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Renormalization and Hyperscaling for Self-Avoiding Manifold Models

François David, Bertrand Duplantier, and Emmanuel Guitter
Service de Physique Théorique, Centre d'Etudes de Saclay, F-91191 Gif-sur-Yvette, France
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The renormalizability of the self-avoiding manifold Edwards model is established. We use a new short distance multilocal operator product expansion, which extends methods of local field theories to a large class of models with nonlocal singular interactions. This validates the direct renormalization method introduced before, as well as scaling laws. A new general hyperscaling relation is derived. Manifolds at the Θ point and long-range Coulomb interactions are briefly discussed.

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The statistical mechanics of fluctuating surfaces has attracted much attention in recent years with applications in many areas of physics from string theories in high energy physics to interface and membrane problems in soft condensed matter physics and biophysics [1]. In particular, *tethered surfaces*, which model polymerized flexible membranes, have unusual and interesting elastic properties. While these properties are now well understood theoretically for "phantom membranes," that is, when self-avoidance (SA) interactions are ignored, the consequence of incorporating SA constraints to describe real membranes is still an open problem. In practice, the search for a consistent theoretical treatment of SA interactions raises the fundamental question of applying renormalization group (RG) methods to *extended* objects, which is the issue addressed here.

The theoretical study of SA polymerized membranes is centered around a model of tethered self-avoiding manifolds (SAM) [2,3] directly inspired by the Edwards model for polymers [4]. The surfaces are generalized to intrinsically D-dimensional *manifolds*, representing D-dimensional connected networks, whose nodes, labeled by internal continuous coordinates $x \in \mathbb{R}^D$, are embedded in external d-dimensional space with position vector $\mathbf{r}(x) \in \mathbb{R}^d$. The associated continuum Hamiltonian $\mathcal{H}$ is

$$\mathcal{H}/k_BT = \frac{1}{2}\int d^Dx\,[\nabla_x \mathbf{r}(x)]^2 + \frac{b}{2}\int d^Dx \int d^Dx'\,\delta^d(\mathbf{r}(x)-\mathbf{r}(x')), \quad (1)$$

with an elastic Gaussian term and a self-avoidance two-body δ potential with excluded volume parameter $b>0$, *nonlocal* in "manifold space" $\mathbb{R}^D$.

A finite upper critical dimension (UCD) d^* for the SA interaction exists only for manifolds with a continuous internal dimension $0<D<2$. Phantom manifolds ($b=0$) are *crumpled* with a finite Hausdorff dimension $d_H = 2D/(2-D)$, and $d^* = 2d_H$. In [2,3,5] an ϵ expansion about d^* was performed via a *direct renormalization* (DR) method adapted from polymer theory [6]. But many issues remain unanswered: The consistency of the DR method is proven only for $D=1$ by the famous mapping of (1) onto a (zero component) $(\Phi^2)^2(\mathbf{r})$ field theory in external d-dimensional space [7]. When $D\neq 1$, model (1) can no longer be mapped onto a local field theory, and the validity of RG methods and of scaling laws has been justified only at leading order through explicit partial resummations [8]. The questions of a proper treatment for boundaries and of the value of the configuration exponent γ [5] are also open.

In this Letter, we introduce a flexible formalism that allows us to prove the validity of the RG approach to self-avoiding manifolds, as well as to a larger class of manifold models with nonlocal interactions. It broadly extends a recent work by the authors [9] for a simpler model [10], with a *local* singular interaction, of a phantom manifold interacting with a single impurity [11]. The present formalism is based on a new operator product expansion involving *multilocal singular operators*, and allows for a systematic analysis of the short distance ultraviolet (uv) singularities of the model. At the critical dimension d^*, we can classify all the relevant operators

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and show that the model (1) is *renormalizable to all orders* by renormalizations (i) of the coupling b and (ii) of the position field $\mathbf{r}$. As a consequence, we establish the validity of scaling laws for *infinite* membranes, as well as the existence of finite size scaling laws for *finite* membranes. The latter result ensures the consistency of the DR approach. A surprising result, which distinguishes manifolds with noninteger D from open linear polymers, is the absence of *boundary* operator renormalization, leading to the general *hyperscaling relation*

$$\gamma = 1 - \nu d \,, \tag{2}$$

valid for finite SAM with $D < 2$, $D \neq 1$. Another surprise when considering SAM at the Θ point is the appearance of a new relevant interaction term, which can supersede the usual three-body term.

