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Counting rational curves on rational surfaces

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To Vladimir Rittenberg on his sixthieth birthday

1. INTRODUCTION

Results from topological field theories such as intersections on moduli spaces or mirror symmetry for Calabi Yau threefolds allowing a classification of their curves according to genus and degree, are quite astounding. Field theory sheds a new light on classical enumerative geometry with roots in the work of Chasles, Schubert, Halphen, Zeuthen to name a few, bringing to the fore new techniques - to be justified - as well as connections with integrable systems, singularity theory, BRS-cohomology and the like.

M. Kontsevich has recently described such an application, which looks simple in that it requires little technical tools, yet remarkable in that it yields a wealth of information on genus zero curves on rational surfaces. With his permission the purpose of this note is to describe and comment on (part of) his work. For those of us who, like the author, have little familiarity with twisted - perturbed - $N = 2$ superconformal topological σ-models, it is instructive to see some arguments reduced to their backbones and how they relate to classical geometry. Therefore we will have no scrupule at recalling some elementary facts, hoping that they might be useful to some readers. For a deeper and more complete discussion of the conjectured interpretation of the computations presented below, one may be referred to some forthcoming publications by M. Kontsevich and Yu. Manin.

One is interested here in enumerating rational curves (and perhaps extending this to higher genus) on a rational surface - starting with the projective plane - required to go through a sufficient number of points - ensuring that, given some invariants such as degree (or multidegree), the number of curves be finite.

We assume our basic field to be $\mathbb{C}$. However one cannot avoid remarking some similarity between enumerating curves (over $\mathbb{C}$) and points over rational fields in other contexts, an analogy which would require elaboration (and imagination).

Rational surfaces such as the plane, the quadrics or cubics in $\mathbb{P}_3$ - to which we limit ourselves here, being more familiar to most of us than Calabi-Yau threefolds, one needs at first only minimal appeal to homology and cohomology, and can rely on basic high school geometry (at least in the sense of old fashioned education). It will appear that methods from topological field theories provide us with very effective means to perform what look at first sight like rather formidable computations.

2. PLANE RATIONAL CURVES

An algebraic curve in $\mathbb{P}_2$, the projective plane, is described by an irreducible homogeneous polynomial in three variables x, y, z,

$$f(x, y, z) = 0 \tag{2.1}$$

Its first natural invariant under projectivities (the 8-parameter group PGL_3) is its degree d. Let the curve be smooth except at δ ordinary double points (i.e. with distinct tangents) and denote by c its class, the number of tangents through an arbitrary point M (X, Y, Z) in the plane. This is the number of common roots between (2.1) and

$$X f_x(x, y, z) + Y f_y(x, y, z) + Z f_z(x, y, z) = 0 \tag{2.2}$$

expressing that a secant through M has a double intersection with the curve, from which one should subtract 2δ, as common solutions of (2.1) and (2.2) include the lines from M to the double points each counted twice. From Bezout's theorem

$$c = d(d-1) - 2\delta \tag{2.3}$$

To compute the genus, following Riemann and Hurwitz, consider the curve as a ramified cover of $\mathbb{P}_1$ by projecting it from a point N outside the curve on a line L. We find a d-fold cover ramified over the intersections with L of the c tangents from N (each one with $d-1$ points as there is generically only one coïncidence). The intersections with lines joining N to double points do not give ramification points, since they are distinct on the curve. Dissect $L \simeq \mathbb{P}_1$ (the Riemann sphere) into two polygonal faces with c vertices and edges and lift it on the curve. The preimage is a decomposition with $2d$ faces, cd edges and $c(d-1)$ vertices. From the Euler characteristic, $2-2g =$ vertices $-$ edges $+$ faces. Therefore

$$g = \frac{(d-1)(d-2)}{2} - \delta \tag{2.4}$$

This shows that there are at most $\frac{(d-1)(d-2)}{2}$ double points. When this limit is reached we have a "generic" rational curve ($g=0$) of degree d. Formulas (2.3) and (2.4) have to be amended when the curve has higher singularities. These families being of higher codimension among rational curves of degree d, we will limit ourselves to rational curves with ordinary double points which are the relevant ones for the present purpose.

Homogeneous polynomials of degree d in $n+1$ variables form a projective space $\mathbb{P}^D$ where

$$D(n,d) = \begin{pmatrix} d+n \\ \\ n \end{pmatrix} - 1 \tag{2.5a}$$

For plane curves, $n=2$, this is

$$D(2,d) = \frac{1}{2}d(d+3) \tag{2.5b}$$

Irreducible smooth curves only form an open subset in this space. The condition that a fixed point in the plane belongs to the curve represents a single linear condition on the coefficients of the corresponding polynomial, so that there is a single curve of degree d through $\frac{1}{2}d(d+3)$ points in general position. Here the meaning of "general position" is clear. The corresponding linear system should be of maximal rank. Similarly to have a double point imposes a single condition, so that one expects a curve of degree d, genus $g \le \frac{(d-1)(d-2)}{2}$ and δ ordinary double points as singularities, to depend on

$$\frac{d(d+3)}{2} - \delta = 3d - 1 + g \tag{2.6}$$

parameters. Hence there are finitely many such curves through $3d-1+g$ generic points of the plane. For rational curves of degree d, described by three homogeneous polynomials of degree d in two indeterminates (homogeneous coordinates on $\mathbb{P}_1$), this is confirmed since this set depends on $3(d+1)$ parameters up to scale (subtracting 1 parameter) and up to a projective transformation in $\mathbb{P}_1$ (subtracting $3 =$ order of PGL_2 parameters). Altogether we recover $3d-1$ parameters.

Let therefore N_d denote the number of rational plane curves through $3d-1$ points in general position. There is a single line through two points and a single conic through five points, so

$$N_1 = N_2 = 1 \tag{2.7}$$

When $d = 3$ we look at uninodal cubics through eight points while a general cubic depends on nine parameters. Through the given points we therefore find a pencil of cubics (i.e. a linear family over $\mathbb{P}_1$) of the form

$$\lambda_0 f_0 + \lambda_1 f_1 = 0 \tag{2.8}$$

with f_1, f_2 cubic polynomials in x, y, z, which intersect at nine points, the eight preassigned ones and an extra ninth point common to all curves in the pencil. Rational cubics will have a double point, and this occurs for pairs (λ_0, λ_1) up to a factor, if simultaneously

$$\lambda_0 \frac{\partial f_0}{\partial x} + \lambda_1 \frac{\partial f_1}{\partial x} = \lambda_0 \frac{\partial f_0}{\partial y} + \lambda_1 \frac{\partial f_1}{\partial y} = \lambda_0 \frac{\partial f_0}{\partial z} + \lambda_1 \frac{\partial f_1}{\partial z} = 0 \tag{2.9a}$$

which from homogeneity entails (2.8). Each solution of this system yields a value $\lambda_0 : \lambda_1$, provided

$$\begin{vmatrix} \frac{\partial f_0}{\partial x} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_0}{\partial z} & \frac{\partial f_1}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial f_0}{\partial y} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_0}{\partial z} & \frac{\partial f_1}{\partial z} \end{vmatrix} = 0 \tag{2.9b}$$

