

Non-perturbative effects in matrix models and vacua of two dimensional gravity

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The most general large N eigenvalues distribution for the one-matrix model is shown to consist of tree-like structures in the complex plane. For the $m=2$ critical point, such a solution describes the strong coupling phase of 2D quantum gravity ($c=0$ non-critical string). It is obtained by taking combinations of complex contours in the matrix integral, and the relative weight of the contours is identified with the non-perturbative “ θ -parameter” that fixes uniquely the solution of the string equation (Painlevé I). This allows to recover by instanton methods results on the non-perturbative effects obtained by the Isomonodromic Deformation Method, and to construct for each θ -vacuum the observables (the loop correlation functions) which satisfy the loop equations. The breakdown of analyticity of the large N solution is related to the existence of poles for the loop operators.

The discovery of the “double scaling solutions” of the matrix models [1–3] led to important progress in the understanding of string theories in $d \leq 2$ backgrounds and of 2D gravity (see refs. [4,5] for reviews). However, the important issue of the non-perturbative status of some of these theories remains unclear, in particular for 2D gravity coupled to unitary matter for $c \leq 1$. In this letter, we discuss some of these questions in the framework of the hermitian one-matrix models. We shall show that a simple generalization of the complex integration contour prescription [6,7], which allows to construct non-perturbative – but in general complex – solutions of the string equations and of the continuous loop equations, leads to real non-perturbative solutions of these equations. This generalization, which consists in taking combinations of inequivalent integration contours, has been already discussed by Fokas, Its and Kitaev [8] in the framework of the isomonodromic deformation method (IDM) approach to the string equations [9], but does not seem to have attracted much attention. Our treatment is based on the BIPZ solution of the one-matrix model [10], and follows

our previous analysis of ref. [11]. We shall show that in the limit $N \rightarrow \infty$, new solutions for the eigenvalues (EVs) distribution exist, which have not been discussed before. They correspond to a distribution of EVs along tree-like structures in the complex plane. Moreover, these solutions depend non-analytically on the coupling constant of the matrix model, and will be associated with the sectors with an infinite number of poles of the string equation solutions. The non-perturbative parameter which characterizes the non-perturbative solutions is simply related to the different weights chosen for the contours, and our treatment allows to recover easily by instanton methods some results of refs. [7,8]. In addition, we show that to each real solution of the string equation is associated a prescription for the asymptotics of the loop operators which defines uniquely observables (i.e. macroscopic loop VEV) which obey the loop equations. Finally we shall show that these new solutions allow to explain the properties of the solutions for the double well matrix models recently discussed by Brower, Deo, Jain and Tan [12].

In the matrix model formulation of 2D gravity, the partition function F (sum over orientable connected 2D riemannian spaces) is discretized into a sum over triangulations, and is written as the logarithm of the partition function Z for the hermitian one-matrix

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model ($F = \ln Z$), which after diagonalization of the matrix Φ can be written as an EV integral,

$$Z_N = C_N \int \prod_{i=1}^N d\lambda_i \exp[-NV(\lambda_i)] \Delta_N(\lambda_i)^2, \quad (1)$$

$$\Delta_N(\lambda_i) = \prod_{i < j} (\lambda_i - \lambda_j),$$

where C_N is a normalization factor, Δ_N the Vandermonde determinant and V the matrix potential. The integral (1) can be calculated in terms of the matrix elements of the operator Q : $\pi_n(\lambda) \mapsto \lambda \pi_n(\lambda)$, where the π_n are orthonormal polynomials for the measure $d\lambda \exp[-NV(\lambda)]$. In the double scaling limit, $N \rightarrow \infty$ and $V \rightarrow V_{\text{critical}}$ while $x = 1 - n/N$ becomes a continuous parameter. Then Q becomes a second order differential operator of the form

$$Q = -\frac{d^2}{dx^2} + 2u(x), \quad (2)$$

u is the string susceptibility

$$u(t) = -\frac{\partial^2 F}{\partial t^2}, \quad (3)$$

where t is the renormalized cosmological constant. For the $m=2$ critical point (pure $c=0$ gravity), t scales with N as $t \sim N^{4/5}$, and u satisfies the Painlevé I string equation

$$-\frac{1}{6} \frac{\partial^2 u}{\partial t^2} + u^2 = t, \quad u \sim \sqrt{t}, \quad t \rightarrow +\infty. \quad (4)$$

