu_{11}] - s^2 t\, I_{12*34*}[1]\right) + \mathcal{O}(\epsilon)$$

$$\int \Delta^{-+++}_{12*34*} = \frac{1}{[12]\langle 23\rangle\langle 34\rangle[41]\langle 13\rangle^2}\Big((D_s-2)\Big(2s^2tu\, I_{12*34*}\left[\mu_{11}^2\right]$$
$$+\frac{1}{2}s^3\, I_{12*34*}\left[\mu_{11}t^2 + 8\left(\mu_{11}x_{13}^2 + \mu_{11}x_{21}x_{13}\right) + 4\left(\mu_{12}+\mu_{22}\right)x_{13}^2\right]$$
$$+2s^2t\, I_{12*34*}\left[-\mu_{11}x_{13}\left(s+2x_{13}+4x_{21}\right) + 4\mu_{11}x_{13}^2 + 8\mu_{11}x_{21}x_{13}\right] + 4st^2\, I_{12*34*}\left[\mu_{11}x_{13}^2\right]\Big)$$
$$+s^3\left(D_s-6\right)\, I_{12*34*}\left[\mu_{22}\left(t-2x_{13}\right)^2\right] + 8s^2(s+2t)\, I_{12*34*}\left[\mu_{12}x_{13}x_{21}\right]\Big) + \mathcal{O}(\epsilon)$$

Amplitudes in self-dual Yang-Mills

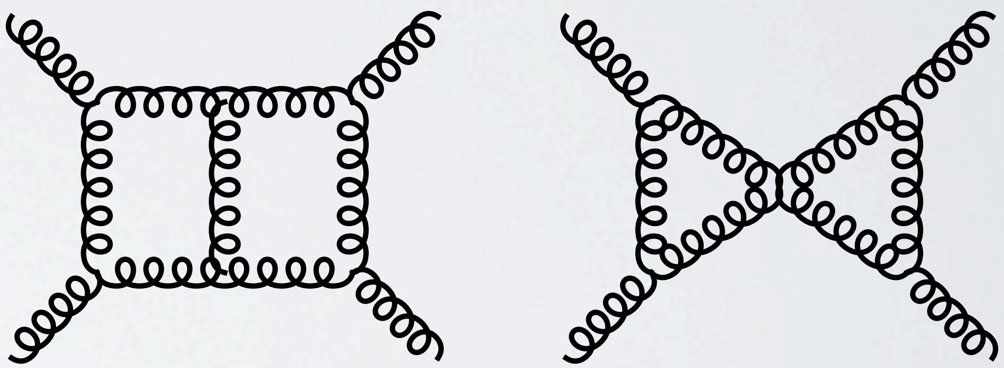
one-loop amplitudes only contain boxes. e.g.

$$A_4^{(1)}(1^+,2^+,3^+,4^+) = \frac{i\mathrm{tr}_+(1234)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} I_{4;1234}[(D_s-2)\mu_{11}^2]$$

$$\mathrm{tr}_+(1234) = [12]\langle 23\rangle[34]\langle 41\rangle$$

[Bern, Dixon, Dunbar, Kosower (1996)]

two-loop four-point also has simple form



$$A_4{}^{(2)}(1^+,2^+,3^+,4^+) = \frac{-i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\Big($$
$$s^2 t\, I_{7;12*34*}[(\mu_{11}\mu_{22}+\mu_{22}\mu_{33}+\mu_{33}\mu_{11}) + 4(\mu_{12}^2 - 4\mu_{11}\mu_{22})]$$
$$+\,t\, I_{6;12*34}[(D_s-2)(\mu_{11}+\mu_{22})\mu_{12}s + (D_s-2)^2\mu_{11}\mu_{22}((k_1+k_2)^2+s)/s]\Big)$$