Perturbation theory.— For infinite SA manifolds, physical observables are expressible in terms of the P-point correlation functions, whose perturbative expansions are formally

$$\left\langle \prod_{l=1}^{P} e^{i\mathbf{q}_l\cdot\mathbf{r}(z_l)} \right\rangle = \frac{1}{\mathcal{Z}} \sum_{N=0}^{\infty} \frac{(-b)^N}{2^N N!} \int \prod_{i=1}^{2N} d^D x_i \left\langle \prod_{l=1}^{P} e^{i\mathbf{q}_l\cdot\mathbf{r}(z_l)} \prod_{a=1}^{N} \delta^d(\mathbf{r}(x_{2a}) - \mathbf{r}(x_{2a-1})) \right\rangle_0 . \tag{3}$$

The right hand side average $\langle \cdots \rangle_0$ is performed with respect to the ideal Gaussian manifold ($b=0$). The partition function $\mathcal{Z}$ in the denominator has a similar perturbative expansion in b, but with no external points. The product of δ functions in (3) can be written in terms of exponential operators as

$$\prod_{a=1}^{N} \delta^d(\mathbf{r}(x_{2a}) - \mathbf{r}(x_{2a-1})) = \int \prod_{i=1}^{2N} \frac{d^d\mathbf{k}_i}{(2\pi)^d} \prod_{a=1}^{N} \mathcal{C}_a\{\mathbf{k}_i\} \prod_{i=1}^{2N} e^{i\mathbf{k}_i\cdot\mathbf{r}(x_i)} , \tag{4}$$

with N "dipolar constraints" $\mathcal{C}_a\{\mathbf{k}_i\} = (2\pi)^d \delta^d(\mathbf{k}_{2a-1} + \mathbf{k}_{2a})$ for momenta $\mathbf{k}_i \in \mathbb{R}^d$ (later called "charges") assigned to the points x_i. The correlation function (3) is defined as translationally invariant in external space, i.e., with the "neutrality rule" $\sum_{l=1}^{P} \mathbf{q}_l = \mathbf{0}$, a condition which is necessary when dealing with infinite membranes to avoid infrared (ir) singularities. The Gaussian average in (3) is easily performed, using the identity

$$\left\langle \prod_i e^{i\mathbf{k}_i\cdot\mathbf{r}(x_i)} \right\rangle_0 = \exp\left(-\frac{1}{2} \sum_{i,j} \mathbf{k}_i\cdot\mathbf{k}_j G_{ij} \right) , \tag{5}$$

where $G_{ij} = -|x_i - x_j|^{2-D}/(2-D)S_D$ is the massless propagator (Coulomb potential in D dimensions), with $S_D = 2\pi^{D/2}/\Gamma(D/2)$. Integration over the momenta $\mathbf{k}_i$ then gives for the Nth term of (3) the manifold integral

$$\int \prod_{i=1}^{2N} d^D x_i \, \Delta^{-d/2} \exp\left(-\frac{1}{2} \sum_{l,m=1}^{P} \mathbf{q}_l\cdot\mathbf{q}_m \frac{\Delta_{lm}}{\Delta} \right) , \tag{6}$$

where $\Delta\{x_i\}$ is the determinant associated with the quadratic form (now on $\mathbb{R}$) $Q\{k_i\} = \sum_{i,j=1}^{2N} k_i k_j G_{ij}$ restricted to the vector space defined by the N neutrality constraints $\mathcal{C}_a\{k_i\}$, $k_{2a} + k_{2a-1} = 0$, and Δ_{lm} is a similar determinant involving also the external points z_l and z_m [8].

Note that a proper analytic continuation in D of (6) is ensured from [9] by the use of distance geometry, where the Euclidean measure over the x_i is understood as the corresponding measure over the mutual squared distances $a_{ij} = |x_i - x_j|^2$, a distribution analytic in D.

