Harmony of Scattering Amplitudes: From $N = 4$ Super-Yang-Mills Theory to $N = 8$ Supergravity

Itzykson WorkShop, June 19, 2009

Zvi Bern, UCLA

Will present results from papers with:
J.J. Carrasco, L. Dixon, D. Forde, H. Ita, H. Johansson,
D. Kosower, V. Smirnov, M. Spradlin, R. Roiban and A. Volovich.

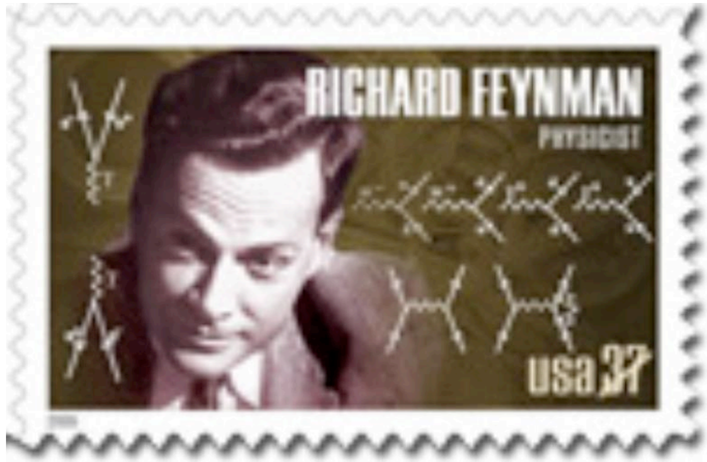
Outline

This talk will present recent work on scattering amplitudes of maximally supersymmtric gauge and gravity theories.

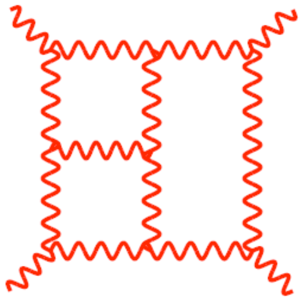
- **Supersymmetric gauge theory:** resummation of certain planar $N = 4$ super-Yang-Mills scattering amplitudes to *all* loop orders.
- **Quantum gravity:** reexamination of standard wisdom on ultraviolet properties of quantum gravity. Four-loop demonstration of novel UV cancellations.

Why are Feynman diagrams clumsy for high-loop or high-multiplicity processes?

- Vertices and propagators involve gauge-dependent off-shell states. An important origin of the complexity.



$$\int \frac{d^4 p}{(2\pi)^4}$$



$$p^2 \neq m^2$$

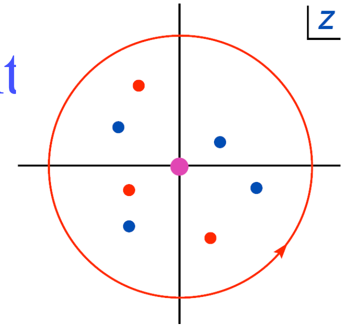
- To get at root cause of the trouble we must rewrite perturbative quantum field theory.

• All steps should be in terms of gauge invariant on-shell states. On-shell formalism. $p^2 = m^2$

On-Shell Recursion for Tree Amplitudes

Britto, Cachazo, Feng and Witten

Consider amplitude under complex shifts of the moment

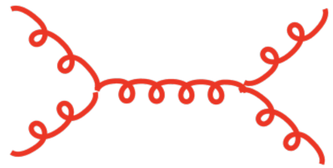


$$p_1^\mu(z) = p_1^\mu - z q^\mu \quad p_n^\mu(z) = p_n^\mu + z q^\mu \qquad q^2 = 0, \; p \cdot q = 0$$

$$(p_i^\mu(z))^2 = 0$$ complex momenta

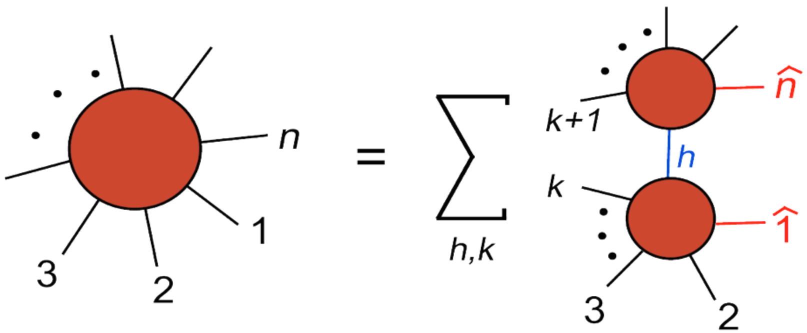
If $A(z) \to 0, \quad z \to \infty$ *A(z)* is amplitude with shifted momenta

$$\oint_{C_\infty} \frac{A(z)}{z} dz = 0 \quad \Rightarrow A(z=0) = -\sum_\alpha \text{Res}_\alpha \frac{A(z)}{z}$$



$$A(z) = \sum_\alpha \frac{c_\alpha}{z - z_\alpha}$$

on-shell amplitude



Sum over residues gives the on-shell recursion relation

Poles in *z* come from kinematic poles in amplitude.

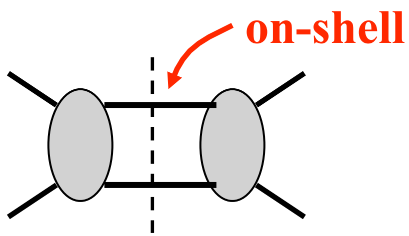
Same construction works in gravity

Brandhuber, Travaglini, Spence; Cachazo, Svrcek;
Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

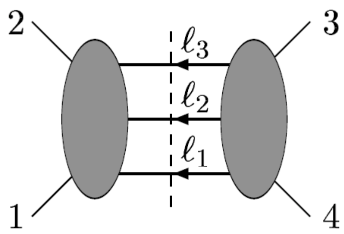
Bern, Dixon, Dunbar and Kosower

Modern Unitarity Method

Two-particle cut:



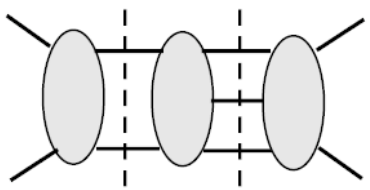
Three-particle cut:

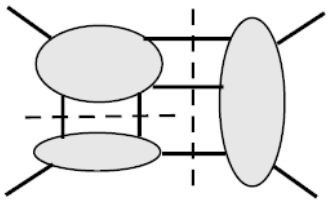


Systematic assembly of complete amplitudes from cuts for any number of particles or loops.

Generalized unitarity as a practical tool:

Bern, Dixon and Kosower
Britto, Cachazo and Feng





Different cuts merged to give an expression with correct cuts in all channels.

Generalized cut interpreted as cut propagators not canceling.

Method of Maximal Cuts

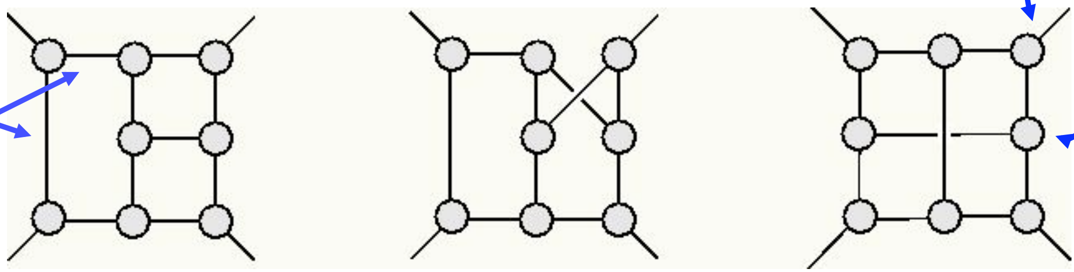
ZB, Carrasco, Johansson, Kosower

A refinement of unitarity method for constructing complete higher-loop amplitudes is "Method of Maximal Cuts". Systematic construction in any massless theory.

