More Zeta Functions for the Riemann Zeros

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Summary. Another family of generalized zeta functions built over the Riemann zeros $\{\rho\}$, namely $\mathscr{Z}(s,x) = \sum_\rho (x-\rho)^{-s}$, has its analytic properties and (countably many) special values listed in explicit detail.

This work is a partial expansion of our first paper [20] on zeta functions built over the *Riemann zeros* $\{\rho\}$, i.e., the nontrivial zeros of the Riemann zeta function $\zeta(s)$. While our oral presentation was more introductory, here we will pursue a fully parallel treatment, begun in [20], for *two* such generalized (i.e., parametric) zeta functions:

$$\mathcal{Z}(\sigma, v) \stackrel{\rm def}{=} \sum_{k=1}^{\infty} ({\tau_k}^2 + v)^{-\sigma} \qquad \text{(and } \mathcal{Z}(\sigma) \stackrel{\rm def}{=} \mathcal{Z}(\sigma, 0)), \quad (1)$$

$$\mathscr{Z}(s,x) \stackrel{\rm def}{=} \sum_\rho (x-\rho)^{-s} \equiv \sum_\rho (\rho+x-1)^{-s} \quad \text{(and } \mathscr{Z}(s) \stackrel{\rm def}{=} \mathcal{Z}(s,1)), \quad (2)$$

$$\text{where } \{\rho\} = \{\tfrac{1}{2} \pm \mathrm{i}\tau_k\}_{k=1,2,\ldots} = \{\text{the Riemann zeros}\} \quad (3)$$

(or, in a latest extension, the zeros of arithmetic zeta or L-functions [21]).

The two families (1) and (2) are truly inequivalent except for one function,

$$\mathcal{Z}(\sigma, 0) \equiv \mathcal{Z}(\sigma) \equiv (2\cos\pi\sigma)^{-1}\, \mathscr{Z}(2\sigma, \tfrac{1}{2}), \quad (4)$$

already considered in [8, Sect. 4 ex. (A)], [4]. Other previous results appear in [11, 15] for the functions $\mathcal{Z}(\sigma, \frac{1}{4})$, in [5, 18] for the family $\{\mathscr{Z}\}$ and earlier [16, 12, 10] for the specific sums $\mathscr{Z}(n) \equiv \sum_\rho \rho^{-n}$ (often denoted σ_n).

In [20], we mainly strived at exhausting explicit results for the family (1), handling the family (2) in lesser detail. The present work will in turn provide a *thoroughly explicit* description for the family (2), in parallel to (1), but now based on a *parametric* analytical-continuation formula, (42). At the same

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time we will switch from a Hadamard to a zeta-regularized product formalism, definitely simpler for the family (2). This zeta-regularization technique is adapted from spectral theory and quantum mechanics, where it serves to define *spectral* (or *functional*) *determinants* [19, 17]. However, our analysis remains wholly decoupled from any actual spectral meaning whatsoever for the Riemann zeros.

We recapitulate the results of [20] in Sect. 1, but refer to that article for further details. We basically keep the same notations, with (2) subsuming the main few changes: the second family used in [20] was $\xi(s,x) \equiv (2\pi)^s \mathscr{Z}(s,x)$, and $\mathscr{Z}(n) = \sum_\rho \rho^{-n}$ was formerly $\mathscr{Z}_n$; we also slightly renormalize the function called $\mathbf{D}$, cf. (38) below. The other essential notations are [1, 7, 3, 6]:

$$\begin{Bmatrix} B_n \\ E_n \end{Bmatrix} : \begin{Bmatrix} \text{Bernoulli} \\ \text{Euler} \end{Bmatrix} \text{ numbers;} \qquad B_n(\cdot) : \text{Bernoulli polynomials;}$$

$$\gamma: \text{ Euler's constant;} \quad \gamma^{\rm c}_{n-1} : \text{“Stieltjes } \textit{cumulants}\text{”, defined by:} \tag{5}$$

$$\log\left[s\,\zeta(1+s)\right] \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!}\,\gamma^{\rm c}_{n-1}\, s^n \quad (e.g.,\ \gamma^{\rm c}_0 = \gamma)\,;$$

the $\gamma^{\rm c}_{n-1}$ are cumulants [20] for the more classic *Stieltjes constants* γ_{n-1} [1, 12]; see also $\eta_{n-1} \equiv (-1)^n n\, \gamma^{\rm c}_{n-1}/(n-1)!$ in [2] – notations are not standardized (the so denoted constants and cumulants all truly have degree n, anyway);

$$\varXi(s) \stackrel{\rm def}{=} s(s-1)\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s), \tag{6}$$

which is an entire function, even under $s \longleftrightarrow (1-s)$, normalized to $\varXi(0) = \varXi(1) = 1$, and only keeping the nontrivial zeros of $\zeta(s)$;

$$\beta(s) \stackrel{\rm def}{=} \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s} : \text{the Dirichlet } \beta\text{-function,} \tag{7}$$

which is a particular L-series of period 4;

$$\zeta(s,a) \stackrel{\rm def}{=} \sum_{n=0}^{\infty} (n+a)^{-s} : \text{the Hurwitz zeta function,} \tag{8}$$

which has a single pole at $s=1$, of polar part $1/(s-1)$, and the special values

$$\zeta(-m,a) = -B_{m+1}(a)/(m+1) \quad (m \in \mathbb{N}), \quad (e.g.,\ \zeta(0,a) = \tfrac{1}{2} - a) \tag{9}$$

$$\mathrm{FP}_{s=1}\,\zeta(s,a) = -\,\Gamma'(a)/\,\Gamma(a) \qquad (\mathrm{FP} \stackrel{\rm def}{=} \text{finite part at a pole}) \tag{10}$$

$$\zeta'(0,a) = \log\left[\Gamma(a)/(2\pi)^{1/2}\right]; \tag{11}$$

upon generalized zeta functions as in (1), (2), (11), $'$ will always mean differentiation with respect to the principal variable: the exponent, s or σ.

