

The periodic Schur process and free fermions at finite temperature

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Abstract

We revisit the periodic Schur process introduced by Borodin in 2007. Our contribution is threefold. First, we provide a new simpler derivation of its correlation functions via the free fermion formalism. In particular, we shall see that the process becomes determinantal by passing to the grand canonical ensemble, which gives a physical explanation to Borodin’s “shift-mixing” trick. Second, we consider the edge scaling limit in the simplest nontrivial case, corresponding to a deformation of the poissonized Plancherel measure on partitions. We show that the edge behavior is described, in a certain crossover regime different from that for the bulk, by the universal *finite-temperature Airy kernel*, which was previously encountered by Johansson and Le Doussal *et al.* in other models, and whose extreme value statistics interpolates between the Tracy–Widom GUE and the Gumbel distributions. We also define and prove convergence for a stationary extension of our model. Finally, we compute the correlation functions for a variant of the periodic Schur process involving strict partitions, Schur’s P and Q functions, and neutral fermions.

1 Introduction

The Schur process, introduced by Okounkov and Reshetikhin [OR03] but also appearing more or less implicitly in the works of Johansson [Joh02, Joh03, Joh05], is in many aspects a discrete analogue of a random matrix model such as Dyson’s Brownian motion. It is therefore not surprising that it can be analyzed by the same techniques and admits scaling limits in the same universality classes (e.g. sine processes in the bulk, Airy processes at the edge). See for instance [Oko02, Joh06] and references therein.

In this paper we revisit the periodic Schur process introduced by Borodin [Bor07]. One of his motivations was that it allows to study random cylindric partitions in the same way as the original Schur process allows to study random plane partitions [OR03, OR07]. See [GK97] for the definition of cylindric partitions and Figure 1 for an illustration.

The periodic Schur process is a measure on periodic sequences of integer partitions of the form

$$\mu^{(0)} \subset \lambda^{(1)} \supset \mu^{(1)} \subset \dots \supset \mu^{(N-1)} \subset \lambda^{(N)} \supset \mu^{(N)} = \mu^{(0)} \quad (1.1)$$

such that

$$\text{Prob}(\vec{\lambda}, \vec{\mu}) \propto u^{|\mu^{(0)}|} \prod_{k=1}^N (s_{\lambda^{(k)}/\mu^{(k-1)}}(\rho_k^+) s_{\lambda^{(k)}/\mu^{(k)}}(\rho_k^-)). \quad (1.2)$$

Here u is a nonnegative real parameter smaller than 1, $s_{\lambda/\mu}$ is a skew Schur function, and the ρ_k^\pm are collections of variables or specializations—see the beginning of Section 3 for a summary of the relevant definitions. Note that the constraint $u < 1$ arises because constant sequences would contribute an infinite mass otherwise. For $u = 0$ the measure is concentrated on sequences such that $\mu^{(0)}$ is the empty partition, and we recover the original Schur process [OR03].

Taking $u \neq 0$ brings an extra level of complexity: as shown by Borodin [Bor07, Theorem A], the point process naturally associated with $(\vec{\lambda}, \vec{\mu})$ requires a nontrivial “shift-mixing” transformation to be determinantal, and its correlation functions are given by an elliptic deformation of those for the original process. Here

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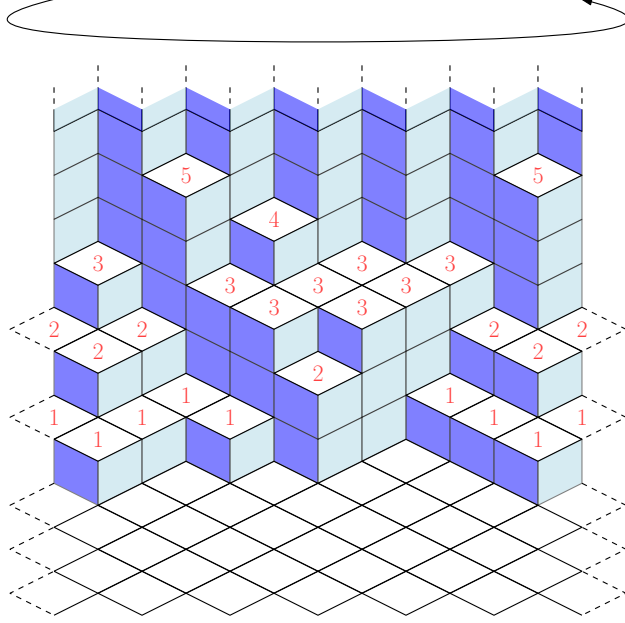


Figure 1: A cylindric partition represented as a lozenge tiling on the cylinder (the left/right sides are identified). The nonzero entries of the cylindric partition (shown in red) correspond to the displacements of the horizontal lozenges with respect to the “ground state”. The corresponding sequence of integers partitions of the form (1.1) is obtained by reading the entries along vertical lines, namely we have $\mu^{(0)} = (2, 1)$, $\lambda^{(1)} = (3, 2, 1)$, $\mu^{(1)} = (2, 1)$, $\lambda^{(2)} = (5, 1)$, etc (here $N = 6$). Note the constraint that two consecutive partitions λ, μ must be interlaced—i.e., satisfy $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots$. In the measure (1.2) this constraint is implemented by taking each ρ_k^\pm to be a specialization in a single variable—see (3.5). Here the profile—in the sense of [Bor07]—is the alternating sequence $(1, -1, 1, -1, \dots)$.

we will rederive this result using the free fermion formalism¹, which was also used recently to analyze the Schur process with free boundary conditions [BBNV18]. We believe that our derivation is more transparent. In particular the shift-mixing transformation follows from a passage to the grand canonical ensemble in the fermionic picture. For pedagogical purposes, we illustrate in Section 2 the key idea of our approach for the most elementary setting $N = 0$, where the measure (1.2) reduces to a single uniform² random partition.

We also investigate the edge behavior of the periodic Schur process, the bulk case having been already analyzed by Borodin. We will concentrate on the simplest nontrivial instance [Bor07, Example 3.4] where $N = 1$ and both ρ_1^+, ρ_1^- are exponential specializations. In other words, we consider two random partitions $\lambda \supset \mu$ such that

$$\text{Prob}(\lambda, \mu) = \frac{1}{Z_{u, \gamma}} u^{|\mu|} \left(\frac{\gamma^{|\lambda| - |\mu|} \dim(\lambda/\mu)}{(|\lambda| - |\mu|)!} \right)^2 \quad (1.3)$$

where $u \in [0, 1)$, $\gamma > 0$, $Z_{u, \gamma} := e^{\frac{\gamma^2}{1-u}} / \prod_{n \geq 1} (1 - u^n)$, and $\dim(\lambda/\mu)$ denotes the number of standard Young tableaux of skew shape λ/μ . The marginal distribution of λ interpolates between the uniform measure ($\gamma = 0$) and the poissonized Plancherel measure ($u = 0$) on partitions, and we call it the *cylindric Plancherel measure*.

We are interested in the thermodynamic limit ($u \rightarrow 1$ or $\gamma \rightarrow \infty$) where the partition λ becomes large, and consider the behavior of the first part of λ . It is well-known that the fluctuations of λ_1 are asymptotically given by the Gumbel distribution in the uniform case [EL41], and by the Tracy–Widom GUE distribution

¹Borodin’s proof involves so-called L -ensembles. As stated in [Bor07], his initial derivation was also based on the formalism of (fermionic) Fock space, but our approach is different (personal communication with Alexei Borodin).