To construct the amplitude we use cuts with maximum number of on-shell propagators:

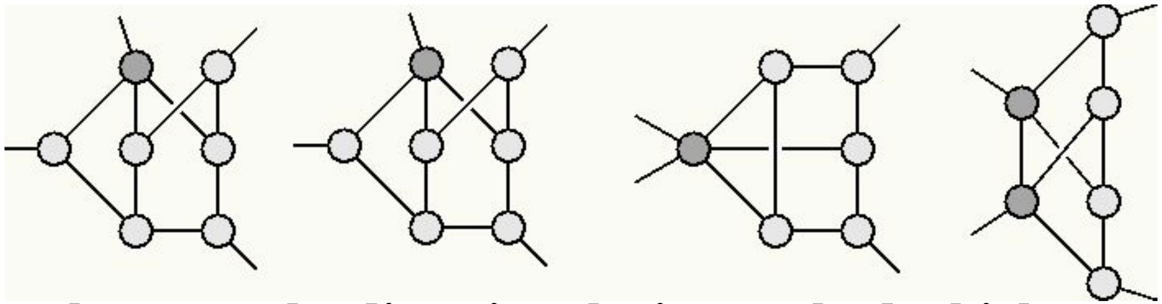
tree amplitudes

on-shell



Maximum number of propagator placed on-shell.

Then systematically release cut conditions to obtain contact terms:

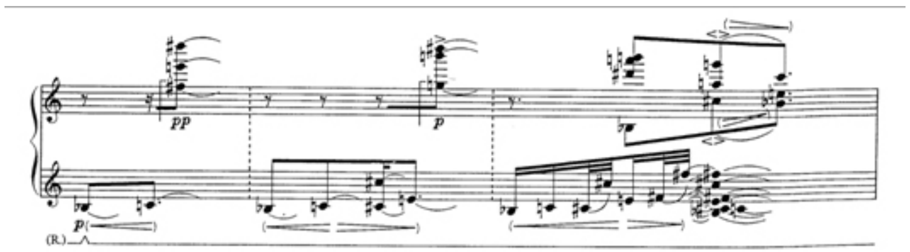


Fewer propagators placed on-shell.

Related to subsequent leading singularity method which uses hidden singularities.

Cachazo and Skinner; Cachazo; Cachazo, Spradlin, Volovich; Spradlin, Volovich, Wen

Examples of Harmony



Gravity vs Gauge Theory

Consider the gravity Lagrangian

flat metric

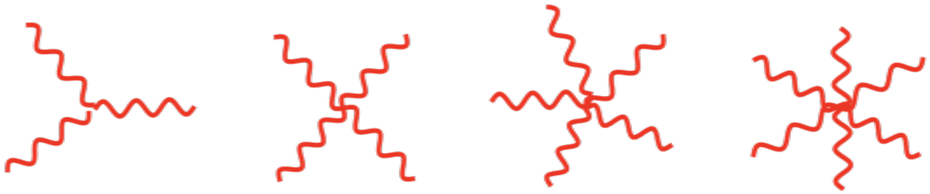
graviton field

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

Infinite number of complicated interactions

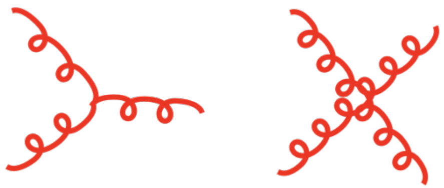


$+\cdots$

bloody mess

Compare to Yang-Mills Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$



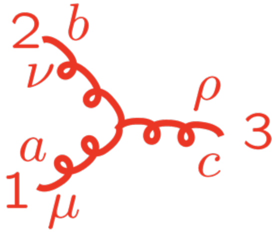
Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Does not look harmonious!

Three Vertices

Three gluon vertex:

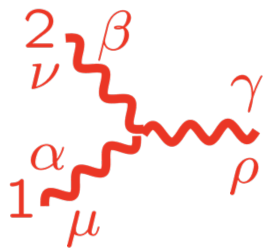


$$V_{3\,\mu\nu\sigma}^{abc} = -gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \eta_{\nu\rho}(k_1-k_2)_\mu + \eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

Three graviton vertex:

$$\begin{aligned}
&G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \\
&\quad \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\quad\quad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\quad\quad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\quad\quad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



About 100 terms in three vertex

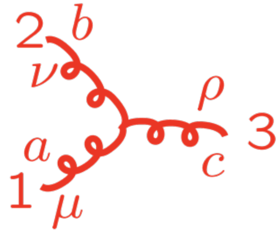
Naïve conclusion: Gravity is a nasty mess.

Not very harmonious!

Simplicity of Gravity Amplitudes

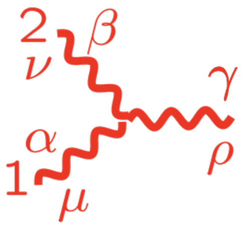
On-shell **three vertices contains all information:** $k_i^2 = 0$

gauge theory:



$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_1)_\rho + \text{cyclic})$$
$$\times\,(\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

"square" of Yang-Mills vertex.

Any gravity scattering amplitude constructible solely from *on-shell* 3 vertex.

- **BCFW on-shell recursion for tree amplitudes.**

  Britto, Cachazo, Feng and Witten; Brandhuber, Travaglini, Spence; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

- **Unitarity method for loops.**

  ZB, Dixon, Dunbar and Kosower; ZB, Dixon, Kosower; Britto, Cachazo, Feng; ZB, Morgan; Buchbinder and Cachazo; ZB, Carrasco, Johansson, Kosower; Cachzo and Skinner.

Gravity vs Gauge Theory

Consider the gravity Lagrangian

flat metric

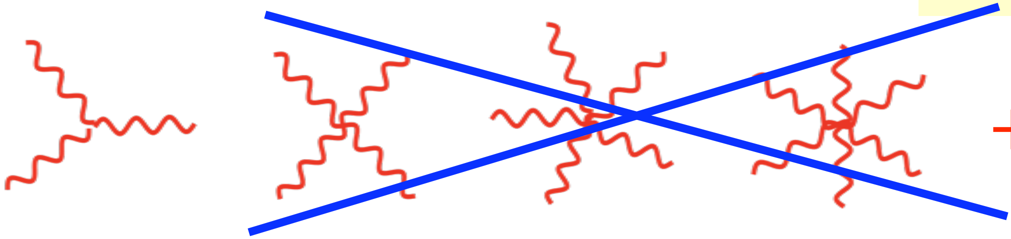
graviton field

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

Infinite number of irrelevant interactions!

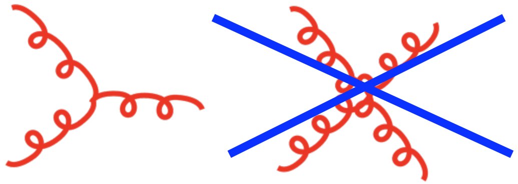


$+ \cdots$

Simple relation to gauge theory

Compare to Yang-Mills Lagrangian

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$



Only three-point interactions

Gravity seems ~~so much~~ no more complicated than gauge theory.