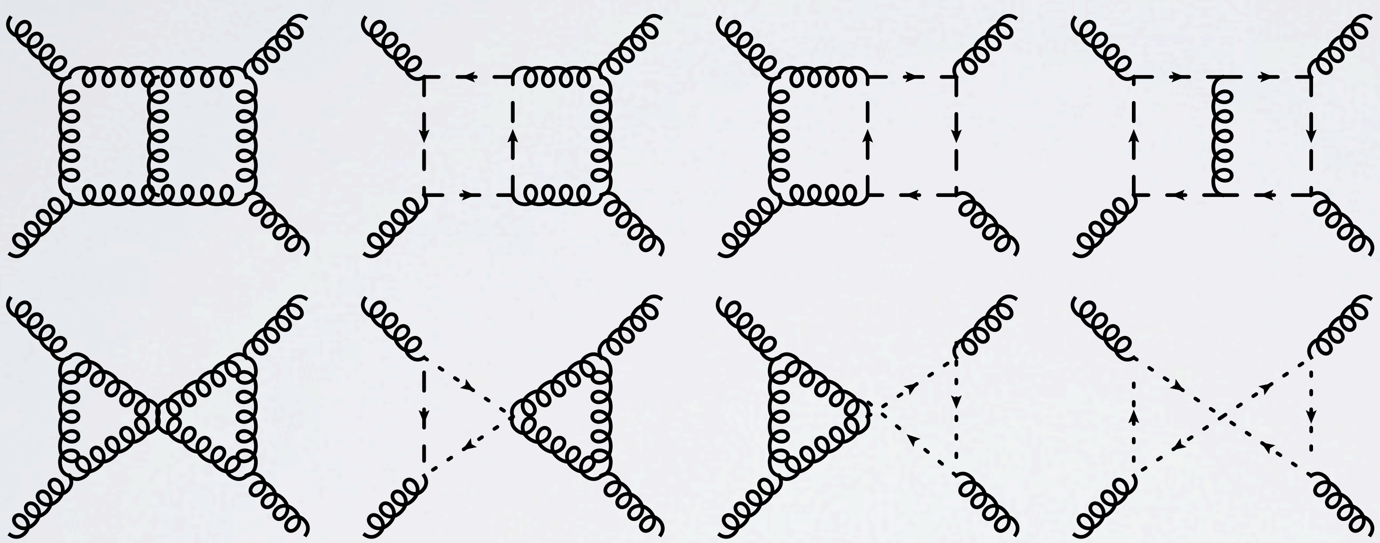
$$\mu_{33} = \mu_{11}+\mu_{22}+\mu_{12}$$

[Bern, Dixon, Kosower (2000)]

Numerator construction

FDH scheme from Feynman diagrams

[Bern, De Freitas, Dixon, Wong (2002)]



$$g^\mu_\mu = D_s$$

Tree-amplitudes using
six-dimensional helicity method

[Cheung, O'Connell (2009)]
[Bern, Carrasco, Dennen, Huang, Ita (2011)]

need to capture $\mu_{11},\ \mu_{22},\ \mu_{12}$

whichever way we choose we need a good way
to deal with complicated kinematics

Momentum twistors

[Hodges (2009)]

$$Z_{i,a} = (\lambda_{i,a}, \mu_{i,a})$$

momentum conservation automatically satisfied for any 4 x n matrix, Z

$$W_{i,\dot{a}} = (\tilde{\mu}_{\dot{a}}, \tilde{\lambda}_{\dot{a}}) = \frac{\epsilon_{\dot{a},b,c,d} Z_{i-1,b} Z_{i,c} Z_{i+1,d}}{\langle i-1 i\rangle\langle i i+1\rangle}$$

3n-10 independent variables

includes all Schouten identities

$$Z = \begin{pmatrix} 1 & 0 & -\frac{1}{s} & -\frac{1}{s}-\frac{1}{t} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

complex phase should be evaluated separately

det. $\rightarrow$ $\langle 12\rangle = 1 \qquad [12] = -s$

Momentum twistors

$s_{12},\ s_{23},\ s_{34},\ s_{45},\ s_{15}$

5 point kinematics contains one Gram matrix relation

$$\mathrm{tr}_5(1,2,3,4)^2 = \left| G\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_1 & p_2 & p_2 & p_4 \end{pmatrix} \right|$$

$$\mathrm{tr}_5^2 = 2s_{12}\left(s_{15}^2(-s_{45}) + s_{15}(s_{23}(s_{34}+s_{45}) + s_{34}s_{45}) + s_{23}s_{34}(s_{45}-s_{23})\right) + \left(s_{45}(s_{15}-s_{34}) + s_{23}s_{34}\right)^2 + s_{12}^2(s_{15}-s_{23})^2$$

$$Z = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1}+\frac{1}{x_1 x_2} & \frac{1}{x_1}+\frac{1}{x_1 x_2}+\frac{1}{x_1 x_2 x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{x_4}{x_2} & 1 \\ 0 & 0 & 1 & 1 & 1-\frac{x_5}{x_4} \end{pmatrix}$$

