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SKEWNESS AND KURTOSIS IN LARGE-SCALE COSMIC FIELDS

by

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ABSTRACT

In this paper, I present the calculation of the third and fourth moments of both the distribution function of the large–scale density and the large–scale divergence of the velocity field, θ. These calculations are made by the mean of perturbative calculations assuming Gaussian initial conditions and are expected to be valid in the linear or quasi linear regime. The moments are derived for a top–hat window function and for any cosmological parameters Ω and Λ. It turns out that the dependence with Λ is always very weak whereas the moments of the distribution function of the divergence are strongly dependent on Ω. A method to measure Ω using the skewness of this field has already been presented by Bernardeau et al. (1993). I show here that the simultaneous measurement of the skewness and the kurtosis allows to test the validity of the gravitational instability scenario hypothesis. Indeed there is a combination of the first three moments of θ that is almost independent of the cosmological parameters Ω and Λ,

$$\frac{\left(\langle\theta^4\rangle - 3\langle\theta^2\rangle^2\right)\ \langle\theta^2\rangle}{\langle\theta^3\rangle^2} \approx 1.5,$$

(the value quoted is valid when the index of the power spectrum at the filtering scale is close to -1) so that any cosmic velocity field created by gravitational instabilities should verify such a property.

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1. INTRODUCTION

At large scales, say for scales larger than 10 h^{-1} Mpc, the amplitudes of the rms fluctuation observed in the cosmic fields are still below unity showing that these fields have undergone only a moderate evolution. The perturbation theory is then well adapted to address many problems associated with these scales. For instance the growing rate of the fluctuations can be accurately described by the linear approximation (*e.g.*, Efstathiou *et al.* 1988). This is a crucial result for the determination of the initial spectrum. However, the statistical properties of the large–scale cosmic fields cannot be reduced to the behavior of the fluctuation spectrum, as it would have been the case if the fluctuations remain Gaussian. For instance the skewness observed in the density distribution at such large scales (see for example Bouchet *et al.* 1993) is in agreement with the theoretical predictions based on calculations using the second order perturbation theory (Peebles 1980, Goroff *et al.* 1986, Juszkiewicz, Bouchet & Colombi 1993a).

The properties of the cosmic velocity field so far have only been considered in relation with the density field. In the linear theory, there is a well known relationship (Peebles 1980) between the density and the peculiar velocity field, $\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) - H_0\mathbf{r}$,

$$\mathbf{u}(\mathbf{x}) = -f(\Omega, \Lambda)\frac{H_0}{4\pi}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\,\delta(\mathbf{x})\,\mathrm{d}^3\mathbf{x}' \tag{1}$$

where $f(\Omega, \Lambda)$ is a function depending on the cosmological parameters and $\delta(\mathbf{x})$ is the matter overdensity field ($\delta(\mathbf{x}) = \rho(\mathbf{x})/\overline{\rho} - 1$). According to Peebles (1980), the function f can be accurately approximated by $f(\Omega) = \Omega^{0.6}$ and Lahav *et al.* (1991) recently showed that its Λ dependence was extremely weak. So in principle a comparison of the velocity field with the density field should lead to a determination of Ω. This analysis has been done using the dipole measurement (that gives the absolute velocity of our galaxy with respect to the microwave background) and assuming that the local density is given by the galaxy field as given by the 1.2 Jy IRAS survey (Strauss & Davis 1988) or by the QDOT survey (Rowan-Robinson *et al.* 1990). These analysis lead to the conclusion that Ω is close to unity (Kaiser & Lahav 1989). However this method raised criticisms since the convergence of the integral in (1) may be quite slow (see Peacock 1992). This difficulty has been partly overcome by Yahil (1991) who uses a local form of the relation (1),

$$\theta(\mathbf{x}) \equiv \frac{1}{H_0}\nabla.\mathbf{u}(\mathbf{x}) = -f(\Omega, \Lambda)\,\delta(\mathbf{x}) \tag{2}$$

to compare the velocity field as determined by Dekel *et al.* (1991) with the density field. Using the IRAS galaxy sample, he also concludes that Ω should be close to unity. However, Shaya, Tully & Pierce (1992) using optical-based study give a slightly lower value for f leading to $\Omega \lesssim 0.1$. In any case, we must have in mind that the light distribution may be a biased indicator for the actual mass distribution. For small fluctuations the existence of such a bias can be parametrized by the relationship

$$\delta_{\rm galaxy}({\bf x}) = b\,\delta({\bf x}). \tag{3}$$

As a result the methods previously mentionned only give the value of the ratio $f(\Omega)/b$. It then would be of great interest to find a method that would separate these two quantities. Yahil (1991) proposed a way to do it but he assumed some arbitrary form for the nonlinear relationship between the galaxy density field and the matter density field that has no reason to be true. The nonlinear corrections between the matter density field and the velocity field have been investigated either by numerical simulations (Nusser & Dekel 1991) or on theoretical basis (Bernardeau 1992b). However these corrections do not give a way to separate the measurement of Ω from the bias effects (see section 4.2 of this paper).

One way to overcome the problem of galaxy biasing is simply to use only the velocity field as a probe of the large–scale structures. The linear theory (eq. [1] or equivalently [2]) is obviously of no help. However some statistical properties of the velocity field can be derived from perturbative calculations at higher orders. It is noticeable that the divergence of the peculiar velocity field is a scalar field that can be analyzed in a similar way than the density field. And as data on large–scale velocity field begin to be available (Lynden-Bell *et al.* 1988, Dekel, Bertschinger & Faber 1991) it becomes crucial to explore the various statistical indicators that could be of cosmological interests in such a field. Following the work of Dekel *et al.* (1991) it seems possible to build the full 3D velocity field from the mere knowledge of the line of sight velocities of a sample of galaxies. The reconstruction is based on the assumption that the large–scale velocity field must be only potential - non–rotational - so that it derives from a simple scalar field. The divergence of the velocity field then contains as much information as the peculiar velocity field apart from a possible uniform flow which just corresponds to the invariance of the equations of motion under a Galilean transformation.

In the following I will assume that the large–scale structures formed by gravitational instabilities from initial Gaussian fluctuations. There are indications that the initial fluctuations were actually Gaussian in the galaxy distribution (see further in this introduction). Then quantities such as the skewness, $\langle\delta^3\rangle$ and $\langle\theta^3\rangle$, or the kurtosis, $\langle\delta^4\rangle - 3\langle\delta^2\rangle^2$ and $\langle\theta^4\rangle - 3\langle\theta^2\rangle^2$, of the cosmic fields are then of great interest ($\langle.\rangle$ is an ensemble average over the initial fluctuations). Indeed, because of the general properties of Gaussian variables, these quantities are zero for the initial fields, and the use of the linear theory would conclude that they remain zero. But in fact gravity, because of the nonlinearities it contains, induces specific non-Gaussian features that, even at very large–scales, will be exhibited in these parameters in a very precise way. The first example of such a calculation was given by Peebles (1980) for the skewness. He showed that its leading term for small σ

can be derived from a second order perturbative calculation (and higher orders give only negligible corrections). Fry (1984) extended these results and showed that the kurtosis can be obtained from third order calculations. These results were extended afterwards in several directions. A method was presented by Bernardeau (1992a) to get the whole series of the cumulants. The Ω dependence and the smoothing effects have been calculated for the density skewness (Bouchet *et al.* 1992, Goroff *et al.* 1986, Juszkiewicz, Bouchet & Colombi 1993) and a recent paper (Bernardeau *et al.* 1993) extends these results to the velocity divergence presenting the dependence of its skewness with Ω and the shape of the power spectrum for various window functions. The result reads,

$$\frac{\langle\theta^3\rangle}{\langle\theta^2\rangle^2} \approx \frac{-1}{\Omega^{0.6}}\left[\frac{26}{7}-(3+n)\right], \tag{4}$$

for a top–hat window function (volume–weighted filtering) and a power law spectrum of index n. As can be seen in the relation (4) the skewness has a dependence with the variance –proportional to its square– that is specific of Gaussian initial conditions. The generic relation for non–Gaussian fluctuations would have been $\langle\theta^3\rangle \propto \langle\theta^2\rangle^{3/2}$. Such a property is also true for the density field and for the same reason. It has been successfully checked in the IRAS galaxy survey (Bouchet *et al.* 1993), bringing strong support to the Gaussian fluctuations hypothesis. The relation (4) has been proposed to determine Ω from the observed velocity field (Bernardeau *et al.* 1993). The purpose of this paper is to extend the result (4) to non-zero values of Λ and to derive a similar property for the kurtosis for both the density and the velocity fields. In part 2, I present the principle of the calculations and in part 3, the results. They are restricted to a top–hat window function due to its special geometrical properties presented in the appendix. Part 4 is devoted to comments and comparisons with numerical simulations.

2. THE METHOD

2.1. *The initial conditions*

At large scale the divergence of the velocity field, as the density field, has fluctuations, the distribution of which can be examined by perturbative calculations. The first order solution in $\delta(\mathbf{x})$ is directly related to the first order in $\theta(\mathbf{x})$ through the continuity equation (Eqs. [1-2]). The variance of this field is then, in the linear regime, driven by the first order approximation. It reads

$$\langle\theta(\mathbf{x})^2\rangle = f(\Omega)^2\langle\delta(\mathbf{x})^2\rangle \tag{5}$$

when the growing mode only is taken into account.

In the following, I assume initial Gaussian fluctuations. The density field, at first order, can then be written

$$\delta^{(1)}(\mathbf{x},t) = D_1(t)\epsilon(\mathbf{x}) = D_1(t)\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\,\epsilon_{\mathbf{k}}\,e^{\mathrm{i}\mathbf{k}.\mathbf{x}} \tag{6}$$

where $D_1(t)$ if the growing factor (see part 3 for the equation the solution of which $D_1(t)$ is) and the random variables $\epsilon_{\mathbf{k}}$ follow the rules,

$$\begin{aligned}\left\langle\epsilon_{\mathbf{k}}\epsilon_{\mathbf{k}'}\right\rangle &= \delta_D(\mathbf{k}+\mathbf{k}')P(k);\\ \left\langle\epsilon_{\mathbf{k}_1}\dots\epsilon_{\mathbf{k}_{2p+1}}\right\rangle &= 0;\\ \left\langle\epsilon_{\mathbf{k}_1}\dots\epsilon_{\mathbf{k}_{2p}}\right\rangle &= \frac{1}{2^p p!}\sum_{\text{permutations }(s)}\left\langle\epsilon_{\mathbf{k}_{s_1}}\epsilon_{\mathbf{k}_{s_2}}\right\rangle\dots\left\langle\epsilon_{\mathbf{k}_{s_{2p-1}}}\epsilon_{\mathbf{k}_{s_{2p}}}\right\rangle.\end{aligned} \tag{7}$$

where the function δ_D is the 3D Dirac distribution, and $P(k)$ is the spectrum of the fluctuations. The divergence of the velocity field now reads,

$$\theta^{(1)}(\mathbf{x},t) = -f(\Omega)D_1(t)\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\,\epsilon_{\mathbf{k}}\,e^{\mathrm{i}\mathbf{k}.\mathbf{x}}, \tag{8}$$

at the first order in $\epsilon_{\mathbf{k}}$. At this order, the fields (6) and (8) are purely Gaussian and their whole statistics is determined by the shape of the power spectrum. Specific features induced by gravity appear when terms of greater orders are taken into account.