It is known that (4) fixes uniquely the terms of the asymptotic expansion of u (in powers of $t^{(1-5k)/2}$) as $t \rightarrow +\infty$, but that the corresponding series is not Borel summable, and that the solutions of (4) form a one-parameter family of “simply truncated solutions” [13], which differ by exponentially small terms of the form

$$\delta u \propto t^{-1/8} \exp(-\frac{4}{3} \cdot 2\sqrt{3} t^{5/4}). \quad (5)$$

The real solutions of (4) have an infinite series of double poles (with residues 1) on the negative real axis. If one divides the complex plane into 5 sectors $s=I, \dots, V$ (which correspond to $(s-1) \cdot \frac{2}{5}\pi < \text{Arg}(t) < s \cdot \frac{2}{5}\pi$), these poles extend to an infinite network of poles in the sectors II, III and IV, so that the asymptotics $u \sim \sqrt{t}$ hold only in the two pole-free sectors I and V.

It was suggested in ref. [6], and shown more precisely in refs. [11,7,8], that, if one constructs the $m=2$ theory by starting from a cubic potential of the form

$$V(\lambda) = -\lambda^3 + \dots, \quad (6)$$

and defines the integral (1) by choosing as integration contour for the λ_i 's the complex contour \mathcal{C}_+ (respectively \mathcal{C}_-) which goes from $-\infty$ to $j\infty$ ($j = e^{i\pi/3}$) (respectively $\bar{j}\infty$), one obtains the “simply truncated solution” u_+ (respectively u_-) of (4) which has poles only in the sector II (respectively IV) and satisfies the asymptotics $u \sim \sqrt{t}$ in the remaining four pole-free sectors. From these two solutions, which are analytic on the real axis, one can construct without ambiguity the operator Q (2), which is not hermitian anymore, and the loop operators $w(p)$ [14] (where p is the loop momentum), which satisfy the loop equations [6,15,16].

In fact a straightforward generalization of this prescription is to consider linear combinations of the two contours, i.e. to replace

$$\int_{\mathcal{C}_\pm} d\lambda \rightarrow c_+ \int_{\mathcal{C}_+} d\lambda + c_- \int_{\mathcal{C}_-} d\lambda, \quad (7)$$

(see fig. 1). Indeed, the partition function Z will still be real (for real V) if the weight c_\pm are complex conjugate

$$c_\pm = \frac{1}{2} \pm i\theta. \quad (8)$$

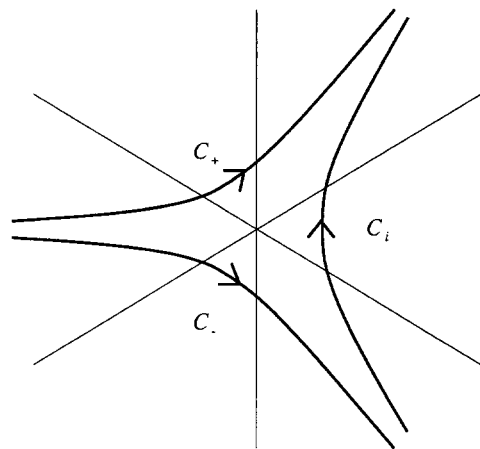


Fig. 1. The three contours for the cubic potential.

With this prescription, the orthogonal polynomial method still works, and the recurrence relations (discrete string equations) still hold. Therefore, if the double scaling limit exists, one should still obtain some solution of the string equation (4).