Singular configurations.— The integrand in (6) is singular when the determinant $\Delta\{x_i\} \leq 0$. The associated quadratic form $Q\{k_i\}$, restricted by the neutrality constraints $\mathcal{C}_a\{k_i\}$, is the electrostatic energy of a gas of charges k_i located at x_i, and constrained to form N neutral pairs a of charges (dipoles). For such a globally neutral gas, the Coulomb energy is minimal when the charge density is zero everywhere, i.e., when the nonzero charges k_i aggregate into neutral "atoms." When $0 < D < 2$, the corresponding minimal energy is furthermore *zero*, which implies that the quadratic form Q is non-negative and thus $\Delta \geq 0$. Singular $\{x_i\}$ configurations, with $\Delta = 0$, still exist when Q is degenerate, which happens when some dipoles are assembled in such a way that, with appropriate nonzero charges, they still can build neutral atoms. This requires some of the points x_i to coincide *and* the corresponding dipoles to form at least one closed loop (Fig. 1). This ensures that the only sources of divergences are *short distance singularities*, and extends the Schoenberg theorem used in [9].

Multilocal operator product expansion.— A singular configuration can thus be viewed as a connected "molecule," characterized by a set $\mathcal{M}$ of atoms p with assigned positions x_p, and by a set $\mathcal{L}$ of links a between these atoms, representing the dipolar constraints $\mathcal{C}_a$. For each p, we denote by $\mathcal{P}_p$ the set of charges i, at x_i, which build

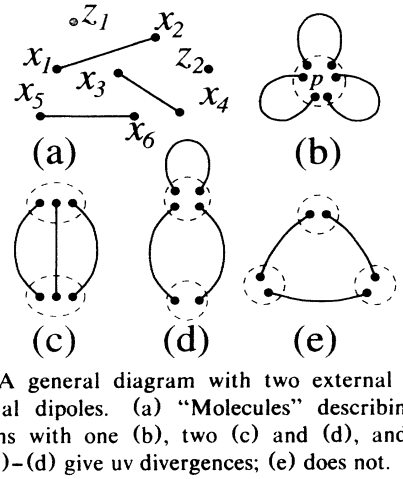

FIG. 1. A general diagram with two external points and three internal dipoles. (a) "Molecules" describing singular configurations with one (b), two (c) and (d), and three (e) "atoms." (b)–(d) give uv divergences; (e) does not.

the atom p and define $y_i = x_i - x_p$ for $i \in \mathcal{P}_p$. The short distance singularity of $\Delta^{-d/2}$ is analyzed by performing a small y_i expansion of the product of the bilocal operators $\varphi(x,x') \equiv \delta^d(\mathbf{r}(x) - \mathbf{r}(x'))$ for the links $a \in \mathcal{L}$, in the Gaussian manifold theory [Eq. (3)]. As will be shown below, this expansion around $\mathcal{M}$ can be written as a *multilocal operator product expansion* (MOPE)

$$\prod_{a \in \mathcal{L}} \varphi(x_{2a}, x_{2a-1}) = \sum_{\Phi} \Phi\{x_p\} \underbrace{C_{\varphi \cdots \varphi}^{\Phi}}_{|\mathcal{L}|}\{y_i\} , \tag{7}$$

where the sum runs over all multilocal operators Φ of the form

$$\Phi\{x_p\} = \int d^d\mathbf{r} \prod_{p \in \mathcal{M}} \{:[(\nabla_r)^{q_p} \delta^d(\mathbf{r} - \mathbf{r}(x_p))] A_p(x_p):\} . \tag{8}$$

Here $A_p(x_p) \equiv A^{(r_p,s_p)}(\nabla_x, \mathbf{r}(x_p))$ is a local operator at point x_p, which is a product of x derivatives of the field $\mathbf{r}$, of degree s_p in $\mathbf{r}(x_p)$ and degree $r_p \geq s_p$ in ∇_x. $(\nabla_{\mathbf{r}})^{q_p}$ denotes a product of q_p derivatives with respect to $\mathbf{r}$, acting on $\delta^d(\mathbf{r} - \mathbf{r}(x_p))$. The symbol ": :" denotes the *normal product* subtraction prescription at x_p (which, in a Gaussian average, amounts to setting to zero any derivative of the propagator G_{ij} at coinciding points $x_i = x_j = x_p$). For $\mathrm{Card}(\mathcal{M}) \equiv |\mathcal{M}| > 1$, (8) describes the most general $|\mathcal{M}|$-body contact interaction between the points x_p, with possible inserted local operators $A_p(x_p)$ at each point x_p. For $|\mathcal{M}| = 1$, it reduces to a local operator $A_p(x_p)$.