2.2. *The equations of the dynamics*

Throughout the paper the matter field is approximated by a nonrelativistic and self-gravitating fluid at zero pressure. I do not exclude the existence of a non-zero cosmological constant Λ. The expansion factor, $a(t)$, is then solution of the equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi\,G\,\overline{\rho}}{3} + \frac{\Lambda}{3} \tag{9}$$

For convenience I will characterize the (time dependent) cosmological parameters by Ω and by the Hubble constant, H, and its time derivative. Let me recall that Ω is defined by

$$4\pi\,G\,\overline{\rho} = \frac{3\Omega}{2}H^2. \tag{10}$$

(At the present time $H = H_0$.) The fluid is described by the overdensity field $\delta(\mathbf{x},t)$ and by the peculiar velocity field $\mathbf{u}(\mathbf{x},t)$ assumed to be non-rotational so that

$$\nabla_{\mathrm{x}}\times\mathbf{u}(\mathbf{x},t) = 0. \tag{11}$$

The equations of the dynamics are then given by (*e.g.*, Peebles 1980),

$$\begin{aligned} \frac{\partial}{\partial t}\delta(\mathbf{x},t)+\frac{1}{a}\nabla_{\mathrm{X}}\,.\,[(1+\delta(\mathbf{x},t))\mathbf{u}(\mathbf{x},t)]=0 \\ \frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t)+\frac{\dot a}{a}\mathbf{u}(\mathbf{x},t)+\frac{1}{a}(\mathbf{u}(\mathbf{x},t).\nabla_{\mathrm{X}}\,)\mathbf{u}(\mathbf{x},t)=-\frac{1}{a}\nabla_{\mathrm{X}}\,\psi(\mathbf{x},t) \\ {\nabla_{\mathrm{X}}}^2\psi(\mathbf{x},t)=4\pi G\overline{\rho}a^2\delta(\mathbf{x},t), \end{aligned} \tag{12}$$

where the spatial derivatives are taken with respect to $\mathbf{x}$. The first step is to take the divergence of the second equation so that the system now reads, using the definition of Ω,

$$\begin{aligned} \frac{1}{H}\dot\delta(\mathbf{x},t)+[1+\delta(\mathbf{x},t)]\theta(\mathbf{x},t)+\frac{1}{aH}\nabla_{\mathrm{X}}\,.[\delta(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)]=0 \\ \left(2+\frac{\dot H}{H^2}\right)\theta(\mathbf{x},t)+\frac{1}{H}\dot\theta(\mathbf{x},t)+\frac{3}{2}\Omega\;\delta(\mathbf{x},t)+\frac{1}{aH}\nabla_{\mathrm{X}}\,.[\mathbf{u}(\mathbf{x},t).\nabla_{\mathrm{X}}\,]\mathbf{u}(\mathbf{x},t)=0 \end{aligned} \tag{13}$$

where a dot is for a time derivative.

It is convenient to introduce the Fourier transforms of these fields, $\delta(\mathbf{k},t)$ and $\theta(\mathbf{k},t)$, defined by

$$\begin{aligned} \delta(\mathbf{x},t)&=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\delta(\mathbf{k},t)\exp(\mathrm{i}\mathbf{x}.\mathbf{k}), \\ \theta(\mathbf{x},t)&=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\theta(\mathbf{k},t)\exp(\mathrm{i}\mathbf{x}.\mathbf{k}). \end{aligned} \tag{14}$$

Due to the absence of rotational part in the velocity field, $\mathbf{u}(\mathbf{x})$ simply reads

$$\mathbf{u}(\mathbf{x},t)=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\frac{-\mathrm{i}\,\mathbf{k}}{k^2}\,\theta(\mathbf{k},t)\exp(\mathrm{i}\mathbf{x}.\mathbf{k}). \tag{15}$$

Using the relations (14-15) I can now write the equation of the dynamics in the $\mathbf{k}$-space:

$$\begin{aligned} &\frac{1}{H}\dot\delta(\mathbf{k},t)+\theta(\mathbf{k},t)+\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^{3/2}}\,\mathcal{P}(\mathbf{k}',\mathbf{k}-\mathbf{k}')\,\delta(\mathbf{k}-\mathbf{k}',t)\,\theta(\mathbf{k}',t)=0 \\ &\left(2+\frac{\dot H}{H^2}\right)\theta(\mathbf{k},t)+\frac{1}{H}\,\dot\theta(\mathbf{k},t)+\frac{3}{2}\,\Omega\delta(\mathbf{k},t)= \\ &-\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^{3/2}}\left[\mathcal{P}(\mathbf{k}',\mathbf{k}-\mathbf{k}')+\mathcal{P}(\mathbf{k}-\mathbf{k}',\mathbf{k}')-2\mathcal{Q}(\mathbf{k}',\mathbf{k}-\mathbf{k}')\right]\theta(\mathbf{k}-\mathbf{k}',t)\,\theta(\mathbf{k}',t), \end{aligned} \tag{16}$$

where

$$\mathcal{P}(\mathbf{k},\mathbf{k}') = 1 + \frac{\mathbf{k}.\mathbf{k}'}{k^2}, \tag{17}$$

$$\mathcal{Q}(\mathbf{k},\mathbf{k}') = 1 - \frac{\left(\mathbf{k}.\mathbf{k}'\right)^2}{k^2 k'^2}. \tag{18}$$

These functions will be extensively used in the following. The system (16) is the basic system of equations used for the perturbative calculations.

2.3. *The filtering of the fields*

In practice we have to consider the density and the velocity field after they have been filtered by a particular window function. Indeed perturbative calculations are only valid at (very) large scale, so that the strong small–scale fluctuations should be smoothed out. I make the assumption that the existence of nonlinear fluctuations at small scales will not change the result. This assumption has been verified for instance for the time dependence of the variance at large scale in various numerical simulations (see for instance Efstathiou *et al.* 1988). There are no reasons to think that this is not true for the purpose of these calculations and the comparison with numerical simulations presented in part 3 confirm the validity of this hypothesis.

For technical reason I only consider the case of the top–hat window function. However, nothing in practice prevents to use any other window function. For instance Goroff *et al.* (1986) present results for the density field using a Gaussian window function but in such a case the smoothing corrections have been to be calculated numerically for each power spectrum. For a top–hat window function all the results can be given analytically as shown in the next part.

As it has been noticed by Bernardeau *et al.* (1993) it is equivalent to apply the filter to the velocity field or to its divergence: both are linear operations that commute each other. Then I focus my interest on the filtered density and divergence of the velocity field. I define $\delta(R_0)$ and $\theta(R_0)$ by

$$\delta(R_0) = \int \mathrm{d}^3\mathbf{x}\ W_{\mathrm{TH}}(\mathbf{x})\ \delta(\mathbf{x}), \tag{19}$$

$$\theta(R_0) = \int \mathrm{d}^3\mathbf{x}\ W_{\mathrm{TH}}(\mathbf{x})\ \theta(\mathbf{x}), \tag{20}$$

with

$$\begin{aligned} W_{\mathrm{TH}}(\mathbf{x}) &= 1 \text{ if } |\mathbf{x}| \le R_0; \\ W_{\mathrm{TH}}(\mathbf{x}) &= 0 \text{ otherwise}; \end{aligned} \tag{21}$$

for a scale R_0. Note that, for the velocity field, it corresponds to a volume weighted filtering: the velocity is equally weighted in each point of space, regardless of the local density. In practice, the velocity is known only at the positions of the galaxies (or the matter points for a numerical simulation) so that it is necessary to smooth the field in two steps: the local field has to be defined in each point by a small scale (mass-weighted) filtering and then filtered with a volume weighted procedure. This is what have been done by Juszkiewicz *et al.* (1993b). In (21) the filter is applied at the origin but the statistics of the filtered fields obviously does not depend on the position at which the filter is applied. I also make the hypothesis that a volume average is equivalent to an ensemble average so that the moments of distribution functions of δ and θ are in principle measurable in a large enough sample.

The quantities (21) can be expressed with the Fourier transforms of respectively the density field and the divergence field. They read,

$$\delta(R_0) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} W_{\mathrm{TH}}(\mathbf{k}\,R_0)\,\delta(\mathbf{k}), \tag{22}$$

$$\theta(R_0) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} W_{\mathrm{TH}}(\mathbf{k}\,R_0)\,\theta(\mathbf{k}), \tag{23}$$

where $W_{\mathrm{TH}}(k\,R_0)$ is the Fourier transform of the top–hat window function,

$$W_{\mathrm{TH}}(k\,R_0) = \frac{3}{(kR_0)^3}\left(\sin(kR_0) - kR_0\cos(kR_0)\right).$$

The variance of these smoothed fields is then given by,

$$\begin{aligned}\sigma^2(R_0) &= \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\,\frac{\mathrm{d}^3k'}{(2\pi)^{3/2}} W_{\mathrm{TH}}(k\;R_0)\,\left\langle\delta(\mathbf{k})\delta(\mathbf{k}')\right\rangle\,W_{\mathrm{TH}}(k'\;R_0)\\ &= D_1^2(t)\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} W_{\mathrm{TH}}^2(k\;R_0)\;P(k),\end{aligned} \tag{24}$$

$$\sigma_\theta^2(R_0) = f^2(\Omega)\;D_1^2(t)\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} W_{\mathrm{TH}}^2(k\;R_0)\;P(k). \tag{25}$$

These results are valid at large scale where, by definition, the linear approximation can be applied to the determination of the variance of the density field. The validity domain of the results given in the following is in principle the same as the validity domain of (24-25).