As already mentioned, it has been shown in ref. [8], within the IDM approach, that this is indeed the case, and that there is a one to one correspondence between the weight ratio c_+/c_- and the simply truncated solution of (4) which is obtained in the double scaling limit. To recover this result in the BIPZ approach, let us consider the matrix model integral (1) in the large N limit. The eigenvalue probability density $(1/N)\sum_{i=1}^N \delta(\lambda-\lambda_i)$ becomes the classical density $v(\lambda)$. It is convenient to consider the function

$$F(\lambda) = \int d\mu \frac{v(\mu)}{\lambda-\mu} = \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \text{Tr} \left(\frac{1}{\lambda-\Phi} \right) \right\rangle. \quad (9)$$

It can be shown, from the saddle point equations for the effective action for v , or through the loop equations [6], that F must be of the form

$$F(\lambda) = \frac{1}{2} [V'(\lambda) + \sqrt{Q(\lambda)}],$$

$$Q(\lambda) = V'(\lambda)^2 + 4N(\lambda), \quad (10)$$

where $N(\lambda)$ is some polynomial of degree

$$\text{degree } N = \text{degree } V - 2 = m - 1. \quad (11)$$

Generically, Q has $2m$ complex zeros, which correspond to square root cuts for F . From (9) the EV density v is proportional to the discontinuity of F along the cuts, and can be reconstructed from F . The normalization $\int d\lambda v = 1$ implies that $F \sim \lambda^{-1}$ at ∞ and this fixes the coefficient of the leading term of N . Requiring that F has only one cut (as done for instance in ref. [10]) implies that Q has $m-1$ double zeros and this fixes uniquely N . However, in general F may have several cuts.

Let us label by α the cuts and by x_α the fraction of EV along each cut ($x_\alpha \geq 0$ and $\sum x_\alpha = 1$). The EV density v must minimize the action

$$S = \int d\lambda v(\lambda) V(\lambda) - \iint d\lambda d\mu v(\lambda) v(\mu) \ln |\lambda - \mu|$$

$$+ \sum_\alpha \Gamma_\alpha \left(x_\alpha - \int_\alpha d\lambda v(\lambda) \right), \quad (12)$$

where Γ_α are Lagrange multipliers. In fact (10) is the

most general solution when one extremizes S with respect to variations of the density which do not change the x_α 's. As a consequence, the effective potential $\Gamma(\lambda)$ for one EV in the background created by the $N-1$ other EVs,

$$\Gamma(\lambda) = V(\lambda) - 2 \int d\mu v(\mu) \ln(\lambda - \mu) \quad (13)$$

is constant along each cut α , and equal to Γ_α

$$\Gamma(\lambda) = \Gamma_\alpha \quad \text{if } \lambda \in \alpha \quad (14)$$

so that the total action is

$$S = \frac{1}{2} \sum_\alpha \left(\int_\alpha d\lambda v(\lambda) V(\lambda) + x_\alpha \Gamma_\alpha \right). \quad (15)$$

The remaining constraints which fix N are the following:

(a) the EV fractions x_α must be real.

(b) One must minimize $\text{Re}(S)$ with respect to variations of the x_α 's, subjected to the constraints that $x_\alpha \geq 0$ and that $\sum x_\alpha = 1$. Since from (12) $\partial S / \partial x_\alpha = \Gamma_\alpha$, this implies that the real part of the effective potential is the same along all the cuts:

$$\text{Re}(\Gamma_\alpha) = \Gamma_0, \quad \text{unless } x_\alpha = 0. \quad (16)$$

In fact the constraints (a) and (b) can be recast in the same form:

$$\text{Im}(\mathcal{I}_\alpha) = 0, \quad \mathcal{I}_\alpha = \oint_{\mathcal{C}_\alpha} \frac{d\lambda}{2i\pi} F(\lambda), \quad (17)$$