²By a slight abuse of language, the model for which $\text{Prob}(\lambda) \propto u^{|\lambda|}$ will be called uniform as it is the “macrocanonical” ensemble associated with uniform random partitions of fixed size [Ver96]. Note that this is not the same notion as the fermionic grand canonical ensemble that we will consider.

in the Plancherel case [BDJ99]. We find that, in a suitably tuned thermodynamic limit, our model provides a “crossover” between these two behaviors. The interpolating distribution depends on a positive parameter α and was previously encountered by Johansson in the so-called MNS random matrix model [Joh07], and by Le Doussal *et al.* for free fermions in a confining trap [DLDMS16, LDMS17]—see also [LW17] and the discussion in Section 8. It is given explicitly by a Fredholm determinant

$$F_\alpha(s) := \det(I - M_\alpha)_{L^2(s, \infty)}, \quad M_\alpha(x, y) := \int_{-\infty}^{\infty} \frac{e^{\alpha v}}{1 + e^{\alpha v}} \text{Ai}(x + v) \text{Ai}(y + v) dv, \quad s, x, y \in \mathbb{R} \quad (1.4)$$

where Ai is the Airy function and α a positive parameter. Johansson proved that F_α is well-defined and indeed interpolates between the Gumbel ($\alpha \rightarrow 0$) and the Tracy–Widom GUE distributions ($\alpha \rightarrow \infty$). The limit $\alpha \rightarrow \infty$ is immediate since the “Fermi factor” $\frac{e^{\alpha v}}{1 + e^{\alpha v}}$ reduces to the indicator function $\mathbb{1}_{v > 0}$, and M_α reduces to the well-known Airy kernel. The limit $\alpha \rightarrow 0$ is more subtle as it requires a rescaling. The kernel M_α has been called the *finite-temperature Airy kernel* [LDMS17]. Our main result may then be stated as follows.

Theorem 1.1. *Consider the cylindric Plancherel measure (1.3) and let $u \rightarrow 1$ and $\gamma \rightarrow \infty$ in such a way that $\gamma(1 - u)^2 \rightarrow \alpha^3 > 0$ (with possibly $\alpha = \infty$). Then we have*

$$\text{Prob} \left(\frac{\lambda_1 - 2L_{u, \gamma}}{L_{u, \gamma}^{1/3}} \leq s \right) \rightarrow F_\alpha(s), \quad \text{where } L_{u, \gamma} := \frac{\gamma}{1 - u} \sim \left(\frac{\gamma}{\alpha} \right)^{3/2}. \quad (1.5)$$

A more general statement holds for the joint convergence of $\lambda_1, \dots, \lambda_k$ for any $k \geq 1$. Let us comment on the “physical” intuition behind Theorem 1.1. On the one hand, for $u \rightarrow 1$, λ_1 will have “thermal” fluctuations whose order of magnitude is $(1 - u)^{-1}$. On the other hand, for $\gamma \rightarrow \infty$, the area $|\lambda|$ concentrates around the value $L_{u, \gamma}^2$ hence $L_{u, \gamma}$ is the natural length scale, and we expect “quantum” Tracy–Widom-type fluctuations of order $L_{u, \gamma}^{1/3}$ for λ_1 . The edge crossover regime corresponds to the situation where thermal and quantum fluctuations are of the same order of magnitude, and we expect the resulting distribution F_α to be universal. Note that the edge crossover regime is *not* the same as that for the bulk, where one takes $u \rightarrow 1$ keeping γ fixed. As noted by Borodin, as soon as we take γ large then we recover in the bulk the usual discrete sine process, with the same density profile as in the case $u = 0$ corresponding to the Logan–Shepp–Vershik–Kerov limit shape [LS77, VK77]. Therefore, there exists an intermediate regime $1 \ll \gamma \ll (1 - u)^{-2}$ where the bulk behaves as at zero temperature ($u = 0$) while the edge behavior is still purely governed by thermal fluctuations. We have a precise counterpart of Theorem 1.1 in the whole “high temperature” regime.

Theorem 1.2. *For the cylindric Plancherel measure (1.3), set $u = e^{-r}$ and assume that $r \rightarrow 0$ and $\gamma r^2 \rightarrow 0$ (with γ possibly remaining finite). Then the rescaled fluctuations of λ_1 follow the Gumbel distribution. To wit:*

$$\text{Prob} \left(r\lambda_1 - \ln \frac{I_0(2\gamma + \gamma r)}{r} \leq s \right) \rightarrow e^{-e^{-s}} \quad (1.6)$$

where $I_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \cos \phi} d\phi$ is the modified Bessel function of the first kind and order zero.

Intuitively speaking, for $0 < \gamma \ll (1 - u)^{-2}$, there is still a crossover, but at the level of the deterministic leading-order contribution to λ_1 rather than at the level of fluctuations. The deterministic term, which we do not expect to be universal, is consistent with both Borodin’s density profile in the regime $\gamma = O(1)$, and with Theorem 1.1 in the intermediate regime $1 \ll \gamma \ll (1 - u)^{-2}$. Indeed, in the latter case we have $r^{-1} \ln \frac{I_0(2\gamma + \gamma r)}{r} = 2L_{u, \gamma} - \frac{\ln(4\pi\gamma r^2)}{2r} + o(1)$, the logarithmic correction matching that in Johansson’s formula for the limit $\alpha \rightarrow 0$ of the F_α distribution [Joh07, Theorem 1.3].

Interestingly, the cylindric Plancherel measure admits a stationary continuous-time periodic extension, which is the periodic analogue of the stationary process of Borodin and Olshanski [BO06b], and which we call the *cylindric Plancherel process*—see Figure 2. We show that its correlation kernel converges in the edge crossover regime to the extended finite-temperature Airy kernel [LDMS17]. The cylindric Plancherel process can be thought as a certain “poissonian” limit of a measure on cylindric partitions, as described below.

Finally, we consider an extension of our approach to the shifted/strict setting. Vuletić [Vul07] defined the so-called shifted Schur process, which is a measure on sequences of *strict* partitions (an integer partition is