2.4. *Calculation of the skewness*

The principle of this calculation has been presented in previous papers (Goroff *et al.* 1986, Bouchet *et al.* 1992 for the density field, Bernardeau *et al.* 1993 for the divergence of the velocity field). The aim is to calculate the leading order of the third moment of the distribution functions of $\delta(R_0)$ or of $\theta(R_0)$. The calculation is based on an expansion of these quantities with respect to the random variables $\epsilon_{\mathbf{k}}$ that describe the initial density and velocity fluctuations. So I write

$$\delta(R_0) = \delta^{(1)}(R_0) + \delta^{(2)}(R_0) + \delta^{(3)}(R_0) + \dots \tag{26}$$

where $\delta^{(1)}(R_0)$ is proportional to $\epsilon_{\mathbf{k}}$, $\delta^{(2)}(R_0)$ is quadratic in $\epsilon_{\mathbf{k}}$ and $\delta^{(3)}(R_0)$ is cubic in $\epsilon_{\mathbf{k}}$... A similar expansion can be written for $\theta(R_0)$,

$$\theta(R_0) = \theta^{(1)}(R_0) + \theta^{(2)}(R_0) + \theta^{(3)}(R_0) + \dots \tag{27}$$

Then the ensemble averages of $[\delta(R_0)]^3$ and $[\theta(R_0)]^3$ read

$$\begin{aligned}\big\langle\delta(R_0)^3\big\rangle &= \Big\langle \Big[\delta^{(1)}(R_0) + \delta^{(2)}(R_0) + \delta^{(3)}(R_0) + \dots\Big]^3 \Big\rangle \\ &= \Big\langle \Big[\delta^{(1)}(R_0)\Big]^3 \Big\rangle + 3\Big\langle \Big[\delta^{(1)}(R_0)\Big]^2 \delta^{(2)}(R_0)\Big\rangle + \dots, \end{aligned}\tag{28}$$

$$\begin{aligned}\big\langle\theta(R_0)^3\big\rangle &= \Big\langle \Big[\theta^{(1)}(R_0) + \theta^{(2)}(R_0) + \theta^{(3)}(R_0) + \dots\Big]^3 \Big\rangle \\ &= \Big\langle \Big[\theta^{(1)}(R_0)\Big]^3 \Big\rangle + 3\Big\langle \Big[\theta^{(1)}(R_0)\Big]^2 \theta^{(2)}(R_0)\Big\rangle + \dots \end{aligned}\tag{29}$$

The first terms of these expansions are zero due to the general properties of Gaussian variables. The leading terms will be respectively $3\big\langle \big[\delta^{(1)}(R_0)\big]^2 \delta^{(2)}(R_0)\big\rangle$ and $3\big\langle \big[\theta^{(1)}(R_0)\big]^2 \theta^{(2)}(R_0)\big\rangle$ that involve the product of four Gaussian variables. Then the ratios,

$$S_3 = \frac{\big\langle\delta(R_0)^3\big\rangle}{\big\langle\delta(R_0)^2\big\rangle^2}, \tag{30}$$

$$S_{3\theta} = \frac{\big\langle\theta(R_0)^3\big\rangle}{\big\langle\theta(R_0)^2\big\rangle^2}, \tag{31}$$

are finite at large scale. The purpose of the next part is to calculate these limits S_3 and $S_{3\theta}$ as a function of the cosmological parameters Ω and Λ and the shape of the power spectrum.

2.5. *Calculation of the kurtosis*

The skewness is a sort of measure of the asymmetry of the distribution function. The kurtosis measures the flatness of the distribution function compared to what would be expected from a Gaussian distribution. The leading order of this term involves both the second order and third order of the dynamics. It reads

$$\begin{aligned}\big\langle\delta^4(R_0)\big\rangle - 3\big\langle\delta^2(R_0)\big\rangle^2 \approx\ & 6\Big\langle \Big[\delta^{(1)}(R_0)\Big]^2 \Big[\delta^{(2)}(R_0)\Big]^2 \Big\rangle + 4\Big\langle \Big[\delta^{(1)}(R_0)\Big]^3 \delta^{(3)}(R_0)\Big\rangle \\ & - 6\Big\langle \Big[\delta^{(1)}(R_0)\Big]^2 \Big\rangle \Big\langle \Big[\delta^{(2)}(R_0)\Big]^2 \Big\rangle - 12\Big\langle \Big[\delta^{(1)}(R_0)\Big]^2 \Big\rangle \big\langle\delta^{(1)}(R_0)\delta^{(3)}(R_0)\big\rangle \end{aligned}\tag{32}$$

for the density field and

$$\big\langle\theta^4(R_0)\big\rangle - 3\big\langle\theta^2(R_0)\big\rangle^2 \approx\ 6\left\langle\,\left[\theta^{(1)}(R_0)\right]^2\left[\theta^{(2)}(R_0)\right]^2\right\rangle + 4\left\langle\,\left[\theta^{(1)}(R_0)\right]^3\theta^{(3)}(R_0)\right\rangle$$
$$- 6\left\langle\,\left[\theta^{(1)}(R_0)\right]^2\right\rangle\left\langle\,\left[\theta^{(2)}(R_0)\right]^2\right\rangle - 12\left\langle\,\left[\theta^{(1)}(R_0)\right]^2\right\rangle\big\langle\theta^{(1)}(R_0)\theta^{(3)}(R_0)\big\rangle \tag{33}$$

for the divergence of the velocity field. In the calculation of these differences, we can notice that the terms that are subtracted actually appear in the expression of the fourth moment. They contain the non–connected part of the moment (so called since they factorize in products of moments) and will exactly vanish when the fourth order cumulant is calculated. For instance the fourth cumulant of the density formally reads

$$\big\langle\delta^4\big\rangle_c \approx 6\left\langle\,\left[\delta^{(1)}(R_0)\right]^2\left[\delta^{(2)}(R_0)\right]^2\right\rangle_c + 4\left\langle\,\left[\delta^{(1)}(R_0)\right]^3\delta^{(3)}(R_0)\right\rangle_c,$$

where the subscript c means that only the connected terms are retained. We call then S_4 and $S_{4\theta}$ the large–scale limits of the ratios,

$$S_4 = \frac{\big\langle\delta^4\big\rangle - 3\big\langle\delta^2\big\rangle^2}{\big\langle\delta^2\big\rangle^3}, \tag{34}$$

$$S_{4\theta} = \frac{\big\langle\theta^4\big\rangle - 3\big\langle\theta^2\big\rangle^2}{\big\langle\theta^2\big\rangle^3}. \tag{35}$$

The next part is devoted to the derivation of the expressions of S_3, $S_{3\theta}$, S_4 and $S_{4\theta}$.

3. THE CALCULATIONS

3.1. *The dynamics of the spherical collapse*

To calculate the previous limits, the dynamics of the spherical collapse is quite helpful. In general the equation describing the evolution of the size of spherical symmetric perturbation reads,

$$\ddot{R} = -\frac{G\,M(<R)}{R^2} + \frac{\Lambda}{3}R, \tag{36}$$

where R is the size of the perturbation and $M(<R)$ is the constant mass within the radius R. The dot stands for a time derivative. This equation can be written in term of the overdensity δ defined by, $\delta = (R/R_0)^{-3} - 1$, where R_0 is the initial

(comoving) radius of the perturbation. The equation (36), is equivalent to (using Eq. [10])

$$-\frac{1}{2}H^2\Omega\,\frac{1}{1+\delta}-\frac{2}{3}\frac{H\dot{\delta}}{(1+\delta)^2}-\frac{1}{3}\frac{\ddot{\delta}}{(1+\delta)^2}+\frac{4}{9}\frac{\dot{\delta}^2}{(1+\delta)^3}=-\frac{1}{2}H^2\Omega. \tag{37}$$

The latter equation can be linearized in δ. Let me write this linear term as

$$\delta^{(1)}(t)=D_1(t)\delta_i$$

where δ_i gives the strength of the initial fluctuation and is supposed to be small. The function $D_1(t)$ is the time dependence of the growing mode and it is the solution of the equation,

$$\ddot{D}_1+2H\,\dot{D}_1-\frac{3}{2}H^2\,\Omega\,D_1=0, \tag{38}$$

which is a growing function of time. The function D_1 is proportional to the expansion factor when t is small or for an Einstein–de Sitter universe, but this is not true in general. The second order in δ (when δ_i is assumed to be the small parameter) reads

$$\delta^{(2)}(t)=D_2(t)\,\frac{\delta_i^2}{2}$$

where the function $D_2(t)$ is solution of the equation,

$$\ddot{D}_2+2H\,\dot{D}_2-\frac{3}{2}H^2\,\Omega\,D_2=3H^2\,\Omega\,D_1^2+\frac{8}{3}\dot{D}_1^{\,2}, \tag{39a}$$

and verifies

$$D_2(t)\sim\frac{34}{21}D_1^2(t)\quad\text{when}\quad t\to 0. \tag{39b}$$

Similarly the third order reads

$$\delta^{(3)}(t)=D_3(t)\,\frac{\delta_i^3}{6}$$

with

$$\ddot{D}_3+2H\,\dot{D}_3-\frac{3}{2}H^2\,\Omega\,D_3=9H^2\,\Omega\,D_1\,D_2+8\dot{D}_1\,\dot{D}_2-8D_1\,\dot{D}_1^{\,2} \tag{40a}$$

and

$$D_3(t)\sim\frac{682}{189}D_1^3(t)\quad\text{when}\quad t\to 0. \tag{40b}$$

In the following I will use the functions D_1, D_2 and D_3 and their time derivatives to build the second and third order of the density and velocity fields. There are two other functions that are convenient to consider. They are given by the evolution of the divergence of the velocity field, given by $\theta(t) = 3\dot{R}/HR - 3$ all along the collapse. This function can be expanded with respect to δ_i in the form,

$$\theta(t) = -f(\Omega,\Lambda)\left[E_1(t)\delta_i + E_2(t)\frac{\delta_i^2}{2} + E_3(t)\frac{\delta_i^3}{6} + \dots\right], \tag{41}$$

Where the factor $f(\Omega,\Lambda)$ is given by definition by

$$f(\Omega,\Lambda) = \frac{a}{D_1}\frac{\mathrm{d}D_1}{\mathrm{d}a}. \tag{42}$$

From the equation of the dynamics we get

$$E_1(t) = D_1(t);$$

$$E_2(t) = D_1\frac{\mathrm{d}}{\mathrm{d}D_1}\left[D_2 - D_1^2\right]; \tag{43}$$

$$E_3(t) = D_1\frac{\mathrm{d}}{\mathrm{d}D_1}\left[D_3 - 3D_2\,D_1 + 2D_1^3\right]. \tag{44}$$

The exact analytic form of these functions can be obtained from the solution of the spherical collapse (*e.g.*, Peebles 1980), but they are complicated and it is unnecessary to give them explicitly. For a non-zero Λ they have to be integrated numerically from the equations (39-40). The time variation of the ratios $D_2(t)/D_1(t)$, $D_3(t)/D_1(t)^3$,... appears simply as a variation of these ratios with the cosmological parameters (Ω,Λ) as they cover a given trajectory in the (Ω,Λ) plan. Here, I present the results of numerical integrations only in the case of zero curvature, $\Omega+\Lambda/3H^2 = 1$ and the analytical results for $\Lambda = 0$. The ratios are given as a function of Ω in Figs. 1-2.

