Renormalization group analysis

The theoretical physicist aims to elaborate theories at the microscopic scale, from which observed phenomena can be explained. Renormalization Group (RG) analysis allows one to determine effective theories at each length scale, from microscopic to macroscopic, by averaging over "degrees of freedom" of the previous scale: for instance, given a system defined on a lattice L of spacing a, (e.g. $\mathbb{Z}^d$ is a lattice of spacing $a = 1$ in $\mathbb{R}^d$) and a Lagrangian, or a Hamiltonian, involving an interaction between sites, a new Lagrangian is obtained on a lattice of spacing $2a$ by summing over possible values of the initial variables at each site of L, for given values of variables on the new lattice, and so forth. In some theories, e.g. few-body Newtonian mechanics, "degrees of freedom" at very different scales "decouple": as a consequence,particles (or planets in the solar system) can just be approximated by point-like objects in large scale analysis. This is no longer true in quantum field theory in particle physics, nor in the related study of phase transitions and critical phenomena in classical statistical physics, domains in which RG analysis has been mainly developed. The RG, which is more precisely a semi-group, is the set of transformations, in an infinite dimensional space of Lagrangians, that give the effective Lagrangian of each scale from those of shorter scales. In cases studied, effective Lagrangians tend to fixed points; moreover, there will exist "universality classes" such that effective Lagrangians will all tend to a common fixed point or to a set of fixed points of finite dimension. The dominant long-distance physics will then depend only on a finite number of parameters.

φ^4*-models.* We illustrate below RG ideas on the φ^4-field theories. Related physical theories are mentioned at the end. A model is defined by a probability measure $d\mu$ on the space of fields φ. Fields are functions or distributions defined on $\mathbb{R}^d$, or on a lattice L of $\mathbb{R}^d$. They have N real-valued components, $N \geq 1$. Correlation functions are the moments of $d\mu$, e.g. $\int \varphi(x_1)...\varphi(x_n)d\mu(\varphi)$ for $N = 1$. In applications, $\mathbb{R}^d$ is either the usual space or euclidean space-time. (In the latter case, real time physics follows by analytic continuation in time variables : $t \to it$.)

Given a Lagrangian $\mathcal{L}, \varphi \to \mathcal{L}(\varphi)$, $d\mu$ has the heuristic ill-defined form $Z^{-1} [\exp -\mathcal{L}(\varphi)] \Pi_{x\in\mathbb{R}^d} \, d\varphi_x$, or $Z^{-1} [\exp -\mathcal{L}(\varphi)] \Pi_{i\in L} \, d\varphi_i$. In the φ^4-model on

$\mathbb{R}^d$, for $N = 1$, $\mathcal{L}(\varphi) = \lambda \int \varphi^4(x)dx + (c/2)\int \varphi^2(x)dx + (b/2)\int (\nabla\varphi)^2(x)dx$ with integrals replaced by discrete sums (and derivatives by differences) on a lattice. At $\lambda = 0$, $c = m^2 \geq 0$, it describes a free theory of mass m. To define the model on a lattice L, one may first restrict L to a finite volume Λ of $\mathbb{R}^d$, and then consider the $\Lambda \to \mathbb{R}^d$ limit. On $\mathbb{R}^d$, one may initially consider in Λ lattices L_j with spacings $2^{-j}\ a$ and parameters λ_j, c_j, b_j, and consider the continuous $j \to \infty$ limit: as explained below, this will allow one, for $d = 2$ and 3, to define the φ^4-model and will yield an effective Lagrangian on a lattice of spacing a in Λ. The $\Lambda \to \infty$ limit is treated in turn.

φ^4-models in a finite volume Λ of $\mathbb{R}^d$

For any given j, RG analysis yields Lagrangians $\mathcal{L}_{j,k}, k = 1, 2, \ldots$, on lattices with spacings $2^{-j+k}a$, including terms in $\varphi^4, \varphi^2, (\nabla\varphi)^2$ with coefficients $\lambda_{j,k}, c_{j,k}, b_{j,k}$, *plus* (already at $k = 1$) infinitely many other terms in φ^6, φ^8... However, the latter will be "irrelevant". Given $\lambda_0 \geq 0, c_0, b_0$, it is then expected at $d = 2$ or $d = 3$ that $d\mu_j$ will be well defined in the $j \to \infty$ limit, with $\lambda_0 = \lim \lambda_{j,j}, c_0 = \lim c_{j,j}, b_0 = \lim b_{j,j}$ if λ_j, c_j, b_j are suitably chosen; λ_j and b_j remain close to λ_0 and b_0, while $c_j \to -\infty$. Results are confirmed by a rigorous analysis in which a truncated version of effective Lagrangians (involving only first terms) is used.