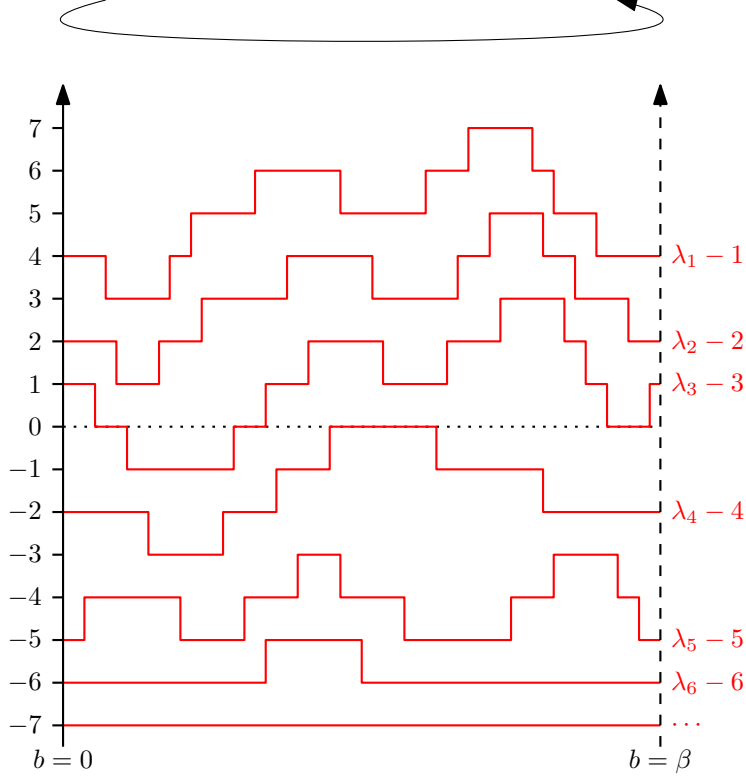


Figure 2: A possible sample of the cylindric Plancherel process. Following the notations of Section 6, the horizontal axis represents the (imaginary) continuous time b , β is the period, and the i -th path represents the time-evolution of $\lambda_i \in \{0, 1, 2, \dots\}$ which performs unit jumps up or down at random times (we rather represent $\lambda_i - i$ so that the paths are nonintersecting). For i large enough (here $i \geq 7$), λ_i remains equal to 0 at all time. For any fixed time b , the law of the partition $\lambda(b)$ is the cylindric Plancherel measure.

called strict if all its parts are distinct), and whose definition involves Schur's P and Q functions instead of the ordinary Schur functions. We naturally introduce the periodic variant of this process, which we prefer to call the *periodic strict Schur process*: to a sequence of the form (1.1) where each element is a strict partition, we assign a weight

$$\text{Prob}(\vec{\lambda}, \vec{\mu}) \propto u^{|\mu^{(0)}|} \prod_{k=1}^N (Q_{\lambda^{(k)}/\mu^{(k-1)}}(\rho_k^+) P_{\lambda^{(k)}/\mu^{(k)}}(\rho_k^-)) \quad (1.7)$$

with u a real parameter smaller than 1, $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$ the Schur P and Q functions, and the ρ_k^\pm strict specializations—see the beginning of Section 7 for definitions. Vuletić's definition is recovered in the case $u = 0$, where the measure is concentrated on sequences such that $\mu^{(0)}$ is the empty partition. We compute the correlation functions of the periodic strict Schur process using again the free fermion formalism that now involves so-called neutral fermions as already observed in the $u = 0$ case [Vul07]. The approach is even simpler than in the non strict case, because the shift-mixing transformation becomes trivial. We find that the periodic strict Schur process is a pfaffian point process, whose kernel involves elliptic functions.

Outline. Our paper is organized as follows. In Section 2 we discuss the connection between uniform random partitions and Fermi–Dirac statistics, which illustrates the key idea of our approach in the most elementary setting. In Section 3, we recall the fundamental results of [Bor07] on the correlation functions of the periodic Schur process. Our new derivation via the free fermion formalism is then given in Section 4. We then turn to the asymptotic analysis of the edge behavior in Section 5, and provide the proof of Theorem 1.1 (for Theorem 1.2, some technical estimates are left to Appendix D). The stationary cylindric Plancherel process is treated in Section 6. Section 7 is devoted to the periodic strict Schur process and the derivation

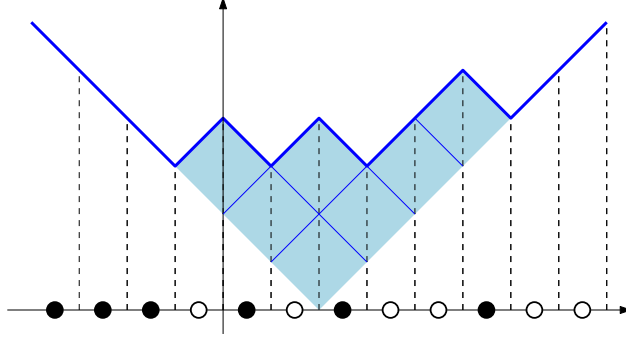


Figure 3: Illustration of the correspondence between a partition and a Maya diagram. The Young diagram of the partition $(4, 2, 1)$ appears in blue. The corresponding Maya diagram is represented as a sequence of “particles” (\bullet) and “holes” (\circ). There must be as many particles on the right of the bottom corner of the diagram as holes on its left. We may lift this constraint on the Maya diagram by moving the origin (displayed at the intersection between the axes) to an arbitrary position on the horizontal axis. By cutting the Maya diagram at the origin, we obtain two strict partitions (parts correspond to particles on the right, holes on the left), whose number of parts differ by the *charge* (the abscissa of the bottom corner of the diagram, here equal to 2).

of its correlation functions via neutral fermions. Finally Section 8 gathers some concluding remarks and discussion. A paper would not be complete without appendices: in Appendix A we recall the definition of the elliptic functions that we use; since it seems that basic facts about fermions should always be recalled in an appendix (see e.g. [Oko01, OR03, OR07, Vul09]), we abide by the rule in Appendices B and C that deal with charged and neutral fermions respectively; finally, in Appendix D we perform some further analysis of the discrete finite-temperature Bessel kernel arising in Section 5, which we use to complete the proof of Theorem 1.2 and also to give a short rederivation of the bulk limiting kernel.

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2 Warm-up: integer partitions and Fermi–Dirac statistics

An (integer) *partition* λ is a nonincreasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \dots$ which vanishes eventually. The strictly positive λ_i ’s are called the *parts*, and their number is the *length* of the partition, denoted by $\ell(\lambda)$. The *size* of λ is $|\lambda| := \sum_{i \geq 1} \lambda_i$. A partition is called *strict* if all its parts are distinct. We denote by \mathcal{P} (respectively \mathcal{SP}) the set of all (respectively strict) partitions. In physics, arbitrary partitions describe “bosons”, and strict partitions “fermions” [Ver96].

There are several classical bijections between various classes of partitions, strict or not. For instance, \mathcal{P} is in bijection with pairs of strict partitions of the same length. Such pairs may conveniently be seen as Maya diagrams (or excited “Dirac seas”), and there is more generally a bijection between $\mathcal{P} \times \mathbb{Z}$ (*charged*

partitions) and $\mathcal{SP} \times \mathcal{SP}$ —see Figure 3. This bijection yields a beautiful proof of the Jacobi triple product identity, see e.g. [Cor03] and references therein. We now recall this proof as it is at the core of our approach.

A *Maya diagram* is a bi-infinite binary sequence $\underline{n} = (n_i)_{i \in \mathbb{Z}'} \in \{0, 1\}^{\mathbb{Z}'}$ such that $n_i = 0$ for $i \gg 0$ and $n_i = 1$ for $i \ll 0$. Here $\mathbb{Z}' := \mathbb{Z} + 1/2$ denotes the set of half-integers. This latter choice is purely conventional and makes some formulas more symmetric. On Figure 3, 0's and 1's are represented as holes and particles respectively. To a Maya diagram \underline{n} we associate its *charge* $C(\underline{n})$ and its *energy* $H(\underline{n})$ defined by

$$C(\underline{n}) := \sum_{i>0} n_i + \sum_{i<0} (n_i - 1), \quad H(\underline{n}) := \sum_{i>0} i n_i + \sum_{i<0} i (n_i - 1). \quad (2.1)$$

Clearly, the energy is nonnegative and vanishes if and only if $\underline{n} = \{\mathbb{1}_{i<0}\}$ (*vacuum*).

