Modern Physics Letters A, Vol. 3, No. 17 (1988) 1651–1656

CONFORMAL FIELD THEORIES COUPLED TO 2-D GRAVITY IN THE CONFORMAL GAUGE

F. DAVID*

Service de Physique Théorique† de Saclay, 91190-Gif sur Yvette, Cedex, France

Received 20 July 1988

The coupling of conformal field theories to 2-*d* gravity may be studied in the conformal gauge. As an application, the results of Knizhnik, Polyakov and Zamolodchikov for the scaling dimensions of conformal fields are derived in a simple way. Their conjecture for the susceptibility exponent γ of strings is proven and extended to arbitrary genus surfaces. The result agrees with exact results from random lattice models.

Recently, Polyakov proposed a new treatment of 2-dimensional gravity[1] that relies upon a light cone gauge fixing procedure for dealing with the invariance of the theory under local diffeomorphisms. The emerging effective theory contains a SL(2, $\mathbb{R}$) current algebra. Knizhnik, Polyakov and Zamolodchikov developed this approach[2] to study conformal field theories coupled to 2-*d* gravity, and to derive a relation between the conformal dimension Δ of any (primary and spinless) field φ coupled to gravity and its original conformal dimension $\Delta^{(0)}$ (with no coupling to gravity). They also conjectured a formula for the "string susceptibility" exponent γ related to the large area behaviour of the partition function. The SL(2, $\mathbb{R}$) current algebra plays a central role in the derivation of those results. They are in striking agreement with exact results for discretized models of 2-*d* gravity, where the internal metric is described by random triangulations, both for "pure string models" in $d = 0$ and $d = -2$ dimensions[3–5] and for critical statistical systems on random lattices, such as Q states Potts models for $Q = 2$ (Ising),[6] $Q \to 1$ (percolation),[7] $Q \to 0$ (tree like polymers),[8] and self-avoiding polymers.[9] The purpose of this letter is to show that the coupling of 2-*d* conformal theories to gravity may be studied in a quite simple way in the conformal gauge.[10] The resulting effective action is indeed the Liouville field theory.[11] At the critical point where the physical cosmological constant vanishes, the Liouville theory is nothing but a free field theory. This makes the calculation of the scaling dimensions straightforward and allows one to recover in a simple way the results of Ref. 2. In addition, this method allows us to compute the susceptibility exponent γ as a function of the topology of the 2-*d* "universe", and thus to prove and to extend the conjecture of Ref. 2.

Let us start from the action for 2-*d* gravity on a manifold M

* Physique Théorique CNRS.

† Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique.

$$S[g_{ab}] = K \int_M d^2\sigma \sqrt{g} \tag{1}$$

coupled in a minimal way to some 2-d conformal field theory characterized in particular by its central charge C and its set of primary fields φ_i. As in the seminal paper,[11] we fix a conformal gauge by choosing a family $\{g^0_{ab}(m)\}$ of conformally inequivalent metrics labelled by the moduli m of the surface M, and by writing a general metric as

$$g_{ab}(\sigma) = g^0_{ab}(\sigma) e^{\Phi(\sigma)}. \tag{2}$$

Integrating out the conformal fields and computing the Faddeev-Popov determinant, one gets the Liouville effective action for[11,12] the conformal degrees of freedom Φ

$$S_L[\Phi; g^0_{ab}] = \frac{\lambda}{48\pi} \int_M d^2\sigma \sqrt{g^0} \left[\frac{1}{2} g^{0ab} \partial_a \Phi \partial_b \Phi + R^0 \Phi\right]$$
$$+ (K - K_0) \int_M d^2\sigma \sqrt{g^0} e^{\Phi} + \text{function (moduli)} \tag{3}$$

with

$$\lambda = 26 - C. \tag{4}$$

K_0 is a (ultraviolet divergent) non-universal term. Writing $K - K_0 = \dfrac{\lambda}{48\pi}\mu^2$, one recovers the standard Liouville action with mass μ. At $K = K_0$ ($\mu = 0$) there is no mass scale. Thus K_0 corresponds to the critical value of K where the area of the 2-d universe is expected to diverge. Then (3) becomes a free field theory.