These two quartics intersect in 16 points, but 4 of them, corresponding to the simultaneous vanishing of $\frac{\partial f_0}{\partial z}$ and $\frac{\partial f_1}{\partial z}$ have to be discarded. So we have

$$N_3 = 12 \tag{2.10}$$

Remark

Uninodal cubics have a modular invariant j equal to infinity. It is interesting to note that there are also 12 smooth cubics ($g = 1$) through 8 points in the plane with fixed $j \neq \infty$ except for $j = 0$ (resp. 1728) where this number is replaced by $12/3 = 4$ (resp. $12/2 = 6$), taking into account automorphisms (P. Aluffi).

As we reach 3-nodal quartics through 11 points the situation becomes already wild. The number has been obtained by Zeuthen as recorded on page 186 of Schubert's treatise on enumerative geometry, where one reads

$$N_4 = 620 \tag{2.11}$$

It would be instructive to retrace Zeuthen's steps to compare them with the forthcoming arguments, as techniques used by classical geometers - involving the study of degeneracies - are after all not so different from those to be discussed below.

3. QUANTUM RING FOR $\mathbb{P}_2$

The object to be introduced is a parametric deformation of the cohomology (or intersection) ring of the plane. The non zero Betti numbers of the plane are

$$b_{0,0} = b_{1,1} = b_{2,2} = 1 \tag{3.1}$$

The corresponding, associative, commutative, ring with multiplication corresponding to wedge product of representatives in each class, is spanned by three elements, t_0, t_1, t_2 so that the multiplication table reads

	t_0	t_1	t_2
t_0	t_0	t_1	t_2
t_1	t_1	t_2	0
t_2	t_2	0	0

(3.2)

where t_0 plays the role of unit and t_1, t_2 are nihilpotent, $t_1^3 = t_2^2 = 0$. Dual to each cohomology class is a class of cycles - here algebraic ones. So t_2 is dual to a point, t_1 to a line, t_0 to the plane. As a point intersects the plane in a point and two (distinct) lines intersect at a point, we have a symmetric intersection map on the ring $\langle\ ,\ \rangle \longrightarrow \mathbb{Z}$ with

$$\langle t_0, t_2 \rangle = 1 \qquad \langle t_1, t_1 \rangle = 1 \tag{3.3}$$

all other intersections on generators being zero. In other words $\langle t_i, t_j \rangle$ is the coefficient of t_2 in $t_i \circ t_j$. The above ring can be identified with the ring of polynomials in one indeterminate $\mathbb{C}[x]$ modulo the gradient of

$$W(x) = \frac{x^4}{4} \tag{3.4}$$

through

$$t_k \longrightarrow x^k \mod W' \qquad 0 \leq k \leq 2 \tag{3.5}$$

understanding that we extend the ring to an algebra over $\mathbb{C}$ (or else restrict the scalars to integers). For two such polynomials $P(x), Q(x)$ one has then

$$\langle P, Q \rangle = \frac{1}{2i\pi} \oint \frac{PQ\mathrm{d}x}{W'(x)} = \mathrm{Res}\frac{PQ}{W'} = \frac{1}{2}\, \frac{\mathrm{d}^2}{\mathrm{d}x^2}(PQ)\Big|_{x=0} \tag{3.6}$$

Let us denote the (non-singular) "metric" and its inverse by

$$\eta_{ij} = \langle t_i, t_j \rangle \qquad \eta^{ij} = \left(\eta^{-1}\right)_{ij}$$

Numerically η^{ij} is equal to $\eta_{ij} = \delta_{i+j,2}$.

Consider $\bigoplus_{i=0}^{2} H^{i,i}$ as a vector space over $\mathbb{C}$ with coordinates y^i (conjugate to t_i). It is suggested that a quantum string field theory depending on parameters y^i underlies the construction, so that brackets denote averages. One therefore inquires about the existence of a partition function Z (the expected "τ -function" of the possibly corresponding integrable system) such that connected correlations are derivatives of a free energy

$$F = \ln Z \tag{3.7}$$

with respect to y's. One also assumes the existence of a parameter playing the role of a "cosmological constant" conjugate to the Euler characteristic in such a way that one can expand the free energy in contributions according to genus and one limits oneself to genus zero. So in this section F will stand for the "tree level" leading term $F^{(0)}$. Derivatives of F with respect to y^i's correspond to insertions of operators T_i, implicitly depending on the parameters. One requires the identification

$$F_{0ij} = \langle T_0 T_i T_j \rangle = \eta_{ij} = \delta_{i+j,2} \tag{3.8}$$

We use a short hand notation for partial derivatives

$$F_{i_1 i_2 \ldots} \equiv \frac{\partial}{\partial y^{i_1}} \frac{\partial}{\partial y^{i_2}} \ldots F$$

From (3.8) one can split F into a second degree polynomial in y^0 and a part independent of y^0 as

$$\begin{aligned} F &= f_0 + f \\ f_0 &= \frac{1}{2}\left(\left(y^0\right)^2 y^2 + y^0 \left(y^1\right)^2\right) \\ f &\equiv f\left(y^1, y^2\right) \end{aligned} \tag{3.9}$$

A deformation of the multiplication table (3.2) becomes the fusion rules for the "quantum cohomology ring" with T's substituted for t's as

$$T_i \circ T_j = F_{ij\ell}\eta^{\ell k} T_k = F_{ij\ell} T_{2-\ell} \tag{3.10}$$

satisfying the requirements

(i) commutativity

(ii) associativity

(iii) existence of a unit T_0

Commutativity follows from the definition, while condition (3.8) (equivalently (3.9)) expresses that T_0 plays the role of unit. The crucial assumption is associativity, which imposes conditions on f, the expected contribution of "instantons", i.e. at tree level, holomorphic maps $\mathbb{P}_1 \longrightarrow \mathbb{P}_2$. These are algebraic plane curves of genus zero of general type i.e. with ordinary double points as only singularities (higher types of singularities are of lower dimension in the corresponding "spaces" of curves).