Given two parameters t, u , we associate to each Maya diagram a weight $t^{C(\underline{n})} u^{H(\underline{n})}$. The total weight, the sum over all Maya diagrams, clearly factorizes as an infinite product over $i \in \mathbb{Z}'$. On the other hand, if we consider the charged partition (λ, c) associated with \underline{n} , then it is not difficult to check that

$$C(\underline{n}) = c, \quad H(\underline{n}) = |\lambda| + \frac{c^2}{2}. \quad (2.2)$$

Therefore, the weight can be rewritten as $t^c u^{|\lambda| + c^2/2}$. By summing over all pairs (λ, c) , we end up with the identity

$$\prod_{k=0}^{\infty} (1 + tu^{k+1/2})(1 + t^{-1}u^{k+1/2}) = \left(\sum_{\lambda \in \mathcal{P}} u^{|\lambda|} \right) \left(\sum_{c \in \mathbb{Z}} t^c u^{c^2/2} \right) = \frac{1}{(u; u)_{\infty}} \theta_3(t; u) \quad (2.3)$$

which is equivalent to the Jacobi triple product identity—see Appendix A for reminders and notations.

Let us now reflect on the probabilistic meaning of this construction. Viewing $t^{C(\underline{n})} u^{H(\underline{n})}$ as a Boltzmann weight for \underline{n} and normalizing by the partition function (2.3), we get a well-defined probability distribution for $t \in (0, \infty)$ and $u \in [0, 1)$. In the Maya diagram picture we learn that the n_i are independent and that

$$\text{Prob}(n_i = 1) = \frac{tu^i}{1 + tu^i} = \frac{1}{1 + t^{-1}u^{-i}}. \quad (2.4)$$

This is nothing but the *Fermi–Dirac distribution* for a system of noninteracting fermions with equally spaced energy levels (write $t^{-1}u^{-i} = e^{\beta(\epsilon_i - \mu)}$ to recognize more physical variables). In the charged partition picture, we learn that λ and c are independent, with λ a uniform random partition and c distributed as

$$\text{Prob}(c) = \frac{t^c u^{c^2/2}}{\theta_3(t; u)}. \quad (2.5)$$

This fact was observed by Borodin [Bor07, Corollary 2.6] and we suspect that it might appear elsewhere in the literature under various forms.

If we condition on $c = 0$, then the n_i 's are no longer independent. Therefore, passing to the “grand canonical ensemble”—as we do by letting c fluctuate—makes it much easier to study random partitions, at least for the observables we can capture in the Maya diagram picture. We argue that this is the fundamental reason why the shift-mixed periodic Schur process (to be defined below) is determinantal: *fermions are only “free” in the grand canonical ensemble*.

Before proceeding, let us briefly discuss the thermodynamic limit $u \rightarrow 1^-$. From (2.4) we see that, if we rescale $i = x/|\ln u|$, then the Maya diagram has a limiting density profile $1/(1 + t^{-1}e^x)$. It is not difficult to check that this is consistent with Vershik’s limit shape for uniform random partitions [Ver96]. Note that the parameter t only induces a shift of the profile. This is because the charge c concentrates around its mean value of order $1/|\ln u|$.

Lemma 2.1. *Let c be an integer-valued random variable distributed as in (2.5). Then, as $u \rightarrow 1^-$, the random variable $\tilde{c} := (c + \ln t / \ln u) \sqrt{|\ln u|}$ converges weakly to the standard normal distribution.*

Proof. Observe that, for any interval I , $\text{Prob}(\tilde{c} \in I)$ is a Riemann sum converging to $\frac{1}{\sqrt{2\pi}} \int_I e^{-x^2/2} dx$. \square

This shows in some sense the equivalence of the canonical and grand canonical ensembles for our fermions. Note that the equivalence with the microcanonical ensemble (partitions of fixed size) is another matter [Ver96], and that the equivalence between the canonical and grand canonical ensembles in the MNS model was established in [LW17]. We now turn to the edge behavior of the Maya diagram—see [Oko01, Section 3.4] for the bulk behavior. It is elementary to check that

$$\text{Prob}(M < k) = \frac{1}{(-tu^k; u)_\infty}, \quad M := \max\{i : n_i = 1\}. \quad (2.6)$$

Recognizing a q -exponential function, and using the asymptotics $((1-u)z; u)_\infty \rightarrow e^{-z}$ [GR04, p.11], we easily deduce that

$$\lim_{u \rightarrow 1^-} \text{Prob} \left(M < -\frac{\ln(1-u)}{\ln u} + \frac{\xi}{|\ln u|} \right) = e^{-te^{-\xi}}. \quad (2.7)$$

In other words, the fluctuations of M are given by the Gumbel distribution. By Lemma 2.1, we deduce that λ_1 has asymptotically the same distribution (with $t = 1$), which is consistent with the result of Erdős and Lehner [EL41]. Note that we could have alternatively started from the expression $\text{Prob}(\lambda_1 < k) = (u^k; u)_\infty$ to arrive at the same result.

3 Correlations functions of the periodic Schur process

In this section we recall the fundamental results of [Bor07], for which we give a new proof in the next section. Here we follow the notational conventions of [BBNV18] regarding Schur functions, which we briefly recall now. Given a sequence of numbers $(h_n(\rho))_{n \in \mathbb{Z}}$ such that $h_0(\rho) = 1$ and $h_n(\rho) = 0$ for $n < 0$, and two partitions λ, μ , the *skew Schur function of shape λ/μ specialized at ρ* is given by

$$s_{\lambda/\mu}(\rho) := \det_{1 \leq i, j \leq \ell(\lambda)} h_{\lambda_i - i - \mu_j + j}(\rho). \quad (3.1)$$

It vanishes unless λ/μ is a *skew shape* (i.e. we have $\lambda \supset \mu$, i.e. $\lambda_i \geq \mu_i$ for all i). The “specialization” ρ is conveniently encoded into the generating function

$$H(\rho; z) := \sum_{n \geq 0} h_n(\rho) z^n. \quad (3.2)$$

Thoma’s theorem [Tho64, AESW51] states that $s_{\lambda/\mu}(\rho)$ is nonnegative for all λ, μ if and only if the generating function $H(\rho; z)$ is of the form

$$H(\rho; z) = e^{\gamma z} \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z} \quad (3.3)$$

with $\gamma, \alpha_1, \beta_1, \alpha_2, \beta_2, \dots$ a summable collection of nonnegative parameters (the specialization ρ is then called *nonnegative*). In particular, when only γ is nonzero, we obtain the *exponential specialization* denoted ex_γ , for which

$$s_{\lambda/\mu}(\text{ex}_\gamma) = \frac{\gamma^{|\lambda| - |\mu|} \dim(\lambda/\mu)}{(|\lambda| - |\mu|)!} \quad (3.4)$$

with $\dim(\lambda/\mu)$ the number of standard Young tableaux of shape λ/μ . When only $\alpha_1 = q$ is nonzero, we obtain the *specialization in a single variable q* for which

$$s_{\lambda/\mu}(q) = \begin{cases} q^{|\lambda| - |\mu|} & \text{if } \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots, \\ 0 & \text{otherwise.} \end{cases} \quad (3.5)$$

For simplicity, we will assume in the following that all specializations are nonnegative and such that their generating functions are analytic and nonzero in a disk of radius $R > 1$ (this is clearly the case for the exponential specialization). In that case, the measure (1.2) can be normalized into a probability distribution which is the *periodic Schur process*—see [Bor07] or the discussion below for more on the normalization.