1 Summary of previous results

1.1 Zeta functions and zeta-regularized products

We first recall some needed results on zeta and infinite-product functions built over certain abstract numerical sequences $\{x_k\}_{k=1,2,\ldots}$ ($0 < x_1 \le x_2 \le \cdots$, $x_k \uparrow +\infty$ as in [19]; or $x_k \in \mathbb{C}^*$ with $|x_k| \uparrow \infty$, $|\arg x_k|$ sufficiently bounded as in [17, 14, 9]). Such a sequence is deemed *admissible of order* μ_0 for some $\mu_0 \in (0, +\infty)$ if, essentially, the series

$$Z(s) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} {x_k}^{-s} \quad \textit{converges in } \{\mathrm{Re}\ s > \mu_0\}, \tag{12}$$

and this *zeta function* $Z(s)$ (analytic for Re $s > \mu_0$) admits a *meromorphic extension to the whole s-plane*, with poles lying in a real sequence $\mu_0 > \mu_1 > \cdots$ ($\mu_n \downarrow -\infty$). The smaller details are better fine-tuned to each context: thus, the zeta functions $\mathcal{Z}$ in (1) could be treated earlier [20] using a very low order $\mu_0 < 1$ but *double* poles, which are handled in [14, 9]; now, the functions $\mathscr{Z}$ in (2) will require $\mu_0 = 1$ but only *simple* poles, as in [19, 17].

As a consequence of (12), the Weierstrass infinite product

$$\Delta(x) \stackrel{\text{def}}{=} \prod_{k=1}^{\infty} \Big(1 + \frac{x}{x_k}\Big) \exp\Bigg\{ \sum_{1 \le m \le \mu_0} \frac{1}{m} \Big(-\frac{x}{x_k}\Big)^m \Bigg\} \tag{13}$$

converges $\forall x \in \mathbb{C}$, to an entire function. In the context of the Riemann zeros, the above meromorphic continuation requirements for $Z(s)$ are more easily enforced through a *controlled large-x behavior of* $\log \Delta(x)$ [20]; here we impose

$$\log \Delta(x) \sim \sum_{n=0}^{\infty} (\tilde a_{\mu_n} \log x + a_{\mu_n}) x^{\mu_n} \qquad (x \to \infty,\ |\arg x| < \theta) \tag{14}$$

uniformly in x for some $\theta > 0$, with $\tilde a_{\mu_n} \ne 0$ only for the (finitely many) $\mu_n \in \mathbb{N}$ [19]: this will fit the family $\{\mathscr{Z}\}$, which only features simple poles (any $\mu_n \notin \mathbb{N}$ with $\tilde a_{\mu_n} \ne 0$ would give a *double* pole).

At the same time, $\log \Delta(x)$ has a specially simple *Taylor series at* $x = 0$:

$$-\log \Delta(x) = \sum_{m > \mu_0} \frac{Z(m)}{m} (-x)^m \quad (\text{converging for } |x| < \inf_k |x_k|) \tag{15}$$

$$= O(|x|^{m_0}) \quad \text{for} \quad m_0 \stackrel{\text{def}}{=} \text{the least integer} > \mu_0 .$$

The latter bound and (14) allow these Mellin representations for $Z(s)$:

$$\frac{\pi}{s \sin \pi s} Z(s) = \int_0^{\infty} \log \Delta(y)\ y^{-s-1} \mathrm{d}y \qquad (\mu_0 < \mathrm{Re}\ s < m_0) \tag{16}$$

$$\equiv \cdots \equiv \frac{(-1)^{m_0}\, \Gamma(-s)}{\Gamma(m_0 - s)} \int_0^{\infty} (\log \Delta)^{(m_0)}(y)\ y^{m_0 - s - 1} \mathrm{d}y. \tag{17}$$

[Proof: equations (16) and (17) are equivalent through integrations by parts; now to verify (17), expand $(\log \Delta)^{(m_0)}(y) = (-1)^{m_0-1}(m_0-1)! \sum_k (y+x_k)^{-m_0}$ and integrate term by term.] Then, repeated integrations by parts, as in [20, Sect. 2.2 and App. A] but pushed further, likewise imply that $Z(s)$ is meromorphic in $\mathbb{C}$, with poles lying in the sequence $\{\mu_n\}$ and polar parts

$$Z(\mu_n + \varepsilon) = \mu_n \big[\pi^{-1} \sin \pi\mu_n \ a_{\mu_n} + \cos \pi\mu_n \ \tilde{a}_{\mu_n}\big]\, \varepsilon^{-1} \ + O(1)_{\varepsilon \to 0}\,, \qquad (18)$$

by specializing formula (23) in [20]. Thus for $Z(s)$, all the poles are *simple*, and $s = 0$ is a *regular* point (as well as all points $s \in -\mathbb{N}$).

All previous results transfer to shifted admissible sequences $\{x + x_k\}$ up to reasonable limitations on the shift parameter x (e.g., $(x + x_k) \notin \mathbb{R}^- \ \forall k$), and hence to the *generalized zeta function* $Z(s, x) \stackrel{\text{def}}{=} \sum_k (x + x_k)^{-s}$. Then, the *zeta-regularized product* $D(x)$ (formally " $\prod_k (x + x_k)$ ") can be defined as

$$D(x) \stackrel{\text{def}}{=} \exp\,[-Z'(0, x)] \qquad \text{(recalling that } ' \equiv \partial/\partial s, \text{ as in (11))}. \qquad (19)$$

It can also be uniquely characterized in several concrete ways [19]. On the one hand, it relates to $\Delta(x)$ through a definite multiplicative factor, trivial in the sense that $D(x)$ stays entire and keeps the same zeros (and order) as $\Delta(x)$:

$$D(x) \equiv \exp\Big[-Z'(0) - \sum_{1 \le m \le \mu_0} \frac{Z_m}{m} (-x)^m\Big]\ \Delta(x), \qquad (20)$$

$$\text{with} \quad Z_1 = \mathrm{FP}_{s=1} Z(s) \quad \text{(finite part)} \qquad (21)$$

$$\text{and} \quad Z_m = Z(m) \qquad \text{if } Z(s) \text{ is regular at } m,$$

otherwise Z_m ($m \ge 2$) is more contrived [19, eq. (4.12)] but unneeded when $\mu_0 = 1$; in which case (15), (20) and (21) finally simplify to

$$-\log D(x) \equiv Z'(0) - [\mathrm{FP}_{s=1} Z(s)]\, x - \log \Delta(x) \qquad (22)$$

$$= Z'(0) - [\mathrm{FP}_{s=1} Z(s)]\, x + \sum_{m=2}^{\infty} \frac{Z(m)}{m} (-x)^m \quad (|x| < \inf_k |x_k|). \qquad (23)$$