The coefficient associated with the operator Φ in the MOPE, $C_{\varphi \cdots \varphi}^{\Phi}\{y_i\}$, can be written as an integral over the momenta $\mathbf{k}_i$,

$$C_{\varphi \cdots \varphi}^{\Phi}\{y_i\} = \int \prod_{a \in \mathcal{L}}{}' \mathcal{C}_a\{\mathbf{k}_i\} \prod_{p \in \mathcal{M}} \left\{ \prod_{i \in \mathcal{P}_p} d^d\mathbf{k}_i \left\{ (\nabla_{\mathbf{k}})^{q_p} \delta^d \left(\sum_{i \in \mathcal{P}_p} \mathbf{k}_i \right) \right\} C^{A_p}\{y_i, \mathbf{k}_i\} \exp\left[-\frac{1}{2} \sum_{i,j \in \mathcal{P}_p} \mathbf{k}_i \cdot \mathbf{k}_j G_{ij} \right] \right\} , \tag{9}$$

where $C^{A_p}\{y_i, \mathbf{k}_i\}$ is a monomial associated with the operator A_p, of similar global degree r_p in the y_i, and s_p in the $\mathbf{k}_i$. The product $\prod'$ is over all constraints $a \in \mathcal{L}$ but one.

The MOPE (7) follows from the expression (4) in terms of free field exponentials plus constraints. For each p, we use the general small y_i local operator product identity

$$\prod_{i \in \mathcal{P}_p} e^{i\mathbf{k}_i \cdot \mathbf{r}(x_i)} = : \prod_{i \in \mathcal{P}_p} \exp\left[y_i \frac{\partial}{\partial x_i} \right] \exp[i\mathbf{k}_i \cdot \mathbf{r}(x_i)]|_{x_i = x_p} : \exp\left[-\frac{1}{2} \sum_{i,j \in \mathcal{P}_p} \mathbf{k}_i \cdot \mathbf{k}_j G(y_i, y_j) \right] . \tag{10}$$

When expanded in the y_i, the normal product $:( )|_{x_i = x_p}:$ in (10) gives a sum $\sum_A C^A\{y_i, \mathbf{k}_i\} : A(x_p) e^{i\mathbf{k}_p \cdot \mathbf{r}_p} :$ [denoting $\mathbf{k}_p = \sum_{i \in \mathcal{P}_p} \mathbf{k}_i$ and $\mathbf{r}_p \equiv \mathbf{r}(x_p)$] which generates the local operators $A(x_p)$ and the monomials C^A of (8) and (9). We insert the identity $1 \equiv \int d^d\mathbf{k}_p \, \delta^d(\mathbf{k}_p - \sum_{i \in \mathcal{P}_p} \mathbf{k}_i)$ in (4) for each atom $p \in \mathcal{M}$, rewrite one of the dipolar constraints as a global neutrality constraint $\delta^d(\sum_{p \in \mathcal{M}} \mathbf{k}_p)$ on the $\mathbf{k}_p$, and expand each $\delta^d(\mathbf{k}_p - \sum_{i \in \mathcal{P}_p} \mathbf{k}_i)$ in powers of $\mathbf{k}_p$. Finally by integrating over the $\mathbf{k}_p$, the constraint $\delta^d(\sum_{p \in \mathcal{M}} \mathbf{k}_p)$ builds the multilocal $|\mathcal{M}|$-body operator $\Phi\{x_p\}$ and we obtain the MOPE (7), (8), and (9).