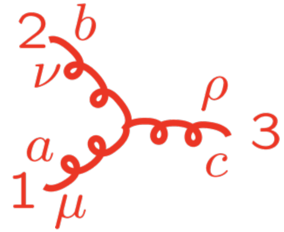
Does ~~not~~ look harmonious!

Harmony of Color and Kinematics

ZB, Carrasco, Johansson

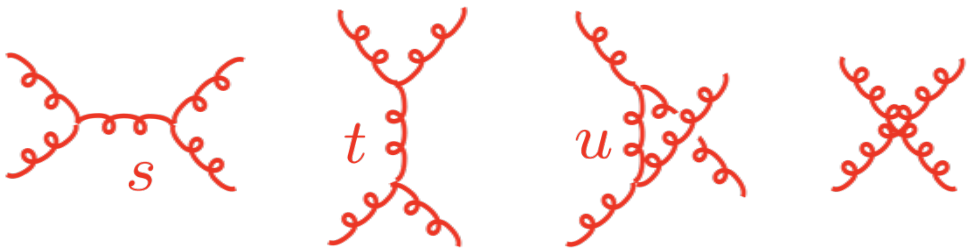
coupling constant **color factor** **momentum dependent kinematic factor**

$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = i f^{abc} T^c$

Jacobi identity $[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$



Use 1 = ***s/s*** = ***t/t*** = ***u/u***
to assign 4-point diagram
to others.

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$s = (k_1 + k_2)^2$ $u = (k_1 + k_3)^2$
$t = (k_1 + k_4)^2$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Color and kinematics are singing same tune!

Harmony of Color and Kinematics

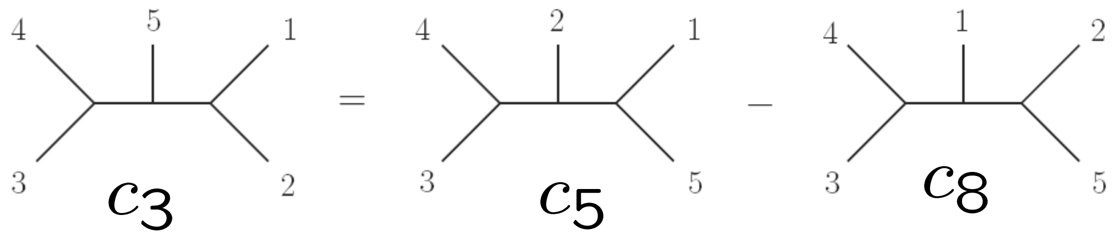
At higher points similar structure:

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i \, n_i}{D_i}$$

color factor

kinematic numerator factor

Feynman propagators



$$c_3 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}\,, \qquad c_5 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}\,, \qquad c_8 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$\boxed{c_3 - c_5 + c_8 = 0 \quad \Leftrightarrow \quad n_3 - n_5 + n_8 = 0}$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* Jacobi constraint equations.

- **Color and kinematics sing same tune!**
- **Nontrivial constraints on amplitudes.**

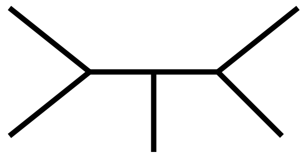
Higher-Point Gravity and Gauge Theory

ZB, Carrasco, Johansson

QCD: $$\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i\, n_i}{D_i}$$

sum over diagrams with only 3 vertices

Einstein Gravity: $$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$



Claim: This is correct though unproven!

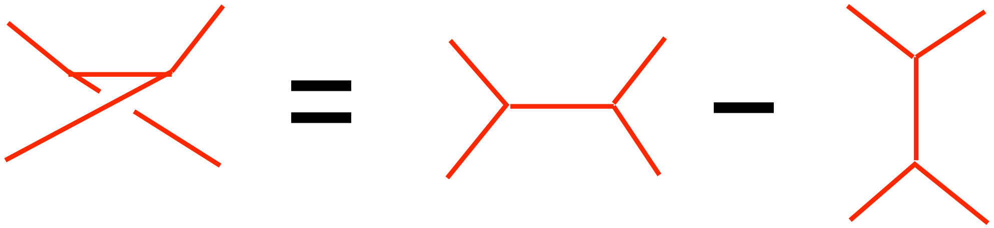
Gravity and QCD kinematic numerators sing same tune!

Cries out for a unified description of the sort given by string theory.

Non-Planar from Planar

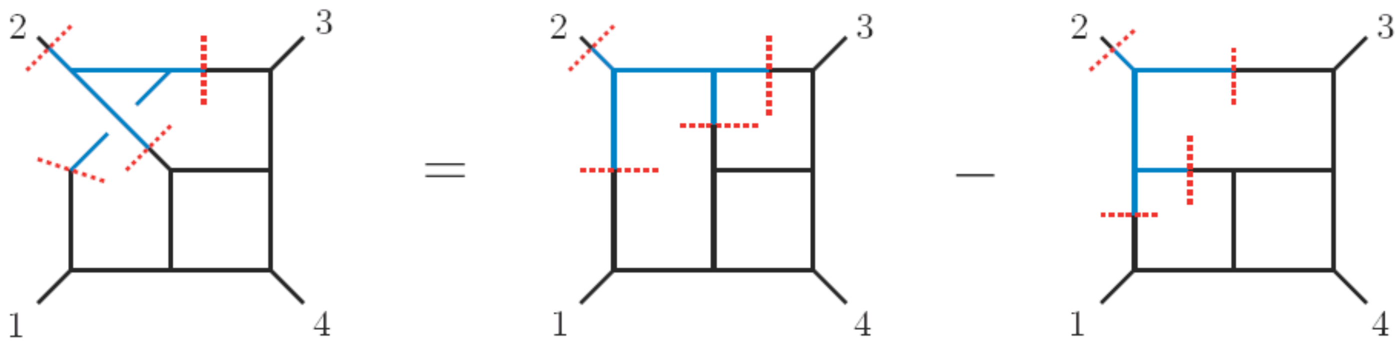
ZB, Carrasco, Johansson

We can use Jacobi-like identities to obtain non-planar from planar contributions at higher loops!



Satisfied by Yang-Mills numerator factors

For example for three loop (super) Yang-Mills



key tool for high loop calculations

non-planar **planar**

All planar symmetries of planar obviously restrict the structure of non-planar as well (*e.g.* dual conformal symmetry)

Applications to AdS/CFT

$\mathcal{N} = 4$ Super-Yang-Mills to All Loops

Since 't Hooft's paper thirty years ago on the planar limit of QCD we have dreamed of solving QCD in this limit. This is too hard. $\mathcal{N} = 4$ sYM is much more promising.

- **Special theory because of AdS/CFT correspondence.**
- **Maximally supersymmetric.**
- **Simplicity both at strong and weak coupling.**

Remarkable relation

Alday and Maldacena

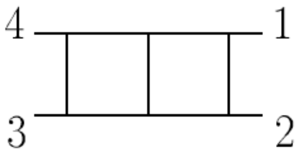
scattering at strong coupling in $\mathcal{N} = 4$ sYM $\longleftrightarrow$ classical string theory in AdS space

To make this link need to evaluate $\mathcal{N} = 4$ super-Yang-Mills amplitudes to *all* loop orders. Seems impossible even with modern methods.