The modified multiplication table defining the ring reads

	T_0	T_1	T_2
T_0	T_0	T_1	T_2
T_1	T_1	$T_2 + f_{111}T_1 + f_{112}T_0$	$f_{112}T_1 + f_{122}T_0$
T_2	T_2	$f_{112}T_1 + f_{122}T_0$	$f_{122}T_1 + f_{222}T_0$

(3.11)

Associativity implies a single condition

$$f_{222} = f_{112}^2 - f_{111} f_{122} \tag{3.12}$$

a third order non linear partial differential equation the solution of which would at first seem hopeless. What are possible further requirements to be imposed on f? The trivial solution reverts back to the initial cohomology ring with $T_i \longrightarrow t_i$. Let us assume that F is only meaningful up to a second degree polynomial in the y's corresponding to a multiplicative renormalization of the partition function and that Z is not scalar but a "section of bundle" over the affine space of y's. Assume that, as suggested by the form of f_0, we have under a affine transformation for some constant χ

$$y^i \longrightarrow \tilde{y}^i = \lambda^{1-i} y^i + \chi \delta_1^i \ln \lambda$$

$$F\left(\tilde{y}^i\right) = \lambda F\left(y^i\right) \qquad \text{mod. a second degree polynomial} \tag{3.13}$$

i.e.

$$f\left(y^1 + \chi \ln \lambda, \lambda^{-1} y^2\right) = \lambda f\left(y^1, y^2\right) \tag{3.14}$$

In effect this reduces f to a function of a single variable and is readily seen to be compatible with (3.12). The idea underlying topological field theories is that the action in the path integral involves a term proportional to the area which under proper normalization becomes the Kähler class, i.e. the degree for holomorphic maps, while y^1 is the corresponding conjugate variable. Thus an expansion of f in terms of degree d rational curves has the formal expansion

$$f\left(y^1, y^2\right) = \sum_{d=1}^{\infty} f_d\left(y^2\right) \exp\, d y^1 \tag{3.15}$$

From (3.14) $f_d\left(y^2\right)$ has to be proportional to $\left(y^2\right)^{\chi d - 1}$. From section 2 we know that a rational curve of degree d is stabilized by $3d-1$ points while each factor y^2 conjugate to T_2 is interpreted as marking a point in target space. It is thus consistent to choose

$$\chi = 3 \tag{3.16}$$

and as the points are indistinguishable, to write

$$f\left(y^1, y^2\right) = \sum_{d=1}^{\infty} N_d \frac{\left(y^2\right)^{3d-1}}{(3d-1)!} \exp\, d y^1 \tag{3.17}$$

The sum is over the degree d of rational maps $\mathbb{P}_1 \longrightarrow \mathbb{P}_2$, and the non negative integer N_d is as before the number of such maps through undistinguished $3d-1$ points in $\mathbb{P}_2$ in general position. Inserting this expansion in (3.12) one obtains the following recursive mean to compute the unknown coefficients N_d.

Proposition 1 (Kontsevich)

$$\begin{aligned} &N_1 = 1 \qquad \text{and for} \qquad d \geq 2 \\ &N_d = \sum_{\substack{d_1, d_2 \geq 1 \\ d_1 + d_2 = d}} N_{d_1} N_{d_2} \left[d_1^2 d_2^2 \begin{pmatrix} 3d-4 \\ 3d_1 - 2 \end{pmatrix} - d_1^3 d_2 \begin{pmatrix} 3d-4 \\ 3d_1 - 1 \end{pmatrix} \right] \end{aligned} \tag{3.18}$$

A short table is

d	$3d-1$	N_d
1	2	1
2	5	1
3	8	12
4	11	620
5	14	87304
6	17	26312976
7	20	14616808192
8	23	13525751027392
9	26	19385778269260800
10	29	40739017561997799680
11	32	120278021410937387514880
12	35	482113680618029292368686080

(3.19)

This agrees perfectly with the known results.

In spite of the growth of N_d it is not excluded that the series for f converges (cf. eq.(6.1)).

4. QUADRICS

Let us now look at the simplest of all examples of a rational surface beyond $\mathbb{P}_2$ namely a smooth quadric $\mathcal{F}_2$ in $\mathbb{P}_3$. From its double system of lines with a pair meeting at a single point, we have the isomorphism $\mathcal{F}_2 \sim \mathbb{P}_1 \times \mathbb{P}_1$. The space $\bigoplus H^{i,i}(\mathbb{C})$ is therefore 4-dimensional with $b^{0,0} = b^{2,2} = 1$ and $b^{1,1} = 2$. We denote the corresponding generators t_0, t_A, t_B and $t_2 = t_A t_B$. The non vanishing elements of the intersection form are

$$\langle t_0 t_2 \rangle = \langle t_A t_B \rangle = 1 \tag{4.1}$$

The deformation into a quantum ring $t \longrightarrow T$ with parameters y^0, y^A, y^B and y^2, is given in terms of the tree level free energy

$$\begin{aligned} F &= f_0 + f \\ f_0 &= \frac{1}{2}\left(y^0\right)^2 y^2 + y^0 y^A y^B \\ f &\equiv f\left(y^A, y^B, y^2\right) \end{aligned} \tag{4.2}$$

and f records as before the counting of rational curves on $\mathcal{F}_2$. The existence of two generators in the middle cohomology agrees with the notion of a multidegree for a curve on $\mathbb{P}_1 \times \mathbb{P}_1$ which we denote by (a, b). Such a curve is described by the vanishing of a polynomial of degree a (resp. b) in a first (resp. second) pair of homogeneous coordinates for $\mathbb{P}_1$, thus depends on

$$A = (a+1)(b+1) - 1 = ab + a + b \tag{4.3}$$

parameters. This is 1 for $(a, b) = (1, 0)$ or (0,1), indeed there is a single line in each system through a point of $\mathcal{F}_2$, 3 for $(a, b) = (1, 1)$ since through 3 points on $\mathcal{F}_2$ a plane cuts it into a conic and 5 for $(a, b) = (2, 1)$ or (1,2) a twisted cubic (a rational curve). This last number being perhaps not so familiar we recall an elementary proof. For this purpose intersect $\mathcal{F}_2$ with a second quadric in $\mathbb{P}_3$ (depending as the first on 9 parameters) passing through one of its lines. The residual intersection is a twisted cubic. For the second quadric to go through a line imposes 3 conditions leaving a six parameters family of quadrics of which a linear pencil goes through a pair (line, twisted cubic). Therefore we are indeed left with a 5 parameter family of twisted cubics (of type (2,1) say).