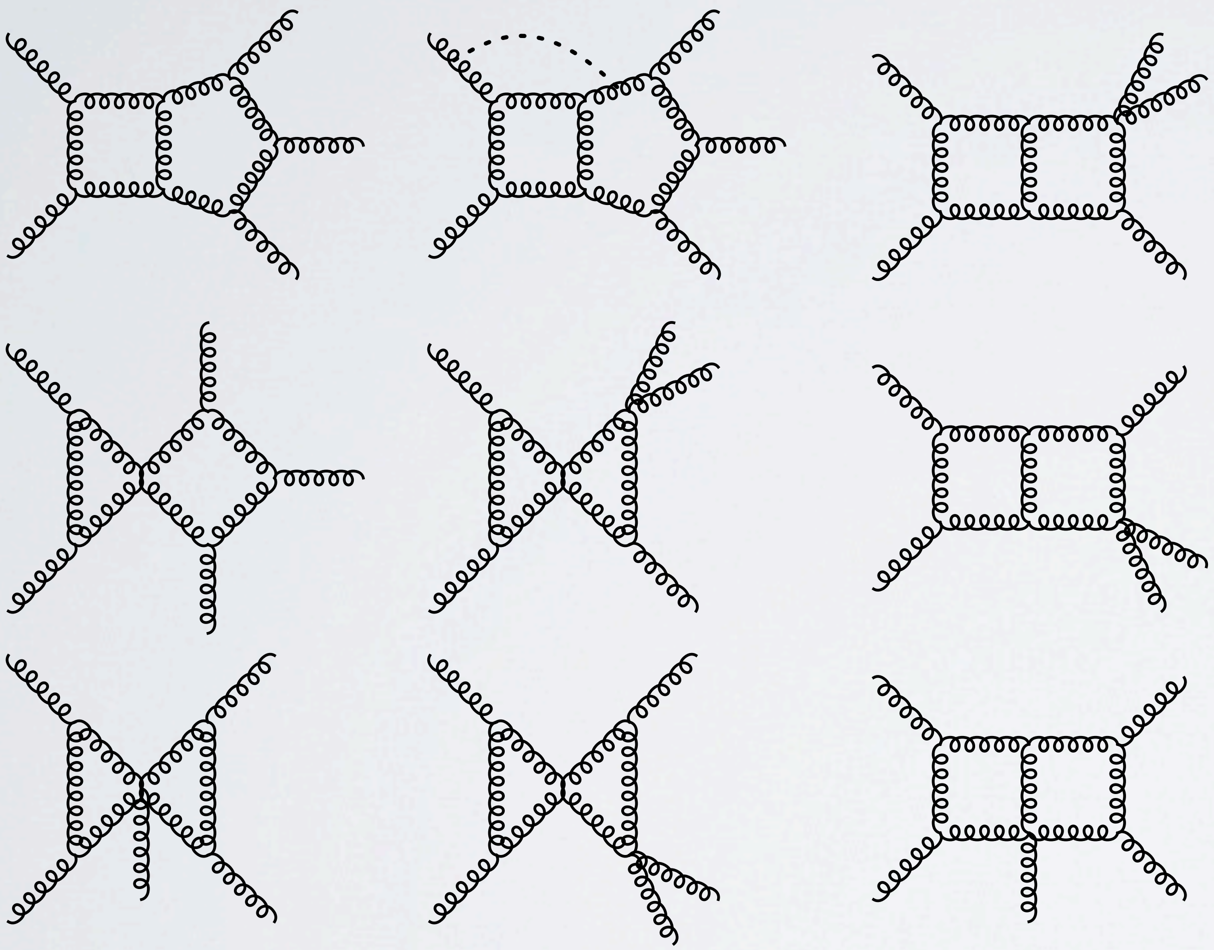
$$x_1 = s_{12}$$
$$x_2 = \frac{\langle 23\rangle\langle 14\rangle}{\langle 12\rangle\langle 34\rangle}$$
$$x_3 = \frac{\langle 34\rangle\langle 15\rangle}{\langle 13\rangle\langle 45\rangle}$$
$$x_4 = \frac{s_{23}}{s_{12}}$$
$$x_5 = \frac{s_{45}}{s_{12}}$$

no square roots

$$\mathrm{tr}_5^2 = \frac{x_1^4}{x_2^2}\left(x_2^2 x_3(x_5-1) + x_2(2x_3x_4 + x_4) - (x_3+1)x_4(x_4-x_5)\right)^2$$