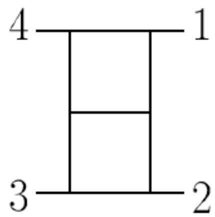
Loop Iteration of the $N = 4$ Amplitude

The planar four-point two-loop amplitude undergoes fantastic simplification.

$$- st\, A_4^{\text{tree}} \Big\{ s$$



$$+ \; t$$



$$\Big\}$$

ZB, Rozowsky, Yan

$$M_4^{\text{2-loop}}(s,t) = \frac{1}{2}\Big(M_4^{\text{1-loop}}(s,t)\Big)^2 + f(\epsilon) M_4^{\text{1-loop}}(s,t)\big|_{\epsilon \to 2\epsilon} - \frac{1}{2}\zeta_2^2$$

$$M_4^{\text{loop}} = A_4^{\text{loop}}/A_4^{\text{tree}} \qquad f(\epsilon) = -\zeta_2 - \zeta_3 \epsilon - \zeta_4 \epsilon^2$$

Anastasiou, ZB, Dixon, Kosower

$f(\epsilon)$ is universal function related to IR singularities $\qquad D = 4 - 2\epsilon$

This gives two-loop four-point planar amplitude as iteration of one-loop amplitude.

Three loop satisfies similar iteration relation. Rather nontrivial.

ZB, Dixon, Smirnov

All-Loop Generalization

Why not be bold and guess scattering amplitudes for all loop and all legs, at least for simple helicity configurations?

$$\mathcal{A}_n = A_n^{\text{tree}} A_n^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_n^{\text{1-loop}} + C\right]$$

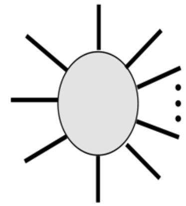
constant independent of kinematics.

all-loop resummed amplitude

IR divergences

cusp anomalous dimension

finite part of one-loop amplitude



"BDS conjecture"

Anastasiou, ZB, Dixon, Kosower
ZB, Dixon and Smirnov

- **IR singularities agree with Magnea and Sterman formula.**
- **Limit of collinear momenta gives us key analytic information, at least for MHV amplitudes. Warning: This argument has a loophole.**

Gives a definite prediction for *all* values of coupling given BES integral equation for the cusp anomalous dimension.

Beisert, Eden, Staudacher

Alday and Maldacena Strong Coupling

For MHV amplitudes: $F_4^{\text{1-loop}} = \frac{1}{2}\ln^2(s/t) + \frac{2\pi^2}{3}$ ZB, Dixon, Smirnov

$$\mathcal{A}_4 = A_4^{\text{tree}} A_4^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_4^{\text{1-loop}} + C\right]$$

constant independent of kinematics.

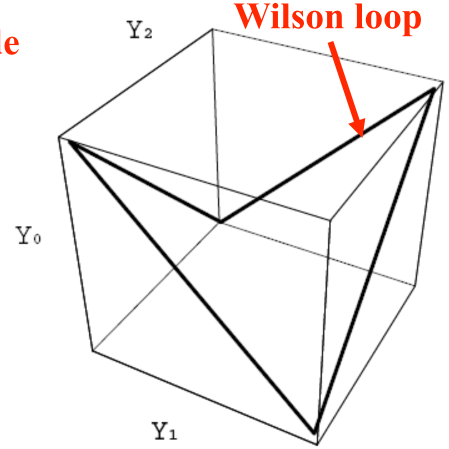
all-loop resummed amplitude

IR divergences

cusp anomalous dimension

finite part of one-loop amplitude

In a beautiful paper Alday and Maldacena confirmed the conjecture for 4 gluons at *strong coupling* from an AdS string theory computation. Minimal surface calculation.



Very suggestive link to Wilson loops even at weak coupling.

Drummond, Korchemsky, Sokatchev ; Brandhuber, Heslop, and Travaglini
ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich

- **Identication of new symmetry: "dual conformal symmetry"**
- **Link to integrability**
- **Yangian structure!**

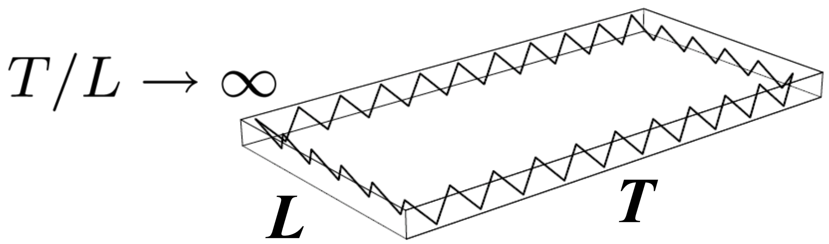
Drummond, Henn, Korchemsky, Sokatchev ;Berkovits and Maldacena; Beisert, Ricci, Tseytlin, Wolf Brandhuber, Heslop, Travaglini; Drummond, Henn, Plefka; Bargheer, Beisert, Galleas, Loebbert, McLoughlin.

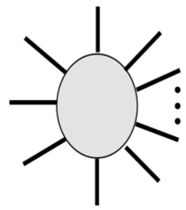
Trouble at Higher Points

For various technical reasons it is hard to solve for minimal surface for large numbers of gluons.

Alday and Maldacena realized certain terms can be calculated at strong coupling for an infinite number of gluons.

Disagrees with BDS conjecture

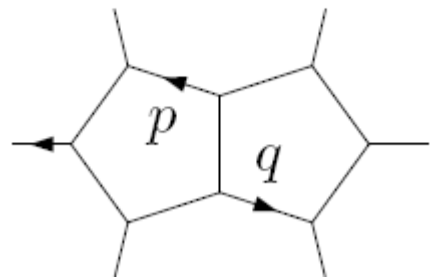




$s/t \to \infty$

Trouble also in the Regge limit.

Bartels, Lipatov, Sabio Vera

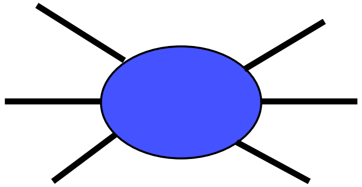


Explicit computation at 2-loop six points.
Need to modify conjecture!

ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich
Drummond, Henn, Korchemsky, Sokatchev

Can the BDS conjecture be repaired for six and higher points?

In Search of the Holy Grail



$$A^{\text{truth}} = A^{\text{div}} + A^{\text{BDS}} + \text{Discrepancy}$$

Can we figure out the discrepancy?

Important new information from regular polygons should serve as a guide.

Explicit solution at eight points

$$k_i = x_{i+1} - x_i$$

$$A_{BDS} = -\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1, j\neq i, i-1}^{n} \log\frac{x_j^+ - x_i^+}{x_{j+1}^+ - x_i^+}\log\frac{x_j^- - x_{i-1}^-}{x_j^- - x_i^-}$$

Alday and Maldacena (2009)

$$A = A_{div} + A_{BDS} + R$$

$$R = -\frac{1}{2}\log(1+\chi^-)\log(1+\frac{1}{\chi^+}) + \frac{7\pi}{6} + \int_{-\infty}^{\infty} dt \frac{|m|\sinh t}{\tanh(2t+2i\phi)}\log\left(1+e^{-2\pi|m|\cosh t}\right)$$

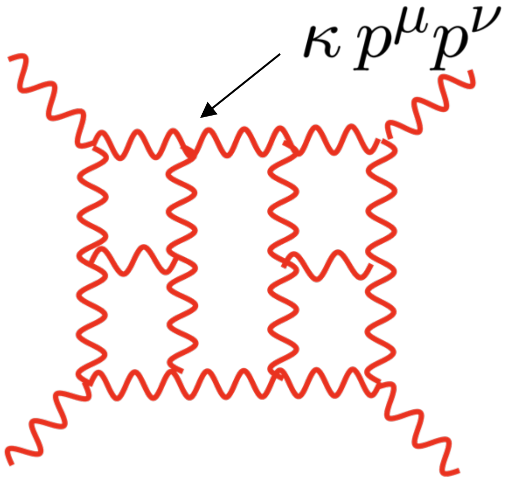
Solution only valid for strong coupling and special kinematics

Together with the conformal and dual conformal symmetry, this may provide the information we need.