On the other hand, $\log D(x)$ has a characteristic large-x asymptotic behavior as well: a *generalized Stirling expansion*, of a very specific or "canonical" form,

$$-\log D(x) \sim \sum_{n=0}^{\infty} \hat{a}_{\mu_n} \{x^{\mu_n}\} \qquad (x \to +\infty), \qquad (24)$$

$$\text{where} \quad \{x^{\mu_n}\} = x^{\mu_n} \qquad \text{for } \mu_n \notin \mathbb{N}$$

$$\{x^{\mu_n}\} = x^{\mu_n} (\log x - C_{\mu_n}) \quad \text{for } \mu_n \in \mathbb{N}, \quad C_0 = 0, \ C_1 = 1$$

(higher C_m [19, eq. (5.1)] are again unneeded when $\mu_0 = 1$); conversely, the constrained form of expansion (14) for $\log \Delta(x)$ is implied by (20) and (24).

A basic feature of the zeta-regularized product prescription is, by construction, its full *invariance under pure translations* $\{x_k\} \mapsto \{x_k + y\}$ (but under no other change of variables in general). As an application, we now express $Z(s,x)$ as a Mellin transform over $\log D$. First, for integer $m > \mu_0$, the formulae (15), (20), (21) shifted by y yield

$$Z(m,y) = \sum_k (y+x_k)^{-m} \equiv -\frac{1}{(m-1)!}\Big(-\frac{\mathrm{d}}{\mathrm{d}y}\Big)^m \log D(y); \tag{25}$$

whereas for $m = 1$, they yield the *finite part* value

$$\mathrm{FP}_{s=1} Z(s,y) = (\log D)'(y). \tag{26}$$

Then, since $(\log D)^{(m)} \equiv (\log \varDelta)^{(m)}$ for $m > \mu_0$ by (20), it follows that (25) can be substituted into (17) shifted by x, giving

$$Z(s,x) = \frac{(m_0-1)!}{\varGamma(s)\,\varGamma(m_0-s)} \int_0^\infty Z(m_0, x+y)\, y^{m_0-s-1} \mathrm{d}y \qquad (\mu_0 < \mathrm{Re}\ s < m_0), \tag{27}$$

Remarks: – (27) actually defines an extension of (25) to $m \equiv s$ no longer an integer; – the rightmost pole of $Z(s,x)$ remains $s = \mu_0$ for any x.

The above results will be invoked later for $\mu_0 = 1$, hence $m_0 = 2$; except that we will actually need a formula analogous to (27) but *for some* Re $s < 1$: this just requires a couple of integrations by parts upon (27), as

$$Z(s,x) = \frac{\sin \pi s}{\pi(1-s)} \int_0^\infty Z(2, x+y)\, y^{1-s} \mathrm{d}y \qquad (1 < \mathrm{Re}\ s < 2) \tag{28}$$

$$= -\frac{\sin \pi s}{\pi(1-s)^2} \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}y}[yZ(2,x+y)]y^{1-s} \mathrm{d}y \quad (0 < \mathrm{Re}\ s < 2) \tag{29}$$

$$= \frac{\sin \pi s}{\pi(1-s)} \int_0^\infty \tilde{Z}_x(2,y)\, y^{1-s} \mathrm{d}y \qquad (0 < \mathrm{Re}\ s < 1), \tag{30}$$

$$\text{with} \quad y\tilde{Z}_x(2,y) \stackrel{\mathrm{def}}{=} yZ(2,x+y) + \tilde{a}_1 \qquad (\text{vanishing at } y = +\infty). \tag{31}$$

1.2 The family $\{\mathcal{Z}(\sigma, v)\}$

We make a digression to recall earlier formulae for that family [20]. The main primary result was the integral representation (72) therein for $\mathcal{Z}(\sigma) \equiv \mathcal{Z}(\sigma, 0)$,

$$\mathcal{Z}(\sigma) \;=\; \frac{-\mathbf{Z}(2\sigma) + 2^{2\sigma}\mathrm{e}^{\mp 2\pi \mathrm{i}\sigma}}{2\cos \pi\sigma} + \frac{\sin \pi\sigma}{\pi} \int_0^{+\mathrm{e}^{\pm \mathrm{i}\varepsilon}\infty} t^{-2\sigma} \frac{\zeta'}{\zeta}\big(\tfrac12 + t\big)\, \mathrm{d}t, \tag{32}$$

$$\mathbf{Z}(2\sigma) \stackrel{\mathrm{def}}{=} \mathbf{Z}(2\sigma, \tfrac12), \qquad \text{where} \tag{33}$$

$$\mathbf{Z}(s,x) \stackrel{\mathrm{def}}{=} \sum_{k=1}^{\infty} (x+2k)^{-s} \equiv 2^{-s}\zeta(s, 1+x/2) \tag{34}$$

(admitting real variants [20, eqs. (73)–(74)]); $\mathbf{Z}(2\sigma)$ is the "shadow" zeta function of $\mathcal{Z}(\sigma)$, i.e., the precise counterpart of $\mathcal{Z}(\sigma)$ for the *trivial* zeros of $\zeta(s)$; the more general form $\mathbf{Z}(s,x)$ will enter in Sect. 2, and (34) writes it as a variant of the Hurwitz zeta function (8).