where \mathcal{C}_α is any contour encircling a pair of zeros of Q . Indeed, if \mathcal{C} encircles a cut α , $\mathcal{I} = x_\alpha$, while if \mathcal{C} encircles the end points of two different cuts α and β , it follows from the fact that away from the cuts, $\Gamma'(\lambda) = V'(\lambda) - 2F(\lambda)$ (obtained by taking the derivative of (13)), that $\mathcal{I} = (1/i\pi)(\Gamma_\beta - \Gamma_\alpha)$ (see fig. 2). Since there are $2m-2$ independent contours, (17) fixes the $m-1$ remaining coefficients of N . Let us note that if Q has a double zero, two independent constraints (17) are automatically satisfied, since for any contour \mathcal{C} which encircles the double zero and at most one single zero, $\mathcal{I} = 0$. Therefore we expect in general to have one solution of (17) with no double zeroes, $2m-1$ solutions with one double zero, etc. Some of these different solutions will be excluded because:

(c) Some x_α 's are < 0 .

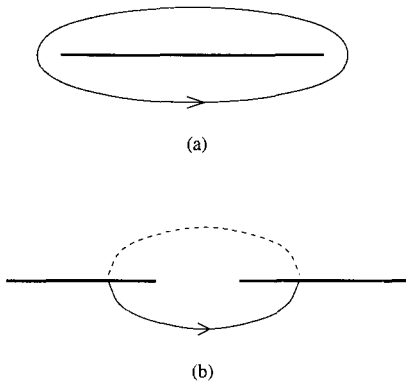


Fig. 2. The contours in (17) corresponding to constraints (a) and (b).

(d) The contours of integration for the EV cannot pass through one of the cuts α . Finally among the remaining solutions, it is the one with minimal $\text{Re}(S)$ (real part of the action (12)) which is the physical saddle point.

Let us discuss explicitly the case of the $m=2$ critical point. We start from the potential $V(\lambda) = -\frac{1}{3}\lambda^3 + g\lambda$. In the critical regime we rescale $g_c - g \simeq a^2 t$, $\lambda - \lambda_c \simeq ap$ [11]. t is the renormalized cosmological constant and p the loop momentum. The scaling parameter (short-distance cut-off) a is defined so that the double scaling limit is obtained by taking $N \rightarrow \infty$, $g_{\text{string}}^{-2} = N^2 a^5 = 1$. In the planar scaling limit ($N = \infty$, then $a \rightarrow 0$), the general solution (10) for $F(\lambda)$ becomes

$$F(\lambda) \rightarrow w(p) = a^{5/2} \cdot \frac{2}{3} \sqrt{p^3 - 3tp + c}, \quad (18)$$

where c is the only parameter to be determined in the polynomial $N(\lambda)$ which is relevant in the scaling limit. It corresponds to the VEV of the puncture operator [6]. In the weak coupling phase $t > 0$, the saddle point is the standard one cut solution [10,6]

$$c = 2t^{3/2},$$

$$w(p) = a^{5/2} \cdot \frac{2}{3} (t^{1/2} - p) \sqrt{p + 2t^{1/2}}. \quad (19)$$

The EVs are located on the real axis along the cut $]-\infty, -2t^{1/2}]$ (see fig. 3a). The action for this solution scales as

$$S \propto a^{5/2} t^{5/2}. \quad (20)$$

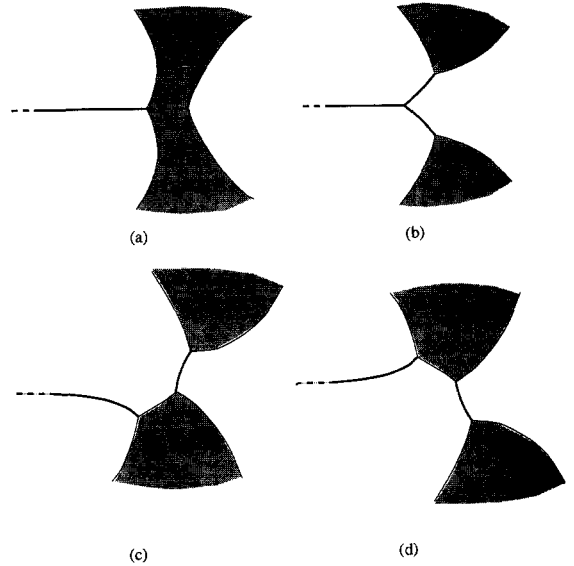


Fig. 3. Schematic picture of the EV distribution (black line) and the $\text{Re}(\Gamma) > 0$ domains (grey) in the p complex plane for the generic solution of the $m=2$ critical point: (a) real $t > 0$ (and sectors I and V); (b) real $t < 0$ (and sector III); (c) t in sector II; (d) t in sector IV.