UV Properties of $N = 8$ Supergravity

Is a UV finite theory of gravity possible?

$\kappa = \sqrt{32\pi G_N}$ ← Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV

Much more sophisticated power counting in supersymmetric theories but this is the basic idea.

Reasons to focus on $N = 8$ maximal supergravity: Cremmer and Julia

- **With more susy suspect better UV properties.**
- **High symmetry implies simplicity. Much simpler than expected. May be "simplest theory"** **See Nima's talk**

Finiteness of Point-Like Gravity Theory?

We are interested in UV finiteness of $N = 8$ supergravity because it would imply a new symmetry or non-trivial dynamical mechanism.

The discovery of either would have a fundamental impact on our understanding of gravity.

- **Here we only focus on order-by-order UV finiteness.**
- **Non-perturbative issues and viable models of Nature are *not* the goal for now.**

Opinions from the 80's

Unfortunately, in the absence of further mechanisms for cancellation, the analogous $N = 8$ $D = 4$ supergravity theory would seem set to diverge at the three-loop order.

Howe, Stelle (1984)

There are no miracles… It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions. … The final word on these issues may have to await further explicit calculations.

Marcus, Sagnotti (1985)

The idea that *all* supergravity theories diverge has been widely accepted for over 25 years

Divergences in Gravity

One loop: $R^2, \; R_{\mu\nu}^2, \; R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$

Vanish on shell

vanishes by Gauss-Bonnet theorem

Pure gravity 1-loop finite (but not with matter)

't Hooft, Veltman (1974)

Two loop: Pure gravity counterterm has non-zero coefficient:

$$R^3 \equiv R^{\lambda\rho}{}_{\mu\nu} R^{\mu\nu}{}_{\sigma\tau} R^{\sigma\tau}{}_{\lambda\rho}$$

Goroff, Sagnotti (1986); van de Ven (1992)

Any supergravity:

R^3 is ***not*** a valid supersymmetric counterterm.

Produces a helicity amplitude $(-,+,+,+)$ forbidden by susy.

Grisaru (1977); Tomboulis (1977)

The first divergence in *any* supergravity theory can be no earlier than three loops.

R^4 **squared Bel-Robinson tensor expected counterterm**

Deser, Kay, Stelle (1977); Kaku, Townsend, van Nieuwenhuizen (1977), Ferrara, Zumino (1978)

Reasons to Reexamine This

1) The number of *established* divergences for *any* pure supergravity theory in $D = 4$ is zero!

2) Discovery of remarkable cancellations at 1 loop – the "no-triangle property". Nontrivial cancellations!
ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bjerrum-Bohr, Vanhove; Arkani-Hamed Cachazo, Kaplan

3) *Every* explicit loop calculation to date finds $N = 8$ supergravity has identical power counting as $N = 4$ super-Yang-Mills theory, which is UV finite. Green, Schwarz and Brink; ZB, Dixon, Dunbar, Perelstein, Rozowsky; Bjerrum-Bohr, Dunbar, Ita, PerkinsRisager; ZB, Carrasco, Dixon, Johanson, Kosower, Roiban.

4) Interesting hint from string dualities. Chalmers; Green, Vanhove, Russo

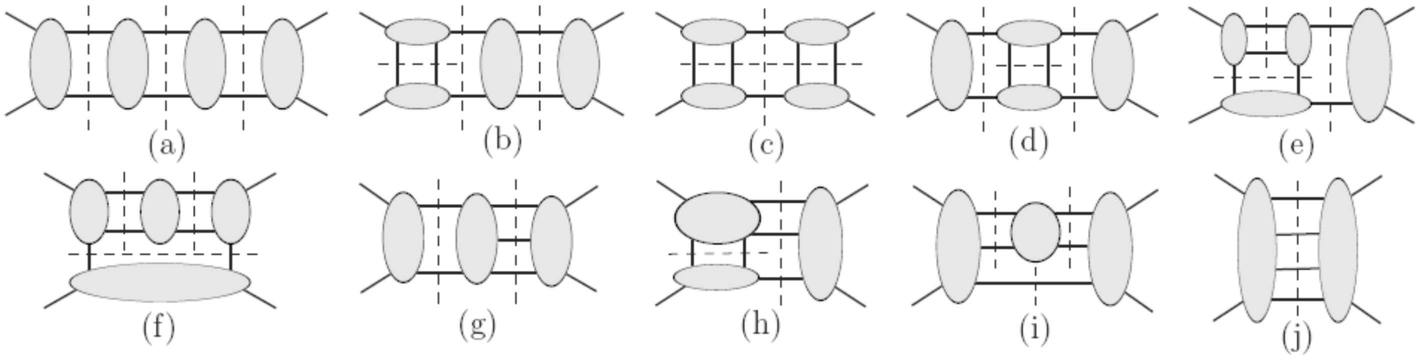
– Dualities restrict form of effective action. May prevent divergences from appearing in $D = 4$ supergravity, athough issues with decoupling of towers of massive states and indirect.

5) Interesting string non-renormalization theorem from Berkovits. Suggests divergence delayed to nine loops, but needs to be redone in field theory not string theory. Green, Vanhove, Russo **Green's Talk**

Full Three-Loop Calculation

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban

Need following cuts:



reduces everything to product of tree amplitudes

For cut (g) have:

$$\sum_{N=8 \text{ states}} M_4^{\rm tree}(1,2,l_3,l_1) \times M_5^{\rm tree}(-l_1,-l_3,q_3,q_2,q_1) \times M_5^{\rm tree}(3,4,-q_1,-q_2,-q_3)$$

Use Kawai-Lewellen-Tye tree relations

$$M_4^{\rm tree}(1,2,l_3,l_1) = -is_{12} A_4^{\rm tree}(1,2,l_3,l_1) A_4^{\rm tree}(2,1,l_3,l_1)$$

$$M_5^{\rm tree}(-l_1,-l_3,q_3,q_2,q_1) = i\, s_{l_1 q_1} s_{l_3 q_3} A_5^{\rm tree}(-l_1,-l_3,q_3,q_2,q_1)\, A_5^{\rm tree}(-l_1,q_1,q_3,-l_3,q_2) + \{l_1 \leftrightarrow l_3\}\,,$$