As shown in [20], (32) supplies an explicit analytical continuation of $\mathcal{Z}(\sigma)$ to a meromorphic function in the whole complex σ-plane, plus exhaustive explicit results and special values for $\mathcal{Z}(\sigma)$. Their extension to the full family $\{\mathcal{Z}(\sigma,v)\}$ then follows using the expansion formula (91) in [20], i.e.,

$$\mathcal{Z}(\sigma,v) = \sum_{\ell=0}^{\infty} \frac{\Gamma(1-\sigma)}{\ell!\,\Gamma(1-\sigma-\ell)}\,\mathcal{Z}(\sigma+\ell)\,v^{\ell} \qquad (|v|<{\tau_1}^2). \tag{35}$$

The following explicit formulae for $\{\mathcal{Z}(\sigma,v)\}$ resulted [20].
a) the full polar parts (of order 2):

$$\mathcal{Z}(\tfrac12 - n + \varepsilon, v) = \frac{1}{8\pi}\frac{\Gamma(n+\frac12)}{n!\,\Gamma(\frac12)} v^n\,\varepsilon^{-2} + \mathcal{R}_n(v)\,\varepsilon^{-1} + O(1)_{\varepsilon\to 0} \quad \text{for } n\in\mathbb{N},$$

$$\text{with}\quad \mathcal{R}_n(v) = -\frac{\Gamma(n+1/2)}{n!\,\Gamma(1/2)}\Big[\frac{1}{4\pi}\sum_{j=1}^{n}\frac{1}{2j-1} + \frac{\log 2\pi}{4\pi}\Big]v^n + \sum_{j=1}^{n}\frac{\Gamma(n+1/2)}{(n-j)!\,\Gamma(j+1/2)}\Big[\frac{(-1)^j}{8\pi j}(1-2^{1-2j})B_{2j}\Big]\,v^{n-j}; \tag{36}$$

b) special values at integer σ, compiled in Table 1; these evaluations can still be pushed further ([20, Table 1] for $v=0$ and $\frac14$; [21, Table 2] for general v).

σ	$\mathcal{Z}(\sigma,v) = \sum_{k=1}^{\infty}({\tau_k}^2+v)^{-\sigma}$
$-m \le 0$	$\sum_{j=0}^{m}\binom{m}{j}(-1)^j\,2^{-2j}(1-\frac18 E_{2j})\,v^{m-j}$
0	$7/8$
derivative at 0	$\mathcal{Z}'(0,v) = \frac14\log 8\pi - \log\Xi(\frac12 \pm v^{1/2})$
$+m \ge 1$	$\frac{(-1)^{m-1}}{(m-1)!}\frac{\mathrm{d}^m}{\mathrm{d}v^m}\log\Xi(\frac12 \pm v^{1/2})$

Table 1. Special values of $\mathcal{Z}(\sigma,v)$ (upper half: algebraic, lower half: transcendental [20, Sect. 4]). Notations: see (5)–(6); m is an integer.

Finally, as an extra result (useful for comparison with (45) below), we now recast the Hadamard product for the Riemann zeta function,

$$\zeta(x) = \frac{\exp\,(\log 2\pi - 1 - \gamma\,/2)x}{2(x-1)\,\Gamma(1+x/2)}\prod_{\rho}(1-x/\rho)\mathrm{e}^{x/\rho}, \tag{37}$$

in terms of *zeta-regularized* factors related to $\mathcal{Z}(\sigma, v)$.

First, the zeta-regularized product underlying the Gamma factor for the trivial zeros, $\Gamma(1+x/2)^{-1}$, i.e. the spectral determinant $\mathbf{D}(x)$ for the sequence $\{2k\}_{k=1,2,\ldots}$, can be specified using (19), (34), (9) and (11), as

$$\mathbf{D}(x) = \exp\left[-\mathbf{Z}'(0,x)\right] = 2^{-x/2}\pi^{1/2} / \, \Gamma(1+x/2) \tag{38}$$

(warning: the determinant called $\mathbf{D}$ in [20] was normalized differently). Check: $\log \mathbf{D}(x)$ has a large-x asymptotic behavior of the *canonical* form (24) for the order $\mu_0 = 1$ (this also being the order of the entire function $\Gamma(1+x/2)^{-1}$),

$$\log \mathbf{D}(x) \sim -\tfrac{1}{2}x(\log x - 1) - \tfrac{1}{2}\log x \quad \Big[+\sum_{1}^{\infty} c_n x^{-n}\Big]. \tag{39}$$

The other factor in (37) essentially contains the function $\Xi(x)$ of (6): it can be related to the zeta-regularized product $\mathcal{D}(v)$ for the sequence $\{{\tau_k}^2\}$, which is admissible of order $\mu_0 = \frac{1}{2}$ [20], through (cf. Table 1)

$$\mathcal{D}(v) = \exp\left[-\,\mathcal{Z}'(0,v)\right] = (8\pi)^{-1/4}\,\Xi(\tfrac{1}{2} + v^{1/2}). \tag{40}$$

The factorization formula (37) thus admits a zeta-regularized form as

$$\zeta(\tfrac{1}{2}+t) = (2\pi)^{t/2}\frac{\mathbf{D}(\frac{1}{2}+t)\mathcal{D}(t^2)}{t-\frac{1}{2}}\ . \tag{41}$$

This is quite analogous to an earlier decomposition of hyperbolic *Selberg zeta functions* over spectral determinants [19, eq. (7.18)]. In (41), the denominator also has the zeta-regularized normalization for an elementary factor; as for the prefactor $(2\pi)^{t/2}$, it corrects for the discrepancy between the zeta-regularizations with respect to t (as in $\mathbf{D}$) and t^2 (in $\mathcal{D}$).

2 The family $\{\mathscr{Z}(s,x)\}$: analytical continuation formula

Apart from the special values $\mathscr{Z}(n)$, $n \in \mathbb{N}^*$ [16, 12, 10], functions equivalent to $\{\mathscr{Z}(s,x)\}$ of (2) were considered first (to our knowledge) by Deninger for $\mathrm{Re}\, x > 1$ [5], then proved by Schröter and Soulé [18] to be meromorphic in the whole s-plane over the larger domain $x \in \Omega \stackrel{\text{def}}{=} \{x \in \mathbb{C} \mid (x+\rho) \notin \mathbb{R}^- \ (\forall \rho)\}$.

