INFRARED DIVERGENCES OF MEMBRANE MODELS

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The perturbative expansion of membrane models describing a $d - n$ dimensional membrane in a d-dimensional space with surface tension is proved to be free of infrared divergences for $d \approx n$ when looking at euclidean invariant quantities.

Membrane models have been introduced by Wallace and Zia [1] in order to study the critical behaviour of the interface between two pure phases of a thermodynamic system. Such models have been generalized by Lowe and Wallace [2] to the case of a $d - n$ dimensional membrane fluctuating in a d-dimensional space. The field $\boldsymbol{\phi}(x)$ is given by the n last coordinates of the membrane expressed (locally) as a function of the $d - n$ first coordinates $\boldsymbol{x}$. Keeping only relevant terms for the long-distance behaviour and from euclidean invariance the effective action has to be proportional to the hypervolume of the membrane

$$A = \frac{1}{T} \int \mathrm{d}^{d-n}x \, (\det g)^{1/2} + \tfrac{1}{2} m^2 \boldsymbol{\phi}^2 , \tag{1}$$

The mass term is introduced in order to stabilize the membrane and acts as an infrared cutoff. $g(x)$ is the $n \times n$ matrix given by

$$g_{ab}(x) = \delta_{ab} + \partial_\mu \phi_a(x) \partial_\mu \phi_b(x) . \tag{2}$$

The renormalization properties of this model have been studied in $d = n + \epsilon$ dimensions [1–3]. It has an UV fixed point $t_c = O(\epsilon)$, an IR fixed point at $t = 0$, and is asymptotically free at $\epsilon = 0$, as for non-linear σ models in $2 + \epsilon$ dimensions [4], or gauge theories in $4 + \epsilon$ dimensions. As for these models, there is a non-abelian symmetry group, here the euclidean group of displacements in the d-dimensional space, which acts in a non-linear way in the space of field configurations.

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In this letter we establish another similarity with non-linear σ models [5–7]: near the critical dimension (here $d = n$), the weak coupling expansion of invariant quantities is infrared finite. In fact, since the field $\boldsymbol{\phi}$ is massless, the propagator itself is divergent for $d - n \leqslant 2$. The relationship of this divergence with the roughening transition has been discussed by Lüscher [4]. Much more important divergences arise from interaction terms of the action for $d \leqslant n$. Indeed, integration over internal loops gives integrals as $\int \mathrm{d}^\epsilon k \sim \int_0^\infty \mathrm{d}k^{\epsilon - 1}$ (where $\epsilon = d - n$) which diverge logarithmically at zero when $\epsilon = 0$. As for non-linear σ models, such divergences are related to the disappearance of the spontaneously broken phase (that is of the existence of a well-defined, although delocalized membrane) at $d = n$.

Let us first check the infrared finiteness of the model in a simple example of invariant observable. Since the euclidean group acts in a non-local way, mixing field and coordinates variables, an invariant observable has in general to be non-local. A two-points observable is for instance

$$\mathcal{O}[F] = \int \mathrm{d}^{d-n}y \, [\det g(y)]^{1/2} \times F\{(y - x)^2 + [\boldsymbol{\phi}(y) - \boldsymbol{\phi}(x)]^2\} , \tag{3}$$

where $F(r^2)$ is some function with sufficient decrease at infinity. Taking for F the function $F(r^2) = \theta(r_0^2 - r^2)$, we get for $\mathcal{O}$ the volume of the membrane contained in the sphere of radius r_0 around the point $(\boldsymbol{x}, \boldsymbol{\phi}(x))$ of the membrane. Various n-points invariant quantities may be constructed in the same way, or by incorporating curvature or invariants of higher dimensionality. Now let us compute first orders of $\bar{O}[F]$ at a dimension d just below n ($d = n + \epsilon$ with ϵ negative). As previously explained, the propagator $G(x)$ is divergent as $m \to 0$ as

$$G(x) = (4\pi)^{-\epsilon/2}\Gamma(1-\epsilon/2)[m^{\epsilon-2} + (x^2/2\epsilon)m^\epsilon] + D(x) + O(m)\,, \qquad (4)$$

$D(x)$ is the finite part of the massless propagator

$$D(x) = \tfrac{1}{4}\pi^{-\epsilon/2}\Gamma(\epsilon/2-1)|x|^{2-\epsilon}\,. \qquad (5)$$

Computing $\bar{O}[F]$ at first order, divergences as $m^{\epsilon-2}$ cancel immediately between graphs and divergences as m^ϵ are proportional to the integral

$$\int d^\epsilon x\,[F(x^2) + (2x^2/\epsilon)F'(x^2)]\,, \qquad (6)$$

which vanishes after integration by parts.

