RANDOMLY TRIANGULATED SURFACES IN −2 DIMENSIONS

F. DAVID [1]

Service de Physique Théorique, CEN Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 4 June 1985

A model of planar random surface in d-dimensional space is studied in the special case $d = -2$. Using Parisi–Sourlas dimensional reduction, this model may be mapped onto a zero-dimensional supersymmetric planar field theory, or equivalently onto a planar stochastic equation. Previous results by Kazakov, Kostov and Migdal are rederived and some new results are obtained. In particular the critical behaviour for open surfaces is the same for $d = -2$ as for $d = 0$.

Random surfaces are expected to play a role in many areas of field theory and they have been actively studied during the last years. Many different discrete models of random surfaces have been proposed. Among them, models based on randomly triangulated planar surfaces seem particularly interesting [1–7]. They are related by duality to some planar ϕ^3 field theories [5–8], and exact and numerical results indicate a non trivial critical behaviour. For $d = 0$ (where d is the dimension of the euclidian space in which the surface is embedded) the problem of computing the partition function may be reduced to counting problems of Feynman diagrams in the planar ϕ^3 theory, and it may be solved by combinatoric [9] as well as functional [10] methods.

Recently, Kazakov, Kostov and Migdal have succeeded in computing the partition function for a closed surface in $d = -2$ dimensions [6]. Their method is in essence combinatoric, since they reduce this problem to counting problems of (maximal) connected trees on planar triangulations. The purpose of this letter is to show that it is possible to rederive their results by functional methods. As we shall see, the model of random triangulation in $d = -2$ may be simply reduced, by the so-called "dimensional reduction" trick [11], to a supersymmetric planar field theory in $d = 0$, or equivalently to a "planar spin" in a random external field. This zero-dimensional effective theory may easily be solved by the methods developed in ref. [10].

Besides providing a check of the result of ref. [6], our method allows to obtain other quantities, such as the partition functions for open surfaces with free boundary, and for triangulations without "tadpoles and self-energy insertions"[‡1], which should be much more difficult to obtain by the combinatoric arguments of ref. [6]. The consequences of those results will be discussed at the end of this paper.

For a closed planar surface, the partition function of the model is

$$F(\beta) = \sum_T \exp(-\beta|T|)\frac{1}{C(T)}\int \prod_{i=1}^{N_v - 1} \frac{\mathrm{d}^d X_i}{\pi^{d/2}} \times \exp\left(-\sum_{l=(i,j)} (X_i - X_j)^2\right). \tag{1}$$

In (1) the sum runs over all triangulations T with the topology of the sphere S_2. $C(T)$ is a combinatoric factor [5,6]. The integral runs over the position X_i of the $N_v - 1$ first vertices i of T.[‡2]. The position of the last one, X_{N_v}, is fixed to be zero. The gaussian action in the exponential is the sum over the links $l = (i,j)$ of T of the square of the link length. The "intrinsic area" $|T|$ is simply the number of triangles in T. β is the coupling constant.

[1] Physique Théorique CNRS.

‡1 In the dual language.

‡2 Let us note that the measure for X_i in (1) is slightly different from the one used in ref. [5].

As shown in refs. [5,6], this model is equivalent, for any d, to the planar ϕ_d^3 theory defined by the action:

$$S = N \int d^dx \, \mathrm{Tr}[\tfrac{1}{2}\phi(x)(e^{-\Delta}\phi)(x) + \tfrac{1}{3}e^{-\beta}\phi^3(x)], \quad (2)$$

where ϕ is an $N \times N$ hermitian matrix ($\phi = \phi^+$), in the planar limit $N \to \infty$. $F(\beta)$ is the vacuum energy of this model and we have:

$$\partial F(\beta)/\partial\beta = \lim_{N\to\infty} (e^{-\beta}/3N)\langle \mathrm{Tr}\,\phi^3\rangle_d, \quad (3)$$

Moreover, since the propagator is gaussian in (2), the contribution I_G of a Feynman graph G in (3) was shown in refs. [5,6] to be simply

$$I_G = (n_G)^{-d/2}, \quad n_G = \text{number of trees in G}. \quad (4)$$

Let us now consider the stochastic equation:

$$dV(\phi)/d\phi = h,$$

$$V(\phi) = \mathrm{Tr}(\tfrac{1}{2}\phi^2 + \tfrac{1}{3}e^{-\beta}\phi^3), \quad (5)$$

where ϕ denotes an N × N hermitian matrix and where h is an $N \times N$ hermitian *random* matrix with *gaussian* probability distribution such that

$$\overline{h_{ab}h_{cd}} = \delta_{ad}\delta_{bc}. \quad (6)$$

(This corresponds to the probability distribution $dP(h) = dh\,\exp(-\mathrm{Tr}\,h^2/2)$). In the following the bar denotes the average with respect to h. Given any function $F(\phi)$, and provided that the stochastic equation (5) has a unique solution for any h, dimensional reduction states that:

$$\overline{F(\phi)} = \langle F(\phi(x_0))\rangle_{-2} \quad (7)$$

where $\langle\ \rangle_{-2}$ denotes the expectation value in the −2-dimensional theory defined by the action (2).