The theory is asymptotically free at short distances (the weight of the interaction, in $\lambda_j b_j^2 2^{j(d-4)}$, tends to 0 as $j \to \infty$).

There is no result at $d = 4$, physical dimension in particle physics; a conjecture is that only more refined "gauge" theories can then exist.

φ^4-models on a lattice ($\Lambda \to \infty$ limit)

Effective theories are obtained on lattices with spacings $2^k a, k = 1, 2, \ldots$. Two-point correlations decrease exponentially as $|x_1 - x_2| \to \infty$ in "massive" cases ($\lim c_k = m^2 > 0$), as inverse powers of $|x_1 - x_2|$ in "critical" massless cases ($\lim c_k = m^2 = 0$).

(i) There exists $c(\lambda, b)$ such that the theory is critical. For $d = 4, \lambda_k = [\beta_2 k + \beta_3 \ln k + A(\lambda)]^{-1} \to 0$ as $k \to \infty$: asymptotic freedom at large distances follows with a trivial (gaussian) fixed point and a dominant long-distance behaviour in $|x_1 - x_2|^{-(d-2)}$ as in the free $\lambda = c = 0$ theory.

For $d < 4$, and N large, a non trivial fixed point (NTFP) is obtained as $k \to \infty$, through $(1/N)$ expansions : dominant behaviour, at $\lambda > 0$, in $1/|x_1 - x_2|^{d-2+\eta(N,d)}$, with a critical exponent $\eta > 0$ independent of λ ("anomalous dimension"). These results can be proved under suitable conditions.

On the other hand, a heuristic analysis for any N can be made through ε-expansions. One considers a space dimension $4-\varepsilon$. For $\varepsilon > 0$ small, a NTFP, close to the gaussian fixed point for $d = 4$, is obtained. Results are then extended to $\epsilon = 1$ ($d = 3$) and $\epsilon = 2$ ($d = 2$). Critical indices obtained at $d = 3$ or $d = 2$ are close to experimental ones in related situations, or to those obtained from numerical calculations, or also, at $d = 2$, to exact results on the related Ising model.

(ii) For $c > c(\lambda)$, one obtains a massive gaussian fixed point.

(iii) For $c < c(\lambda)$, there is a discrete symmetry breaking at $N = 1$: according e.g. to boundary conditions on Λ, different NTFP which are mixtures of 2 pure "phases" are obtained in the $\Lambda \to \infty$ limit (symmetry then relates different solutions).

For $N \geq 2$, there is for $d > 2$ a continuous symmetry breaking: infinite number of NTFP linked by a "Goldstone boson".

Physical theories

Applications in statistical physics related to previous models include phase transitions: liquid-vapour (N=1), superfluid helium (N=2), ferromagnetic systems (N=3), and statistical properties of long polymers ("N=0"). Among other applications of RG analysis, the BCS theory of superconductivity. In particle physics, the theory of strong interactions (QCD) makes sense as an asymptotically free theory at short distances; the situation is different for theories of strong and electroweak interactions, which might be viewed as effective theories arising from a more fundamental theory at "ultrashort" distances.

Applications of RG ideas have been made in many other domains; of particular mathematical interest there have been recent studies of partial differential equations using this method.

For general references on RG analysis, see [A1][A2] and [A3] for rigorous results.

[A1] Wilson K.G., and Kogut J., *The renormalization group and the ϵ-expansion, Phys.Rep.* **12**, 75-200 (1974)

[A2] Zinn-Justin J., *Quantum Field Theory and Critical Phenomena*, 3$^{\mathrm{rd}}$ ed., Oxford Univ. Press, Oxford (1996)

[A3] Magnen J., in *XI$^{\mathrm{th}}$ International Congress of Mathematical Physics,* Iagolnitzer D. (ed.), p. 121-141, International Press, Boston (1995)