The same kind of cancellations occurs at second order, so that we get the infrared finite result

$$\langle\bar{O}(F)\rangle = \int d^\epsilon x\; F(x^2) - TF'(x^2)\,2nD(x) + T^2F''(x^2)\,2(n^2+\epsilon n)D^2(x) + O(T^3) \qquad (7)$$

[the ultraviolet poles at $\epsilon = 0$ are contained in $D(x)$, see eq. (5)].

To prove the infrared finiteness at any order at dimension $d = n + \epsilon$ (with ϵ negative sufficiently close to zero), we have used technics developed in ref. [7] for studying two-dimensional non-linear σ models. We shall simply point out the main steps of the proof for the "interface model" ($n = 1$) which has some peculiar simplifications.

First, let us notice that in any invariant operator, the field ϕ appears only as a difference between two points, or as spacial derivatives.

Writing such a difference as

$$\phi(x) - \phi(y) = \int_y^x dx^\mu\,\partial_\mu\phi(x)\,, \qquad (8)$$

any invariant operator may be decomposed into integrals (in position space) of products of local operators involving only derivatives of the field ϕ (that is of positive dimension). Short-distance divergences are eliminated by dimensional regularization.

The infrared behaviour of such a product of local operators $A(x_1 \ldots x_p)$ may be extracted as in ref. [7]. We get

$$A(x_1 \ldots x_p) = \sum_{K=0}^{\infty} F_{(\mu_i)}(x_1 \ldots x_p)\left(\prod_{i=1}^{K}\partial_{\mu_i}\phi(0)\right) + O(m) \qquad (9)$$

The operators $\Pi_{i=1}^K\,\partial_{\mu_i}\phi(0)$ are the divergent parts of A. The operator $F_{(\mu_i)}(x_1 \ldots x_p)$ is infrared finite and is defined by inserting the K operators $\int d^\epsilon x\; m^2 x_{\mu_i} \times \phi(x)(i\,1 \ldots K)$ in A (disconnected graphs where there are only such insertions being forbidden) and by retaining the infrared finite part of it, namely

$$F_{(\mu_i)}(x_1 \ldots x_p) = \text{finite part}\left(\frac{1}{K!}\prod_{i=1}^{K} m^2\int d^\epsilon x\, x_{\mu_i}\,\phi(x)\right)_{\text{"conn"}} \times A(x_1 \ldots x_p). \qquad (10)$$

From eq. (9), the finiteness of an invariant observable $\bar{O}$ will be proved if any such "connected insertion" into $\bar{O}$ gives zero.

Let us perform a finite rotation θ in the plane ($\boldsymbol{u}$, ϕ) ($\boldsymbol{u}$ is some unitary vector in $\boldsymbol{x}$). Coordinates and field are transformed, respectively, as

$$\boldsymbol{x} \to \boldsymbol{x} + (\cos\theta - 1)(\boldsymbol{x}\cdot\boldsymbol{u})\boldsymbol{u} + \sin\theta\,\phi\boldsymbol{u}\,,$$

$$\phi \to \cos\theta\,\phi + \sin\theta\,(\boldsymbol{x}\cdot\boldsymbol{u})\,. \qquad (11)$$