Up to degree $d = 3$ all curves on $\mathcal{F}_2$ are rational, but this is not true beyond. In general since a pair of lines of each system defines a tangent plane at their intersection, the degree of a curve is given by

$$d = a + b \tag{4.4}$$

the number of points of the curve in this plane. As a and b increase we do not expect a rational curve of bi-degree (a, b) to depend on as many as A parameters, but only on

$$A_0 = 2d - 1 = 2(a + b) - 1 \tag{4.5}$$

parameters. Indeed let $M_0 \in \mathcal{F}_2$ and project $\mathcal{F}_2$ on a plane blowing down the two lines through M_0 at M_1 and M_2 in the plane (while the line L_{12} through M_1 and M_2 in the plane is blown down to M_0 on $\mathcal{F}_2$). Plane sections of $\mathcal{F}_2$ become the 3-dimensional family of conics in the plane through M_1 and M_2 as was known to the ancients; and a curve of bi-degree (a, b) is projected in the plane into a curve of degree $d = a + b$ having an a-fold (resp. b-fold) - a priori ordinary - multiple point at M_1 (resp. M_2). Assume generically that the only other singularities of the projection are δ ordinary double points and that the curve is of genus zero. From Plucker's formula

$$\begin{aligned} g = 0 &= \frac{(a+b-1)(a+b-2)}{2} - \frac{a(a-1)}{2} - \frac{b(b-1)}{2} - \delta \\ \delta &= (a-1)(b-1) \end{aligned} \tag{4.6}$$

and δ vanishes when either a or b is equal to one as one would guess from the projection on either $\mathbb{P}_1$ in $\mathcal{F}_2 \sim \mathbb{P}_1 \times \mathbb{P}_1$. Going through a given point with multiplicity a imposes $1 + 2 + ... + a = \frac{a(a+1)}{2}$ conditions by expanding the equation of the curve in inhomogeneous coordinates x, y around the point. Thus we recover (4.3) when $\delta = 0$ for a generic (a, b) curve as indeed with $d = a + b$

$$A = \frac{(a+b)(a+b+3)}{2} - \frac{a(a+1)}{2} - \frac{b(b+1)}{2} = ab + a + b$$

For the curve to be rational we have to impose $\delta = (a-1)(b-1)$ conditions leaving as claimed $A_0 = A - \delta = 2(a+b) - 1$ parameters. Let us therefore denote by $N_{a,b} = N_{b,a}$ the number of rational curves of bidegree (a, b) through $2(a+b)-1$ points on $\mathcal{F}_2$. With similar reasoning as in section 3 we therefore set

$$f\left(y^A, y^B, y^2\right) = \sum_{\substack{a,b \geq 0 \\ a+b \geq 1}} N_{a,b} \frac{\left(y^2\right)^{2(a+b)-1}}{(2(a+b)-1)!} \exp\left(ay^A + by^B\right) \tag{4.7}$$

With a self explanatory notation for partial derivatives we define the deformed ring generated by T_0, T_A, T_B, T_2 through

	T_0	T_A	T_B	T_2
T_0	T_0	T_A	T_B	T_2
T_A	T_A	$f_{AAB}T_A+f_{AAA}T_B+f_{AA2}T_0$	$T_2+f_{ABB}T_A+f_{AAB}T_B+f_{AB2}T_0$	$f_{AB2}T_A+f_{AA2}T_B+f_{A22}T_0$
T_B	T_B	$T_2+f_{ABB}T_A+f_{AAB}T_B+f_{AB2}T_0$	$f_{BBB}T_A+f_{ABB}T_B+f_{BB2}T_0$	$f_{BB2}T_A+f_{AB2}T_2+f_{B22}T_0$
T_2	T_2	$f_{AB2}T_A+f_{AA2}T_B+f_{A22}T_0$	$f_{BB2}T_A+f_{AB2}T_B+f_{B22}T_0$	$f_{B22}T_A+f_{A22}T_B+f_{222}T_0$

(4.8)

From the 8 conditions for associativity

$$\begin{array}{ccc} \left(T_A^2\right) T_B = T_A\left(T_A T_B\right) & \text{and} & A \longleftrightarrow B \\ \left(T_A^2\right) T_2 = T_A\left(T_A T_2\right) & \text{and} & A \longleftrightarrow B \\ \left(T_2^2\right) T_A = T_2\left(T_2 T_A\right) & \text{and} & A \longleftrightarrow B \\ T_A\left(T_B T_2\right) = T_B\left(T_A T_2\right) = T_2\left(T_A T_B\right) & & \end{array} \tag{4.9}$$

we obtain the following result.

Proposition 2

The integers $N_{a,b}$ satisfy (and are determined) by the following relations

0)

$$N_{0,1} = N_{1,0} = 1 \qquad N_{a,b} = N_{b,a}$$

1)

$$2abN_{a,b} = \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} N_{a_1b_1}N_{a_2b_2} \begin{pmatrix} 2(a+b)-2 \\ 2\left(a_1+b_1\right)-1 \end{pmatrix} a_1^2\left(a_1b_2 - a_2b_1\right)b_2^2$$

2a)

$$aN_{ab} = \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} N_{a_1b_1}N_{a_2b_2} \begin{pmatrix} 2(a+b)-3 \\ 2\left(a_1+b_1\right)-1 \end{pmatrix} a_1\left(a_1^2b_2^2 - b_1^2a_2^2\right)$$

2b)

$$\text{exchange} \quad a \longleftrightarrow b, a_1 \longleftrightarrow b_1, a_2 \longleftrightarrow b_2$$

3a)

$$0 = \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} N_{a_1b_1} N_{a_2b_2} a_1^2 \left\{ (a_2+b_2-1)(b_1a_2+a_1b_2) - [2(a_1+b_1)-1] a_2b_2 \right\} \times \begin{pmatrix} 2(a+b)-3 \\ 2(a_1+b_1)-1 \end{pmatrix} \tag{4.10}$$

3b)

$$\text{exchange} \quad a_1 \longleftrightarrow b_1, a_2 \longleftrightarrow b_2$$

4)

$$N_{a,b} = \sum_{\substack{a_1+a_2=a \\ b_1+b_2=b}} N_{a_1 b_1} N_{a_2 b_2} \left\{ a_1 \binom{2(a+b)-4}{2(a_1+b_1)-2} - a_2 \binom{2(a+b)-4}{2(a_1+b_1)-3} \right\} (a_1 b_2 + a_2 b_1) b_2$$

In all sums a_i and b_i run over non negative integers and $a_i + b_i \geq 1$. This overdetermined system is however consistent, so that it is sufficient to use the last equation. One readily sees that $N_{d,1} = N_{d-1,1}$, hence

$$N_{d,1} = N_{1,d} = 1 \qquad d \geq 0 \tag{4.11a}$$

and one obtains the following table

a / b	0	1	2	3	4	5	6
0	0	1	0	0	0	0	0
1	1	1	1	1	1	1	1
2	0	1	12	96	640		
3	0	1	96	3.510			
4	0	1	640				
5	0	1					
6	0	1					

(4.11b)