3.2. *Expressions of* $\delta^{(1)}(R_0)$ *and* $\theta^{(1)}(R_0)$

The fields at the first order are given by the linear solution (Eqs. [6, 8]). As a result we have

$$\delta^{(1)}(R_0) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\epsilon_{\mathbf{k}}\,W_{\mathrm{TH}}(k\,R_0)\,D_1(t) \tag{45}$$

and

$$\theta^{(1)}(R_0) = -f(\Omega,\Lambda)\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\epsilon_{\mathbf{k}}\,W_{\mathrm{TH}}(k\,R_0)\,D_1(t). \tag{46}$$

3.3. *Expressions of* $\delta^{(2)}(R_0)$ *and* $\theta^{(2)}(R_0)$

We have to write the system (18) at the second order in $\epsilon_{\mathbf{k}}$. It leads to the system,

$$
\begin{aligned}
&\frac{1}{H}\dot{\delta}^{(2)}(\mathbf{k},t)+\theta^{(2)}(\mathbf{k},t)=a\frac{\mathrm{d}D_1}{\mathrm{d}a}D_1\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^{3/2}}\mathcal{P}(\mathbf{k}',\mathbf{k}-\mathbf{k}')\epsilon_{\mathbf{k}-\mathbf{k}'}\,\epsilon_{\mathbf{k}'}\\
&\left(2+\frac{\dot{H}}{H^2}\right)\theta^{(2)}(\mathbf{k},t)+\frac{1}{H}\,\dot{\theta}^{(2)}(\mathbf{k},t)+\frac{3}{2}\,\Omega\delta^{(2)}(\mathbf{k},t)=\\
&-\left[a\frac{\mathrm{d}D_1}{\mathrm{d}a}\right]^2\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^{3/2}}\left[\mathcal{P}(\mathbf{k}',\mathbf{k}-\mathbf{k}')+\mathcal{P}(\mathbf{k}-\mathbf{k}',\mathbf{k}')-2\mathcal{Q}(\mathbf{k}',\mathbf{k}-\mathbf{k}')\right]\epsilon_{\mathbf{k}-\mathbf{k}'}\,\epsilon_{\mathbf{k}'}
\end{aligned}
\tag{47}
$$

It is convenient to define the notations $\mathcal{P}_{i,j}$ and $\mathcal{Q}_{i,j}$ by

$$
\mathcal{P}_{i,j}=\mathcal{P}(\mathbf{k}_i,\mathbf{k}_j),\quad \mathcal{Q}_{i,j}=\mathcal{Q}(\mathbf{k}_i,\mathbf{k}_j).
$$

In the following the indices (i,j) will run between 1 and 3. As a result of Eq. (47) we obtain

$$
\begin{aligned}
\delta^{(2)}(R_0)=&\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^{3/2}}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^{3/2}}\epsilon_{\mathbf{k}_1}\epsilon_{\mathbf{k}_2}\,W_{\rm TH}\left(|\mathbf{k}_1+\mathbf{k}_2|\;R_0\right)\\
&\times\left(D_1^2\left[\mathcal{P}_{1,2}-\frac{3}{2}\mathcal{Q}_{1,2}\right]+\frac{3}{2}\,D_2\,\mathcal{Q}_{1,2}\right)
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
\theta^{(2)}(R_0)=-&\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^{3/2}}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^{3/2}}\epsilon_{\mathbf{k}_1}\epsilon_{\mathbf{k}_2}W_{\rm TH}\left(|\mathbf{k}_1+\mathbf{k}_2|\;R_0\right)\\
&\times f(\Omega,\Lambda)\left(D_1^2\left[\mathcal{P}_{1,2}-\frac{3}{2}\mathcal{Q}_{1,2}\right]+\frac{3}{4}\,E_2\,\mathcal{Q}_{1,2}\right).
\end{aligned}
\tag{49}
$$

These results have already been obtained by Bouchet *et al.* (1992) for the density field and by Bernardeau *et al.* (1993) for the velocity field in case of $\Lambda=0$.

3.4. *Expressions of $\delta^{(3)}(R_0)$ and $\theta^{(3)}(R_0)$*

The calculation of these functions is based on the same principle as for the second order. The system (16) has now to be written at the third order in $\epsilon_{\mathbf{k}}$,

$$\begin{aligned}
&\frac{1}{H}\dot{\delta}^{(3)}(\mathbf{k},t)+\theta^{(3)}(\mathbf{k},t)= \\
&-\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^{3/2}}\mathcal{P}(\mathbf{k}',\mathbf{k}-\mathbf{k}')\left[\delta^{(1)}(\mathbf{k}-\mathbf{k}')\,\theta^{(2)}(\mathbf{k}')+\delta^{(2)}(\mathbf{k}-\mathbf{k}')\,\theta^{(1)}(\mathbf{k}')\right] \\
&\left(2+\frac{\dot{H}}{H^2}\right)\theta^{(3)}(\mathbf{k},t)+\frac{1}{H}\,\dot{\theta}^{(3)}(\mathbf{k},t)+\frac{3}{2}\,\Omega\delta^{(3)}(\mathbf{k},t)= \\
&-\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^{3/2}}\left[\mathcal{P}(\mathbf{k}',\mathbf{k}-\mathbf{k}')+\mathcal{P}(\mathbf{k}-\mathbf{k}',\mathbf{k}')-2\,\mathcal{Q}(\mathbf{k}',\mathbf{k}-\mathbf{k}')\right]\;\theta^{(1)}(\mathbf{k}-\mathbf{k}')\,\theta^{(2)}(\mathbf{k}')
\end{aligned} \tag{50}$$

I have to define an extra function that will be useful for the intermediate calculations, $F_3(t)$, defined by the equation,

$$\ddot{F}_3+2H\,\dot{F}_3-\frac{3}{2}H^2\,\Omega\,F_3=\frac{3}{2}H^2\,\Omega\,D_1^3 \tag{51}$$

and which behaves like $9\,D_1^3(t)/10$ when t is small. This function will not enter in the expressions of the kurtosis so this equation has not to be solved. Then we can search a solution of the system (50) having the form,

$$\begin{aligned}
\delta^{(3)}(\mathbf{k})=(2\pi)^{3/2}\int&\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^{3/2}}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^{3/2}}\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^{3/2}}\epsilon_{\mathbf{k}_1}\epsilon_{\mathbf{k}_2}\epsilon_{\mathbf{k}_3}\,\delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \\
&\times\left[D_1^3\,\mathcal{R}_1(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)+D_1\,D_2\,\mathcal{R}_2(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\right. \\
&\quad\left.+D_3\,\mathcal{R}_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)+F_3\,\mathcal{R}_4(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\right]
\end{aligned} \tag{52}$$

$$\begin{aligned}
\theta^{(3)}(\mathbf{k})=(2\pi)^{3/2}\int&\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^{3/2}}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^{3/2}}\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^{3/2}}\epsilon_{\mathbf{k}_1}\epsilon_{\mathbf{k}_2}\epsilon_{\mathbf{k}_3}\,\delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \\
&\times\left[a\frac{\mathrm{d}D_1}{\mathrm{d}a}\,D_1^2\,\mathcal{S}_1(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)+a\frac{\mathrm{d}D_1}{\mathrm{d}a}\,D_2\,\mathcal{S}_2(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\right. \\
&\quad+a\frac{\mathrm{d}D_2}{\mathrm{d}a}D_1\mathcal{S}_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)+a\frac{\mathrm{d}D_3}{\mathrm{d}a}\,\mathcal{S}_4(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \\
&\quad\left.+a\frac{\mathrm{d}F_3}{\mathrm{d}a}\,\mathcal{S}_5(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\right]
\end{aligned} \tag{53}$$

where the functions $\mathcal{R}_1\ldots\mathcal{S}_5$ are homegeneous functions of the wave vectors $(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ and have to satisfy the system (50). From the properties of the functions D_1, D_2, D_3 and F_3 (Eqs. [38, 39, 40, 51]), one can find a natural solution of this system. Its writing is a bit complicated so that I introduce new simplified notations,

$$\mathcal{P}_{ij,k}=\mathcal{P}(\mathbf{k}_i+\mathbf{k}_j,\mathbf{k}_k),\qquad\mathcal{P}_{i,jk}=\mathcal{P}(\mathbf{k}_i,\mathbf{k}_j+\mathbf{k}_k).$$

and similar definitions for $\mathcal{Q}_{ij,k}$ and $\mathcal{Q}_{i,jk}$. Note that $\mathcal{Q}_{ij,k} = \mathcal{Q}_{k,ij}$ but in general $\mathcal{P}_{ij,k} \neq \mathcal{P}_{k,ij}$. The homogeneous functions $\mathcal{R}_1, \ldots, \mathcal{S}_5$ then read

$$
\begin{aligned}
\mathcal{R}_1 &= \left(\frac{1}{2}\mathcal{P}_{3,12} + \frac{1}{2}\mathcal{P}_{12,3} - \frac{1}{3}\mathcal{Q}_{3,12}\right)\mathcal{P}_{1,2} + \left(-\frac{3}{2}\mathcal{P}_{12,3} - \frac{4}{3}\mathcal{P}_{3,12} + \frac{5}{2}\mathcal{Q}_{3,12}\right)\mathcal{Q}_{1,2};\\
\mathcal{R}_2 &= \frac{3}{4}\left[\mathcal{P}_{3,12} + \mathcal{P}_{12,3} - 3\mathcal{Q}_{3,12}\right]\mathcal{Q}_{1,2};\\
\mathcal{R}_3 &= \frac{3}{8}\mathcal{Q}_{3,12}\,\mathcal{Q}_{1,2};\\
\mathcal{R}_4 &= \frac{2}{3}\mathcal{Q}_{3,12}\,\mathcal{P}_{1,2} - \left(\frac{1}{3}\mathcal{P}_{3,12} + \frac{1}{2}\mathcal{Q}_{3,12}\right)\mathcal{Q}_{1,2};\\
\mathcal{S}_1 &= \left(-\frac{1}{2}\mathcal{P}_{3,12} - \frac{1}{2}\mathcal{P}_{12,3} + \mathcal{Q}_{3,12}\right)\mathcal{P}_{1,2} + \left(\frac{5}{2}\mathcal{P}_{12,3} + \frac{3}{2}\mathcal{P}_{3,12} - \frac{15}{2}\mathcal{Q}_{3,12}\right)\mathcal{Q}_{1,2};\\
\mathcal{S}_2 &= \frac{3}{4}\left(-\mathcal{P}_{12,3} + 3\,\mathcal{Q}_{3,12}\right)\mathcal{Q}_{1,2};\\
\mathcal{S}_3 &= \frac{3}{4}\left(-\mathcal{P}_{3,12} + 3\,\mathcal{Q}_{3,12}\right)\mathcal{Q}_{1,2};\\
\mathcal{S}_4 &= -\mathcal{R}_3;\\
\mathcal{S}_5 &= -\mathcal{R}_4.
\end{aligned}
$$

So we now have the expression of $\delta^{(3)}(R_0)$ and $\theta^{(3)}(R_0)$ (using eventually the functions E_2 and E_3 for the latter). These expressions seem rather complicated but further simplications are unnecessary. The functions $\mathcal{P}$ and $\mathcal{Q}$ defined in (17-18) are well adapted for a convolution with a top–hat window function as shown in the appendix.