When the action (3) is quantized and Φ treated as a dynamical field, both λ and the interaction term $\mu^2 e^{\Phi}$ are renormalized. This has been first noticed[13,14] in the full quantization of the interacting theory ($\mu > 0$) but may be seen in a much simpler way by considering the massless theory ($\mu = 0$) as an effective theory and by using the fact that the background metric g^0_{ab} is a gauge fixing parameter.[15] Therefore the effective theory must satisfy the consistency condition that there is no Weyl anomaly when a classical conformal transformation

$$g^0_{ab}(\sigma) \to g^0_{ab}(\sigma) e^{\varphi_0(\sigma)} \tag{5}$$

is performed. The most general effective action involving Φ, the conformal field theory and the ghosts is simply[15]

$$S_{\text{eff}} = \frac{\lambda}{48\pi} \int d^2\sigma \sqrt{g^0} \left[\frac{1}{2} g^{0ab} \partial_a \Phi \partial_b \Phi + R^0 \Phi\right] + S_{\text{CFT}}[g^0_{ab}] + S_{\text{Ghosts}}[g^0_{ab}] \tag{6}$$

where S_{CFT} and S_{G} contain the minimal coupling of the CFT and of the ghosts to the classical background metric g^0_{ab}. This action is in general not Weyl invariant even at the classical level. Indeed the classical equations of motion for Φ are

$$-\Delta_0 \Phi_{\mathrm{cl}} + R^0 = 0 \tag{7}$$

and under (5)

$$\Phi_{\mathrm{cl}} \to \Phi_{\mathrm{cl}} - \varphi_0 \,. \tag{8}$$

This gives a classical contribution proportional to the coupling constant λ to the trace of the stress energy tensor which writes

$$\langle T^a_a(\sigma) \rangle = \frac{\delta \Gamma}{\delta \varphi_0(\sigma)} = -\frac{C_{\mathrm{tot}}}{48\pi} \sqrt{g^0}(\sigma) R^0(\sigma) \,. \tag{9}$$

A standard calculation gives

$$C_{\mathrm{tot}} = \lambda + 1 + C - 26 \tag{10}$$

where 1 is the contribution of the quantum fluctuations of Φ, which is a free bosonic field, C that of the CFT, -26 that of the ghosts.[11]

Thus the consistency condition $C_{\mathrm{tot}} = 0$ fixes[10]

$$\lambda = 25 - C \,. \tag{11}$$

In order to move away from the critical point, one has to add to the action (6) interaction terms $\Delta_i S$ obtained by integrating some scaling fields φ_i over the space M. The most natural ansatz is

$$\Delta_i S = \int d^2\sigma \sqrt{g^0}(\sigma) \varphi_i(\sigma) e^{A_i \Phi(\sigma)} \,. \tag{12}$$

Classically ($\lambda = \infty$),

$$A_i = A_i^{(0)} = (1 - \Delta_i^{(0)}) \tag{13}$$

where $\Delta_i^{(0)}$ is the scaling dimension of the field φ_i (as in Refs. 1 and 2, we consider only spinless primary fields). The term corresponding to the unit operator

$$\varphi_0 = \mathbb{1} \,, \qquad \Delta_0^{(0)} = 0 \tag{14}$$

is nothing but the Liouville interaction term $\int \sqrt{g^0} e^{\Phi}$. In the quantized theory, A_i must differ from (13). Indeed $\Delta_i S$ has to be Weyl invariant and the conformal weight of the operator $\varphi_i e^{A_i \Phi}$ has to be one. This weight is easily obtained by calculating the g^0 dependence of the vacuum expectation value of this operator with action (6). This gives the constraint

$$1 = \Delta_i^{(0)} + A_i - \frac{6}{\lambda} A_i^2 \tag{15}$$

where A_i comes from the classical part of Φ and A_i^2 from the quantum fluctuations. In the classical limit, one must recover (13) and this fixes the solution of (15) to be