We recover $N_{2,1} = N_{1,2} = 1$ for the number of twisted cubics on $\mathcal{F}_2$ through 5 points. The first non trivial number is $N_{2,2} = 12$ rational quartics through 7 points. There are two equivalent ways to recover this number. One is to use the Zeuthen-Segré formula

$$n = \chi(\mathcal{F}) - 4 + 4g + s \tag{4.12}$$

for the number n of singular elements in a linear pencil of curves of genus g on a surface $\mathcal{F}$ with Euler character $\chi(\mathcal{F})$ through s base points. Here we can take the linear pencil of genus 1 quartic curves through 7 points of $\mathcal{F}_2$. These have an eight's unassigned point in common as can be seen by projection on the plane as before. Two such projected, genus 1 quartics having double points at M_1 and M_2 intersect at $4 \times 4 = 16$ points, out of which $2 \times 4 = 8$ correspond to M_1 and M_2, 7 to the projections of the assigned points, leaving an eight's unassigned point, through which pass all curves of the corresponding pencil. Hence $n = 4 - 4 + 4 \times 1 + 8 = 12$. Alternatively one can look at the Jacobian of a pencil of genus 1 quartics in the plane having double points at M_1 and M_2. Reasoning as in section 2 we find that the number of quartics in this pencil having an extra double point is $3(4-1)^2 = 27$. The multiplicities of M_i are $3^2 - 2^2 = 5$, leaving 17 rational quartics. But this includes the reducible case of a cubic through $M_1 M_2$ together with the line L_{12} through these points, to be counted with the same multiplicity $3^2 - 2^2$, which is not to be counted as a quartic when lifted on $\mathcal{F}_2$. So altogether

$$n = 3(4-1)^2 - 3\left(3^2 - 2^2\right) = 12$$

as before.

5. CUBIC SURFACES

We investigate here the case of cubic surfaces with a system of 27 lines and intersection form invariant under the Weyl group of E_6, denoted G, of order $72 \times 6!$ (exhibiting the group as transitive on the 72 sextets of non intersecting lines, each one invariant under a subgroup of permutations on six objects).

A classical description is based on the linear system of plane cubics through 6 points, no three on a line, the six of them not on a conic. Since plane cubics form a $\mathbb{P}_9$ and we impose six linear (independent) constraints, this defines a rational map $\mathbb{P}_2 \longrightarrow \mathcal{F}_3 \subset \mathbb{P}_3$, blowing up the six points into six non intersecting lines. Any two cubics in a linear pencil through the six points intersect in $3^2 - 6 = 3$ variable points, hence the image is indeed a cubic surface. The remaining lines are the 15 images $L_{ij} = L_{ji}$ of lines in the plane through pairs of distinct points and the 6 images $\tilde{L}_i$ of conics through 5 of the six points.

In the sequel we identify the second cohomology group of the surface $H^2 \equiv H^{1,1}(\mathbb{C})$ and the divisor class group (equivalence being linear equivalence), isomorphic to $\mathbb{Z}^7$, tensored with $\mathbb{C}$. Indeed divisor classes are generated by the six L_i and the image of a line in the plane, call it L_0. On $\mathcal{F}_3$ this is a twisted cubic, since in the plane a line intersects a cubic of the family in 3 points. A plane section of $\mathcal{F}_3$ (generically a genus one cubic) is linearly equivalent to $3L_0 - \sum_{i=1}^{6} L_i$.

An effective divisor, i.e. a linear combination of algebraic cycles with non negative multiplicities of the form

$$\operatorname{div}(\underline{a}) = a^0 L_0 - \sum_{i=1}^{6} a^i L_i \tag{5.1}$$

has degree

$$d_{\underline{a}} = 3a^0 - \sum_{i=1}^{6} a^i \tag{5.2}$$

and self intersection

$$(\underline{a}, \underline{a}) = \left(a^0\right)^2 - \sum_{i=1}^{6} \left(a^i\right)^2 \tag{5.3}$$

This may be interpreted as corresponding to the image of a degree a^0 curve in the plane having an ordinary multiple point (i.e. with distinct tangents) of multiplicity a^i at each of the six base points.

The arithmetic genus - a less stringent concept than the topological one - (but an upper bound for irreducible curves) denoted $p_{\underline{a}}$ is given by

$$p_{\underline{a}} = \frac{1}{2}\left(a^0 - 1\right)\left(a^0 - 2\right) - \frac{1}{2}\sum_{i=1}^{6} a^i \left(a^i - 1\right) \tag{5.4}$$

where we recognize a generalization of (2.4). It does not include contributions from other possible singularities.

Let us set up the quantum cohomology ring. We use notations from Minkowski space with greek indices μ, ν, ρ... running over seven values, while "space" latin indices i, j, k... take six values from 1 to 6. To avoid confusion the 9 dimensional deformation will be parametrized by $x, \underline{y}, z$ where $\underline{y} \equiv \{y^\mu\}$. Correspondingly the ring is generated by T_x (instead of T_0) the identity, T_μ, and T_z (instead of T_2). The "metric" η inherited from the intersection formula is given by $\eta_{xz} = 1$, $\eta_{\mu\nu} = g_{\mu\nu}$ the diagonal Lorentzian metric of signature $(+------)$ numerically equal to its inverse and $\underline{y}^2$ will stand for $(\underline{y}, \underline{y})$.

We are interested in counting rational irreducible curves (effective irreducible divisors) of degree d given by (5.2). This requires that

$$\underline{a} = (0, -1, 0, 0, 0, 0, 0) \qquad \text{up to space permutations}$$

or else

$$\left.\begin{array}{lll} (i) & 0 \le a^i < a^0 & 1 \le i \le 6 \\ (ii) & a^i + a^j \le a^0 & 1 \le i < j \le 6 \\ (iii) & \sum_{i=1}^{6} a^i \le 2a^0 + a^k & 1 \le k \le 6 \end{array}\right\} \quad \text{except if } d = 1$$

These conditions are necessary (but not sufficient) for irreducibility. Their interpretation for an irreducible curve of degree a^0 in the plane is that a point cannot be of multiplicity equal to the degree, that no line through two points on the curves cuts it in more that a^0 points (for $a^0>1$) and finally that a conic through 5 points cannot cut the curve in more than $2a^0$ points (for $a^0>2$). There are higher conditions but they are all subsumed under the condition that

$$p_{\underline{a}}\geq 0$$

Let us call $\Delta_p\equiv\sum_d \Delta_{p,d}$ the set of effective rational irreducible divisors of arithmetic genus p which we split according to the degree d and

$$\Delta=\sum_{p\geq 0}\Delta_p=\sum_{p,d}\Delta_{p,d} \tag{5.5}$$

Rational curves in $\Delta_{p,d}$ have p extra double points generically and form a set invariant under the Weyl group G of $E(6)$. In particular Δ_0 corresponds to smooth rational curves on $\mathcal{F}_3$.