2.1 The primary result

For the family (2), the “shadow” zeta function over the trivial zeros (definable just as before, but now with x as second argument) is just the function $\mathbf{Z}(s,x)$ of (34). It governs an integral representation for $\mathscr{Z}(s,x)$ similar to (32), but simpler and *now available for all* $x \in \Omega \setminus (-\infty, +1]$:

$$\mathscr{Z}(s,x) = -\mathbf{Z}(s,x) + \frac{1}{(x-1)^s} + \frac{\sin \pi s}{\pi} \int_0^\infty \frac{\zeta'}{\zeta}(x+y)\, y^{-s} \mathrm{d}y \qquad (\mathrm{Re}\ s < 1); \tag{42}$$

here, $(x-1)^s$ is given its standard determination in $\mathbb{C} \setminus (-\infty, +1]$; *this* cut is not a singularity for $\mathscr{Z}(s,x)$, indeed the discontinuities across it of the three right-hand side terms in (42) can be seen to precisely cancel out when added.

Alternative real forms can be built; a very simple one for real $x > -2$ is

$$\mathscr{Z}(s,x) = -\mathbf{Z}(s,x) + \frac{\sin \pi s}{\pi} \int_0^\infty \Big[\frac{\zeta'}{\zeta}(x+y) + \frac{1}{x+y-1}\Big] y^{-s} \mathrm{d}y \qquad (0 < \mathrm{Re}\ s < 1); \tag{43}$$

this form only converges in the stated s-plane strip, but contrary to (42), it enjoys a well defined $x \to +1$ limit:

$$\mathscr{Z}(s)\ (\equiv \mathscr{Z}(s,1)) = 1-(1-2^{-s})\,\zeta(s) + \frac{\sin \pi s}{\pi} \int_0^\infty \Big[\frac{\zeta'}{\zeta}(1+y) + \frac{1}{y}\Big] y^{-s} \mathrm{d}y. \tag{44}$$

Equation (42) (plus (43) for x real) is the new basic result here, extending an earlier formula by Deninger valid only for $\mathrm{Re}\, x > 1$ [5, p. 149]. It is a genuine analog for $\mathscr{Z}(s,x)$ to the Joncquière–Lerch functional relation for $\zeta(s,a)$ [7, Sect. 1.11 (16)], itself generalizing the functional equation of $\zeta(s)$. At $x = \frac{1}{2}$, (42) also restores our previous formula (32) for $\mathcal{Z}(\sigma)$ by virtue of the relation (4). Every explicit consequence that (32) implied for $\mathcal{Z}(\sigma)$ alone will extend here *to the whole family* $\mathscr{Z}(s,x)$ solely by (42).

2.2 Derivation of the main formula (42)

As a preliminary step, we transform the Hadamard product (37) for $\zeta(s)$ into a zeta-regularized factorization even simpler than (41).

We now just factor out the previous "shadow" determinant $\mathbf{D}(x)$, as

$$\zeta(x) \equiv \frac{\mathbf{D}(x)\mathscr{D}(x)}{x-1}, \tag{45}$$

$$\mathscr{D}(x) = (x-1)\, 2^{x/2} \pi^{-1/2}\, \Gamma(1+x/2)\zeta(x) \equiv \tfrac{1}{2}\pi^{-1/2}(2\pi)^{x/2}\, \Xi(x). \tag{46}$$

We can then anticipate that this factor $\mathscr{D}(x)$ must be a *zeta-regularized product in* x over the Riemann zeros $\{\rho\}$. Indeed, $\mathscr{D}(x)$ has precisely the $\{\rho\}$ as zeros, *and* $\log \mathscr{D}(x)$ has a large-x expansion of the canonical form (24) in x because all other factors present in (45) have that property. We will accordingly confirm that $\log \mathscr{D}(x) \equiv -\mathscr{Z}'(0,x)$ below: see (54), and earlier in a variant form, [5, thm 3.3] (for $\mathrm{Re}\ x > 1$) and [18] (for general x).

Now to prove (42), we specialize the results of Sect. 1.1 to the sequences $\{-\rho\}$ and $\{2k\}$: both of these *are admissible* of order $\mu_0 = 1$ [5, 18, 9], mainly because $\log \mathbf{D}(x)$, and hence $\log \mathscr{D}(x)$, comply with (14) (cf. (39), then (45)).

Specifically here, the factorization (45), together with (25) at $m = 2$ (first with $Z = \mathscr{Z}$, then with $Z = \mathbf{Z}$) and with (31), entail

$$\tilde{\mathscr{Z}}_x(2,y) \equiv -\tilde{\mathbf{Z}}_x(2,y) + (x+y-1)^{-2} - [\zeta'/\zeta]'(x+y) \tag{47}$$

$$\text{with} \quad \tilde{\mathscr{Z}}_x(2,y) \equiv \mathscr{Z}(2,x+y) + \frac{1}{2y}, \quad \tilde{\mathbf{Z}}_x(2,y) \equiv \mathbf{Z}(2,x+y) - \frac{1}{2y}\,; \tag{48}$$

the last line comes from generalized Stirling expansions for $\mathscr{D}$ and $\mathbf{D}$, cf. (39). Upon the specific decomposition (47), it is allowed to apply the Mellin transformation (30) term by term on both sides, at fixed $x \in \Omega \setminus (-\infty, +1]$. Then, the left-hand side yields $\mathscr{Z}(s,x)$; as for the right-hand side, the first term yields $-\mathbf{Z}(s,x)$ by exactly the same argument, the second term trivially evaluates to $(x-1)^{-s}$, and the last term can be subjected to an ultimate integration by parts now valid in the whole half-plane $\{\mathrm{Re}\ s < 1\}$, using

$$\mathscr{J}_\zeta(s,x) \stackrel{\text{def}}{=} \frac{1}{1-s}\int_0^\infty -\Big[\frac{\zeta'}{\zeta}\Big]'(x+y)\ y^{1-s}\mathrm{d}y = \int_0^\infty \frac{\zeta'}{\zeta}(x+y)\,y^{-s}\mathrm{d}y; \tag{49}$$

all that yields the desired formula (42). If the last two terms in (47) are kept together instead, (43) can be obtained likewise. The structure of the representation (42) thus clearly stems from the simple factorization formula (45).