After some partial integrations and elimination of terms independent of ϕ the quadratic term $\int d^\epsilon x\;\phi(x)^2$ is transformed into

$$\int d^\epsilon x\,\phi^2(x) \to \int d^\epsilon x\,[\cos\theta\,\phi^2(x) + 2\sin\theta\,\phi(\boldsymbol{x}\cdot\boldsymbol{u})]\,, \qquad (12)$$

so that the mass term mixes with $m^2\int d^\epsilon x\; x\phi(x)$. Performing such a transformation in the functional integral of an invariant observable $\bar{O}$ we get

$$\langle\bar{O}\rangle = \frac{\langle\bar{O}\exp[m^2\sin\theta\,\boldsymbol{u}\int d^\epsilon x\,\boldsymbol{x}\phi(x)]\rangle}{\langle\exp[m^2\sin\theta\,\boldsymbol{u}\int d^\epsilon x\,\boldsymbol{x}\phi(x)]\rangle}\,. \qquad (13)$$

Expanding the rhs of eq. (13) in powers of $\sin\theta$ it is easy to see that we generate the "connected insertions" of eq. (10). Eq. (13) ensures that such insertions into $\tilde{O}$ give zero, and so that $\tilde{O}$ is infrared finite.

In the general case ($n \neq 1$) the infrared divergences have the same structure but under a general euclidean rotation, the mass term ϕ^2 mixes not only with $x\phi$ but with other possible symmetry breaking terms such as $\phi^2\,\partial\phi$... . One has to consider the mixing between all possible such terms and there is no simple transformation law such as eq. (12).

Finally we discuss some consequences of this IR finiteness. Of course such a result has no direct implications at the critical dimension $\epsilon = 0$, since the model is then trivial. However, as for the non-linear σ model at $2 + \epsilon$ dimensions [8], it allows some simplifications in calculations of critical exponents at $d = n + \epsilon$. Let us consider the observable $\tilde{O}$ defined by eq. (3). As shown by Wallace and Zia, there is a coupling constant but no wavefunction renormalization [11]. Using dimensional renormalization, the renormalized quantity $\tilde{O}_R$ (expressed in terms of the renormalized coupling constant t) obeys to the renormalization group equation

$$[\mu\partial/\partial\mu + \beta(t)\partial/\partial t + \gamma(t)]\ \tilde{O}_R = 0\ , \qquad (14)$$

where $\gamma(t)$ is related to the β function via

$$\gamma(t) = \epsilon + \mathrm{d}\beta/\mathrm{d}t - 2\beta/t\ , \qquad (15)$$

since there is only one renormalisation in t.

From the one-loop calculation of eq. (7), we get β and γ at order t^2, but eq. (15) gives us the third order of β,

$$\beta(t) = \epsilon t - 4nt^2 - 8n^2t^3 + \ldots\ , \qquad (16)$$

so that we get the index ν at second order in ϵ

$$\nu = 1/\beta'(t_c) = 1/\epsilon - \tfrac{1}{2} + O(\epsilon)\ , \qquad (17)$$

in agreement with known results [2,9]. Let us recall that ν is defined in ref. [1] from the "bulk correlation length" ξ, which diverges at t_c as

$$\xi \sim (t - t_c)^{-\nu}\ , \quad t \to t_{c_-}\ . \qquad (18)$$

It is interesting to look at the long-distance behaviour of $\tilde{O}$ (which is related to the volume of the membrane enclosed in a sphere of radius R). From eq. (14), since $t = 0$ is an IR stable fixed point, for $t < t_c$, $\tilde{O} \sim R^{\epsilon}$ as $R \to \infty$ (that is, at large scale, the membrane is a well-defined object). But at the critical temperature t_c (that is at short distance) $\tilde{O} \sim R^{1/\nu}$, so the membrane becomes a critical object with dimension

$$d_c = 1/\nu = \epsilon + \tfrac{1}{2}\epsilon^2 + \ldots\ . \qquad (19)$$

Such a behaviour is very similar to the classical problem of a polymer with a chemical potential associated to monomers [10]. This similarity is enforced by results of the limit $n \to \infty$ [2], which shows that for a one-dimensional surface ($\epsilon = 1$), $\nu = \frac{1}{2} + O(1/n)$, which is the result for the polymers in a space of large dimension ($\geqslant 4$). However, in this model of a fluctuating membrane, effects of self interaction of the membrane or non-planar configurations (bubbles, ...) are not taken into account.

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