Curves of degree $a^0\geq 1$ in the plane having assigned multiplicities $a^i\geq 0$ at the base points form a projective space of dimension

$$\frac{a^0\left(a^0+3\right)}{2}-\sum_{i=1}^{6}\frac{a^i\left(a^i+1\right)}{2}=p_{\underline{a}}+d_{\underline{a}}-1 \tag{5.6}$$

Imposing $p_{\underline{a}}$ extra conditions to reduce their topological genus to zero leaves a space of rational curve of dimension $d_{\underline{a}}-1$. Requiring further that the curve passes through $d_{\underline{a}}-1$ "generic" points, leaves finitely many possibilities, the number of which will be denoted $N_{\underline{a}}$. Whenever $p_{\underline{a}}=0$ the set of conditions reduces to $d_{\underline{a}}-1$ linear ones (generically independent) so that

$$p_{\underline{a}}=0 \qquad\qquad N_{\underline{a}}=1 \tag{5.7}$$

It is convenient to introduce the (anti-) canonical divisor

$$\underline{\omega}=(3;1,1,1,1,1,1) \tag{5.8}$$

in such a way that

$$d_{\underline{a}}=\underline{\omega}.\underline{a} \qquad\qquad p_{\underline{a}}=\frac{\underline{a}^2-\underline{\omega}.\underline{a}}{2}+1 \tag{5.9}$$

The last formula is also called the "adjunction formula". As is to be expected, and as we shall see explicitly below, if $\underline{b}=\gamma\underline{a}\in\Delta$ where γ denotes an element of the Weyl group G, then

$$N_{\underline{b}}=N_{\underline{a}} \tag{5.10}$$

A useful presentation of G is the following. It is generated by permutations of the space indices as well as reflections (involutions)

$$\underline{a}\longrightarrow a+(\underline{e}.\underline{a})\underline{e} \tag{5.11a}$$

where $\underline{e}$ is one of the 20 vectors of "square length" -2, of the form $e^0=1$ and having 3 non vanishing space components equal to 1. For the specific choice $\underline{e}=(1,1,1,1,0,0,0)$ this reads

$$\begin{array}{llll} a^0\longrightarrow 2a^0-a^1-a^2-a^3 & a^1\longrightarrow a^0-a^2-a^3, & & a^2\longrightarrow a^0-a^1-a^3, \\ a^3\longrightarrow a^0-a^1-a^2, & a^4\longrightarrow a^4, & a^5\longrightarrow a^5, & a^6\longrightarrow a^6 \end{array} \tag{5.11b}$$

This group leaves $\underline{\omega}$ invariant, and preserves scalar products. Hence it is a finite subgroup of the Lorentz group (in fact a subgroup of orthogonal transformations since $\underline{w}$ is time-like). Invariance under G gives a check on the forthcoming computations.

Let us write the free energy

$$\begin{aligned} F(x,\underline{y},z)&=f_0(x,\underline{y},z)+f(\underline{y},z)\\ f_0&=\frac{1}{2}\left[x^2z+x\underline{y}^2\right]\\ f&=\sum_{a\in\Delta}N_{\underline{a}}\frac{z^{d_{\underline{a}}-1}}{(d_{\underline{a}}-1)!}\exp\,\underline{a}.\underline{y}\end{aligned} \tag{5.12}$$

The factorial in the denominator restricts $d_{\underline{a}} = \underline{w}.\underline{a}$ to be larger or equal to one.

The multiplication in the commutative associative ring $\{T\}$ is described by ($T^\rho = g^{\rho\nu} T_\nu$)

$$
\begin{aligned}
T_x^2 = T_x \qquad T_x T_\mu &= T_\mu \qquad T_x T_z = T_z \\
T_\mu T_\nu &= g_{\mu\nu} T_z + \sum_\rho f_{\mu\nu\rho} T^\rho + f_{\mu\nu z} T_x \\
T_\mu T_z &= \sum_\rho f_{\mu\rho z} T^\rho + f_{\mu z z} T_x \\
T_z^2 &= \sum_\rho f_{\rho z z} T^\rho + f_{zzz} T_x
\end{aligned} \tag{5.13}
$$

The triples to be considered in imposing associativity are $T_\mu (T_\nu T_\rho) = (T_\mu T_\nu) T_\rho$, $T_\mu (T_\nu T_z) = (T_\mu T_\nu) T_z$ and $T_\mu (T_z T_z) = (T_\mu T_z) T_z$, leading to the conditions

a)

$$g_{\mu\nu} f_{\nu\delta z} - g_{\mu\rho} f_{\mu\delta z} + g_{\nu\delta} f_{\mu\rho z} - g_{\mu\delta} f_{\rho\nu z} = \sum_{\sigma,\sigma'} g^{\sigma\sigma'} \left(f_{\mu\delta\sigma} f_{\nu\rho\sigma'} - f_{\nu\delta\sigma} f_{\mu\rho\sigma'} \right)$$

b)

$$g_{\mu\rho} f_{\nu z z} - g_{\nu\rho} f_{\mu z z} = \sum_{\sigma,\sigma'} g^{\sigma\sigma'} \left(f_{\mu\sigma z} f_{\nu\rho\sigma'} - f_{\nu\sigma z} f_{\mu\rho\sigma'} \right)$$

c)

$$2 g_{\mu\nu} f_{\rho z z} - g_{\mu\rho} f_{\nu z z} - g_{\nu\rho} f_{\mu z z} = \sum_{\sigma,\sigma'} g^{\sigma\sigma'} \left[f_{\mu\sigma\rho} f_{\nu\sigma' z} + f_{\nu\sigma\rho} f_{\mu\sigma' z} \right] \tag{5.14}$$

d)

$$g_{\mu\nu} f_{zzz} = \sum_{\sigma,\sigma'} g^{\sigma\sigma'} \left[f_{\mu\sigma z} f_{\nu\sigma' z} - f_{\mu\nu\sigma} f_{\sigma' z z} \right]$$

Since f is a Lorentz scalar as well as z, these equations are covariant under the Weyl group G, so is the following overdetermined but consistent system for the integers $N_{\underline{a}}$, from a), b), c), d) successively.