Power counting and renormalization.—The MOPE (7) allows us to determine those singular configurations which give rise to actual uv divergences in the manifold integral (6). Indeed, given a singular configuration $\mathcal{M}$ and integrating over the domain where the relative positions $y_i = x_i - x_p$ are of order $|y_i| \lesssim \rho$, we can use the MOPE of (4) to obtain an expansion of the integrand in (3) in powers of ρ. Each coefficient $C_{\varphi \cdots \varphi}^{\Phi}$ gives a contribution of order ρ^{ω}, with degree ω given by power counting,

$$\omega = D\{2|\mathcal{L}| - |\mathcal{M}|\} + d\nu_0\{|\mathcal{M}| - |\mathcal{L}| - 1\} + \sum_{p \in \mathcal{M}} \{\nu_0(q_p - s_p) + r_p\} , \tag{11}$$

with $\nu_0 \equiv (2 - D)/2 < 1$ and $r_p \geq s_p$. Whenever $\omega \leq 0$, a uv divergence occurs, as a factor multiplying the insertion of the corresponding operator Φ. At the upper critical dimension $d^* = 2D/\nu_0$, ω becomes independent of the number $|\mathcal{L}|$ of dipoles, and is equal to the canonical dimension ω_{Φ} of $\int \prod_{\mathcal{M}} d^D x \, \Phi$ in the Gaussian theory. Three relevant operators with $\omega_{\Phi} \leq 0$ and such that the corresponding coefficient does not vanish by symmetry, are found by simple inspection. Two of these operators are *marginal* ($\omega_{\Phi} = 0$): (i) the two-body SA interaction term $\delta^d(\mathbf{r}_p - \mathbf{r}_{p'})$ itself, obtained through singular configurations with $|\mathcal{M}| = 2$ atoms (and with $q = r = s = 0$ for p and p'); and (ii) the one-body local elastic term $:(\nabla \mathbf{r}_p)^2:$, obtained for $|\mathcal{M}| = 1$ ($q = 0$, $r = s = 2$). The third operator is *relevant* with $\omega_{\Phi} = -D$, and is just the identity operator $\mathbf{1}$ obtained when $|\mathcal{M}| = 1$ ($q = r = s = 0$). It gives "free energy" divergences proportional to the manifold volume, which cancel out in ir finite observables (3).

The above analysis deals with *superficial uv divergences* only. A complete analysis of the general uv singularities associated with successive contractions toward "nested" singular configurations can be performed, using the techniques of [9] and the fact that an iteration of the MOPE only generates multilocal operators of the type (8). The results are as follows: (i) that the observables (3) are uv finite for $d < d^*(D)$, and are meromorphic functions in d with poles at $d = d^*$; (ii) that a renormalization operation, similar to the subtraction operation of [9], can be achieved to remove these poles; and (iii) that this operation amounts to a renormalization of the Hamiltonian (1). More explicitly, the renormalized correlation functions $\langle \prod_{l=1}^{P} e^{i\mathbf{q}_l \cdot \mathbf{r}_{\mathbf{R}}(z_l)} \rangle_{\mathbf{R}}$ have a finite perturba-

tive expansion in the renormalized coupling $b_{\mathbf{R}}$, when $\langle\cdots\rangle_{\mathbf{R}}$ is the average with respect to the renormalized Hamiltonian

$$\mathcal{H}_{\mathbf{R}}/k_BT=\frac{Z}{2}\int d^Dx[\nabla_x\mathbf{r}_{\mathbf{R}}(x)]^2+\frac{1}{2}b_{\mathbf{R}}\mu^\epsilon Z_b\int d^Dx\int d^Dx'\delta^d(\mathbf{r}_{\mathbf{R}}(x)-\mathbf{r}_{\mathbf{R}}(x')). \tag{12}$$

μ is a renormalization (internal) momentum scale, $\epsilon=2D-dv_0$, and $Z_b(b_{\mathbf{R}})$ and $Z(b_{\mathbf{R}})$ are, respectively, the coupling constant and the field renormalization factors, singular at $\epsilon=0$. At first order, we find by explicitly calculating $C_\varphi^{(\nabla\mathbf{r})^2}$ and $C_{\varphi\varphi}^{\varphi}$ that $Z=1+b_{\mathbf{R}}(B/\epsilon)(2-D)^2/2D$, $Z_b=1+b_{\mathbf{R}}(B/\epsilon)\Gamma^2(D/(2-D))/\Gamma(2D/(2-D))$, with $B=\frac{1}{2}(4\pi)^{-d/2}S_D^{2+d/2}(2-D)^{-1+d/2}$. For the quantities which are not ir finite, which we discuss later, an additive counterterm proportional to the volume of the manifold (corresponding to the relevant identity operator 1) is also necessary.