supergravity

super-Yang-Mills

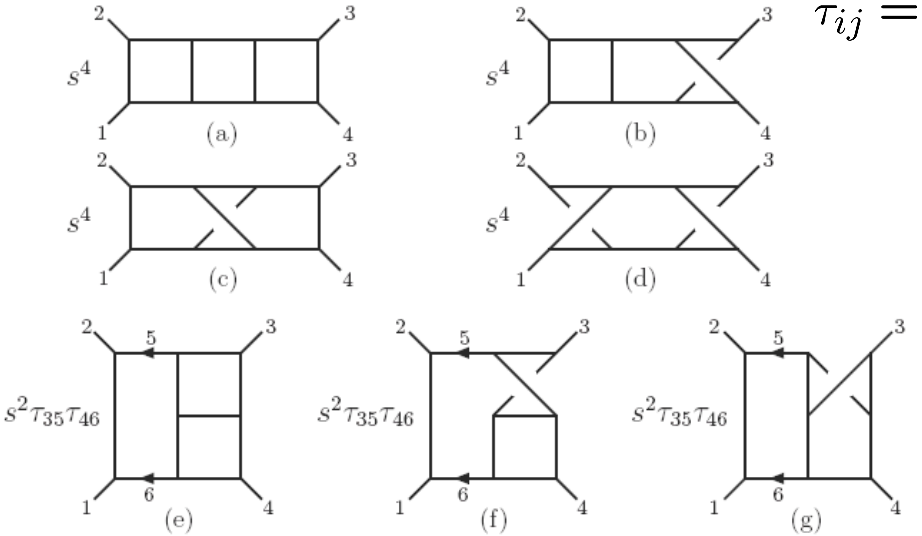
$N = 8$ supergravity cuts are sums of products of $N = 4$ super-Yang-Mills cuts

Complete Three-Loop Result

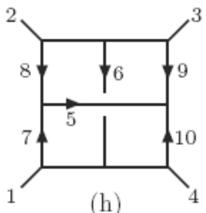
ZB, Carrasco, Dixon, Johansson, Kosower, Roiban; hep-th/0702112
ZB, Carrasco, Dixon, Johansson, Roiban arXiv:0808.4112 [hep-th]

$$M_4^{(3)} = \left(\frac{\kappa}{2}\right)^8 stu M_4^{\text{tree}} \sum_{S_3} \left[I^{(a)} + I^{(b)} + \tfrac{1}{2} I^{(c)} + \tfrac{1}{4} I^{(d)} + 2I^{(e)} + 2I^{(f)} + 4I^{(g)} + \tfrac{1}{2} I^{(h)} + 2I^{(i)} \right]$$

$$\tau_{ij} = 2k_i \cdot k_j$$



$$\begin{aligned}&(s(\tau_{26}+\tau_{36}) + t(\tau_{15}+\tau_{25}) + st)^2\\ &+(s^2(\tau_{26}+\tau_{36}) - t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})\\ &+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10}) + t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})\\ &+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})\end{aligned}$$

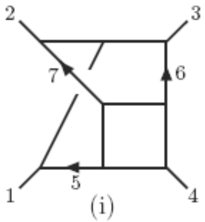


Three loops is not only UV finite it is "superfinite"— cancellations beyond those needed for finiteness in *D* = 4.

Finite for *D* < 6

No more divergent than *N* = 4 sYM amplitude!

$$\begin{aligned}&(s\tau_{45} - t\tau_{46})^2 - \tau_{27}(s^2\tau_{45} + t^2\tau_{46})\\ &-\tau_{15}(s^2\tau_{47} + u^2\tau_{46}) - \tau_{36}(t^2\tau_{47} + u^2\tau_{45})\\ &+l_5^2 s^2 t + l_6^2 s t^2 - \tfrac{1}{3} l_7^2 stu\end{aligned}$$



$$\text{UVpole}_{D=6-2\epsilon} = \frac{1}{\epsilon} \frac{5\zeta_3}{(4\pi)^9} \left(\frac{\kappa}{2}\right)^8 (stu)^2 M_4^{\text{tree}}$$

numerators quadratic in loop momenta

More Recent Opinion

In particular, they [non-renormalization theorems and algebraic formalism] suggest that maximal supergravity is likely to diverge at four loops in $D = 5$ and at five loops in $D = 4$, unless other infinity suppression mechanisms not involving supersymmetry or gauge invariance are at work. Bossard, Howe, Stelle (2009)

D^6R^4 is expected counterterm in $D = 5$.

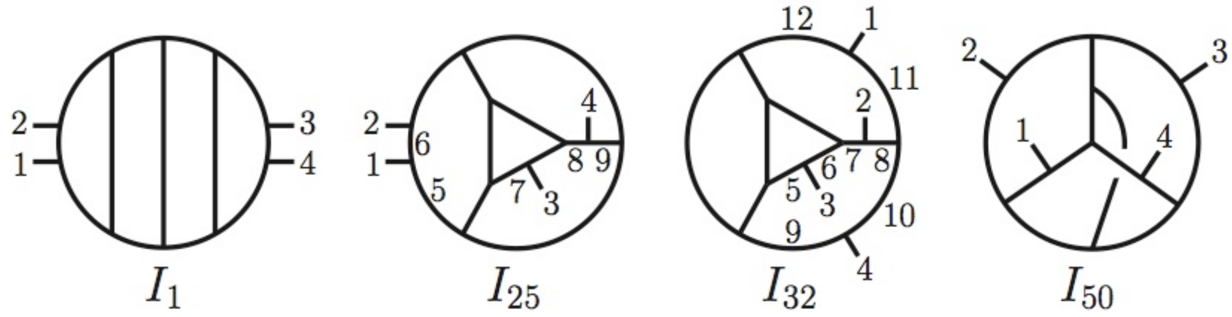
We have the tools to decisively decide this!

Widespread agreement ultraviolet finiteness of maximal supergravity requires a mechanism beyond known one of supersymmetry – little else is agreed upon by the experts.

Four-Loop Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban

Get 50 distinct diagrams or integrals (ones with two- or three-point subdiagrams not needed).



arXiv submission has mathematica files with all 50 diagrams

$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

Integral

leg perms

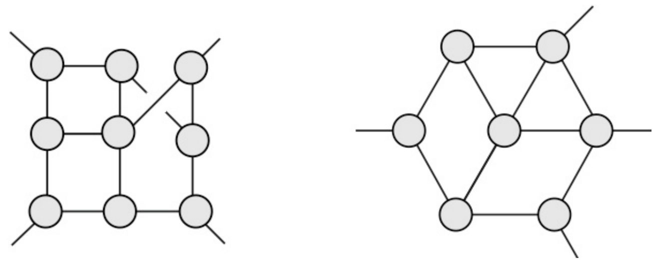
symmetry factor

Four-Loop Construction

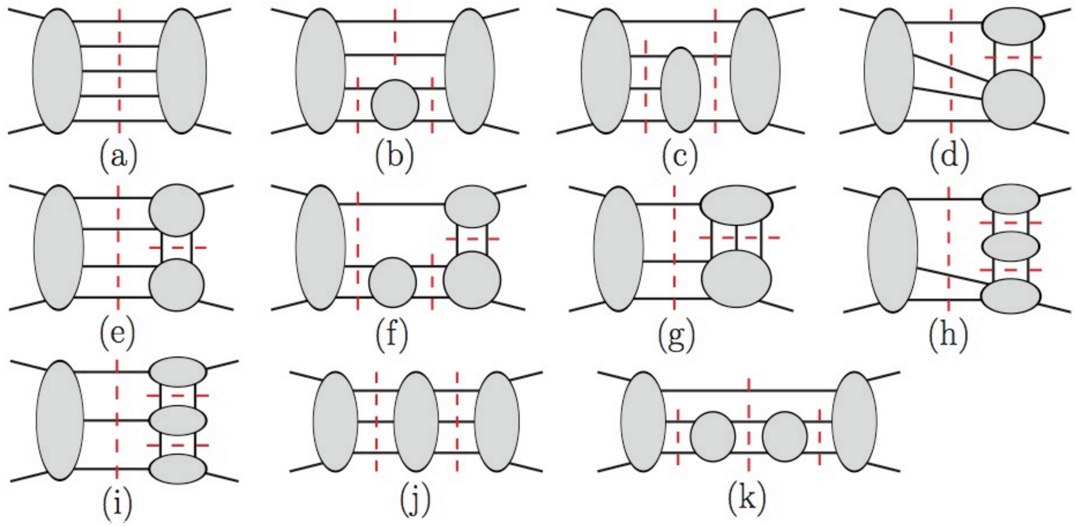
numerator

$$I_i = \int d^D l_1 d^D l_2 d^D l_3 d^D l_4 \, \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 l_6^2 l_7^2 l_8^2 l_9^2 l_{10}^2 l_{11}^2 l_{12}^2 l_{13}^2}$$