Following [OR03], to any sequence of partitions $\vec{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$, we associate the *point configuration*

$$\mathfrak{S}(\vec{\lambda}) := \left\{ \left(i, \lambda_j^{(i)} - j + \frac{1}{2} \right), 1 \leq i \leq N, j \geq 1 \right\} \subset \{1, \dots, N\} \times \mathbb{Z}'. \quad (3.6)$$

Observe that this definition is closely related to the Maya diagrams encountered in Section 2: the $\lambda_j^{(i)} - j + \frac{1}{2}$, $j \geq 1$, are precisely the positions of particles (1's) in the Maya diagram associated with the partition $\lambda^{(i)}$ when the charge is 0. See again Figure 3.

Our aim is to study the point configuration when $\vec{\lambda}$ is the first marginal of the periodic Schur process. There is no loss of generality in discarding $\vec{\mu}$, and doing so simplifies the ensuing expressions. $\mathfrak{S}(\vec{\lambda})$ is a discrete point process, and as such its properties are characterized by the data of the *correlation functions*

$$\varrho(U) := \text{Prob} \left(U \subset \mathfrak{S}(\vec{\lambda}) \right) \quad (3.7)$$

for any finite set $U \subset \{1, \dots, N\} \times \mathbb{Z}'$.

Borodin observed that $\varrho(U)$ is quite complicated in general (though it can be written in the form of a multiple contour integral—see Proposition 3.2 below), but we may transform it into a determinant at the price of “shift-mixing” the process. More precisely, we take an integer-valued random variable c , independent of $\vec{\lambda}$ and distributed according to (2.5) (with t an extra arbitrary parameter), and we define the *shift-mixed correlation function* as

$$\tilde{\varrho}(U) := \text{Prob} \left(U \subset \mathfrak{S}(\vec{\lambda}) + (0, c) \right). \quad (3.8)$$

In other words, we are just moving $\mathfrak{S}(\vec{\lambda})$ by a random vertical shift c . The shift-mixed process is *determinantal* in the sense that there exists a *correlation kernel* $K : (\{1, \dots, N\} \times \mathbb{Z}') \times (\{1, \dots, N\} \times \mathbb{Z}') \rightarrow \mathbb{C}$ such that

$$\tilde{\varrho}(U) = \det_{1 \leq i, j \leq n} K(u_i; u_j), \quad U = \{u_1, \dots, u_n\}. \quad (3.9)$$

See for instance [Joh06] for general background on determinantal processes. Let us now state Borodin’s main result in our current notations.

Theorem 3.1 ([Bor07, Theorem A]). *The shift-mixed periodic Schur process is determinantal with correlation kernel given by*

$$K(i, k; i', k') = \frac{1}{(2\pi i)^2} \oint_{|z|=r} \frac{dz}{z^{k+1}} \oint_{|w|=r'} \frac{dw}{w^{-k'+1}} \cdot \frac{F(i, z)}{F(i', w)} \cdot \kappa(z, w) \quad (3.10)$$

where $i, i' \in \{1, \dots, N\}$, $k, k' \in \mathbb{Z}'$, the radii r, r' satisfy

$$R^{-1} \leq r, r' \leq R, \quad r/r' \in \begin{cases} (u, 1) & \text{if } i > i', \\ (1, u^{-1}) & \text{if } i \leq i', \end{cases} \quad (3.11)$$

F carries the dependence on the specializations ρ_k^\pm as

$$F(i, z) := \frac{\prod_{1 \leq \ell \leq i} H(\rho_\ell^+; z)}{\prod_{i \leq \ell \leq N} H(\rho_\ell^-; z^{-1})} \prod_{n \geq 1} \prod_{1 \leq \ell \leq N} \frac{H(u^n \rho_\ell^+; z)}{H(u^n \rho_\ell^-; z^{-1})} \quad (3.12)$$

and

$$\kappa(z, w) := \begin{cases} \sum_{m \in \mathbb{Z}'} \frac{(z/w)^m}{1 + (tu^m)^{-1}} & \text{for } |z/w| \in (1, u^{-1}), \\ -\sum_{m \in \mathbb{Z}'} \frac{(z/w)^m}{1 + tu^m} & \text{for } |z/w| \in (u, 1). \end{cases} \quad (3.13)$$

Several remarks are now in order. On the one hand, we recover the correlation kernel of the original Schur process [OR03, Theorem 1] by taking $u = 0$, in which case $\kappa(z, w)$ simplifies into $\sqrt{zw}/(z - w)$. Note that $\text{Prob}(c = 0) = 1$ so shift-mixing can be disregarded in that case. On the other hand, if we consider only trivial specializations (so that $F(i, z) = 1$), then $K(i, k; i', k')$ vanishes for $k \neq k'$, and the shift-mixed process is a discrete Poisson process. This is nothing but the phenomenon discussed in Section 2.

As remarked by Borodin [Bor07, Remark 2.4], the two cases in (3.13) correspond to the expansions in two different annuli of the same meromorphic function, namely

$$\kappa(z, w) = \sqrt{\frac{w}{z}} \cdot \frac{(u; u)_\infty^2}{\theta_u(\frac{w}{z})} \cdot \frac{\theta_3(\frac{tz}{w}; u)}{\theta_3(t; u)} = i \frac{\eta(u)^3 \theta_3(\frac{tz}{w}; u)}{\theta_1(\frac{w}{z}; u) \theta_3(t; u)} \quad (3.14)$$

with $(u; u)_\infty$ the infinite Pochhammer symbol, η the Dedekind eta function, θ_u the “multiplicative” theta function, and θ_1, θ_3 the Jacobi theta functions—see Appendix A for conventions. The equality between the sum expression (3.13) and the “theta” form (3.14) is a particular case of Ramanujan’s ${}_1\Psi_1$ summation formula, and may be rederived directly using the boson–fermion correspondence as explained in Section 4. The first theta form of $\kappa(z, w)$ arises naturally from the computations; in the second form we introduce the Dedekind eta function to make the intriguing modular properties more apparent. In fact, $\kappa(z, w)$ may be interpreted as a propagator $\langle \psi(z) \psi^*(w) \rangle$ for the conformal field theory of charged fermions on a torus, in which modular invariance plays a crucial role [DFMS97, Chapter 10].

Finally, Borodin observed that using an elliptic version of the Cauchy determinant due to Frobenius—see Remark 4.1 below—, we may obtain an explicit contour integral representation for $\varrho(U)$:

Proposition 3.2 ([Bor07, Corollary 2.8]). *The general n -point correlation function for the periodic Schur process is given by*

$$\varrho(U) = \frac{\eta(u)^{3n}}{(2i\pi)^{2n}} \oint \cdots \oint \prod_{\ell=1}^n \left(\frac{dz_\ell}{z_\ell^{k_\ell+1}} \cdot \frac{dw_\ell}{w_\ell^{-k_\ell+1}} \cdot \frac{F(i_\ell, z_\ell)}{F(i_\ell, w_\ell)} \right) \cdot \frac{\prod_{1 \leq \ell < m \leq n} \theta_1(z_\ell/z_m; t) \theta_1(w_\ell/w_m; t)}{\prod_{1 \leq \ell, m \leq n} \theta_1(z_\ell/w_m; t)} \quad (3.15)$$

where $U = \{(i_1, k_1), (i_2, k_2), \dots, (i_n, k_n)\}$ and, upon ordering $i_1 \leq i_2 \leq \dots \leq i_n$, the integration is taken over nested circles $\min(u^{1/2}, R) > |z_1| > |w_1| > |z_2| > |w_2| > \dots > |z_n| > |w_n| > \max(u^{-1/2}, R^{-1})$. The n -point correlation $\tilde{\rho}(U)$ for the shift-mixed process is given by simply multiplying the integrand above by $\theta_3(t \frac{z_1 z_2 \cdots z_n}{w_1 w_2 \cdots w_n}; u) / \theta_3(t; u)$.