The proof of (7) is completely standard and may be obtained by the general perturbative argument of ref. [12]. First we expand $F(\phi_h)$ in the coupling constant $e^{-\beta}$, i.e. in ϕ^3 tree diagrams with a h field at the end of any open line. The effect of the average over h is, by Wick tehorem, to close the trees T and reconstruct the diagrams of a ϕ^3 theory. The contribution of some G is simply the sum of trees which can be constructed in G and from (4) may be identified to the contribution of G to $\langle F(\phi)\rangle$ in −2 dimensions. Let us note that dimensional reduction holds in general only with the ordinary propagator $(\Delta + m^2)^{-1}$. It is only in zero dimensions that this argument may be applied to any kind of propagator, in particular the gaussian one of (2).

Representing $F(\phi_h)$ as

$$F(\phi_h) = \int d\phi\, F(\phi)\,\delta(\partial V/\partial\phi - h)\,|\partial^2 V/\partial\phi\partial\phi|, \quad (8)$$

and integrating over h we have thus reduced the calculation of any local observable $\langle F(\phi)\rangle_{-2}$ in $d = -2$ with the action (2) to the calculation of $\langle F(\phi)\rangle_0$ in the zero-dimensional effective theory given by the action

$$\exp[-S_{\mathrm{eff}}(\phi)] = |\partial^2 V/\partial\phi\partial\phi|\,\exp[-\tfrac{1}{2}N\,\mathrm{Tr}(\partial V/\partial\phi)^2]. \quad (9)$$

Representing the determinant $|V''(\phi)|$ in (9) as an integral over grassmannian variables we obtain the supersymmetric effective action of Parisi and Sourlas [11] for this problem.

In particular from (3) the calculation of the partition function $F(\beta)$ for $d = -2$ is reduced to the calculation of $\langle N^{-1}\,\mathrm{Tr}\,\phi^3\rangle_0$ with the effective action (9) in the planar limit $N \to \infty$.

This limit may be obtained by the method of ref. [10]. Diagonalizing ϕ and integrating over the $U(N)$ "radial" degrees of freedom we are left with the average over the eigenvalues $\lambda_1 < \ldots < \lambda_N$ of ϕ with the probability distribution: (we denote $Z = e^{-\beta}$)

$$dP(\lambda_i) = \prod_{i=1}^{N} d\lambda_i \prod_{i<j}|\lambda_i - \lambda_j|^2 \prod_{(i,j)}|1 + Z(\lambda_i + \lambda_j)|$$
$$\times \exp\left(-\frac{N}{2}\sum_i(\lambda_i + Z\lambda_i^2)^2\right). \quad (10)$$

In the limit $N \to \infty$ this measure is dominated by a saddle point. As in ref. [10] we introduce the density of eigenvalues $v(\lambda)$ normalized by

$$\int d\lambda\, v(\lambda) = 2. \quad (11)$$

The saddle point is given by the equation

$$(1 + 2Z\lambda)(\lambda + Z\lambda^2)$$
$$-\int d\mu\, v(\mu)\{1/(\lambda - \mu) + 1/[1/Z + (\lambda + \mu)]\} = 0. \quad (12)$$

It is convenient to perform the change of variable $x = \lambda + 1/2Z$. The solution of (12) is found to have compact support and is given by

$$v(\lambda) = (2Z^2/\pi)\, x\, [(b^2 - x^2)(x^2 - a^2)]^{1/2} \quad \text{if} \quad a \leqslant x \leqslant b$$
$$= 0 \quad \text{otherwise} \tag{13a}$$

with

$$b = (1/4Z^2 + 2/Z)^{1/2}, \qquad a = (1/4Z^2 - 2/Z)^{1/2},$$
$$x = \lambda + 1/2Z. \tag{13b}$$

From the following arguments we have

$$\lim_{N\to\infty} \frac{1}{N} \langle \mathrm{Tr}\, \phi^3 \rangle_0 = \frac{1}{2} \int \mathrm{d}\lambda\, \lambda^3\, v(\lambda), \tag{14}$$

and using (3) and (13) the explicit calculation gives

$$F(\beta) = \sum_{k=0}^{\infty} Z^{2(k+1)} \frac{\Gamma(\frac{5}{2})\, 2^{4k+6}}{\Gamma(-\frac{3}{2} - 2k)\, \Gamma(k+3)\, \Gamma(k+4)}. \tag{15}$$