Proposition 3

The numbers $N_{\underline{a}}$ of rational curves on a cubic surface satisfy - and are determined by - the following conditions

$$N_{\underline{a}} = 1 \qquad \text{for} \qquad d_{\underline{a}} = 1$$

and

a) $d_{\underline{a}} \geq 2$

$$
\begin{aligned}
&2 \left[g_{\mu\rho} a_\nu a_\delta - g_{\nu\rho} a_\mu a_\delta + g_{\nu\delta} a_\mu a_\rho - g_{\mu\delta} a_\rho a_\nu \right] N_{\underline{a}} = \\
&\textstyle\sum_{\underline{b}+\underline{c}=\underline{a}} N_{\underline{b}} N_{\underline{c}} \begin{pmatrix} d_{\underline{a}} - 2 \\ d_{\underline{b}} - 1 \end{pmatrix} (\underline{b}.\underline{c}) \left[b_\mu c_\nu - b_\nu c_\mu \right] \left[b_\delta c_\rho - b_\rho c_\delta \right]
\end{aligned}
$$

b) $d_{\underline{a}} \geq 3$

$$\left(g_{\mu\rho} a_\nu - g_{\nu\rho} a_\mu \right) N_{\underline{a}} = \sum_{\underline{b}+\underline{c}=\underline{a}} N_{\underline{b}} N_{\underline{c}} \begin{pmatrix} d_{\underline{a}} - 3 \\ d_{\underline{b}} - 2 \end{pmatrix} (\underline{b}.\underline{c}) c_\rho \left[b_\mu c_\nu - b_\nu c_\mu \right]$$

c) $d_{\underline{a}} \geq 3$

$$
\begin{aligned}
&\left(2 g_{\mu\rho} a_\rho - g_{\mu\rho} a_\nu - g_{\nu\rho} a_\mu \right) N_{\underline{a}} = \\
&\textstyle\sum_{\underline{b}+\underline{c}=\underline{a}} N_{\underline{b}} N_{\underline{c}} \begin{pmatrix} d_{\underline{a}} - 3 \\ d_{\underline{b}} - 1 \end{pmatrix} (\underline{b}.\underline{c}) \left[b_\mu b_\rho c_\nu + b_\nu b_\rho c_\mu - 2 b_\mu b_\nu c_\rho \right]
\end{aligned} \tag{5.15}
$$

d) $d_{\underline{a}} \geq 4$

$$g_{\mu\nu} N_{\underline{a}} = \sum_{\underline{b}+\underline{c}=\underline{a}} N_{\underline{b}} N_{\underline{c}} (\underline{b}.\underline{c}) \left[b_\mu c_\nu \begin{pmatrix} d_{\underline{a}} - 4 \\ \\ d_{\underline{b}} - 2 \end{pmatrix} - b_\mu b_\nu \begin{pmatrix} d_{\underline{a}} - 4 \\ \\ d_{\underline{b}} - 1 \end{pmatrix} \right]$$

If $a, b, c \in \Delta_0$, since $d_{\underline{a}}$ is linear in $\underline{a}$, one has $d_{\underline{a}} = \underline{a}^2 + 2 = (\underline{b} + \underline{c})^2 + 2 = d_{\underline{b}} + d_{\underline{c}} + 2((\underline{b}.\underline{c}) - 1)$ hence $(\underline{b}.\underline{c}) = 1$ on the r.h.s. of (5.15). Again all these relations exhibit covariance with respect to G so $N_{\underline{a}}$ is indeed constant on its orbits.

For explicit calculations it is better to get scalar (as opposed to tensor) equations. For instance contracting (5.15a) with $\omega^\mu \omega^\rho g^{\nu\delta}$ one gets

$d_{\underline{a}} \geq 2$

$$2 \left[5 d_{\underline{a}}^2 + 3 \underline{a}^2 \right] N_{\underline{a}} = \sum_{\underline{b}+\underline{c}=\underline{a}} N_{\underline{b}} N_{\underline{c}} \begin{pmatrix} d_{\underline{a}} - 2 \\ \\ d_{\underline{b}} - 1 \end{pmatrix} (\underline{b}.\underline{c}) \left[2 d_{\underline{b}} d_{\underline{c}} (\underline{b}.\underline{c}) - d_{\underline{b}}^2 \underline{c}^2 - d_{\underline{c}}^2 \underline{b}^2 \right] \tag{5.16}$$

Starting from the 27 lines on $\mathcal{F}_3$ (an orbit of G), this formula yields a conic ($N_{\underline{a}} = 1$) through a generic point in each of the 27 planes containing one of these lines - as it should (again a single orbit). Using (5.16) we compile the elements of the following table up to and including $d_{\underline{a}} = 3$. All divisors except one are in Δ_0, the exception being $\underline{\omega}$, with $N_{\underline{\omega}} = 12$, corresponding in our case to uninodal cubics, an orbit reduced to a single element. To proceed further it is simpler to contract (5.15d) with $g^{\mu\nu}$ to find

$d_{\underline{a}} \geq 4$

$$7 N_{\underline{a}} = \sum_{\underline{b}+\underline{c}=\underline{a}} N_{\underline{b}} N_{\underline{c}} (\underline{b}.\underline{c}) \left\{ (\underline{b}.\underline{c}) \begin{pmatrix} d_{\underline{a}} - 4 \\ \\ d_{\underline{b}} - 2 \end{pmatrix} - \underline{b}^2 \begin{pmatrix} d_{\underline{a}} - 4 \\ \\ d_{\underline{b}} - 1 \end{pmatrix} \right\} \tag{5.17}$$

Using this formula, the table below displays divisors in the form

$$a^0 > a^1 \geq a^2 \geq a^3 \geq a^4 \geq a^5 \geq a^6$$

The number of distinct equivalent ones obtained by permutations of space indices is recorded in the column before the last. To obtain a full orbit under G one should apply the reflections (5.11). These serve as checks. The last column gives the size of the corresponding orbit.

We observe that the 72 effective divisors of degree 3 and arithmetic genus 0 form a single orbit under G. Indeed (see Y. Manin "Cubic forms") they are in one to one correspondence with the system R of 72 non zero roots of E_6 (the Lie algebra E_6 has dimension $72 + 6 = 78$)

$$R \equiv \left\{ \underline{a} \in \mathbb{Z}^7 \,\middle|\, \underline{\omega}.\underline{a} = 0, \underline{a}^2 = -2 \right\}$$

This correspondence reads

$$\Delta_{3,0} = \underline{\omega} + R$$

Table: rational curves on $\mathcal{F}_3$

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
0	-1	0	0	0	0	0	1	-1	0	1	6	
1	1	1	0	0	0	0	1	-1	0	1	15	27
2	1	1	1	1	1	0	1	-1	0	1	6	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
1	1	0	0	0	0	0	2	0	0	1	6	
2	1	1	1	1	0	0	2	0	0	1	15	27
3	2	1	1	1	1	1	2	0	0	1	6	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
1	0	0	0	0	0	0	3	1	0	1	1	
2	1	1	1	0	0	0	3	1	0	1	20	
3	2	1	1	1	1	0	3	1	0	1	30	72
4	2	2	2	1	1	1	3	1	0	1	20	
5	2	2	2	2	2	2	3	1	0	1	1	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
3	1	1	1	1	1	1	3	3	1	12	1	1