Five gluon integrand representation

only ≥6 propagator topologies



$$c_{431} = -\frac{s_{12}s_{23}s_{34}s_{45}^2s_{15}}{\mathrm{tr}_5}\,, \qquad c_{431}^T = -\frac{s_{12}s_{23}s_{45}\,\mathrm{tr}_+(1345)}{\mathrm{tr}_5}\,,$$

$$c_{331;M_1} = -\frac{s_{34}s_{45}^2\,\mathrm{tr}_+(1235)}{\mathrm{tr}_5}\,, \qquad c_{331;M_2} = -\frac{s_{15}s_{45}^2\,\mathrm{tr}_-(1234)}{\mathrm{tr}_5}\,,$$

$$c_{331;5L} = \frac{s_{12}s_{23}s_{34}s_{45}s_{15}}{\mathrm{tr}_5}\,, \qquad c_{430} = -\frac{s_{12}\,\mathrm{tr}_+(1345)}{2s_{13}s_{45}}\,,$$

$$c_{330;M_1} = -\frac{(s_{45}-s_{12})\,\mathrm{tr}_+(1345)}{2s_{13}s_{45}}\,, \qquad c_{330;M_2} = -\frac{(s_{45}-s_{23})\,\mathrm{tr}_+(1345)}{2s_{13}s_{45}}\,,$$

$$c_{330;5L}^b = \frac{\mathrm{tr}_+(1235)}{2s_{35}s_{12}}\,, \qquad c_{330;5L}^c = \frac{\mathrm{tr}_+(1345)}{2s_{13}s_{45}}\,,$$

$$c_{330;5L}^a = -\frac{1}{2}\left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1235)\,\mathrm{tr}_+(1345)}{s_{13}s_{35}}\right),$$

$$c_{330;5L}^d = c_{330;5L}^a\frac{s_{12}+s_{45}}{s_{12}s_{45}} - s_{12}c_{330;5L}^b - s_{45}c_{330;5L}^c - s_{15}\,,$$

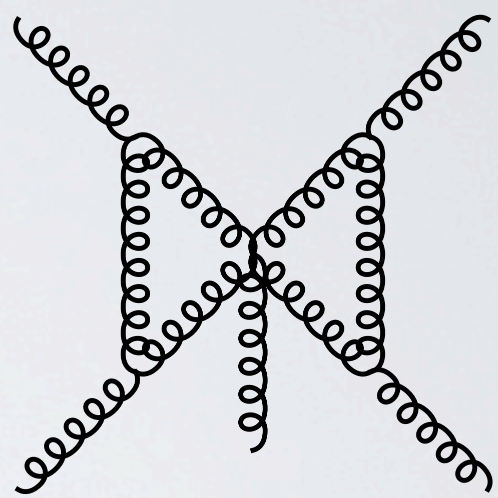
+ spurious terms

choice of basis important to find simplest form

double-box type topologies are N=4 x $(\mu_{11}\mu_{22}+\mu_{22}\mu_{33}+\mu_{33}\mu_{11})+4(\mu_{12}^2-4\mu_{11}\mu_{22})$