Determine numerators from 2906 maximal and near maximal cuts



Completeness of Expression confirmed using 26 generalized cuts



11 most complicated cuts shown

UV Finiteness at Four Loops

$$I_i = \int d^D l_1 d^D l_2 d^D l_3 d^D l_4 \, \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 l_6^2 l_7^2 l_8^2 l_9^2 l_{10}^2 l_{11}^2 l_{12}^2 l_{13}^2}$$

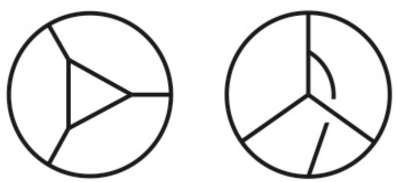
$N_i \sim O(k^4 l^8)$

k_i: external momenta

l_i : loop momenta

Manfestly finite for $D = 4$, but no surprise here.

Leading terms can be represented by two vacuum diagrams which cancel in the sum over all contributions.



coefficients vanish

$O(k^4 l^8)$

- $O(k^5 l^7)$ **vanishes by Lorentz invariance** $O(k^6 l^6)$ **leaving possible**
- **If no further cancellation corresponds to $D = 5$ divergence.**

UV Finiteness in $D = 5$ at Four Loops

ZB, Carrasco, Dixon, Johansson, Roiban

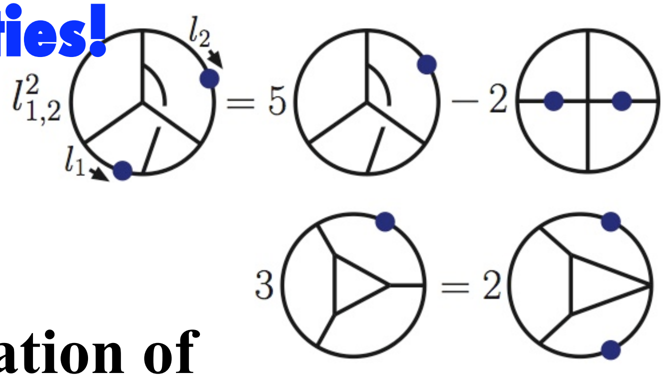
$N \sim O(k^6 l^6)$ **corresponds to** $D = 5$ **divergence.**

Expand numerator and propagators in small k $\quad \dfrac{1}{(l_j + K_n)^2}$

$$N^{(6)} + N^{(7)} \frac{K_i \cdot l_j}{l_j^2} + N^{(8)} \left(\frac{K_i^2}{l_j^2} + \frac{K_i \cdot l_j \; K_m \cdot l_n}{l_j^2 l_n^2} \right)$$

Marcus & Sagnotti
UV extraction method

Cancels after using $D = 5$ integral identities!



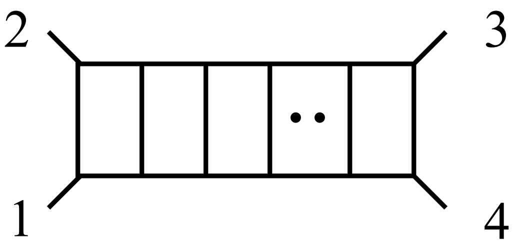
UV finite for $D = 4$ and 5

actually finite for $D < 5.5$

1. Shows potential supersymmetry explanation of Bossard, Howe, Stelle does *not* work!
2. The cancellations are stronger at 4 loops than at 3 loops, which is in turn stronger than at 2 loops. Rather surprising from traditional susy viewpoint.

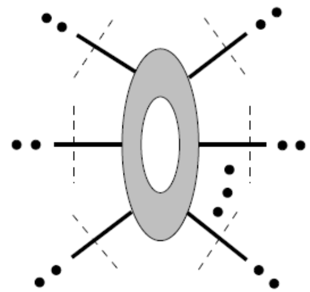
L-Loop UV Cancellations

ZB, Dixon, Roiban

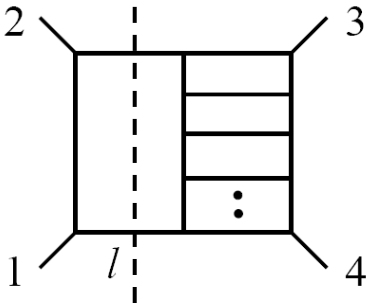


$$[(k_1 + k_2)^2]^{2(L-2)}$$

numerator factor



From 2 particle cut:



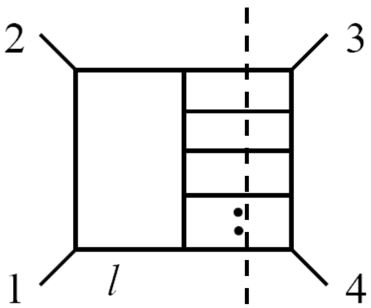
1 in $N = 4$ YM

$$[(l + k_4)^2]^{2(L-2)}$$

numerator factor

Using generalized unitarity and no-triangle property *all* one-loop subamplitudes should have power counting of $N = 4$ Yang-Mills

From L-particle cut:

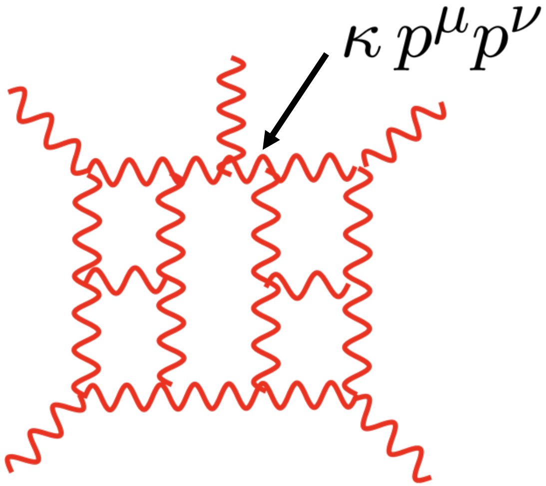


Above numerator violates a one-loop "no-triangle" property. Too many powers of loop momentum in one-loop subamplitude.

ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr and Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bjerrum-Bohr and Vanhove; Arkani-Hamed, Cachazo and Kaplan

UV cancellation exist to *all* loop orders! (not a proof of finiteness)

Higher Point Divergences?