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
2	1	1	0	0	0	0	4	2	0	1	15	
3	2	1	1	1	0	0	4	2	0	1	60	
4	3	1	1	1	1	1	4	2	0	1	6	216
4	2	2	2	1	1	0	4	2	0	1	60	
5	3	2	2	2	1	1	4	2	0	1	60	
6	3	3	2	2	2	2	4	2	0	1	15	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
3	1	1	1	1	1	0	4	4	1	12	6	
4	2	2	1	1	1	1	4	4	1	12	15	27
5	2	2	2	2	2	1	4	4	1	12	6	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
2	1	0	0	0	0	0	5	3	0	1	6	
3	2	1	1	0	0	0	5	3	0	1	60	
4	3	1	1	1	1	0	5	3	0	1	30	
4	2	2	2	1	0	0	5	3	0	1	60	
5	3	2	2	2	1	0	5	3	0	1	120	432
6	4	2	2	2	2	1	5	3	0	1	30	
6	3	3	3	2	1	1	5	3	0	1	60	
7	4	3	3	2	2	2	5	3	0	1	60	
8	4	3	3	3	3	3	5	3	0	1	6	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
3	1	1	1	1	0	0	5	5	1	12	15	
4	2	2	1	1	1	0	5	5	1	12	60	
5	3	2	2	1	1	1	5	5	1	12	60	216
5	2	2	2	2	2	0	5	5	1	12	6	
6	3	3	2	2	2	1	5	5	1	12	60	
7	3	3	3	3	2	2	5	5	1	12	15	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
4	2	1	1	1	1	1	5	7	2	96	6	
5	2	2	2	2	1	1	5	7	2	96	15	27
6	3	2	2	2	2	2	5	7	2	96	6	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
2	0	0	0	0	0	0	6	4	0	1	1	
4	2	2	2	0	0	0	6	4	0	1	20	72
6	4	2	2	2	0	0	6	4	0	1	30	
8	4	4	4	2	2	2	6	4	0	1	20	
10	4	4	4	4	4	4	6	4	0	1	1	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
3	2	1	0	0	0	0	6	4	0	1	30	
4	3	1	1	1	0	0	6	4	0	1	60	
5	4	1	1	1	1	1	6	4	0	1	6	
5	3	2	2	2	0	0	6	4	0	1	60	432
6	3	3	3	2	1	0	6	4	0	1	120	
7	5	2	2	2	2	2	6	4	0	1	6	
7	4	3	3	3	1	1	6	4	0	1	60	
8	5	3	3	3	2	2	6	4	0	1	60	
9	5	4	3	3	3	3	6	4	0	1	30	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
3	1	1	1	0	0	0	6	6	1	12	20	
4	2	2	1	1	0	0	6	6	1	12	90	
5	3	2	2	1	1	0	6	6	1	12	180	
6	3	3	3	1	1	1	6	6	1	12	20	
6	3	3	2	2	2	0	6	6	1	12	60	720
6	4	2	2	2	1	1	6	6	1	12	60	
7	4	3	3	2	2	1	6	6	1	12	180	
8	4	4	3	3	2	2	6	6	1	12	90	
9	4	4	4	3	3	3	6	6	1	12	20	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
4	2	1	1	1	1	0	6	8	2	96	30	
5	2	2	2	2	1	0	6	8	2	96	30	
5	3	2	1	1	1	1	6	8	2	96	30	
6	3	3	2	2	1	1	6	8	2	96	90	270
7	4	3	2	2	2	2	6	8	2	96	30	
7	3	3	3	3	2	1	6	8	2	96	30	
8	4	3	3	3	3	2	6	8	2	96	30	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
4	1	1	1	1	1	1	6	10	3	620	1	
5	2	2	2	1	1	1	6	10	3	620	20	
6	3	2	2	2	2	1	6	10	3	620	30	72
7	3	3	3	2	2	2	6	10	3	620	20	
8	3	3	3	3	3	3	6	10	3	620	1	

a^0	a^1	a^2	a^3	a^4	a^5	a^6	d	a^2	$p_{\underline{a}}$	$N_{\underline{a}}$		Orb.
6	2	2	2	2	2	2	6	12	4	2.376	1	1

Some numbers are easily recognizable. For instance for $\underline{a} = \underline{\omega}$, $d_{\underline{a}} = 3$ interpreted in the plane as uninodal cubics through $6 + (3 - 1) = 8$ points we find $N_{\underline{a}} = 12$ as in section 3. Similarly for $\underline{a} = (4, 1, 1, 1, 1, 1, 1)$, $d_{\underline{a}} = 6$ interpreted in the plane as three nodal quartics through $6 + (6 - 1) = 11$ points we recover $N_{\underline{a}} = 620$ again as in section 3, although the deduction is apparently different. At the price of cumbersome calculations the table could be extended beyond degree 6.

While we have discussed in this section cubic surfaces in $\mathbb{P}_3$, we could carry similar calculations for other Del Pezzo surfaces in particular those that exhibit invariance of exceptional divisors under the Weyl groups of $E(7)$ and $E(8)$. It would be interesting to understand if there exists a connection between f and standard constructions for the corresponding affine Lie algebras.

6. OPEN ENDS

Our presentation is very incomplete. The main question is of course to set up the path integral which generates the corresponding free energies and therefore enable one to proceed beyond genus zero curves. Alternatively one might inquire about further axioms needed to relate higher genus to lower ones. This might suggest other properties of the partition functions and their possible connection with integrable systems and/or chiral algebras. M. Kontsevich has already noticed in the case of plane curves such a relation with a Painlevé VI equation. At a more modest level - and returning to genus zero plane curves - it is interesting to try and derive an asymptotic estimate for N_d (as given by (3.18)). Numerical evidence provided by P. Di Francesco suggests a behaviour for $d \longrightarrow \infty$ of the type

$$\frac{N_d}{(3d-1)!} \sim (\alpha)^d d^{-\beta} \gamma \left(1 + 0\left(\frac{1}{d}\right)\right) \tag{6.1}$$

with $\alpha \simeq 0.1380 \qquad \beta \simeq 3.500 \qquad \gamma \simeq 0.18868$. Indeed transforming eq.(3.12) into a differential equation for a function of a single variable using (3.17), one can derive the value $7/2$ for the exponent β.

One would also like to study rational curves on higher dimensional algebraic manifolds (assuming no obstruction to the existence of such curves) as well as the corresponding theory for target dimension equal to one. There is a suggestion that this is partly related to the large N limit of 2-dimensional chromodynamics.

Finally, there is the problem of providing a purely geometric proof of the recurrence relations found on rational surfaces.

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