3.5. *Expressions of the skewness and the kurtosis*

The expressions (28) and (32) are then calculated using the general property (7) of Gaussian variables. It leads to the expressions for S_3 and S_4,

$$
S_3(R_0) = \frac{3}{\sigma^4(R_0)} \int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3} P(k_1)\,P(k_2)\,W_1\,W_2\,W_{12}\,\mathcal{U}_{1,2} \tag{54}
$$

and

$$S_4(R_0) = \frac{1}{\sigma^6(R_0)}\Bigg[6\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3} P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_{12}\,W_{23}\,W_3 \times [\mathcal{U}_{1,2}\mathcal{U}_{2,3}+\mathcal{U}_{1,2}\mathcal{U}_{3,2}+\mathcal{U}_{2,1}\mathcal{U}_{2,3}+\mathcal{U}_{2,1}\mathcal{U}_{3,2}] + 4\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3} P(k_1)\,P(k_2)\,P(k_3)\,W_{123}\,W_1\,W_2\,W_3 \times \left[D_1^3\,\mathcal{R}_1 + D_1\,D_2\,\mathcal{R}_2 + D_3\,\mathcal{R}_3 + F_3\,\mathcal{R}_4\right]\Bigg], \tag{55}$$

where

$$\mathcal{U}_{i,j} = D_1^2\left[\mathcal{P}_{i,j} - \frac{3}{2}\mathcal{Q}_{i,j}\right] + \frac{3}{2}\,D_2\,\mathcal{Q}_{i,j},$$
$$W_i = W_{\mathrm{TH}}(k_i\,R_0),\ \ W_{ij} = W_{\mathrm{TH}}\left(|\mathbf{k}_i+\mathbf{k}_j|\,R_0\right),$$
$$W_{ijk} = W_{\mathrm{TH}}\left(|\mathbf{k}_i+\mathbf{k}_j+\mathbf{k}_k|\,R_0\right).$$

Similar expressions can be obtained for the velocity field. The previous calculations lead to technical results that fortunately reduce to very simple expressions. Indeed the values of the integrals appearing in these expressions are all given at the end of the appendix and are very simple. Careful summations of these terms (that must take into account all symmetry factors) lead to the following results,

$$S_3(R_0) = \left[3\frac{D_2}{D_1^2} + \gamma_1\right]; \tag{56}$$

$$S_{3\theta}(R_0) = \frac{-1}{f(\Omega,\Lambda)}\left[3\frac{E_2}{D_1^2} + \gamma_{1\theta}\right]; \tag{57}$$

$$S_4(R_0) = 4\frac{D_3}{D_1^3} + 12\frac{D_2^2}{D_1^4} + \left(14\frac{D_2}{D_1^2} - 2\right)\gamma_1 + \frac{7}{3}\gamma_1^2 + \frac{2}{3}\gamma_2; \tag{58}$$

$$S_{4\theta}(R_0) = \frac{1}{f^2(\Omega,\Lambda)}\left[4\frac{E_3}{D_1^3} + 12\frac{E_2^2}{D_1^4} + \left(12\frac{E_2}{D_1^2} + 2\frac{D_2}{D_1^2} - 2\right)\gamma_{1\theta} + \frac{7}{3}\gamma_{1\theta}^2 + \frac{2}{3}\gamma_{2\theta}\right], \tag{59}$$

where

$$\gamma_1 = \frac{\mathrm{d}\log[\sigma^2(R_0)]}{\mathrm{d}\log R_0},\quad \gamma_2 = \frac{\mathrm{d}^2\log[\sigma^2(R_0)]}{\mathrm{d}\log^2 R_0},$$
$$\gamma_{1\theta} = \frac{\mathrm{d}\log[\sigma_\theta^2(R_0)]}{\mathrm{d}\log R_0},\quad \gamma_{2\theta} = \frac{\mathrm{d}^2\log[\sigma_\theta^2(R_0)]}{\mathrm{d}\log^2 R_0}.$$

In all cases only the leading term has been given. The expected corrections, due to higher order terms, contain an extra factor $\sigma^2(R_0)$ (or $\sigma_\theta^2[R_0]$) so they are expected to be negligible at large scale.

In Figs. (1-2) it is shown that the Ω and Λ dependences of D_2/D_1^2, D_3/D_1^3, E_2/D_1^2 and E_3/D_1^3 are extremely weak and, for the values of Ω and Λ of interest, can be safely approximated by their value when $\Omega = 1$, $\Lambda = 0$. So we end up with

$$S_3(R_0) = \frac{34}{7} + \gamma_1; \tag{60}$$

$$S_{3\theta}(R_0) = \frac{-1}{\Omega^{0.6}}\left[\frac{26}{7} + \gamma_{1\theta}\right]; \tag{61}$$

$$S_4(R_0) = \frac{60712}{1323} + \frac{62}{3}\gamma_1 + \frac{7}{3}\gamma_1^2 + \frac{2}{3}\gamma_2; \tag{62}$$

$$S_{4\theta}(R_0) = \frac{1}{\Omega^{1.2}}\left[\frac{12088}{441} + \frac{338}{21}\gamma_{1\theta} + \frac{7}{3}\gamma_{1\theta}^2 + \frac{2}{3}\gamma_{2\theta}\right]. \tag{63}$$

One can notice that the effects of smoothing introduce a dependence of these coefficients with the scale through the successive logarithmic derivatives of the variance. If the smoothing were not taken into account one would have obtained $S_3 = 34/7$ and $S_4 = 60712/1323$ (*e.g.*, Bernardeau 1992) and it corresponds to a scale independent σ, not necessarily to a small smoothing radius limit. Moreover 34/7 does not appear as an upper bound for the skewness of the density field. The smoothing actually mixes scales and for a variance decreasing with scale it lowers both the skewness and the kurtosis. However for particular initial power spectra (such the ones given by Hot Dark Matter models) the skewness may take values greater than the one obtained without smoothing. Such a trend is indeed observed in numerical simulations (Bouchet & Hernquist, 1992).

3.6. *Power law spectrum and scale dependence*

The scale dependences that appear in the expressions (60-63) are only due to a possible change of the shape of the power spectrum with scale. A self similar power spectrum, *i.e.* , a power law of k,

$$P(k) \propto k^n, \tag{64}$$

leads to values of S_3, $S_{3\theta}$, S_4 and $S_{4\theta}$ that are independent of scale (as long as the quasilinear regime is valid). The previous results are then very simple since $\gamma_2 = \gamma_{2\theta} = 0$, and read

$$S_3 = \frac{34}{7} - (n+3); \tag{65}$$

$$S_{3\theta} = \frac{-1}{\Omega^{0.6}}\left[\frac{26}{7} - (n+3)\right]; \tag{66}$$

$$S_4 = \frac{60712}{1323} - \frac{62}{3}(n+3) + \frac{7}{3}(n+3)^2; \tag{67}$$

$$S_{4\theta} = \frac{1}{\Omega^{1.2}}\left[\frac{12088}{441} - \frac{338}{21}(n+3) + \frac{7}{3}(n+3)^2\right]. \tag{68}$$

In such a case the results for the skewness have already been given in previous papers (Juszkiewicz *et al.* 1993, Bernardeau *et al.* 1993).

These functions are given in Figs. 3-5 as a function of Ω when $\Lambda = 0$ and when $\Omega + \Lambda/3H^2 = 1$ and for different values of the index of power law spectra. In Fig. 6, I present these functions as a function of the power law index n. In practice it is very useful to approximate the actual scale dependence of the variance of the density field by a power law,

$$\sigma^2(R_0) \propto R_0^{-(3+n)} \tag{69}$$

where

$$n \equiv -\frac{\mathrm{d}\log\sigma^2(R_0)}{\mathrm{d}\log R_0} - 3. \tag{70}$$

With such an approximation one gets the relationships (65-68). For the spectra of interest it turns out to be a good approximation (see Figs. 7-8). In these figures I give the expected behavior of the skewness and the kurtosis as a function of scale using eqs. (60-63) (thick lines). I also give the kurtosis using the approximation (67) and the relations (67, 68) (thin lines). The decrease of the skewness and the kurtosis is due to the variation of the index n with scale (upper panel). This variation is more important for the CDM spectrum (Fig. 7) than for the spectrum proposed by Peacock (1991) (Fig. 8) to fit the observation in the galaxy distribution. It corresponds to the well known discrepancy between the shape of the CDM spectrum and the one derived from the actual galaxy clustering properties.

In table 1, I summarize the results of these perturbative calculations for the skewness and the kurtosis for an Einstein–de Sitter Universe. The parameters obtained in this paper for the density field are somehow lower than the ones obtained by Goroff *et al.* (1986) but this is due to the fact that they used a Gaussian window function instead of a top–hat window function. This effect is particularly important when the dependence of σ with the scale is strong. However, the dependence of the results with the cosmological parameters, Ω and Λ, which has not been considered by Goroff *et al.* , is roughly independent of the shape of the window function.

3.7. *Validity domain and comparison with numerical simulations*

The results that are given throughout the paper are thought to be valid in the linear or quasi-linear regime. Numerical simulations have been done to check the evolution of the variance compared to what is given by perturbative calculations and the agreement with the predictions based on the linear approximation hold for σ^2 up to 0.8 at a 15% precision level (*e.g.*, Efstathiou *et al.* 1988 or Hamilton *et al.* 1991). In principle the results given in this paper are valid in the same domain. Indeed, results of numerical simulations show that the agreement for S_3 and S_4 (Bouchet & Hernquist 1992) are extremely good even for values of σ close to unity.

Table 1: *Predictions of the perturbation theory for S_3, S_4, $S_{3\theta}$ and $S_{4\theta}$ vs. spectral slope for an Einstein–de Sitter Universe.*

n	S_3	S_4	$S_{3\theta}$	$S_{4\theta}$	$S_{4\theta}/S_{3\theta}^2$
−3	34/7	60712/1323	26/7	12088/441	3022/1521
−2	27/7	36457/1323	19/7	6019/441	6019/3249
−1	20/7	18376/1323	12/7	2008/441	251/162
0	13/7	6469/1323	5/7	55/441	11/45
1	6/7	736/1323	−2/7	160/441	–

Juszkiewicz *et al.* (1993) extend this analysis to the behavior of $S_{3\theta}$ for a Gaussian filter showing a good agreement with the theoretical predictions. I present here a new set of results for the skewness and the kurtosis for the density field using a top–hat filter to check, in particular, the accuracy of the scale dependence of the theoretical results for a CDM power spectrum. The measures have been made in an adaptive P^3M simulation, kindly provided by Couchman (Couchman 1991), with CDM initial conditions, $H_0 = 50$ km s^{-1} Mpc^{-1} and $\Omega = 1$. It is done in a cubic box of side 400 Mpc with periodic boundary conditions and contains 2.1 10^6 particles and at the end of the simulation the amplitude of the density fluctuations is $\sigma_8 = 0.97$. The resulting field has been filtered with a top–hat window function at four different radius and at two different timesteps corresponding to an expansion factor of 0.6 and 1 (in units of the final expansion factor). This procedure allows to check the stability of the result for different values of σ. For each radius it is possible to calculate the index n of the power spectrum (Eq. [69]) as well as the parameter γ_2 ($\equiv -\mathrm{d}n/\mathrm{d}\log(R_0)$) and then the expected values of S_3 and S_4. These numbers are given in respectively the second to the fourth column of table 2. In parentheses is also given the value of S_4 when the γ_2 term has been dropped. The remaining columns correspond to the measurement of these quantities together with σ at the two different timesteps.