$$A_i = \frac{25 - C}{12}\left[1 - \sqrt{1 + \frac{24}{25 - C}(\Delta_i^{(0)} - 1)}\right] \tag{16}$$

and in particular

$$A_0 = \tfrac{1}{12}[25 - C - \sqrt{(25 - C)(1 - C)}]. \tag{17}$$

Let us now consider the Liouville theory away from the critical point, with action

$$S[h_i] = S_L(\Phi) + \sum_{i=0}^{\infty} h_i \Delta_i S(\Phi) \tag{18}$$

where $h_0 \propto \mu^2$ is the renormalized cosmological constant and the h_i's ($i > 0$) source terms for the scaling fields φ_i. From the equations of motion for $h_0 \neq 0$, the natural ground states are spaces with non-zero curvature. Let us therefore take for g_{ab}^0 a curved compact metric with Euler characteristics

$$\chi = \frac{1}{4\pi} \int \sqrt{g^0} R_0 \, d^2\sigma. \tag{19}$$

From (6) and (12), a shift on the Liouville field

$$\Phi \to \Phi - \frac{1}{A_0} \ln(h_0) \tag{20}$$

leads to the crucial relation for the effective action

$$\Gamma(h_i) = \Gamma(h_i h_0^{-A_i/A_0}) - \frac{\lambda\chi}{12 A_0} \ln(h_0). \tag{21}$$

From (21), we can easily derive the scaling dimension of any field φ_i coupled to gravity. Indeed from the above considerations, h_0 is proportional to $(K - K_0)$ where the cosmological constant K has dimension (length)$^{-2}$. Thus the scaling dimension of h_0 is 1. This is confirmed by considering the average area of the surface. From (21), for $h_i = 0$ $(i > 0)$, we have

$$\langle \text{Area} \rangle = -\frac{\partial}{\partial h_0}\Gamma \propto \frac{1}{h_0}. \tag{22}$$

From (21), the scaling dimension of h_i is A_i/A_0 and therefore the scaling dimension of the field φ_i is

$$\Delta_i = 1 - \frac{A_i}{A_0}. \tag{23}$$

One can check that (23), with (16) and (17), gives the correct solution of the equations of Knizhnik, Polyakov and Zamolodchikov.[2] Finally, (21) allows us to obtain the susceptibility exponent γ. Indeed from (22), the total vacuum energy diverges with the area A as

$$\Gamma \propto \frac{\lambda\chi}{12A_0}\ln A. \tag{24}$$

From the definition[10] of the susceptibility γ $(\Gamma \simeq (2-\gamma)\ln A)$, we get

$$\gamma = 2 - \chi\frac{\lambda}{12A_0} = (2-\chi)\frac{1}{24}[(25-C) + \sqrt{(1-C)(25-C)}]. \tag{25}$$

This formula agrees with exact results for random lattices obtained for $C = -2$ and 0,[16] and with the semi-classical calculations of Ref. 17. For the sphere ($\chi = 2$), it agrees with the value conjectured by Knizhnik, Polyakov and Zamolodchikov.[2]