From (15), F has a singularity at $Z_c = \frac{1}{8}$ and the singular part of F behaves as $(Z - Z_c)^3 \,\mathrm{Ln}\,|Z - Z_c|$, which confirms the results of ref. [6] for the value of the critical point and of the critical exponent $\gamma = -1$. For $Z > Z_c$, a^2 given by (13b) becomes negative and $v(\lambda)$ is no more real and positive. It is interesting to note that in the effective zero-dimensional theory, this phenomenon simply reflects the fact that the determinant $|V''(\phi)|$ is no more positive, which means that the stochastic equation (5) has more than one solution and the supersymmetry of the effective theory is spoiled [11].

This method may easily be applied to compute the partition function of an *open* planar surface with the topology of a disc. Let $F(P, \beta)$ be defined by (1) but now the sum is made over planar triangulations with a boundary with fixed number of edges P. In (1) the position of the boundary is free so we may view this case as a discretization of an euclidian free open string. From the duality of refs. [5–7], it is easy to see that for $d = 2$,

$$F(P, \beta) = \lim_{N\to\infty} \frac{1}{N} \langle \mathrm{Tr}(-\phi)^P \rangle_{d=-2} = \frac{1}{2} \int \mathrm{d}\lambda\, \lambda^P\, v(\lambda). \tag{16}$$

As done in ref. [3] for $d = 0$, we shall look at the behaviour of the mean area of a surface

$$A(P, \beta) = -\,\partial \ln F(P, \beta)/\partial\beta, \tag{17}$$

close to the critical point. The calculations are straightforward and we get the following results: for fixed P the mean area remains finite at β_c, since $\gamma = -1$ is negative. When the length of the boundary P goes to infinity, the mean area diverges in the same way as in the case $d = 0$ [3]:

$$A(P, \beta) \sim C(\beta) \cdot P, \quad \text{if} \quad \beta > \beta_c,$$
$$A(P, \beta) \sim P^2, \quad \text{if} \quad \beta = \beta_c. \tag{18}$$

The coefficient $C(\beta)$ in (18) diverges at β_c like

$$C(\beta) \sim (\beta - \beta_c)^{-1/2}, \tag{19}$$

and it is possible to define a continuum limit by introducing an elementary length a and defining renormalized quantities as

$$A_R = a^2 A, \quad L_R = aP, \quad \beta(a) = \beta_c + a^2 \lambda_R. \tag{20}$$

In the "continuum limit" ($a \to 0$; L_R and λ_R fixed), A_R is of the form

$$A_R(L_R, \lambda_R) = (1/\lambda R)\, \psi(L_R \sqrt{\lambda_R}). \tag{21}$$

The explicit form of the function ψ in (20) differs from the case $d = 0$ [3] but ψ has a similar asymptotic behaviour:

$$\psi(Z) \sim Z^2, \quad Z \to 0,$$
$$\psi(Z) \sim Z, \quad Z \to \infty. \tag{22}$$

As argued in ref. [3], an appealing interpretation of such a result is that we generate a space with a negative average curvature R proportional to $-\lambda_R$. It is remarkable that this critical behaviour is the same in the two solvable cases $d = 0$ and $d = -2$.

Finally our functional derivation allows easily to deal with triangulations without closed loops of length 1 or 2 (i.e. made of only one or two different links). Let us first consider the case of triangulations without 1 loops. In the dual ϕ^3 theory this corresponds to eliminate ϕ^3 diagrams with tadpoles (see fig. 1). Since the restriction of a connected tree T to a tadpole S, T/S, is a tree in S, contribution of tadpoles may be cancelled in the zero-dimensional stochastic equation (5) by adding a counterterm of the form $X\, \mathrm{Tr}(\phi)$ to $V(\phi)$ and by adjusting $X(Z)$ such that

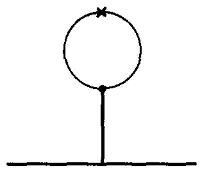

Fig. 1. The contribution of a tadpole graph to the stochastic equation. The cross denotes a contraction of h.