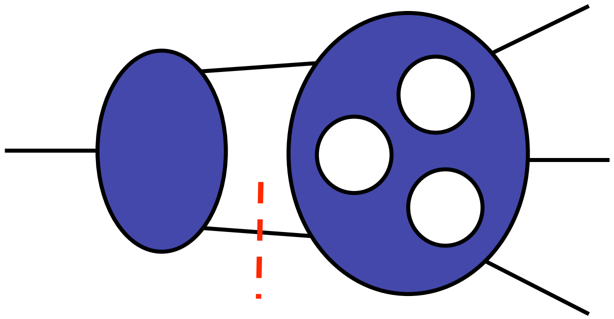


Add an extra leg:

1. extra $\kappa\, p^\mu p^\nu$ in vertex

2. extra $1/p^2$ from propagator

Adding legs generically does not worsen power count.



Cutting propagators exposes lower loop higher-point amplitudes.

- **Higher-point divergences should be visible in high-loop four-point amplitudes.**
- **A proof of UV finiteness would need to systematically rule out higher-point divegences.**

Origin of Cancellations?

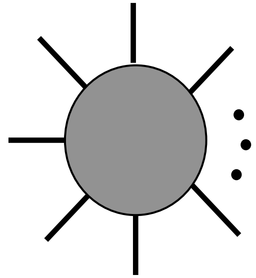
There does not appear to be a supersymmetry explanation for observed cancellations.

If it is *not* supersymmetry what might it be?

$$k_1^\mu \to k_1^\mu + \frac{z}{2}\langle k_1^-|\gamma^\mu|k_2^-\rangle \qquad A^{\text{tree}}_{\text{gravity}}(z) \sim \frac{1}{z^2}$$

$$k_2^\mu \to k_2^\mu - \frac{z}{2}\langle k_1^-|\gamma^\mu|k_2^-\rangle, \qquad A^{\text{tree}}_{\text{gauge}}(z) \sim \frac{1}{z}$$

$$z \longrightarrow \infty$$



Property useful for construction of BCFW recursion relations for gravity,

See Nima's talk

Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrcek;
Benincasa, Boucher-Veronneau , Cachazo; Arkani-Hamed, Kaplan; Hall

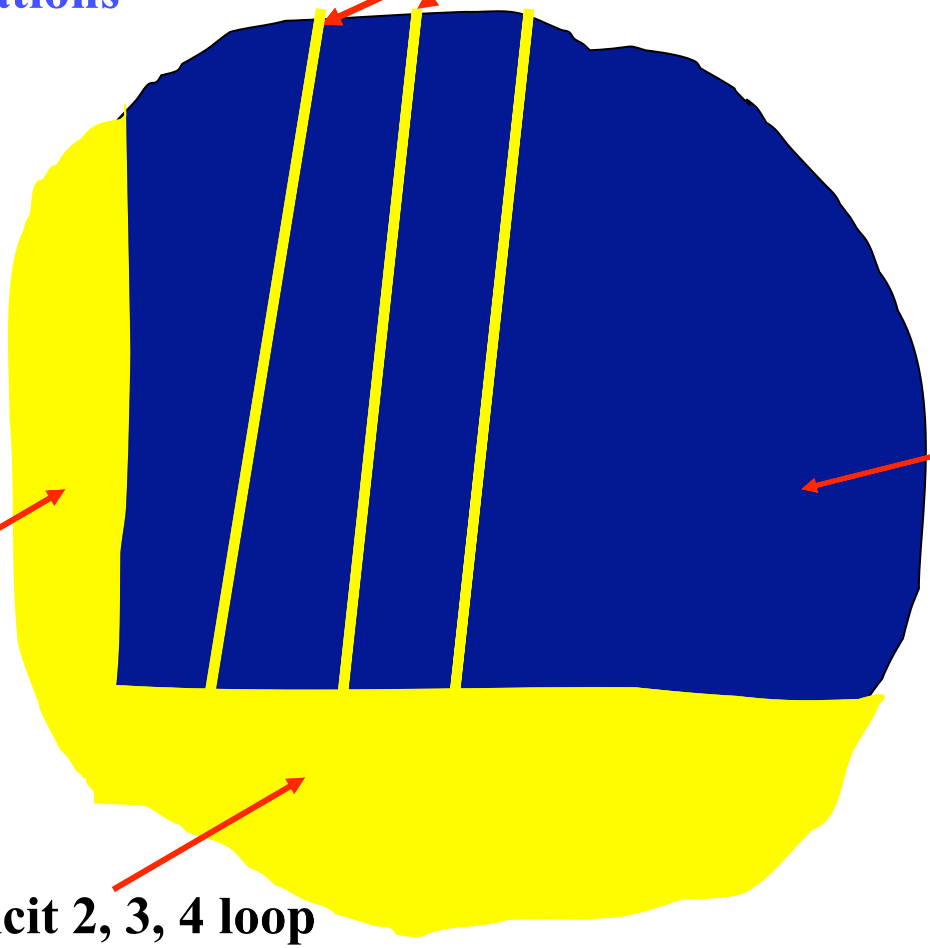
Claim: Same property is directly related to the novel UV cancellations observed in loops. ZB, Carrasco, Forde, Ita, Johansson

Can we prove perturbative finiteness of *N* = 8 supergravity?
Time will tell… but it gets less divergent every year!

Schematic Illustration of Status

Same power count as N=4 super-Yang-Mills

UV behavior unknown



Summary

Scattering amplitudes have a surprising simplicity and rich structure. Remarkable progress in a broad range of topics: AdS/CFT, quantum gravity and LHC physics.

- ***N*=4 supersymmetric gauge theory:**
  - — **Scattering amplitudes open an exciting new venue for studying Maldacena's AdS/CFT conjecture.**
  - — **Examples valid to all loop orders, matching strong coupling!**
  - — **Can we repair BDS conjecture at 6 points and beyond?**
  - — **New symmetries. Dual conformal invariance and Yangians.**
- **Quantum gravity: Surprisingly simple structures emerge.**
  - — **Gravity as the "square" of gauge theory.**
  - — **Is a point-like perturbatively UV finite quantum gravity theory possible? Explicit four-loop evidence!**
  - — **Better descriptions? Holographic description.** **(see Nima's talk)**

Expect many more exciting developments in scattering amplitudes in the coming years.

Extra

Where are the $N = 8$ $D = 4$ Divergences?

Depends on whom you ask and when you ask.

Howe and Lindstrom (1981)
Green, Schwarz and Brink (1982)
Howe and Stelle (1989)
Marcus and Sagnotti (1985)

3 loops: Conventional superspace power counting.

5 loops: Partial analysis of unitarity cuts.

ZB, Dixon, Dunbar, Perelstein, and Rozowsky (1998)

If harmonic superspace with $N = 6$ susy manifest exists.

algebraic susy arguments.

Howe and Stelle (2003)
Bossard, Howe and Stelle (2009)

6 loops: *If* harmonic superspace with $N = 7$ susy manifest exists

Howe and Stelle (2003)

7 loops: If a superspace with $N = 8$ susy manifest were to exist.

Light cone superspace non-locality

Grisaru and Siegel (1982)
Kallosh (2009)

8 loops: Explicit identification of potential susy invariant counterterm with full non-linear susy.

Kallosh; Howe and Lindstrom (1981)

9 loops: *Assume* Berkovits' superstring non-renormalization theorems can be naively carried over to $N = 8$ supergravity. Also need to extrapolate. Should be done in field theory! Superspace gets here with additional speculations.

Green, Vanhove, Russo (2006)
Stelle (2006)

Note: none of these are based on demonstrating a divergence. They are based on arguing susy protection runs out after some point.