Finally, let us discuss briefly the case $1 \leq C \leq 25$. This is the domain where tachyons appear in the Liouville theory[14] and it has been argued by various authors that this is related to the branching of surfaces.[4–5,15,18] For the sphere ($\chi = 2$), in the weak coupling phase $C < 1$ ($\lambda > 24$), γ is negative and it vanishes for $C = 1$ where (25) becomes singular. This fact is in striking similarity with the important result of Durhuus, Fröhlich and Jonsson.[19] They showed that in a class of models of random surfaces on hypercubic lattices, if $\gamma > 0$, then the surface is a branched polymer. Although there is no rigorous proof that this result is valid for other models of random surfaces, one may expect that this is a somewhat universal property. In such a situation, for $\lambda < 24$, a typical surface would be a branched polymer made of "blobs" with typical width of the order of the ultraviolet cut-off a, and the Liouville theory, which is an effective theory describing the metric fluctuations of a 2-d object at scales much larger than this cut-off, would be inadequate to describe such an object. On the contrary, for $\lambda > 24$, the fact that $\gamma < 0$ means that the average area of a planar surface ($\chi = 2$) at the critical point is finite, and therefore of the order of the (squared) ultraviolet cut-off. To obtain a large surface is possible either by fixing the total area to be large (microcanonical ensemble), as in Ref. 17, or by enforcing some special boundary conditions, as in Ref. 14. For $C = 25$ ($\lambda = 0$) and $\mu^2 = 0$, the Liouville field Φ decouples classically from the background metric g^0_{ab}, one recovers the usual bosonic string and Φ plays the role of the missing 26th dimension.[15] The special values of λ (18, 12 and 6) where some consistent string models may be constructed from the interacting Liouville theory ($\mu^2 > 0$) have also tachyons.[20] As for the usual bosonic string, it is not clear whether they are related to "physical" models of random surfaces.

Acknowledgments

I thank A. Bilal, B. Duplantier, E. Guitter, I. Kostov and S. Leibler for their interest and their comments, and especially B. Duplantier for a careful reading of the manuscript.

References

1. A. M. Polyakov, *Mod. Phys. Lett.* **A2** (1987) 899.
2. V. G. Knizhnik, A. M. Polyakov and A. A. Zamolodchikov, *Mod. Phys. Lett.* **A3** (1988) 819.
3. F. David, *Nucl. Phys.* **B257** (1985) 45; **B257** (1985) 543.
4. V. A. Kazakov, *Phys. Lett.* **150B** (1985) 282; V. A. Kazakov, I. K. Kostov and A. A. Migdal, *Phys. Lett.* **157B** (1985) 295; D. V. Boulatov, I. K. Kostov and A. A. Migdal, *Nucl. Phys.* **B275** (1986) 641.
5. J. Frohlich, in *Lecture Notes in Physics*, Vol. 216, ed. L. Garrido (Springer, 1985); J. Ambjorn, B. Durhuus and J. Frohlich, *Nucl. Phys.* **B257** (1985) 433.
6. V. A. Kazakov, *Phys. Lett.* **119A** (1986) 140; D. V. Boulatov and V. A. Kazakov, *Phys. Lett.* **184B** (1987) 247.
7. V. A. Kazakov, Niels Bohr Inst. preprint (1988).
8. V. A. Kazakov, *Nucl. Phys.* **B** (Proc. Supp.) **4** (1988) 93.
9. B. Duplantier and I. K. Kostov, Saclay Preprint (1988).
10. F. David, Saclay Preprint (1988) to appear in C. R. Acad. Sci. Paris.
11. A. M. Polyakov, *Phys. Lett.* **163B** (1981) 207.
12. O. Alvarez, *Nucl. Phys.* **B216** (1983), 125; D. Friedan, in "Recent Advances in Field Theory and Statistical Mechanics" Les Houches 1982, eds. J. B. Zuber and R. Stora (North-Holland, 1984).
13. T. L. Curthright and C. B. Thorn, *Phys. Rev. Lett.* **48** (1982) 1309.
14. J. L. Gervais and A. Neveu, *Nucl. Phys.* **B238** (1984) 125 and 396.
15. F. David and E. Guitter, *Europhys. Lett.* **3** (1987) 1169 and *Nucl. Phys.* **B293** (1988) 332.
16. I. K. Kostov and M. L. Mehta, *Phys. Lett.* **189B** (1987) 118.
17. A. B. Zamolodchikov, *Phys. Lett.* **117B** (1982) 87. I. K. Kostov and A. Krzywicki, *Phys. Lett.* **187B** (1987) 149.
18. M. Cates, Cavendish Laboratory Preprint (1988) Cambridge.
19. B. Durhuus, J. Frohlich and T. Jonsson, *Nucl. Phys.* **B240** (1984) 453.
20. J. L. Gervais and A. Neveu, *Phys. Lett.* **151B** (1985) 59.