4 Derivation via free fermions

This section is devoted to a proof of Theorem 3.1 and Proposition 3.2 via the machinery of free fermions. Again we make use of the same conventions and notations as in [BBNV18]—for convenience the relevant definitions and facts are recalled in Appendix B.

Our starting point is the observation that the *partition function*—i.e., the sum of the unnormalized weights (1.2)—of the periodic Schur process can be represented as

$$Z = \text{tr} \left(\Pi_0 u^H \Gamma_+(\rho_1^+) \Gamma_-(\rho_1^-) \cdots \Gamma_+(\rho_N^+) \Gamma_-(\rho_N^-) \right) \quad (4.1)$$

where tr stands for the trace over the fermionic Fock space and Π_0 denotes the orthogonal projector over the subspace of charge 0 (as we shall sum over ordinary *uncharged* partitions). Similarly, for $U = \{(i_1, k_1), \dots, (i_n, k_n)\} \subset \{1, \dots, N\} \times \mathbb{Z}'$, if the abscissas are ordered as $i_1 \leq \dots \leq i_n$, then the correlation function $\varrho(U)$ as defined in (3.7) is equal to Z_U/Z , where

$$Z_U := \text{tr} \left(\Pi_0 u^H \Gamma_+(\rho_1^+) \cdots \Gamma_+(\rho_{i_1}^+) \psi_{k_1} \psi_{k_1}^* \Gamma_-(\rho_{i_1}^-) \cdots \Gamma_+(\rho_{i_n}^+) \psi_{k_n} \psi_{k_n}^* \Gamma_-(\rho_{i_n}^-) \cdots \Gamma_-(\rho_N^-) \right) \quad (4.2)$$

is the sum of the unnormalized weights of sequences such that $U \subset \mathfrak{S}(\vec{\lambda})$.

Let us denote by \tilde{Z} and \tilde{Z}_U the quantities obtained by replacing respectively in (4.1) and (4.2) the projector Π_0 by the operator t^C . Then, it is not difficult to check that \tilde{Z}_U/\tilde{Z} is precisely the shift-mixed correlation function $\tilde{\rho}(U)$ as defined in (3.8).

Eliminating the Γ -operators. We now rewrite the quantities $Z, Z_U, \tilde{Z}, \tilde{Z}_U$ following the strategy of [OR03], which was already adapted to the periodic setting in [BCC17] (in the case of Z). We call this method *Γ -elimination*: in a nutshell if we consider the trace of a product of operators involving some Γ_\pm ’s and other operators which they quasi-commute with (such as $u^H, \psi(z), \psi^*(w)$), then the quasi-commutation

relations can be exploited to remove the Γ 's from the product, up to a multiplication by the corresponding scalar factors.

Let us illustrate this method on the easiest case of Z . The basic idea is to move each of the $\Gamma_+(\rho_i^+)$ to the right, using the quasi-commutation relations (B.8) with the other operators and the cyclic property of the trace (recall also that Γ -operators commute with C hence Π_0). For instance, performing this operation on each $\Gamma_+(\rho_i^+)$ in (4.1) until it goes back into the same place, we get

$$Z = \prod_{1 \leq i \leq j \leq N} H(\rho_i^+; \rho_j^-) \prod_{1 \leq j < i \leq N} H(u\rho_i^+; \rho_j^-) \times \text{tr}(\Pi_0 u^H \Gamma_+(u\rho_1^+) \Gamma_-(\rho_1^-) \cdots \Gamma_+(u\rho_N^+) \Gamma_-(\rho_N^-)). \quad (4.3)$$

(note that the specializations ρ_i^+ have all been multiplied by u in the trace on the right). By iterating this procedure m times, and noting that the process converges for $m \rightarrow \infty$ because $\Gamma_+(u^m \rho) \rightarrow 1$ for $m \rightarrow \infty$, we may rederive Borodin's expression for the partition function [Bor07, Proposition 1.1]—see [BCC17, proof of Theorem 12] for more details.

For the correlation functions, it is useful to first rewrite

$$Z_U = \left[z_1^{k_1} w_1^{-k_1} \cdots z_n^{k_n} w_n^{-k_n} \right] \text{tr}(\Pi_0 u^H \Gamma_+(\rho_1^+) \cdots \Gamma_+(\rho_{i_1}^+) \psi(z_1) \psi^*(w_1) \Gamma_-(\rho_{i_1}^-) \cdots \Gamma_+(\rho_{i_n}^+) \psi(z_n) \psi^*(w_n) \Gamma_-(\rho_{i_n}^-) \cdots \Gamma_-(\rho_N^-)) \quad (4.4)$$

where $[z_1^{k_1} w_1^{-k_1} \cdots z_n^{k_n} w_n^{-k_n}]$ denotes coefficient extraction in the multivariate Laurent series on the right. We may now apply the same strategy to the trace in (4.4), and we will pick extra factors of the form $H(u^m \rho_i^+; z_\ell) / H(u^m \rho_i^+; w_\ell)$ from the quasi-commutations between Γ_+ 's with $\psi(z_\ell) \psi^*(w_\ell)$'s. Note that the factors $H(u^m \rho_i^+; \rho_j^-)$ coming from quasi-commutations between Γ_+ 's and Γ_- 's will eventually get cancelled when we divide by Z to get $\rho(U)$. Furthermore, after we have “eliminated” the Γ_+ 's, we also need to get rid of the Γ_- 's by moving them similarly but to the left. This will produce factors of the form $H(u^m \rho_i^-; w_\ell^{-1}) / H(u^m \rho_i^-; z_\ell^{-1})$. We end up with the expression

$$\rho(U) = \left[z_1^{k_1} w_1^{-k_1} \cdots z_n^{k_n} w_n^{-k_n} \right] \prod_{\ell=1}^n \frac{F(i_\ell, z_\ell)}{F(i_\ell, w_\ell)} \cdot \langle \psi(z_1) \psi^*(w_1) \cdots \psi(z_n) \psi^*(w_n) \rangle_u^{(0)} \quad (4.5)$$

where $\langle O \rangle_u^{(0)} := \text{tr}(\Pi_0 u^H O) / \text{tr}(\Pi_0 u^H)$ denotes the “canonical” expectation value of the operator O , and where F is as in Theorem 3.1: the $F(i_\ell, z_\ell)$ in the numerator (respectively the $F(i_\ell, w_\ell)$ in the denominator) contains all the factors arising from the quasi-commutations of Γ 's with $\psi(z_\ell)$ (respectively $\psi^*(w_\ell)$) in (4.4).

All this reasoning remains valid if we consider the shift-mixed correlation function $\tilde{\rho}(U)$, which then admits the expression obtained by replacing in (4.5) the canonical expectation value $\langle \cdot \rangle_u^{(0)}$ by the grand canonical one $\langle \cdot \rangle_{u,t} := \text{tr}(t^C u^H \cdot) / \text{tr}(t^C u^H)$.