Choices of integrand basis

$$\Delta_{330;5L}(1^+,2^+,3^+,4^+,5^+) \;=\; -\frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\times$$
$$\Bigg(\frac{1}{2}\left(\mathrm{tr}_+(1245)-\frac{\mathrm{tr}_+(1345)\mathrm{tr}_+(1235)}{s_{13}s_{35}}\right)\Bigg(2(D_s-2)(\mu_{11}+\mu_{22})\mu_{12}$$
$$+(D_s-2)^2\mu_{11}\mu_{22}\frac{4(k_1\cdot p_3)(k_2\cdot p_3)+(k_1+k_2)^2(s_{12}+s_{45})+s_{12}s_{45}}{s_{12}s_{45}}\Bigg)$$
$$+(D_s-2)^2\mu_{11}\mu_{22}\Bigg[(k_1+k_2)^2 s_{15}$$
$$+\mathrm{tr}_+(1235)\left(\frac{(k_1+k_2)^2}{2s_{35}}-\frac{k_1\cdot p_3}{s_{12}}\left(1+\frac{2(k_2\cdot\omega_{453})}{s_{35}}+\frac{s_{12}-s_{45}}{s_{35}s_{45}}(k_2-p_5)^2\right)\right)$$
$$+\mathrm{tr}_+(1345)\left(\frac{(k_1+k_2)^2}{2s_{13}}-\frac{k_2\cdot p_3}{s_{45}}\left(1+\frac{2(k_1\cdot\omega_{123})}{s_{13}}+\frac{s_{45}-s_{12}}{s_{12}s_{13}}(k_1-p_1)^2\right)\right)\Bigg]\Bigg)$$



important to identify spurious direction for each loop integral

these are reducible but with this choice five propagator cuts vanish

Full result

$$A_5^{[P]}(1^+,2^+,3^+;4^+,5^+) = \frac{i}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\Big(c_{431} I_{431}[F_1]$$
$$+ c_{431}^T I_{431}\left[F_1\,(k_1+p_5)^2\right] + c_{331;M_1} I_{331;M_1}[F_1] + c_{331;M_2} I_{331;M_2}[F_1] + c_{331;5L} I_{331;5L}[F_1]$$
$$+ c_{430}\big(s_{23} I_{430}\left[F_3\,((k_1+k_2)^2+s_{45})\right] + I_{430}\left[F_3\,((k_1+k_2)^2+s_{45})\,2(k_1\cdot\omega_{123})\right]\big)$$
$$+ c_{330;M_1} I_{330;M_1}\left[F_3\,((k_1+k_2)^2+s_{45})\right] + c_{330;M_2} I_{330;M_2}\left[F_3\,((k_1+k_2)^2+s_{45})\right]$$
$$+ c_{330;5L}^a I_{330;5L}[F_3\,N_1(k_1,k_2,1,2,3,4,5)] + c_{330;5L}^b I_{330;5L}[F_3\,N_2(k_1,k_2,1,2,3,4,5)]$$
$$+ c_{330;5L}^c I_{330;5L}[F_3\,N_2(k_2,k_1,5,4,3,2,1)] + c_{330;5L}^d I_{330;5L}\left[F_3\,(k_1+k_2)^2\right]\Big)$$

$$c_{431} = -\frac{s_{12}s_{23}s_{34}s_{45}^2 s_{15}}{\mathrm{tr}_5}, \qquad c_{431}^T = -\frac{s_{12}s_{23}s_{45}\,\mathrm{tr}_+(1345)}{\mathrm{tr}_5},$$
$$c_{331;M_1} = -\frac{s_{34}s_{45}^2\,\mathrm{tr}_+(1235)}{\mathrm{tr}_5}, \qquad c_{331;M_2} = -\frac{s_{15}s_{45}^2\,\mathrm{tr}_-(1234)}{\mathrm{tr}_5},$$
$$c_{331;5L} = \frac{s_{12}s_{23}s_{34}s_{45}s_{15}}{\mathrm{tr}_5}, \qquad c_{430} = -\frac{s_{12}\,\mathrm{tr}_+(1345)}{2s_{13}s_{45}},$$
$$c_{330;M_1} = -\frac{(s_{45}-s_{12})\,\mathrm{tr}_+(1345)}{2s_{13}s_{45}}, \qquad c_{330;M_2} = -\frac{(s_{45}-s_{23})\,\mathrm{tr}_+(1345)}{2s_{13}s_{45}},$$
$$c_{330;5L}^b = \frac{\mathrm{tr}_+(1235)}{2s_{35}s_{12}}, \qquad c_{330;5L}^c = \frac{\mathrm{tr}_+(1345)}{2s_{13}s_{45}},$$
$$c_{330;5L}^a = -\frac{1}{2}\left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1235)\,\mathrm{tr}_+(1345)}{s_{13}s_{35}}\right),$$
$$c_{330;5L}^d = c_{330;5L}^a\frac{s_{12}+s_{45}}{s_{12}s_{45}} - s_{12}c_{330;5L}^b - s_{45}c_{330;5L}^c - s_{15},$$