The density has been measured in spherical cells of given radius centered on the 50^3 points of a grid. The actual second, third and fourth moments of the density distributions thus obtained have been calculated, as well as the ratios S_3 and S_4. The estimations of the errors have been made by taking the extreme deviations that are obtained when the volume of the simulation is divided in eight different parts. It provides an estimation of the sampling errors for the measurement of the skewness and the kurtosis expected in a cubic box of size $100h^{-1}$Mpc ($h = H_0/100$ km s^{-1} Mpc^{-1}) which is the order of magnitude of the size of available catalogues. As it can be seen the measurements are in good agreement with the theoretical predictions even for values of σ slightly higher than unity. Unfortunately the determination of the same quantities for the velocity field reveals, for a top–hat filter, to be more complicated and there are no available results. The results for the density field are however quite encouraging since if they are transposed to the divergence of the velocity field, it seems possible to measure the skewness and the kurtosis with enough accuracy to put strong constraint on Ω, for filtering radius up to 15 h^{-1} Mpc. At such a scale the perturbative calculations should be valid (and for the density field the analytical results are valid down to 5 h^{-1}Mpc).

Table 2: *Comparison of the theoretical predictions with numerical simulations for a CDM initial spectrum, $\Omega = 1$ and $H_0 = 50$ km s^{-1} Mpc^{-1}. In parentheses is given the value of S_4 when the γ_2 parameter is neglected, eq. (67).*

			predictions		*measures* $a/a_0 \approx 0.6$			$a/a_0 = 1$		
R_0/Mpc	n	γ_2	S_3	S_4	σ	S_3	S_4	σ	S_3	S_4
10	– 1.34	– 0.46	3.2	17.7 (18.1)	0.92	$3.2^{+0.3}_{-0.3}$	18^{+5}_{-5}	1.52	$3.4^{+0.5}_{-0.5}$	20^{+5}_{-8}
20	– 0.98	– 0.57	2.9	13.3 (13.7)	0.46	$2.8^{+0.4}_{-0.4}$	13^{+5}_{-5}	0.74	$2.8^{+0.5}_{-0.6}$	13^{+6}_{-7}
30	– 0.73	– 0.64	2.6	10.6 (11.0)	0.29	$2.4^{+1.1}_{-0.8}$	11^{+20}_{-7}	0.47	$2.7^{+0.7}_{-0.9}$	13^{+10}_{-12}
40	– 0.54	– 0.71	2.4	8.7 (9.2)	0.21	$2.2^{+1.5}_{-1}$	16^{+23}_{-20}	0.33	$2.6^{+0.6}_{-1.6}$	15^{+14}_{-16}

A lot of numerical works still remain to be done to check the various features expected for the skewness and the kurtosis with regards to the cosmological parameters. Moreover it is not clear whether the validity domain is the same for any power spectrum and whether the errors associated with the finite size of the sample depend only on the local index of the power spectrum or on its global shape. In the latter case the sampling errors would have to be determined by numerical simulations for any model to be tested against the observations.

4. COMMENTS

4.1. *Higher order moments*

Obviously the skewness and the kurtosis do not carry all the information available from the distribution of the divergence of the velocity field. The (connected part of the) moments of higher order also contain extra independent informations on the distribution function. The derivation of these moments requires similar perturbative calculations, and have been calculated (Bernardeau 1992a and b) when the filter function is neglected (which turns out to be equivalent to the limit

$n \to -3$). The results are of the following form

$$\left\langle \delta^p \right\rangle_c \sim S_p \left\langle \delta^2 \right\rangle^{p-1} \tag{71}$$

for the density field and

$$\left\langle \theta^p \right\rangle_c \sim S_{p\theta} \left\langle \theta^2 \right\rangle^{p-1} \tag{72}$$

for the divergence of the velocity field. The Ω and Λ of the coefficients S_p are very weak but for the coefficients $S_{p\theta}$ they read

$$S_{p\theta}(\Lambda,\Omega) = [f(\Lambda,\Omega)]^{2-p}\, S_{p\theta}(\Lambda=0,\Omega=1) \approx \frac{1}{\Omega^{0.6(p-2)}}\, S_{p\theta}(\Lambda=0,\Omega=1). \tag{73}$$

This dependence with the cosmological parameters is expected to hold even with the application of a filter function (as verified for $p = 3, 4$). It implies that whatever the number of moments you can measure it is impossible to give any constraint on Λ. Actually this property comes from the fact that the divergence of the velocity, at any order in the expansion (27), always depends on Ω and Λ like $f(\Omega,\Lambda)$, $\theta^{(p)}(R_0) \propto f(\Omega,\Lambda)$, so that, at large scale, any statistical indicator can only give a constraint on this combination of Ω and Λ.

4.2. *The Zel'dovich approximation*

It has been claimed that the Zel'dovich approximation (Zel'dovich 1970) was an extremely good and powerful approximation for the large scale dynamics (*e.g.*, Kofman *et al.* 1993). This approximation, based on a linear approximation for the displacement field, is a simplified dynamics that however reproduces most of the large-scale features of the gravitational dynamics. Obviously the results that have been obtained in the previous section do take into account corrections to the Zel'dovich approximation when necessary, that is the second or third order of the displacement field (see Bouchet *et al.* 1992 for an interesting discussion on this subject). It is anyway possible to calculate the large scale behaviors of the density and velocity moments in the simplified Zel'dovich dynamics. We obtain for $P(k) \propto k^n$,

$$\begin{aligned} S_3^{\rm Zel} &= 4 - (3+n); \\ S_{3\theta}^{\rm Zel} &= \frac{-1}{\Omega^{0.6}}\left[2 - (3+n)\right]; \\ S_4^{\rm Zel} &= \frac{272}{9} - \frac{50}{3}(3+n) + \frac{7}{3}(3+n)^2; \\ S_{4\theta}^{\rm Zel} &= \frac{1}{\Omega^{1.2}}\left[8 - \frac{26}{3}(3+n) + \frac{7}{3}(3+n)^2\right]. \end{aligned} \tag{74}$$

The Ω and Λ dependences are roughly the same as for the general results. As can be seen from these results the Zel'dovich approximation leads to an underestimation of the correct values, especially for the velocity field since for instance for $n = -1$ the predicted skewness and kurtosis are zero. The use of a reconstruction method starting with the velocity field using the Zel'dovich approximation (Nusser & Dekel 1993, Gramann 1993) to constrain Ω then may lead to a wrong estimation of Ω.

4.3. *The biases*

As mentioned previously the observations in galaxy fields provide good indications in favor of Gaussian initial conditions. However in case of biases between the galaxy distribution and the matter field, one can wonder whether the demonstration still holds. In fact, as pointed out by Fry & Gaztañaga (1992), the scaling between the moments of the galaxy distribution at large scale is expected to hold for a broad range of bias schemes. However, neither S_3 or S_4 can be used to give constraints on the galaxy linear bias, b (Eq. [3]). The reason is that the galaxy field skewness and kurtosis would be not only sensitive to the linear term $b\,\delta$ but also to the quadratic or cubic corrections. Following Fry & Gaztañaga, if we assume that,

$$\delta_g = b\ \delta + \frac{b_2}{2}\delta^2 + \frac{b_3}{3!}\delta^3 + \dots \tag{75}$$

then we have,

$$\begin{aligned} S_3^{\rm gal} &= \frac{S_3}{b} + 3\frac{b_2}{b^2}; \\ S_4^{\rm gal} &= \frac{S_4}{b^2} + 12\frac{S_3 b_2}{b^3} + 4\frac{b_3}{b^3} + 12\frac{b_2^2}{b^4}. \end{aligned} \tag{76}$$

It shows that any measurement of high order moments involves a new parameter in the expansion (75), so that no new constraints on b alone can be given. However a departure between the measured values of $S_3^{\rm gal}$ and $S_4^{\rm gal}$ and the theoretical predictions is the manifestation of biases that is δ_g is not equal to δ. For instance neither the skewness measured in the IRAS galaxy sample by (Bouchet *et al.* 1993) nor the kurtosis are in agreement with the theoretical predictions. These are indications of the existence of biases in the galaxy distribution, although the value for b defined as the linear bias cannot be inferred from these measurements.

4.4. *A universal function to test the gravitational instability scenario*

Unlike the density field, the velocity divergence field is thought to be free of unknown biases. This supposes that it is possible to do a volume weighted filtering of the field as done by Juszkiewicz *et al.* (1993b) in a numerical simulation. For real data, however, it may not be easy (due for instance to undersampling in voids) but it is not the purpose of this paper to test the reliability of these tests on the present data. The skewness has already been proposed as a good indicator for the measurement of Ω and measured in observational data set (Bernardeau *et al.* 1993). In fact it turns out that the kurtosis could be used as well to measure Ω, giving the possibility to measure Ω with two independent methods that are based on the same physical assumptions. Obviously the two results should be in agreement with each other. In fact there is a combination of the first three moments of the divergence of the velocity field that is almost independent of the cosmological parameters,

$$\frac{S_{4\theta}}{S_{3\theta}^2}=\frac{\left(\langle\theta^4\rangle-3\langle\theta^2\rangle^2\right)\langle\theta^2\rangle}{\langle\theta^3\rangle^2}=\frac{\frac{12088}{441}-\frac{338}{21}(n+3)+\frac{7}{3}(n+3)^2}{\left[\frac{26}{7}-(n+3)\right]^2} \tag{77}$$

The values taken by this ratio are given in table 1. The case $n=1$ has not been given since the velocity skewness vanishes for $n\approx 0.7$ leading to an unstable ratio. In practice one should use the index observed in the velocity field rather than the one of the galaxy clustering since the latter may be affected by biases. However in the absence of such determination and to illustrate this result we can use the index observed at 10 h^{-1}Mpc in galaxy catalogues. It equals approximately -1 (there are numerous determinations but see for instance Peacock 1991) so that we expect a ratio of the order of 1.5 (table 1 and Figs. 5-8). *Then, if the measured large scale velocity flows fail to reproduce such a number, they cannot originate from gravitational instabilities with Gaussian initial conditions.*

4.5. *Conclusion*

I gave here a series of results for the dependence of the skewness and the kurtosis of the cosmic fields at large scales with the cosmological parameters. These results have been exactly derived using perturbation theory. For the comparison with observational data however, numerous effects have to be been taken into account such as the redshift space distorsion for the density field (see Bouchet *et al.* 1992 for the skewness), the inhomogeneous Malmquist bias for the velocity field. Anyway the analysis of the statistical properties of the large–scale cosmic fields looks promising and it should be possible to constraint the density of the universe or to check the validity of the gravitational instability scenario.