There are other unphysical solutions which violate (c) or (d).

This solution can be analytically continued into the strong coupling unstable phase $t < 0$. The two complex conjugate solutions describe EVs still located along a single arc [11], and correspond to the triply truncated solutions of (4). From (20) they have a purely imaginary action. However, for $t < 0$ the constraints (17) have another solution, where $w(p)$ has now three branch points. For this solution c is given by

$$c = c(-t)^{3/2}, \quad c > 0 \quad (21)$$

and it corresponds to the branched distribution of the EVs depicted in fig. 3b. The density of EVs along the negative real axis vanishes at the real branch point p_0 as $\sqrt{p_0 - p}$, but there are two arcs starting from p_0 toward the two other complex conjugate branch points p_{\pm} . Therefore equal fractions $x_+ = x_-$ of EVs are sitting along these two arcs. Moreover the action for this solution is real and negative, and therefore it is generically the dominant one. Away from the negative real axis, the branched solution still exists, but with

asymmetric branches ($x_+ \neq x_-$), provided that one stays in the sector III ($-\frac{1}{5}\pi < \text{Arg}(-t) < \frac{1}{5}\pi$), and still has a smaller action than the perturbative one. Moreover, since the constraints (17) are non-analytic, this solution does not depend analytically on t (in other words, the number c in (21) depends on $\text{Arg}(t)$).

Finally, in the sectors II and IV, another kind of solutions, with two cuts, is dominant (see figs. 3c, 3d). These solutions are still non-analytic in t . In the sectors I and V, the perturbative analytic solution (19) is the physical one.

As is clear from fig. 3, for the general integration prescription (7), one can obtain these new, non-analytic, solutions in the sectors II, III and IV. Since they have a smaller action than the analytic solution, they will dominate the large N limit, unless c_+ or c_- is zero.

This new solution for the EV distribution is associated with the simply truncated solutions of (4). This can be seen as follows. First, we have seen that these solutions are non-analytic in t in the sectors where simply truncated solutions have poles. Since double poles of $u(t)$ should correspond to simple zeros of the τ function, which corresponds to the partition function Z for the matrix model, and since in the large N limit Z is obtained from the action S for the saddle point EV configuration by $Z = \exp(N^2 S)$, using the Cauchy formula for the derivative of $\ln(Z)$ and Stoke's formula we obtain the estimate for the number of zeros in a domain D :

$$\begin{aligned} \# \text{ of zeros in domain } D &= \oint_{\partial D} \frac{dg}{2i\pi} \frac{\partial \ln(Z)}{\partial g} \\ &\simeq N^2 \iint_D \frac{dg d\bar{g}}{4\pi} \frac{\partial}{\partial g} \frac{\partial}{\partial \bar{g}} S(g, \bar{g}). \end{aligned} \tag{22}$$

Thus the new solution describes a partition function with a positive density of zeros $\rho \propto \partial \bar{\partial} S$ in the three sectors II, III, IV, which scales as $\rho \propto N^2 a^{5/2} |t|^{1/2} \times f(\phi)$, where $\phi = \text{Arg}(t)$.