$$F_1 = (D_s-2)(\mu_{11}\mu_{22}+\mu_{11}\mu_{33}+\mu_{22}\mu_{33}) + 4(\mu_{12}^2 - 4\mu_{11}\mu_{22}),$$
$$F_3 = (D_s-2)^2\mu_{11}\mu_{22},$$
$$N_1(k_1,k_2,1,2,3,4,5) = \frac{1}{s_{12}s_{45}}\left(4(k_1\cdot p_3)(k_2\cdot p_3) + s_{12}s_{45}\right),$$
$$N_2(k_1,k_2,1,2,3,4,5) = \frac{2}{s_{45}}(k_1\cdot p_3)\left(s_{35}s_{45} - (s_{12}-s_{45})2(k_2\cdot p_5)\right).$$

Numerical evaluation

Pentagon-box integrals and 5-leg double box unknown

Check universal IR pole structure numerically

Mellin-Barnes and Sector decomposition

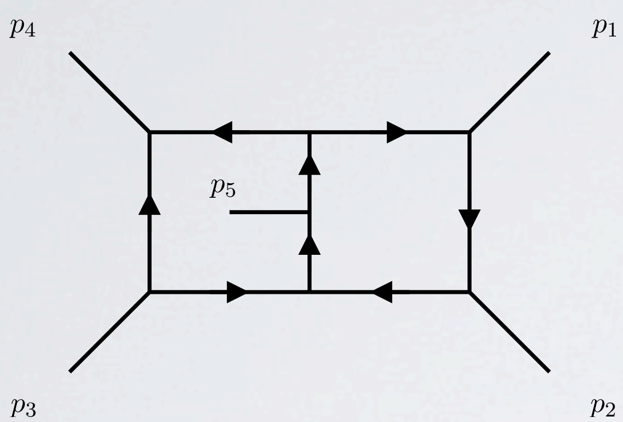
[**Fiesta** Smirnov, Smirnov, Tentyukov]

[**SecDec** Borowka, Carter, Heinrich]

$$A_5^{(2)}(1^+,2^+,3^+,4^+,5^+) = -\left(\frac{1}{\epsilon^2}\sum_{i=1}^{5}\left(\frac{\mu_R^2}{-s_{i\,i+1}}\right)^{\epsilon} + \frac{11}{6\epsilon}\right)A_5^{(1)}(1^+,2^+,3^+,4^+,5^+) + \mathcal{O}(\epsilon)$$

	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$A_5^{[P]}(1,2,3;4,5)$	-145.03 ± 0.01	473.37 ± 0.10	-1643.16 ± 0.60
$A_5^{[P]}(2,3,4;5,1)$	-23.00 ± 0.00	86.54 ± 0.02	-229.22 ± 0.09
$A_5^{[P]}(3,4,5;1,2)$	-70.65 ± 0.00	118.03 ± 0.02	3279.84 ± 0.10
$A_5^{[P]}(4,5,1;2,3)$	5.19 ± 0.00	-15.11 ± 0.00	45.91 ± 0.01
$A_5^{[P]}(5,1,2;3,4)$	-159.87 ± 0.01	625.73 ± 0.10	-794.94 ± 0.90
$A_5^{(2),\mathrm{bare}}(1,2,3,4,5)$	-393.36 ± 0.02	1288.56 ± 0.20	658.43 ± 1.00
$\mathcal{I}_5^{(1),\mathrm{bare}}A_5^{(1)}(1,2,3,4,5)$	-393.35 ± 0.02	1288.50 ± 0.08	-2627.61 ± 0.20