Expressing the observables of the SAM model (1) in terms of renormalized variables $\mathbf{r}=Z^{1/2}\mathbf{r}_{\mathbf{R}}$, $b=b_{\mathbf{R}}\mu^\epsilon Z_b\times Z^{d/2}$, one can derive in the standard way RG equations involving Wilson's functions $W(b_{\mathbf{R}})=\mu(\partial/\partial\mu)b_{\mathbf{R}}|_b$, $\nu(b_{\mathbf{R}})=\nu_0-\frac{1}{2}\mu\,\partial\ln Z/\partial\mu|_b$. A nontrivial ir fixed point $b_{\mathbf{R}}^*\propto\epsilon$ is found for $\epsilon>0$. It governs the large distance behavior of the SA infinite manifold, which obeys scaling laws characterized by the exponent ν, defined for instance through the two-point function $\langle[\mathbf{r}(x)-\mathbf{r}(0)]^2\rangle\propto|x|^{2\nu}$. The value obtained in this approach, $\nu=\nu(b_{\mathbf{R}}^*)$, corroborates that obtained in [2,3,8] at first order in ϵ.

Finite size scaling and direct renormalization.—The DR formalism requires one to consider *finite* manifolds with "internal volume" V, and to express scaling functions in terms of a dimensionless second virial coefficient $g=-R_G^{-d}Z_{2,c}/(Z_1)^2$, where $Z_1(V)$ and $Z_{2,c}(V)$ are, respectively, the one- and two-membrane (connected) partition functions, and R_G is the radius of gyration.

When dealing with a finite closed manifold (for instance the D-dimensional sphere $\mathcal{S}_D$ [9]), characterized by its (in general curved) internal metric, the massless propagator G_{ij} gets modified. Nevertheless, from (10) and the short distance expansion of G_{ij} in a general metric [12], one can show that the short distance MOPE (7) remains valid, provided that the sum is extended to include multilocal operators Φ still of the form (8), but with local operators $A(x)$ involving also the Riemann curvature tensor and its derivatives, with appropriate coefficients $C_{\varphi\cdots\varphi}^{\Phi}$. A crucial point is that in the MOPE the dependence on the geometry of the manifold (size, curvature, etc.) is encoded only in the expectation values $\langle\ldots\Phi\ldots\rangle_0$ of the multilocal operators Φ, while the short distance behavior ($y_i\to 0$) of coefficients $C_{\varphi\cdots\varphi}^{\Phi}\{y_i\}$ is *independent* of the geometry. Thus, at d^*, uv divergences still come with insertions of relevant multilocal operators with $\omega_\Phi\le 0$.

When $0<D<2$, none of the new operators involving the curvature is found to be relevant by power counting. Therefore, the infinite membrane counterterms Z and Z_b still renormalize the finite membrane theory. Since, as for finite size scaling [13], the manifold size is not renormalized, arguments parallel to those of [14] for polymers can be used to justify the DR formalism. Indeed, the second virial coefficient $g(b,V)$ (as well as any *dimensionless* scaling function) must be uv finite when expressed as a function $g_{\mathbf{R}}(b_{\mathbf{R}},V\mu^D)$ of the renormalized coupling $b_{\mathbf{R}}$ (and of μ). As a consequence, (i) the scaling functions are finite when expressed in terms of g and obey RG equations. The existence of a nontrivial ir fixed point $b_{\mathbf{R}}^*$ for $\epsilon>0$ implies that (ii) in the large volume limit $V\to\infty$, g tends toward a finite limit $g^*=g_{\mathbf{R}}(b_{\mathbf{R}}^*)$ (independent of $V\mu^D$), and so do all scaling functions. Points (i) and (ii) are the essence of DR.