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APPENDIX A: Geometrical properties of the top hat filter function

This appendix is devoted to the derivation of general properties of the top hat window function, $W_{\rm TH}(\mathbf{x})$. It is defined by

$$\begin{aligned} W_{\rm TH}(\mathbf{x}) &= 1 \text{ if } |\mathbf{x}| \le R_0; \\ W_{\rm TH}(\mathbf{x}) &= 0 \text{ otherwise}; \end{aligned} \quad (A.1)$$

for a scale R_0. The Fourier transform of this function is then given by

$$W_{\rm TH}(k\,R_0) = \frac{3}{(kR_0)^3}\left(\sin(kR_0) - kR_0\cos(kR_0)\right) \quad (A.2)$$

where k is the norm of $\mathbf{k}$.

A.1. *General properties*

Let me define two functions used in the main part:

$$\mathcal{P}(\mathbf{k}_1,\mathbf{k}_2) = 1 + \frac{\mathbf{k}_1.\mathbf{k}_2}{k_1^2} \quad (A.3)$$

and

$$\mathcal{Q}(\mathbf{k}_1,\mathbf{k}_2) = 1 - \frac{(\mathbf{k}_1.\mathbf{k}_2)^2}{k_1^2\,k_2^2} \quad (A.4)$$

Then if one considers two wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ on which one may have to integrate, a property of interest concerning the integrations over their angular parts, $\mathrm{d}\Omega_1$ and $\mathrm{d}\Omega_2$, have already been given by Bernardeau (1993, Appendix B). It reads

$$\begin{aligned} &\frac{1}{(4\pi)^2}\int \mathrm{d}\Omega_1\mathrm{d}\Omega_2\;\mathcal{Q}(\mathbf{k}_1,\mathbf{k}_2)W_{\rm TH}\left(|\mathbf{k}_1+\mathbf{k}_2|\;R_0\right) \\ &\quad = \frac{2}{3}\;W_{\rm TH}(k_1\,R_0)W_{\rm TH}(k_2\,R_0). \end{aligned} \quad (A.5)$$

For the purpose of this paper we need a few other properties involving other geometrical dependences.

They are the following:

$$\begin{aligned} &\frac{1}{(4\pi)^2}\int \mathrm{d}\Omega_1\mathrm{d}\Omega_2\;\mathcal{P}(\mathbf{k}_1,\mathbf{k}_2)W_{\rm TH}\left(|\mathbf{k}_1+\mathbf{k}_2|\;R_0\right) \\ &= W_{\rm TH}(k_1\,R_0)\left(W_{\rm TH}(k_2\,R_0) + \frac{1}{3}k_2\,R_0\,W'_{\rm TH}(k_2\,R_0)\right) \end{aligned} \quad (A.6)$$

and

$$\frac{1}{(4\pi)^2}\int \mathrm{d}\Omega_1\mathrm{d}\Omega_2\ \mathcal{Q}(\mathbf{k}_1,\mathbf{k}_2)\,|\mathbf{k}_1+\mathbf{k}_2|R_0\,W'_{\mathrm{TH}}\left(|\mathbf{k}_1+\mathbf{k}_2|\ R_0\right)$$
$$=\frac{2}{3}\ \left(k_1\,R_0\,W'_{\mathrm{TH}}(k_1\,R_0)\,W_{\mathrm{TH}}(k_2\,R_0)+W_{\mathrm{TH}}(k_1\,R_0)\,k_2\,R_0\,W'_{\mathrm{TH}}(k_2\,R_0)\right) \tag{A.7}$$

and

$$\frac{1}{(4\pi)^2}\int \mathrm{d}\Omega_1\mathrm{d}\Omega_2\ \mathcal{P}(\mathbf{k}_1,\mathbf{k}_2)\,|\mathbf{k}_1+\mathbf{k}_2|R_0\,W'_{\mathrm{TH}}\left(|\mathbf{k}_1+\mathbf{k}_2|\ R_0\right)$$
$$=k_1\,R_0\,WTH'(k_1\,R_0)\,W_{\mathrm{TH}}(k_2\,R_0)+\frac{1}{3}k_1\,R_0\,W'_{\mathrm{TH}}(k_1\,R_0)\,k_2\,R_0\,W'_{\mathrm{TH}}(k_2\,R_0)$$
$$-\frac{1}{3}W_{\mathrm{TH}}(k_1\,R_0)\,k_2^2\,R_0^2\,W_{\mathrm{TH}}(k_2\,R_0). \tag{A.8}$$

The expression (A.6) can be derived straightforwardly from the expression (A.2) of the Fourier transform of the top hat window function. The integral over the angular part $u=\mathbf{k}_1.\mathbf{k}_2/k_1k_2$ can be done by the change of variable, $x=|\mathbf{k}_1+\mathbf{k}_2|\,R_0$, so that the integral reads,

$$\frac{1}{(4\pi)^2}\int \mathrm{d}\Omega_1\mathrm{d}\Omega_2\ \left[1+\frac{k_1}{k_2}\ \frac{\mathbf{k}_1.\mathbf{k}_2}{k_1k_2}\right]W_{\mathrm{TH}}\left(|\mathbf{k}_1+\mathbf{k}_2|\ R_0\right)$$
$$=\frac{3}{2}\int_{|\mathbf{k}_1-\mathbf{k}_2|\,R_0}^{|\mathbf{k}_1+\mathbf{k}_2|\,R_0}\frac{x\,\mathrm{d}x}{k_1k_2R_0^2}\ \left[1+\frac{k_1}{k_2}\ \frac{x^2/R_0^2-k_1^2-k_2^2}{k_1k_2}\right]\ \frac{[\sin x-x\ \cos x]}{x^3} \tag{A.9}$$

The integration of (A.9) can be done directly and leads to the expression (A.6).

The two other properties can be shown using the properties of the derivative of $W_{\mathrm{TH}}(k)$. Indeed we have

$$\frac{1}{3}\,kW'_{\mathrm{TH}}(k)=-W_{\mathrm{TH}}(k)+\frac{\sin k}{k}. \tag{A.10}$$

The latter term can be written with a Bessel function, $\sin k/k=\sqrt{\pi/2}\,J_{1/2}(k)/k^{1/2}$, and and then decomposed as a sum over products of Bessel functions,

$$\frac{\sin|\mathbf{k}_1+\mathbf{k}_2|}{|\mathbf{k}_1+\mathbf{k}_2|}=\pi\ \sum_{m=0}^{\infty}(\frac{1}{2}+m)\frac{J_{1/2+m}(k_1)}{k_1^{1/2}}\frac{J_{1/2+m}(k_2)}{k_2^{1/2}}\,P_m(-u), \tag{A.11}$$

where

$$u=\frac{\mathbf{k}_1.\mathbf{k}_2}{k_1k_2}. \tag{A.12}$$

and P_m is the Legendre polynomial of order m. The general properties of the Legendre polynomials then lead to the relationships,

$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}u\, P_m(-u) = 1 \;\; \text{if} \;\; m=0, \;\; 0 \;\; \text{otherwise};$$
$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}u\, u\, P_m(-u) = -1/3 \;\; \text{if} \;\; m=1, \;\; 0 \;\; \text{otherwise}; \qquad (A.13)$$
$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}u\, u^2\, P_m(-u) = 1/3 \;\; \text{if} \;\; m=0, \;\; 2/15 \;\; \text{if} \;\; m=2, \;\; 0 \;\; \text{otherwise}.$$

Using these relationships we obtain,

$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}u[1-u^2]\frac{\sin|\mathbf{k}_1+\mathbf{k}_2|}{|\mathbf{k}_1+\mathbf{k}_2|} = \frac{2}{3}\frac{\sin k_1}{k_1}\frac{\sin k_2}{k_2} - \frac{\pi}{3}\frac{J_{5/2}(k_1)}{k_1^{1/2}}\frac{J_{5/2}(k_2)}{k_2^{1/2}} \qquad (A.14)$$
$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}u\left[1+\frac{k_1}{k_2}u\right]\frac{\sin|\mathbf{k}_1+\mathbf{k}_2|}{|\mathbf{k}_1+\mathbf{k}_2|} = \frac{\sin k_1}{k_1}\frac{\sin k_2}{k_2} - \frac{\pi}{2}\frac{k_1}{k_2}\frac{J_{3/2}(k_1)}{k_1^{1/2}}\frac{J_{3/2}(k_2)}{k_2^{1/2}} \qquad (A.15)$$

It can also be noticed that ,

$$J_{3/2}(k) = k^{3/2}\, W_{\mathrm{TH}}(k)\sqrt{\frac{2}{3\pi}} \qquad (A.16),$$
$$J_{5/2}(k) = -k^{3/2}\, W'_{\mathrm{TH}}(k)\sqrt{\frac{2}{3\pi}} \qquad (A.17),$$

so that

$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}u[1-u^2]\frac{\sin|\mathbf{k}_1+\mathbf{k}_2|}{|\mathbf{k}_1+\mathbf{k}_2|} =$$
$$\frac{2}{3}\left(\frac{\sin k_1}{k_1}\frac{\sin k_2}{k_2} - \frac{1}{9}k_1 W'_{\mathrm{TH}}(k_1)\, k_2 W'_{\mathrm{TH}}(k_2)\right) \qquad (A.18)$$
$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}u\left[1+\frac{k_1}{k_2}u\right]\frac{\sin|\mathbf{k}_1+\mathbf{k}_2|}{|\mathbf{k}_1+\mathbf{k}_2|} = \frac{\sin k_1}{k_1}\frac{\sin k_2}{k_2} - \frac{1}{9}k_1^2\, W_{\mathrm{TH}}(k_1) W_{\mathrm{TH}}(k_2) \qquad (A.19)$$

The relations (A.7) and (A.8) can then be deduced from (A.10), (A.18) and (A.19).