Non-planar corrections

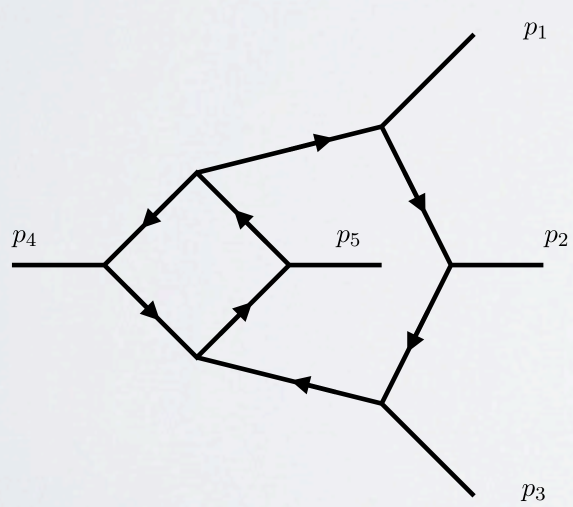


$$\Delta_{8;332} = \frac{2iF_1\ s_{12}s_{34}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle\,\mathrm{tr}_5}\,(b_1 k_1\cdot p_4 + b_2 k_2\cdot p_1 + b_3 k_1\cdot p_5)$$

$$b_1 = -s_{15}\,\mathrm{tr}_-(2345)$$

$$b_2 = s_{45}\,\mathrm{tr}_-(2351)$$

$$b_3 = s_{23}s_{45}s_{15} - s_{15}\,\mathrm{tr}_-(2345) - s_{45}\,\mathrm{tr}_-(2351).$$



$$\Delta_{8;422} = \frac{iF_1\ s_{12}s_{23}s_{45}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle\,\mathrm{tr}_5}\,(a_0 + 2a_1 k_2\cdot p_5)$$

$$a_0 = s_{15}s_{34}s_{45}$$

$$a_1 = -\,\mathrm{tr}_+(1345)$$

Outlook

- Integrand level reduction method for multi-loop amplitudes
  - find integrand parametrizations using polynomial division
  - valid in D dimensions
- First non-trivial application
  - planar two-loop five gluon amplitude in self-dual Yang-Mills

Backup Slides

More momentum twistors

Fitting massive external particles via on-shell decays

e.g. pp → H+2j

$$p_H = p_5 + p_6$$

$$Z = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1} + \frac{1}{x_1 x_2} & \frac{1}{x_1} + \frac{1}{x_1 x_2} + \frac{1}{x_1 x_2 x_3} & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 - x_4 & 1 & 0 \\ 0 & 0 & 1 & 1 & \frac{1 - x_4 x_5}{1 - x_4} & x_5 - \frac{x_6}{x_1 x_2 x_3 x_4} \end{pmatrix}$$

$$x_1 = \frac{s_{12} s_{23}}{s_{12} + s_{23}} \qquad x_2 = \frac{\langle 23 \rangle \langle 14 \rangle}{\langle 12 \rangle \langle 34 \rangle}$$

$$x_3 = \frac{\langle 1 | p_H | 1]}{s_{12} + s_{23}} \qquad x_4 = \frac{(s_{12} + s_{23})[34]}{\langle 12 \rangle [23][14]}$$

$$x_5 = \frac{(s_{12} + s_{23})(m_H^2 + \langle 1 | p_H | 1])}{s_{23} \langle 1 | p_H | 1\rangle} \qquad x_6 = m_H^2$$

Radical ideals

Definition:

for a field $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ and an ideal $I \in k[\mathbf{x}]$

the radical of I is $\sqrt{I} = \{f \in k[\mathbf{x}] \,|\, f^m \in I, m \in \mathbb{N}\}$

An ideal is a ***radical ideal*** if $\sqrt{I} = I$

Algorithms to compute the radical of an ideal are available in Macaulay2

sketch proof that D-dimensional propagator ideals are radical

at 2-loops there are $P-3$ linear relations
leading to $m = 11 - P$ ISPs of the form x_{ij}

$$I = \langle \mu_{11} - f_1(x_1, \ldots x_m), \quad \mu_{12} - f_2(x_1, \ldots x_m), \quad \mu_{22} - f_3(x_1, \ldots x_m) \rangle$$

we have an isomorphism

$$\phi : \mathbb{C}[x_1, \ldots x_m, \mu_{11}, \mu_{12}, \mu_{22}]/I \to \mathbb{C}[x_1, \ldots x_m]$$

with $\mu_{11} \mapsto f_1(x_1, \ldots x_m)$, $\mu_{12} \to f_2(x_1, \ldots x_m)$ and $\mu_{22} \to f_3(x_1, \ldots x_m)$

$\mathbb{C}[x_1, \ldots x_m]$ is a domain $\Rightarrow I$ is a prime ideal

prime ideal are radical ideals