3 Explicit consequences for the family $\{\mathscr{Z}(s,x)\}$

3.1 Analytical results (in the s-variable)

First, (42) gives an explicit one-step analytical continuation of $\mathscr{Z}(s,x)$ to the half-plane $\{\mathrm{Re}\ s < 1\}$. It also implies its analytical continuation in s to all of $\mathbb{C} \setminus \{1\}$, since the Mellin transform $\mathscr{J}_\zeta(s,x)$ of (49) is seen (through repeated integrations by parts, using $[\log\zeta]^{(n)}(x) = o(x^{-N})_{x\to+\infty}\ \forall n, N$) to be meromorphic in the whole s-plane, and to have only simple poles at $s = 1, 2, \ldots$ with residues

$$\mathrm{Res}_{s=n}\mathscr{J}_\zeta(s,x) = -[\log|\zeta|]^{(n)}(x)/(n-1)! \qquad (x \neq 1), \quad n = 1, 2, \ldots \tag{50}$$

(the singularity at $x=1$ is harmless: see after (58), and left part of Table 3). At fixed x, (42) and (50) imply that $\mathscr{Z}(s,x)$ acquires its polar structure solely from $-\mathbf{Z}(s,x)$: it thus has the only pole $s=1$, of polar part $-\frac{1}{2}/(s-1)$ [18].

Still for fixed x, the mere substitution into (42) of the standard Dirichlet series

$$\frac{\zeta'}{\zeta}(z) = -\sum_{n\ge2}\frac{\Lambda(n)}{n^z} \quad (\Lambda(n) \stackrel{\text{def}}{=} \log p \text{ if } n = p^r \text{ for some prime } p, \text{ else } 0) \tag{51}$$

for $z = x+y$, followed by term-by-term y-integration, yields

$$\mathscr{Z}(s,x) + \mathbf{Z}(s,x) - (x-1)^{-s} \sim -\frac{1}{\Gamma(s)}\sum_{n\ge2}\frac{\Lambda(n)}{n^x}(\log n)^{s-1}. \tag{52}$$

The summation in the right-hand side of (52) converges iff the Dirichlet series (51) converges uniformly for $y > 0$: i.e., for $\mathrm{Re}\, x > 1$, where (52) becomes an *identity* – written in [5, p. 148], but it is just a particular case of *Weil's explicit formula*, or equivalently of equation (1.1) in [8], again provided $\mathrm{Re}\, x > 1$. Here, by contrast, (52) is meant for *general* fixed x, albeit only as an *asymptotic expansion* (for $s \to -\infty$) if $\mathrm{Re}\, x \le 1$.

3.2 Special values for general x

Finally, (42) outputs all the special values of $\mathscr{Z}(s,x)$ just by inspection:

$$\mathscr{Z}(-n,x) = -2^n \zeta(-n, 1+x/2) + (x-1)^n \qquad (n \in \mathbb{N}), \quad (53)$$

$$\mathscr{Z}'(0,x) = -\tfrac{1}{2}(\log 2)x + \tfrac{1}{2}\log \pi - \log \Gamma(1+x/2) - \log[(x-1)\zeta(x)]$$

$$\equiv -\log \mathscr{D}(x), \quad (54)$$

$$\mathrm{FP}_{s=1}\, \mathscr{Z}(s,x) = \frac{1}{2}\Big(\log 2 + \frac{\Gamma'}{\Gamma}(1+x/2)\Big) + \Big[\frac{1}{x-1} + \frac{\zeta'}{\zeta}(x)\Big] \quad (55)$$

$$\equiv (\log \mathscr{D})'(x), \quad (56)$$

$$\mathscr{Z}(+n,x) = -2^{-n}\zeta(n, 1+x/2)$$

$$+\Big[(x-1)^{-n} - \frac{(-1)^n}{(n-1)!}[\log|\zeta|]^{(n)}(x)\Big] \quad (n = 2,3,\ldots) \quad (57)$$

$$\equiv \frac{(-1)^{n-1}}{(n-1)!}(\log \mathscr{D})^{(n)}(x) \qquad (n = 2,3,\ldots) \quad (58)$$

using (9)–(11), (34), (46), (50); the quantities in square brackets are apparently singular for $x = +1$ but globally extend there by continuity, using the expansion (5) with the Stieltjes cumulants.

In particular, (54) confirms that the factor $\mathscr{D}(x)$ in (45) *is the zeta-regularized product* in x over the sequence of Riemann zeros $\{\rho\}$ – the argument is not circular, because our derivation of the basic formula (42) does not rely on that fact but purely on the factorization itself. Thereupon, (56), (58) simply repeat the general formulae (26), (25) respectively.

The point $s = 1$ $(= \mu_0)$ deserves extra attention. Upon logarithmic differentiation, one Hadamard product formula for $\Xi(x)$ [6, Sects. 1.10, 2.8] directly yields

$$\Xi(x) = \prod_\rho (1 - x/\rho) \quad \Longrightarrow \quad \mathscr{Z}(1,x) \stackrel{\mathrm{def}}{=} \sum_\rho (x-\rho)^{-1} \equiv (\log \Xi)'(x), \quad (59)$$

where both product and sum (now only *semiconvergent*) are performed with zeros grouped in symmetrical pairs, as usual. Thus, *in spite of the pole of* $\mathscr{Z}(s,x)$ at $s = 1$, (59) yields a *finite* value for $\mathscr{Z}(1,x)$, which however *differs from the finite part* (FP) *of* $\mathscr{Z}(s,x)$ *at* $s = 1$:

$$\mathscr{Z}(1,x) - \mathrm{FP}_{s=1}\, \mathscr{Z}(s,x) \equiv -\tfrac{1}{2}\log 2\pi \quad (60)$$

according to (46), (56). This fixed discrepancy can also be traced to the *nonzero residue of the double pole* in the former zeta function $\mathcal{Z}(\sigma)$, see (69) below.