Hyperscaling.—As mentioned above, the renormalization of partition functions for a finite SAM requires an additional counterterm (shift of the free energy) proportional to the manifold volume V. A consequence of the absence of other geometry dependent relevant operators when $0<D<2$ is the general hyperscaling law (2) valid for *closed* SAM, and relating the configuration exponent γ, defined by

$$Z_1(V)=\int\mathcal{D}[\mathbf{r}]\delta^d(\mathbf{r}(0))e^{-\mathcal{H}/k_BT}\sim V^{(\gamma-1)/D}, \tag{13}$$

to the exponent ν. Indeed, from (13), once the free energy divergent term has been subtracted, Z_1 is simply multiplicatively renormalized as $Z_1(b,V)=Z^{-d/2}Z_1^{\mathbf{R}}(b_{\mathbf{R}},V\mu^D)$. This validates the scaling hypothesis that $Z_1\sim|\mathbf{r}|^{-d}\sim V^{-\nu d/D}$, and leads directly to (2).

For *open* SAM with *free* boundaries, and when $1\le D<2$, the boundary operator $\int_{\text{boundary}}d^{D-1}x\,1$ becomes relevant. Since it is simply a geometrical quantity, it cannot modify the renormalizations of $\mathbf{r}$ and b. Furthermore, it is marginally relevant only for $D=1$ [5] and, therefore, as long as $D\neq 1$ the scaling laws and the hyperscaling relation (2) *remain valid*. Only at $D=1$ does the corresponding end-point counterterm enter the multiplicative renormalization of Z_1, and γ becomes an independent exponent, with an extra contribution from the two end points.

Equation (2) has been checked explicitly at order ϵ for the sphere $\mathcal{S}_D$ and the torus $\mathcal{T}_D$. Previous calculations [2,3], which yield [5] $\gamma=1$ for noninteger D, did not involve the physical massless propagator [12] (valid for a finite manifold with Neumann boundary conditions) used here.

When $D\ge 2$, if the small ϵ RG picture remains valid, i.e., if the large distance properties of SAM are governed by the ir fixed point $b_{\mathbf{R}}^*$, operators involving curvature become relevant, and (2) is not expected to be valid in general; it holds at least for a curvature-free, toroidal polymerized two-dimensional membrane.

Θ point and long-range interactions.—The above for-

malism is directly applicable to a large class of manifold models where the interaction can be expressed in terms of free field exponentials with suitable neutrality constraints $\mathcal{C}_a\{\mathbf{k}_i\}$. Examples of such interactions are the n-body contact potentials but also the two-body long-range Coulomb potential $1/|\mathbf{r}-\mathbf{r}'|^{d-2}$, which can be represented by modified dipolar constraints $\mathcal{C}\{\mathbf{k}_i\}=|\mathbf{k}|^{-2}\delta^d(\mathbf{k}+\mathbf{k}')$. For all these models, the MOPE involves the same multilocal operators as in (8), with modified coefficients [still given by (9), but with new constraints $\mathcal{C}_a$].

As an application, we may ask for the most relevant short-range interaction describing a polymerized membrane at the Θ point, i.e., when the two-body term b in (1) vanishes. It is either the usual three-body contact potential, with UCD $d_3^*=3D/(2-D)$, as for ordinary polymers, or the two-body singular potential $\Delta_{\mathbf{r}}\delta^d(\mathbf{r}-\mathbf{r}')$ with UCD $\tilde{d}_2^*=2(3D-2)/(2-D)$, which indeed is the most relevant one when $D>\frac{4}{3}$.

Finally, the absence of long-range potentials in the MOPE shows that long-range interactions are not renormalized. For instance, when considering charged polymerized membranes with a two-body Coulomb potential, the only (marginally) relevant operator at the UCD is the local operator $:(\nabla_{\mathbf{r}})^2:$, indicating that only $\mathbf{r}$ is renormalized. As a consequence, it is easy to show that $\nu=2D/(d-2)$ exactly in this case.

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