One can make the identification more precise by relating the constraints (17) to the asymptotics of the simply truncated solutions of (4). Following ref. [13] (see refs. [17,18] for more recent discussions) we make the change of variable

$$u = t^{1/2} U, \quad T = \frac{4}{5} t^{5/4} \tag{23}$$

in (4), which becomes

$$U'' - 6U^2 + 6 = -\frac{U'}{T} + \frac{4}{25} \frac{U}{T^2}. \tag{24}$$

As $|T| \rightarrow \infty$, U is asymptotic to a Weierstrass elliptic \wp function $U_0(T, E_0)$, solution of

$$(U'_0)^2 = 4U_0^3 - 12U_0 + E_0, \tag{25}$$

which is doubly periodic (with a lattice of double poles) with periods

$$\Omega_{1,2} = \oint_{\mathcal{C}_{1,2}} dU_0 (4U_0^3 - 12U_0 + E_0)^{-1/2}, \tag{26}$$

where $\mathcal{C}_{1,2}$ are two contours encircling pairs of zeros of the RHS of (25). In a neighborhood of some $T = T_0$, one can treat E_0 as a slowly varying variable $E_0(T)$. From (24), in the local periods coordinates $T - T_0 = \Omega_1 y^1 + \Omega_2 y^2$, E_0 varies as

$$\frac{\partial E_0}{\partial y^{1,2}} \simeq -\frac{2}{T} \mathcal{J}_{1,2}(E_0),$$

$$\mathcal{J}_{1,2}(E_0) = \oint_{\mathcal{C}_{1,2}} dU_0 (4U_0^3 - 12U_0 + E_0)^{1/2}. \tag{27}$$

Solving the flow equations (27) one can check that asymptotically, E_0 depends only on the argument of T , $\Theta = \text{Arg}(T)$, but not on its modulus, and is solution of the two constraints

$$\text{Re}[e^{i\Theta} \mathcal{J}_{1,2}(E_0)] = 0. \tag{28}$$

But these constraints are exactly equivalent to (17) for (18), one we identify $E_0 = t^{-3/2} c$ and use the fact that $\Theta = \frac{3}{4} \text{Arg}(t)$.

Before discussing the asymptotics for the loop operators, let us show how the choice of weight contours c_{\pm} in (7) fixes uniquely the non-perturbative part of the solution of (4). Let us denote by $Z_N(\theta)$ (respectively $F_N(\theta)$) the partition function (1) (respectively its Logarithm) for N EVs with the contour coefficients (8). Taking the derivative of F with respect to θ singles out one of the EVs

$$\frac{dF_N}{d\theta} = i \frac{N}{Z_N} C_N \times \int_{\mathcal{C}_i} d\lambda_0 \int \prod_{i=1}^{N-1} d\lambda_i \Delta_N(\lambda_i)^2 \exp\left(-N \sum V(\lambda_i)\right), \tag{29}$$

where \mathcal{C}_i is the contour going from $\bar{j}\infty$ to $j\infty$. One estimates this integral by first integrating out the $N-1$ last EVs (by using the BIPZ method), in the effective potential

$$\tilde{V}(\lambda) = V(\lambda) + \frac{1}{N} [V(\lambda) - 2 \ln(\lambda_0 - \lambda)]$$

modified by the first EV. The resulting effective potential for the first EV is in general complicated, since it takes into account the backreaction of this EV on the bulk $N-1$ other EVs. However it takes a simple form if λ_0 is close to the end-point λ_e of the EV distribution, since we obtain

$$\frac{dF_N}{d\theta} = \frac{i}{8\pi} \int_{\mathcal{C}_i} d\lambda_0 \frac{1}{\lambda_0 - \lambda_e} \times \exp\{-N[\Gamma(\lambda_0) - \Gamma(\lambda_e)]\} \left[1 + O\left(\frac{1}{N}\right)\right], \tag{30}$$

where $\Gamma(\lambda)$ is the effective potential (13). For the $m=2$ critical point, in the scaling regime $\Gamma'(p) = -2w(p)$, with $w(p)$ given by (19). At large N the integral (30) is dominated by the instanton configuration of ref. [11], where the EV sits at the top of the potential $p = t^{1/2}$. The result, including the contribution of fluctuations around the saddle point, is

$$\frac{dF}{d\theta} = -g_s^{1/2} \frac{3^{-3/4}}{8\sqrt{\pi}} t^{-5/8} \times \exp\left(-\frac{1}{g_s} \frac{4}{3} \cdot 2\sqrt{3} t^{5/4}\right) \left[1 + O\left(\frac{1}{N}\right)\right], \tag{31}$$

where $g_s = N^{-1} a^{-5/2}$ is the string coupling constant. In the double scaling limit $g_s = 1$, and (31) gives the non-perturbative θ -dependence of the sting susceptibility $u = -F''$. Using the fact that the triply truncated solutions u_{\pm} correspond to $\theta = \mp \frac{1}{2}i$, one thus recovers the results of refs. [7,8] for the non-perturbative part of the simply truncated solutions of Painlevé I.