The resulting special values for $\{\mathscr{Z}(s,x)\}$ are fully compiled in Table 2, in their form closest to their analogs for the family $\{\mathcal{Z}(\sigma,v)\}$ in Table 1.

s	$\mathscr{Z}(s,x) = \sum_\rho (x-\rho)^{-s}$
$-n \le 0$	$2^n B_{n+1}(1+\frac{x}{2}) \,/\, (n+1) \;+\; (x-1)^n$
0	$\frac{1}{2}(x+3)$
derivative at 0	$\mathscr{Z}'(0,x) = -\frac{1}{2}(\log 2\pi)\, x + \frac{1}{2}(\log 4\pi) - \log \Xi(x)$
finite part at +1	$\mathrm{FP}_{s=1}\, \mathscr{Z}(s,x) = \frac{1}{2}\log 2\pi + (\log \Xi)'(x)$
$+n \ge 1$	$\dfrac{(-1)^{n-1}}{(n-1)!}(\log \Xi)^{(n)}(x)$

Table 2. Special values of $\mathscr{Z}(s,x)$ (upper part: algebraic, lower part: transcendental [20, Sect. 4]); see also (53)–(62). Notations: see (6); $B_{n+1}(\cdot)$: Bernoulli polynomial; n is an integer.

Finally, we state two sets of linear identities imposed upon the values $\mathscr{Z}(n,x)$ purely by the symmetry ($\rho \longleftrightarrow 1-\rho$) in (2):

$$\mathscr{Z}(n,x) = (-1)^n\, \mathscr{Z}(n,1-x) \qquad \text{for } n=1,2,\ldots; \qquad (61)$$

$$\mathscr{Z}(k,x) = -\frac{1}{2}\sum_{\ell=k+1}^{\infty} \binom{\ell-1}{k-1} (2x-1)^{\ell-k}\, \mathscr{Z}(\ell,x) \quad \text{for each } odd\ k \ge 1; \qquad (62)$$

and the finite, triangular linear relations connecting them to the *other* special values $\mathcal{Z}(m,v)$, $m=1,2,\ldots$, at $v \equiv (x-\frac{1}{2})^2$, as derived in [21, Sect. 3.3]:

$$\mathcal{Z}(m,v) = \begin{cases} \displaystyle\sum_{\ell=0}^{m-1} \binom{m+\ell-1}{m-1} (2x-1)^{-m-\ell}\, \mathscr{Z}(m-\ell,x) & (v \ne 0) \\ \frac{1}{2}(-1)^m\, \mathscr{Z}(2m,\frac{1}{2}) & (v=0) \end{cases}$$

$$\Longleftrightarrow \quad \frac{\mathscr{Z}(n,x)}{n} = \sum_{0\le \ell \le n/2} (-1)^\ell \binom{n-\ell}{\ell} (2x-1)^{n-2\ell}\, \frac{\mathcal{Z}(n-\ell,v)}{n-\ell}\,. \qquad (63)$$

The countably many "sum rules" (62) merely result from the Taylor expansion around $x=\frac{1}{2}$ of $(x-\rho)^{-k} = (x-1+\rho)^{-k} = (-1)^k (x-\rho)^{-k}\left[1-(2x-1)/(x-\rho)\right]^{-k}$ followed by summation over the zeros ρ grouped in pairs. Equations (62) recursively allow to eliminate any *finite* subset of odd values, expressing them as series over all higher values. (In the infinite recursion limit, every odd value ends up as a formal power series around $x = \frac{1}{2}$ over the higher even values only, $\mathscr{Z}(2m+1, x) = \sum_{\ell=m+1}^{\infty} A_{m,\ell}\, \mathscr{Z}(2\ell, x)\,(2x-1)^{2\ell-(2m+1)}\ \forall m \in \mathbb{N}$, but *this* has to be a *divergent* series (exercise!); only for $x = \frac{1}{2}$ is the latter elimination fully effective, giving $\mathscr{Z}(2m+1, \frac{1}{2}) \equiv 0\ \forall m \in \mathbb{N}$, cf. (67) below.)

Remark: the previous argument expanded $\mathscr{Z}(s, y)$ about $\mathscr{Z}(s, x)$ in powers of $(y - x)$, specifically at $s = n$, $y = 1 - x$. Just like the earlier series (35) for $\mathcal{Z}(\sigma, v)$ in powers of v, such expansions could also yield results for $\mathscr{Z}(s, x)$ at general x, but the continuation formula (42) now suffices for this task.

3.3 Special values for $x = 1$ and $\frac{1}{2}$

For half-integer x, the values $\zeta(\pm m, 1 + x/2)$ which arose in (53) and (57) can be made slightly more explicit, and even more so for integer x. The most interesting cases are $x = 1$ and $\frac{1}{2}$: then, (34) implies

$$\mathbf{Z}(s, 1) \equiv (1-2^{-s})\,\zeta(s) - 1, \qquad \mathbf{Z}(s, \tfrac{1}{2}) \equiv \tfrac{1}{2}\big[(2^s - 1)\,\zeta(s) + 2^s \beta(s)\big] - 2^s, \quad (64)$$

and the resulting special values of $\mathscr{Z}(s, 1) \equiv \mathscr{Z}(s)$ and $\mathscr{Z}(s, \frac{1}{2})$ form Table 3. They display many relations with the special values of $\mathcal{Z}(\sigma, \frac{1}{4})$ and $\mathcal{Z}(\sigma, 0) \equiv \mathcal{Z}(\sigma)$ respectively [20, Table 1], as discussed next.