$$\overline{\mathrm{Tr}(\phi_h)} = 0. \tag{23}$$

In order to suppress triangulations with 1 and 2 loops, we have to cancel the contribution of tadpoles and self-energy insertions. This may be done for $d = -2$ in the following way. The restriction of a connected tree T to a self-energy insertion S may be either a connected tree in S, or a tree with two connected components in S. In the first case we may cancel this term by a mass renormalization $\frac{1}{2} Y\,\mathrm{Tr}(\phi^2)$ in $V(\phi)$. For the second case one has to renormalize the variance of the gaussian random field h to $N^{-1}\,\overline{\mathrm{Tr}(h\,h)} = 1 + Z$ (see fig. 2). The cancellation of tadpoles and self-energy is obtained by tuning the counterterms X, Y and Z in such a way that

$$\overline{\mathrm{Tr}(\phi_h)} = 0, \quad \overline{\mathrm{Tr}(\phi_h^2)} = N, \quad \overline{\mathrm{Tr}(\partial\phi_h/\partial h)} = N. \tag{24}$$

In those two cases the value of $-F'(\beta)$ is now given by the solution of an implicit transcendental equation and we have not been able to obtain on explicit expression like in (15). However one can obtain an explicit expression for the critical points $Z_c = \exp(-\beta_c)$ (in the following the subscripts 1, 1 and 2 denote the general case, the case with no 1 loops and the case with no 1 and 2 loops respectively)

$$Z_c^{(0)} = \tfrac{1}{8} = 0.125,$$
$$Z_c^{(1)} = (\tfrac{15}{128}\pi)^2 = 0.13554\ldots,$$
$$Z_c^{(2)} = (128\sqrt{2}/105\pi)[1 - 2048/(15\pi)^2]^{1/2} = 0.15302\ldots. \tag{25}$$

Fig. 2. The two possible contributions of a self-energy graph. Term (a) corresponds to a mass renormalization and term (b) to a renormalization of the variance of h.

In all cases the value of the critical exponent γ is $\gamma = -1$. From those results it is possible to extract the behaviour of the mean number of trees $\langle n_T \rangle$ for closed planar triangulations with fixed area $|T|$[‡3]. From (4) this is the ratio between the term order of $(e^{-\beta})^{|T|}$ in the series (1) for $d = -2$ and $d = 0$. Therefore $\langle n_T \rangle$ has an exponential growth for large $|T|$:

$$\langle n_T \rangle \sim C^{|T|}, \quad \text{with } C = Z_c(d = 0)/Z_c(d = -2). \tag{26}$$

In the cases 0, 1 and 2, we obtain respectively for C (using (25) and the results of ref. [10] for $d = 0$):

$$C^{(0)} = 1.754\ldots, \quad C^{(1)} = 2.008\ldots, \quad C^{(2)} = 2.122\ldots. \tag{27}$$

This shows that the number of trees grows faster with the area for more regular triangulations (case (2)) than for more irregular ones (case (0)). This result corroborates the conjecture, made in refs. [6,13], that for large negative dimension d regular triangulations will dominate the partition function,

The author is grateful to C. Itzykson for his interest, and to J.B. Zuber for a critical reading of this manuscript. He thanks J. Ambjørn and B. Durhuus for fruitful and stimulating discussions.

‡3 This was suggested to us by J. Ambjørn and B. Dorhuus.

[1] V.A. Kazakov, Phys. Lett. 150B (1985) 282.
[2] J. Fröhlich, in: Proc. Sitges Conf. on Statistical mechanics (Sitges, Barcelona, June 1984) (Springer, Berlin, 1985).
[3] F. David, Nucl. Phys. B257 (1985) 45.
[4] J. Ambjørn, B. Durhuus and J. Fröhlich, Diseases of triangulated random surfaces and possible curves. Nordita preprint 85/2, Nucl. Phys. B[FS], to be published.
[5] F. David, Saclay preprint SPhT/85/20, Nuclear Phys. B., to be published.
[6] V.A. Kazakov, I.K. Kostov and A.A. Migdal, Phys. Lett. 157B (1985) 295.
[7] F. David and R. Hong Tuan, Phys. Lett. 158B (1985) 435.
[8] P. Orland, unpublished.
[9] J. Koplick, A. Neveu and S. Nussinov, Nucl. Phys. B123 (1977) 109;
P. Cvitanovic, P.G. Lauwers and P.N. Sarbach, Nucl. Phys. B203 (1982) 385.
[10] E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, Commun. Math. Phys. 59 (1978) 35.
[11] G. Parisi and N. Sourlas, Phys. Rev. Lett. 43 (1976) 744.
[12] E. Brezin and G. Parisi, unpublished;
G. Parisi, in: Recent advances in field theory and statistical mechanics (Les Houches, 1982) eds. J.B. Zuber and R. Stora (North-Holland, Amsterdam, 1984).
[13] J. Ambjørn and B. Durhuus, private communication.