Finally, let us return to the construction of the loop operators. It is very easy to check that with the generic choice of contours (7), the finite- N loop equations of the matrix model are still satisfied by the loop operators. It remains to understand what is the continuum limit of these operators. In string perturbation theory, the operator which creates a macroscopic loop with momentum p (conjugate to the loop length l),

$$w(p) = \int_0^{\infty} dl e^{-pl} w(l),$$

can be expressed in terms of matrix elements of the operator Q given by (2) [14]. For instance the one-loop correlator is

$$\langle w(p) \rangle = \int_t^{\infty} dx \langle x | \frac{1}{p+Q} | x \rangle, \tag{32}$$

and the problem is to define the resolvent

$$G(x; p) = \langle x | (p - Q)^{-1} | x \rangle$$

for the generic real solutions of (4), with poles on the negative real axis. G must satisfy the Gelfand-Dikii equation

$$-2GG'' + G'^2 + 4[p + 2u(x)]G^2 = 1, \tag{33}$$

and generically G has also double poles at the poles of u . In fact there is a unique asymptotic prescription for $G(x)$ in the strong coupling regime $x \rightarrow -\infty$, which is consistent with the contour prescription (7) (defined by the θ -parameter), and the specific u . If we perform the rescaling (similar to (23))

$$X = \frac{4}{3}x^{5/4}, \quad G = x^{-1/4}H, \quad P = x^{-1/2}p, \tag{34}$$

in the limit $|X| \rightarrow \infty$ (33) becomes

$$-2HH'' + H'^2 + 4[P + 2U_0(X)]H^2 = 1 + O\left(\frac{1}{X}\right), \tag{35}$$

where $U_0(X)$ is the elliptic function given by (23), which is solution of (25). There is a unique solution of (35) which is doubly periodic with the same periods $\Omega_{1,2}(E_0)$ than U_0 . It is given explicitly by

$$H(X; P) = \frac{P - U_0(X)}{\sqrt{4P^3 - 12P + E_0}}. \tag{36}$$

This, together with (32) and the constraints (28) which fix E_0 , gives the same asymptotic expression for the one-loop correlator $w(p)$ in the non-perturbative phase $t < 0$ than the expression (18) that we have obtained previously through the BIPZ approach. This achieves the identification of our large N non-perturbative solution of the matrix model with the real solutions of the string equation (4). These loop operators will satisfy the loop equations [6,15,16], at variance with the operators constructed only in the perturbative phase with the prescription of refs. [2,3,14]. Each loop operator will have a *single pole* in t wherever the string susceptibility has a double pole. This is in fact natural, since the string susceptibility is the VEV of two microscopic loops (puncture operator).

Let us summarize and discuss our results for the case of pure gravity.

The proposal of ref. [8] to take a real combination of complex integration contours in the one-matrix model to obtain real solutions of the Painlevé I string equation for pure 2D gravity ($c=0$) has been formulated here in the framework of the large N solution of the matrix model à la BIPZ, i.e. in terms of distribution of eigenvalues. We have shown that the strong coupling phase, which corresponds to negative values of the renormalized cosmological constant t , and in which the string susceptibility has poles, can be described simply in terms of splitting of the mean-field distribution of the eigenvalues into two branches at the end of the EV distribution.