Determinantal structure of $\tilde{\rho}(U)$. At this stage we may employ Wick's lemma in finite temperature (Lemma B.1) in its “determinantal” form to conclude that the shift-mixed correlation function can be rewritten as

$$\begin{aligned} \tilde{\rho}(U) &= \left[z_1^{k_1} w_1^{-k_1} \cdots z_n^{k_n} w_n^{-k_n} \right] \prod_{\ell=1}^n \frac{F(i_\ell, z_\ell)}{F(i_\ell, w_\ell)} \cdot \det_{1 \leq \ell, m \leq n} \langle \mathcal{T}(\psi(z_{i_\ell}), \psi^*(w_{i_m})) \rangle_{u,t} \\ &= \det_{1 \leq \ell, m \leq n} K(i_\ell, k_\ell; i_m, k_m), \quad K(i, k; i'; k') := \left[z^k w^{-k'} \right] \frac{F(i, z)}{F(i', w)} \cdot \begin{cases} \langle \psi(z) \psi^*(w) \rangle_{u,t} & i \leq i' \\ -\langle \psi^*(w) \psi(z) \rangle_{u,t} & i > i' \end{cases} \end{aligned} \quad (4.6)$$

This shows that the shift-mixed process is indeed determinantal. We may rewrite $K(i, k; i'; k')$ in the form (3.10) as follows. From the discussion of Section 2 and particularly (2.4)—or alternatively from (B.14)—we see that

$$\langle \psi_k \psi_k^* \rangle_{u,t} = \frac{tu^k}{1 + tu^k} = \frac{1}{1 + t^{-1}u^{-k}}, \quad \langle \psi_k^* \psi_k \rangle_{u,t} = \frac{1}{1 + tu^k}, \quad k \in \mathbb{Z}' \quad (4.7)$$

while the expectation value of any other product of two fermionic operators vanishes. Passing to the generating series we get

$$\langle \psi(z)\psi^*(w) \rangle_{u,t} = \sum_{m \in \mathbb{Z}'} \frac{(z/w)^m}{1 + (tu^m)^{-1}}, \quad \langle \psi^*(w)\psi(z) \rangle_{u,t} = \sum_{m \in \mathbb{Z}'} \frac{(z/w)^m}{1 + tu^m}. \quad (4.8)$$

Note that, at an analytic level, the first sum converges for $|uz| < |w| < |z|$ and the second for $|uw| < |z| < |w|$, and in fact we have

$$\kappa(z, w) = \begin{cases} \langle \psi(z)\psi^*(w) \rangle_{u,t} & \text{for } |z/w| \in (1, u^{-1}), \\ -\langle \psi^*(w)\psi(z) \rangle_{u,t} & \text{for } |z/w| \in (u, 1) \end{cases} \quad (4.9)$$

(by the canonical anticommutation relations, it is a general fact that fermionic propagators $\langle \psi(z)\psi^*(w) \rangle$ and $-\langle \psi^*(w)\psi(z) \rangle$ should be two Laurent series expansions of a same meromorphic function with a pole of residue 1 at $z = w$). We conclude the proof of Theorem 3.1 by representing the coefficient extraction $[z^k w^{-k}]$ in (4.6) as a double contour integral. Note that, by our analyticity assumptions, $F(i, \cdot)$ is analytic and nonzero in the annulus $R^{-1} < |\cdot| < R$.

Theta form for fermionic expectations. Interestingly, it is possible to derive the “theta” form (3.14) of $\kappa(z, w)$ from the boson—fermion correspondence (B.9). Assuming $|z| > |w|$ we may write

$$\begin{aligned} \text{tr}(t^C u^H \psi(z)\psi^*(w)) &= \sqrt{\frac{w}{z}} \cdot \text{tr} \left(\left(\frac{tz}{w} \right)^C u^H \Gamma_-(z) \Gamma'_+(-z^{-1}) \Gamma'_-(-w) \Gamma_+(w^{-1}) \right) \\ &= \sqrt{\frac{w}{z}} \cdot \frac{(u; u)_\infty^2}{\theta_u(\frac{w}{z})} \cdot \text{tr} \left(\left(\frac{tz}{w} \right)^C u^H \right) = \sqrt{\frac{w}{z}} \cdot \frac{(u; u)_\infty^2}{\theta_u(\frac{w}{z})} \cdot \frac{\theta_3(\frac{tz}{w}; u)}{(u; u)_\infty} \end{aligned} \quad (4.10)$$

where we pass to the second line by Γ -elimination, and to the final form by using (2.3) with $t \rightarrow tz/w$. Upon dividing by the normalization $\text{tr}(t^C u^H) = \theta_3(t; u)/(u; u)_\infty$ we arrive at (3.14).

Remarkably, the same trick allows to evaluate the canonical and grand canonical expectation value of the product of any number of fermionic generating series. We find

$$\langle \psi(z_1)\psi^*(w_1) \cdots \psi(z_n)\psi^*(w_n) \rangle_u^{(0)} = \sqrt{\frac{w_1 \cdots w_n}{z_1 \cdots z_n}} \frac{(u; u)_\infty^{2n} \prod_{1 \leq i < j \leq n} \theta_u(z_j/z_i, w_j/w_i)}{\prod_{1 \leq i \leq j \leq n} \theta_u(w_j/z_i) \prod_{n \geq i > j \geq 1} \theta_u(z_i/w_j)}. \quad (4.11)$$

and

$$\langle \psi(z_1)\psi^*(w_1) \cdots \psi(z_n)\psi^*(w_n) \rangle_{u,t} = \theta_3 \left(t \frac{z_1 \cdots z_n}{w_1 \cdots w_n}; u \right) \frac{\langle \psi(z_1)\psi^*(w_1) \cdots \psi(z_n)\psi^*(w_n) \rangle_u^{(0)}}{\theta_3(t; u)}. \quad (4.12)$$

Note that we should have $u < |w_j/z_i| < 1$ (respectively $1 < |w_j/z_i| < u^{-1}$) for $i \leq j$ (respectively for $i > j$) in order for the expectation values to be well-defined. By plugging these expressions in (4.5) and its shift-mixed counterpart, and writing the coefficient extractions as contour integrals, we obtain Proposition 3.2.

Remark 4.1. In fact, our computation yields a fermionic proof of the Frobenius elliptic determinant identity [Fro82]: on the one hand, using Wick’s lemma, the correlator $\langle \psi(z_1)\psi^*(w_1) \cdots \psi(z_n)\psi^*(w_n) \rangle_{u,t}$ may be written as the determinant $\det_{1 \leq i, j \leq n} \kappa(z_i, w_j)$ whose entries can be put in the theta form (3.14); on the other hand we obtain a product of theta functions through (4.12) and (4.11).