s	$\mathscr{Z}(s) \equiv \sum_\rho \rho^{-s}$ $[x = 1]$	$\mathscr{Z}(s, \frac{1}{2}) \equiv \sum_\rho (\rho - \frac{1}{2})^{-s}$ $[x = \frac{1}{2}]$
$-n<0$	$1 - (2^n - 1)\frac{B_{n+1}}{n+1}$	$\begin{cases} 2^{-n+1}(1 - \frac{1}{8}E_n) & n \text{ even} \\ -\frac{1}{2}(1-2^{-n})\frac{B_{n+1}}{n+1} & n \text{ odd} \end{cases}$
0	2	$7/4$
derivative at 0	$\mathscr{Z}'(0) = \frac{1}{2}\log 2$	$\mathscr{Z}'(0, \frac{1}{2}) = \log\,\big[2^{11/4}\pi^{1/2}\,\Gamma(\frac{1}{4})^{-1}\lvert\zeta(\frac{1}{2})\rvert^{-1}\big]$
finite part at +1	$\mathrm{FP}_{s=1}\,\mathscr{Z}(s) = 1 - \frac{1}{2}(\log 2 - \gamma)$	$\mathrm{FP}_{s=1}\,\mathscr{Z}(s, \frac{1}{2}) = \frac{1}{2}\log 2\pi$
$+1$	$-\frac{1}{2}\log 4\pi + 1 + \frac{1}{2}\gamma$	0
$+n>1$	$\left\{\begin{matrix} 1-(1-2^{-n})\,\zeta(n) + \frac{n}{(n-1)!}\,\gamma^{\mathrm{c}}_{n-1} \\ \equiv \\ 1-(-1)^n 2^{-n}\zeta(n) - \frac{(\log\lvert\zeta\rvert)^{(n)}(0)}{(n-1)!} \end{matrix}\right\}$	$\begin{cases} \left.\begin{matrix} -\frac{1}{2}\big[(2^n - 1)\,\zeta(n) + 2^n\beta(n)\big] \\ +2^{n+1} - \frac{1}{(n-1)!}(\log\lvert\zeta\rvert)^{(n)}(\frac{1}{2}) \end{matrix}\right\} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

Table 3. Special values of the functions $\mathscr{Z}(s, x)$ for $x = 1$ (see also (62), (63), (71)), and for $x = \frac{1}{2}$ (see also (65)–(70)). Notations: see (5); n is an integer. In the bottom line, when n is even, $\zeta(n) \equiv (2\pi)^n \lvert B_n\rvert/(2\,n!)$ while $\beta(n)$ (cf. (7)) remains elusive.

In the case $x = \frac{1}{2}$ (right-hand column), there is a 1–1 correspondence between the latter explicit results and those for $\mathcal{Z}(\sigma)$ (Sect. 1.2, and [20]), through the relation (4):

$$\mathscr{Z}(2m, \tfrac{1}{2}) = 2(-1)^m \, \mathcal{Z}(m) \qquad \text{for } m \in \mathbb{Z} \tag{65}$$

$$\mathscr{Z}(1{+}2m, \tfrac{1}{2}) = (-1)^{m+1} 2\pi \operatorname{Res}_{\sigma=\frac{1}{2}+m} \mathcal{Z}(\sigma) \qquad \text{for } m \in \mathbb{Z}^* \tag{66}$$

$$\equiv 0 \text{ for } m = 1, 2, \ldots \tag{67}$$

$$\operatorname{Res}_{s=1} \mathscr{Z}(s, \tfrac{1}{2}) = -4\pi \lim_{\varepsilon \to 0} [\varepsilon^2 \mathcal{Z}(\tfrac{1}{2} + \varepsilon)] \qquad (= -\tfrac{1}{2}) \tag{68}$$

$$\operatorname{FP}_{s=1} \mathscr{Z}(s, \tfrac{1}{2}) = -2\pi \operatorname{Res}_{\sigma=\frac{1}{2}} \mathcal{Z}(\sigma) \qquad (= \tfrac{1}{2} \log 2\pi) \tag{69}$$

$$\mathscr{Z}'(0, \tfrac{1}{2}) = \mathcal{Z}'(0). \tag{70}$$

Equation (67) otherwise immediately results from the zeros' symmetry (e.g., from (61) or (62)), *now including* $m = 0$ *as well*: i.e., $\mathscr{Z}(1, \frac{1}{2}) = 0$. This and (69) in turn imply, upon setting $x = \frac{1}{2}$ in the formula (60), that the constant discrepancy $[\mathscr{Z}(1, x) - \operatorname{FP}_{s=1} \mathscr{Z}(s, x)]$ relates to the *nonzero residue* in the double pole of $\mathcal{Z}(\sigma)$ at $\sigma = \frac{1}{2}$. The actual value $(-(4\pi)^{-1} \log 2\pi)$ of this residue [20], used in (69), was retrieved here from (36) taken at $n = 0,\ v = 0$.

In the case $x = 1$ (left-hand column), and with $n = 1, 2, \ldots$ henceforth, the special values $\mathscr{Z}(n)$ were already known: for $n = 1$, see [3, ch. 12], [6, Sec. 3.8]; for $n > 1$, we tabulate two equivalent expressions [16, 12, 20] and we refer to [12, Table 5] for numerical values. Furthermore, the $\mathscr{Z}(n)$ satisfy three sets of linear identities:
- the (infinite) sum rules (62) specialized to $x = 1$ [10, eq. (18)] [20] (remark: [10, eq. (18)] states a sum rule for every even index k as well, but this reduces to a finite linear combination of the higher odd-index sum rules that we wrote);
- the (finite, triangular) relations (63) specialized to $x = 1$, $v = \frac{1}{4}$, which then connect the $\mathscr{Z}(n)$ to the other special values $\mathcal{Z}(m, \frac{1}{4})$ [15, 20];
- a similar connection to the sequence $\lambda_n \stackrel{\text{def}}{=} \sum_\rho [1 - (1 - 1/\rho)^n]$ used by *Li's criterion* for the Riemann Hypothesis (i.e., $\lambda_n > 0\ \forall n$ [13]) [10, eq. (27)] [2, thm 2]:

$$\lambda_n = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} \mathscr{Z}(j) \quad \Longleftrightarrow \quad \mathscr{Z}(n) = \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} \lambda_j. \tag{71}$$

(Note: the λ_n of [13], used here, are n times the λ_n of [10].)

Aside from those $\mathscr{Z}(n),\ n = 1, 2, \ldots$ and $\mathscr{Z}'(0, x)$ [5, eq. (3.3.1)] [18], the values in Table 3 seem new to us. Remark: the fully explicit $\mathscr{Z}'(0)$ yields the *zeta-regularized product of all the Riemann zeros*: "$\prod_\rho \rho$" $= \mathrm{e}^{-\mathscr{Z}'(0)} = 2^{-1/2}$.

3.4 Concluding remark

Just as stated for the family (1) [20, Sect. 5.5], the whole foregoing analysis extends straightforwardly to zeros of other zeta and L-functions having functional equations similar to that of $\zeta(s)$ [21].

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