These two different branches can be viewed as two different topological sectors in the integral over the EVs and the non-perturbative θ -parameter which distinguishes the different solutions of the string equation is simply the phase difference between these two sectors, which has to be specified in the functional integral. Therefore, at a formal level, each θ defines a θ -vacuum of 2D gravity, as in field theories with topological sectors, such as 4D non-abelian gauge theories or some 2D σ -models. The non-perturbative effects associated with this θ parameter can be estimated by simple instanton methods.

Finally, we have shown that, for each real solution of the string equation (θ -vacuum), it is possible to construct in a consistent way observables (loop operators), in such a way that the loop equations should be satisfied non-perturbatively.

Of course, many interesting questions are still open.

It is clear that one can define non-perturbatively the one-matrix model for general potential, and probably reconstruct by adequate choice of contours the real non-perturbative solutions of the unstable even m string equations. We shall discuss below the case of the double well potential. Similarly, the same recipe can be applied to the multi-matrix models, and used to study the general (p, q) string equations (although for the multi-matrix models there is no simple picture of the large N limit in terms of EV distribution).

The fact that the loop equations are still valid non-perturbatively in the framework discussed here is quite appealing. These equations can be derived from various points of view: Dyson–Schwinger equations for the matrix models, Virasoro constraints for the KdV hierarchy, recursion relations in 2D topological gravity. This is at variance with other schemes which have been proposed for defining non-perturbatively 2D gravity [19,20].

One important issue has to be properly understood. In the strong coupling phase ($t < 0$) the partition function $Z = \exp(F)$ has zeros, and the loop operators have poles. This implies that the non-perturbative real solutions of 2D gravity that we have discussed here should suffer from non-perturbative violation of positivity, even in the weak coupling regime $t > 0$. It remains to understand what this really means when one formulates these solutions in terms of string field theories in low dimensional backgrounds, in particular for positivity and unitarity.

The $c=1$ matrix model solution studied in ref. [21] does not suffer from the kind of instability of the $c=0$ model, since it corresponds to free fermions in an inverted harmonic potential, with the two wells of the potential filled at the same Fermi level. Consequently, although the string perturbation theory for the $c=1$ model is not Borel summable, there is a well defined summation prescription which allows to reconstruct this non-perturbative solution. Does the kind of ideas discussed here allow to construct other non-perturbative solutions of the $c=1$ model?

Finally let us briefly discuss the case of the double well potential,

$$V(\lambda) = \frac{1}{2}\mu\lambda^2 + \frac{1}{4}\lambda^4. \quad (37)$$

For $\mu < 0$ large enough, the EVs are distributed along

two cuts (symmetric under $\lambda \leftrightarrow -\lambda$). At the critical point μ_c , the two cuts fuse (at $\lambda=0$) into one segment. In the double scaling limit the string equation for this critical point is the Painlevé II equation [22]. Recently, Brower et al. showed that by relaxing the parity condition $\pi_n(\lambda) = (-1)^n \pi_n(-\lambda)$ on the orthonormal polynomial and on the associated solutions of the recurrence equations, new symmetry breaking solutions of the model could be obtained [12]. This can be easily understood by considering the three independent paths of integration for the potential (37). In addition to the real axis \mathcal{C}_r we can also integrate over the paths \mathcal{C}_\pm going from $-\infty$ to $\pm i\infty$. The most general weight factors for these paths which give a real partition function are

$$c_r = 1 - 2x, \quad c_\pm = x \pm i\theta. \quad (38)$$

If $c_\pm \neq 0$, in the strong coupling phase $\mu > \mu_c$ the EV distribution is no more the one cut solution but a cross-shaped distribution with four cuts meeting at the origin. Setting $x \neq 0$ breaks explicitly the symmetry $\lambda \leftrightarrow -\lambda$ and should allow to recover the symmetry breaking solutions of ref. [12] (which differ from the standard solution by subdominant terms of order $1/N^2$). Setting $x=0$ but $\theta \neq 0$ gives solutions which differ non-perturbatively from the standard one, and which correspond to solutions of the Painlevé II equation with (simple) poles on the negative real axis.

These considerations can be extended easily to the multicut matrix models studied in ref. [23].

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