A.2. *Applications*

As an application of the previous results I give the value of few integrals of interest for the large scale cosmic fields. Let me define $\sigma^2(R_0)$ by

$$\sigma^2(R_0) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} P(k)\, W^2_{\mathrm{TH}}(k\, R_0) \qquad (A.20)$$

where $P(k)$ is the power spectrum. We can then notice that

$$R_0\,\frac{\mathrm{d}\sigma^2(R_0)}{\mathrm{d}R_0}=2\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}P(k)\,k\,R_0\,W_{\rm TH}(k\,R_0)W'_{\rm TH}(k\,R_0). \tag{A.21}$$

The second derivative can also be calculated,

$$\begin{aligned}R_0\,\frac{\mathrm{d}}{\mathrm{d}R_0}\left[R_0\,\frac{\mathrm{d}\sigma^2(R_0)}{\mathrm{d}R_0}\right]&=2\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}P(k)\,\left[k\,R_0\,W'_{\rm TH}(k\,R_0)\right]^2\\&+2\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}P(k)\,\left[k\,R_0\right]^2\,W_{\rm TH}(k\,R_0)W''_{\rm TH}(k\,R_0)\\&+2\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}P(k)\,k\,R_0\,W_{\rm TH}(k\,R_0)W'_{\rm TH}(k\,R_0).\end{aligned} \tag{A.22}$$

The expression (A.22) can be simplified by the following property,

$$k^2\,W''_{\rm TH}(k)=-4k\,W'_{\rm TH}(k)-k^2\,W_{\rm TH}(k), \tag{A.23}$$

so that

$$\begin{aligned}R_0\,\frac{\mathrm{d}}{\mathrm{d}R_0}\left[R_0\,\frac{\mathrm{d}\sigma^2(R_0)}{\mathrm{d}R_0}\right]=&2\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}P(k)\,\left[k\,R_0\,W'_{\rm TH}(k\,R_0)\right]^2\\&+3\,R_0\,\frac{\mathrm{d}\sigma^2(R_0)}{\mathrm{d}R_0}+2\,\overline{k^2}\,\sigma^2(R_0)\end{aligned} \tag{A.24}$$

where

$$\overline{k^2}=\frac{1}{\sigma^2(R_0)}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}k^2\,P(k)\,W^2_{\rm TH}(k\,R_0) \tag{A.25}$$

I consider now the integrals,

$$\begin{aligned}I_1(R_0)=&\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\,\mathcal{P}(\mathbf{k}_1,\mathbf{k}_2)\,P(k_1)\,P(k_2)\\&\times W_{\rm TH}\left(|\mathbf{k}_1+\mathbf{k}_2|\;R_0\right)W_{\rm TH}(k_1\,R_0)W_{\rm TH}(k_2\,R_0)\\I_2(R_0)=&\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\,\mathcal{Q}(\mathbf{k}_1,\mathbf{k}_2)\,P(k_1)\,P(k_2)\\&\times W_{\rm TH}\left(|\mathbf{k}_1+\mathbf{k}_2|\;R_0\right)W_{\rm TH}(k_1\,R_0)W_{\rm TH}(k_2\,R_0)\end{aligned} \tag{A.26}$$

The relationships (A.3) and (A.4) give,

$$\begin{aligned}I_1(R_0)&=\sigma^4(R_0)\left(1+\frac{1}{6}\frac{R_0}{\sigma^2(R_0)}\,\frac{\mathrm{d}\sigma^2(R_0)}{\mathrm{d}R_0}\right)\\I_2(R_0)&=\frac{2}{3}\,\sigma^4(R_0)\end{aligned} \tag{A.27}$$

The knowledge of integrals involving three wave vectors are also required. To ease the presentation I introduce simplified notations,

$$\begin{aligned} W_i &= W_{\rm TH}(k_i\,R_0),\ \ W_{ij} = W_{\rm TH}\left(|\mathbf{k}_i+\mathbf{k}_j|\ R_0\right), \\ W_{ijk} &= W_{\rm TH}\left(|\mathbf{k}_i+\mathbf{k}_j+\mathbf{k}_k|\ R_0\right). \\ \mathcal{P}_{i,j} &= \mathcal{P}(\mathbf{k}_i,\mathbf{k}_j),\ \ \mathcal{P}_{ij,k} = \mathcal{P}(\mathbf{k}_i+\mathbf{k}_j,\mathbf{k}_k),\ \ \mathcal{P}_{i,jk} = \mathcal{P}(\mathbf{k}_i,\mathbf{k}_j+\mathbf{k}_k). \end{aligned}$$

and similar notations for $\mathcal{Q}(\mathbf{k}_i,\mathbf{k}_j)$. Then we have the results,

$$\begin{aligned}
&\int \frac{{\rm d}^3\mathbf{k}_1}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_2}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_{12}\,W_{23}\,W_3\,\mathcal{P}_{1,2}\,\mathcal{P}_{2,3} \\
&\qquad\qquad = \sigma^6(R_0)\left(1+\frac{1}{3}\gamma_1+\frac{1}{36}\gamma_1^2\right) \\
&\int \frac{{\rm d}^3\mathbf{k}_1}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_2}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_{12}\,W_{23}\,W_3\,\mathcal{P}_{1,2}\,\mathcal{P}_{3,2} \\
&\qquad\qquad = \sigma^6(R_0)\left(1+\frac{1}{2}\gamma_1+\frac{1}{18}\gamma_1^2+\frac{1}{18}\gamma_2+\frac{1}{9}\overline{k^2}\right) \\
&\int \frac{{\rm d}^3\mathbf{k}_1}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_2}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_{12}\,W_{23}\,W_3\,\mathcal{P}_{2,1}\,\mathcal{P}_{2,3} \\
&\qquad\qquad = \sigma^6(R_0)\left(1+\frac{1}{3}\gamma_1+\frac{1}{36}\gamma_1^2\right) \\
&\int \frac{{\rm d}^3\mathbf{k}_1}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_2}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_{12}\,W_{23}\,W_3\,\mathcal{P}_{1,2}\,\mathcal{Q}_{2,3} \\
&\qquad\qquad = \frac{2}{3}\sigma^6(R_0)\left(1+\frac{1}{6}\gamma_1\right) \\
&\int \frac{{\rm d}^3\mathbf{k}_1}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_2}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_{12}\,W_{23}\,W_3\,\mathcal{P}_{2,1}\,\mathcal{Q}_{2,3} \\
&\qquad\qquad = \frac{2}{3}\sigma^6(R_0)\left(1+\frac{1}{6}\gamma_1\right) \\
&\int \frac{{\rm d}^3\mathbf{k}_1}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_2}{(2\pi)^3}\frac{{\rm d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_{12}\,W_{23}\,W_3\,\mathcal{Q}_{1,2}\,\mathcal{Q}_{2,3} = \frac{4}{9}\sigma^6(R_0),
\end{aligned} \tag{A.28}$$

where

$$\gamma_1 = \frac{{\rm d}\log[\sigma^2(R_0)]}{{\rm d}\log R_0},\quad \gamma_2 = \frac{{\rm d}^2\log[\sigma^2(R_0)]}{{\rm d}\log^2 R_0}. \tag{A.29}$$

Another series of useful integrals are given by,

$$\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_2\,W_3\,W_{123}\,\mathcal{P}_{12,3}\,\mathcal{P}_{1,2}$$
$$= \sigma^6(R_0)\left(1+\frac{1}{3}\gamma_1+\frac{1}{36}\gamma_1^2\right)$$
$$\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_2\,W_3\,W_{123}\,\mathcal{P}_{3,12}\,\mathcal{P}_{1,2}$$
$$= \sigma^6(R_0)\left(1+\frac{1}{3}\gamma_1+\frac{1}{36}\gamma_1^2-\frac{1}{9}\overline{k^2}\right)$$
$$\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_2\,W_3\,W_{123}\,\mathcal{P}_{12,3}\,\mathcal{Q}_{1,2}$$
$$= \frac{2}{3}\sigma^6(R_0)\left(1+\frac{1}{6}\gamma_1\right)$$
$$\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_2\,W_3\,W_{123}\,\mathcal{P}_{3,12}\,\mathcal{Q}_{1,2}$$
$$= \frac{2}{3}\sigma^6(R_0)\left(1+\frac{1}{3}\gamma_1\right)$$
$$\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_2\,W_3\,W_{123}\,\mathcal{Q}_{12,3}\,\mathcal{P}_{1,2}$$
$$= \frac{2}{3}\sigma^6(R_0)\left(1+\frac{1}{6}\gamma_1\right)$$
$$\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}_3}{(2\pi)^3}\,P(k_1)\,P(k_2)\,P(k_3)\,W_1\,W_2\,W_3\,W_{123}\,\mathcal{Q}_{12,3}\,\mathcal{Q}_{1,2} = \frac{4}{9}\sigma^6(R_0). \tag{A.30}$$

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FIGURE CAPTIONS

Fig. 1 : Variation of D_2 (Eq. [39]) and E_2 (Eq. [43]) as a function of Ω. Two cosmological hypothesis are considered: the solid line if for $\Lambda = 0$ and the dashed line for a flat universe, $\Omega + \Lambda/3H^2 = 1$.

Fig. 2 : Variation of D_3 (Eq. [40]) and E_3 (Eq. [44]) as a function of Ω. The line symbols are the same as in Fig. 1.

Fig. 3 : Large–scale behavior of the skewness of the density field (top) and of the divergence of the density field (bottom) as a function of Ω. The lines correspond to various index of the power spectrum index: from top to bottom $n = -3$, -2, -1, 0, $+1$ in each panel and the solid lines are for $\Lambda = 0$ and the dashed lines for $\Omega + \Lambda/3H^2 = 1$.

Fig. 4 : The same as Fig. 3 but for the kurtosis of the density and velocity fields. In the latter case the $n = 1$ case is not presented since $S_{4\theta}$ is close to zero.

Fig. 5 : The same as Fig. 3 but for the combination of the first three moments that is shown to have a very weak dependence with the cosmological parameters for the two fields.

Fig. 6 : The moments as a function of the power spectrum index. The left panels are for the density field and the right for the divergence of the velocity field. The top panels give the skewness, the middle give the kurtosis and the bottom present a combination of these moments (Eq. [77]). The solid lines correspond to $\Omega = 1$, $\Lambda = 0$, the long dashed line to $\Omega = 0.3$, $\Lambda = 0$ and the short dashed lines to $\Omega = 0.3$, $\Lambda/3H^2 = 0.7$.

Fig. 7 : The moments as a function of scale in case of a CDM power spectrum. The top panels give the local index n (Eq. [70], thick lines) and γ_2 ($\equiv -\mathrm{d}n/\mathrm{d}\log(R_0)$) (thin lines) as a function of the smoothing radius. The functions S_3 and S_4 are calculated according to Eqs. [56-59] (thick lines) and thin lines correspond to the approximation (67-68) when the power spectrum is locally approximated by a power law of index n. The other symbols are the same as in Fig. 6. The perturbation theory is thought to be valid for smoothing radius greater than $\sim 10h^{-1}$ Mpc ($h = H_0/100$ km s^{-1}Mpc^{-1}). The squares with the error bars are

the measurements obtained in a CDM numerical simulation with $\Omega = 1$ presented in table 1 (and for $a/a_0 = 1$).

Fig. 8 : The moments as a function of scale in case of the power spectrum proposed by Peacock (1991) to fit the observed galaxy correlation function. Symbols are the same as in Figs. 6-7.