5 Edge behavior of the cylindrical Plancherel measure

The purpose of this section is to establish Theorems 1.1 and 1.2. Recall that we want to study the asymptotic distribution of λ_1 in the cylindric Plancherel measure, which is the λ -marginal of the measure (1.3). By (3.4), this measure is a periodic Schur process of period $N = 1$ with $\rho_1^\pm = \text{ex}_\gamma$ the exponential specialization. We start of course by applying Theorem 3.1, which entails that the (one-dimensional) shift-mixed point configuration

$$\mathfrak{S}(\lambda) + c = \left\{ \lambda_j - j + \frac{1}{2} + c, j \geq 1 \right\} \quad (5.1)$$

is a determinantal point process with correlation kernel

$$K(a, b) = \frac{1}{(2\pi i)^2} \oint_{|z|=u^{-1/4}} \frac{dz}{z^{a+1}} \oint_{|w|=u^{1/4}} \frac{dw}{w^{-b+1}} \cdot \frac{e^{L(z-z^{-1})}}{e^{L(w-w^{-1})}} \cdot \kappa(z, w) \quad (5.2)$$

$$= \sum_{\ell \in \mathbb{Z}'} \frac{J_{a+\ell}(2L) J_{b+\ell}(2L)}{1+t^{-1}u^\ell} \quad (5.3)$$

where $a, b \in \mathbb{Z}'$, $L = \gamma/(1-u)$ (this is the $L_{u,\gamma}$ from Theorem 1.1, but we drop the indices to lighten the notations) and J_n denotes the Bessel function of the first kind. The choice of the integration radii is somewhat arbitrary but will prove convenient for the forthcoming analysis. We pass to the second form by using the definition (3.13) of κ and the Laurent series expansion $e^{L(z-z^{-1})} = \sum_{n \in \mathbb{Z}} J_n(2L) z^n$. Note that, for $u \rightarrow 0^+$, the factor $1/(1+t^{-1}u^\ell)$ becomes the indicator function of \mathbb{Z}'_+ and $K(a, b)$ becomes the celebrated Bessel kernel of Borodin–Okounkov–Olshanski [BOO00] and Johansson [Joh01]. As such and in analogy to Johansson’s “finite-temperature GUE kernel” [Joh07], we call the kernel $K(a, b)$ the *discrete finite-temperature Bessel kernel*.

Elementary properties of the discrete finite-temperature Bessel kernel. We note first that K is symmetric and positive semi-definite. This latter property is most evident from the Bessel representation: for any complex numbers z_1, \dots, z_n we have

$$\sum_{i,j=1}^n z_i \bar{z}_j K(a_i, a_j) = \sum_{i,j=1}^n \sum_{\ell \in \mathbb{Z}'} z_i \bar{z}_j \frac{J_{a_i+\ell}(2L) J_{a_j+\ell}(2L)}{1+t^{-1}u^\ell} = \sum_{\ell \in \mathbb{Z}'} \frac{1}{1+t^{-1}u^\ell} \left| \sum_{i=1}^n z_i J_{a_i+\ell}(2L) \right|^2 \geq 0. \quad (5.4)$$

Second, for any $m \in \mathbb{Z}'$, K is trace-class on $\ell^2(\{m, m+1, \dots\})$: here it is simpler to use the contour integral representation which immediately implies, by crudely bounding the integrand, that $K(a, b) = O(u^{-(a+b)/4})$ for $a, b \rightarrow +\infty$, hence we have $\sum_{i \geq m} |K(i, i)| < \infty$.

It follows that the Fredholm determinant $\det(I - K)_{\ell^2(\{m, m+1, \dots\})}$ is well-defined, and is equal to the probability that $\lambda_1 + c < m$ in the shift-mixed cylindrical Plancherel measure. The parameter t governing the distribution (2.5) of c does not play a significant role in the asymptotic analysis and we will set it to 1 from now on.

Length scales and asymptotics. Before entering into the precise asymptotic analysis of K , it is useful to have a qualitative discussion of length scales. From the expression $Z_{u,\gamma} = e^{\frac{\gamma^2}{1-u}} / (u; u)_\infty$ of the partition function of the cylindrical Plancherel measure (1.3), it is straightforward to check that

$$\mathbb{E}|\lambda| = \frac{\gamma^2}{(1-u)^2} - u \frac{d}{du} \ln(u; u)_\infty, \quad \text{Var}|\lambda| = \frac{\gamma^2(1+u)}{(1-u)^3} - \left(u \frac{d}{du} \right)^2 \ln(u; u)_\infty. \quad (5.5)$$

From the asymptotics $\ln(u; u)_\infty \sim -\frac{\pi^2}{6(1-u)}$ for $u \rightarrow 1^-$, we see that

$$\mathbb{E}|\lambda| \sim \frac{\gamma^2 + \frac{\pi^2}{6}}{(1-u)^2} =: \Lambda_{u,\gamma}, \quad \text{Var}|\lambda| \ll (\mathbb{E}|\lambda|)^2 \quad (5.6)$$

whenever $u \rightarrow 1^-$ or $\gamma \rightarrow \infty$. This shows that $|\lambda|/\Lambda_{u,\gamma}$ tends to 1 in probability. Note that $\Lambda_{u,\gamma} \sim L^2$ for $\gamma \rightarrow \infty$, hence L is the natural length scale for the Young diagram of λ and for the point process $\mathfrak{S}(\lambda)$. For $u \rightarrow 1^-$ and fixed $\gamma > 0$, L remains a good length scale but it is easier to work with r^{-1} , where $u = e^{-r}$.

Borodin has shown the following “bulk” asymptotics for the one-point function [Bor07, Example 3.4]:

$$\lim_{\substack{u=e^{-r}, r \rightarrow 0^+ \\ \gamma \text{ fixed}, t=1}} K\left(\left\lfloor \frac{\tau}{r} \right\rfloor, \left\lfloor \frac{\tau}{r} \right\rfloor\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi}{1 + e^{\tau-2\gamma \cos \phi}} =: \rho(\tau), \quad \tau \in \mathbb{R}. \quad (5.7)$$

In fact he obtained more generally the bulk limiting kernel—see Proposition D.3. For $\tau \rightarrow \infty$ we have $\rho(\tau) \sim I_0(2\gamma)e^{-\tau}$ and we expect to find the “edge” (rightmost particle) at a position $\frac{\tau + O(1)}{r}$ with $\rho(\tau_e) = r$,

i.e. $\tau_e = \ln \frac{I_0(2\gamma)}{r}$. This provides some intuition regarding the spatial rescaling in Theorem 1.2. For $\gamma \rightarrow \infty$, the limiting density ρ tends to that of the Logan–Shepp–Vershik–Kerov limit shape, namely

$$\lim_{\gamma \rightarrow \infty} \rho(\gamma x) = \begin{cases} 0 & \text{if } x \geq 2, \\ \arccos(x) & \text{if } -2 < x < 2, \\ 1 & \text{if } x \leq -2. \end{cases} \quad (5.8)$$

and we now expect to find the edge around the point $2\frac{\gamma}{r} \sim 2L$, consistently with Theorem 1.1. Of course, understanding fluctuations requires more precise asymptotics, which we provide now.

To establish Theorem 1.1, we shall show that K converges to the finite-temperature Airy kernel M_α in the edge crossover regime, as stated in the following.

Proposition 5.1. *For $t = 1$ and in the edge crossover regime*

$$\begin{cases} L \rightarrow \infty \\ u \rightarrow 1^- \\ L^{1/3}(1-u) \rightarrow \alpha > 0 \end{cases} \quad \begin{cases} a = \lfloor 2L + xL^{1/3} \rfloor \\ b = \lfloor 2L + yL^{1/3} \rfloor \\ m = \lfloor 2L + sL^{1/3} \rfloor \end{cases}, \quad (5.9)$$

we have

$$L^{1/3}K(a, b) \rightarrow M_\alpha(x, y) \quad (5.10)$$

and

$$\sum_{i \geq m} K(i, i) \rightarrow \int_s^\infty M_\alpha(s', s') ds'. \quad (5.11)$$

Note that the right-hand side of (5.11) is finite by [Joh07, Proposition 1.1]. For good measure, we will give below *two* proofs of Proposition 5.1. The first proof uses the Bessel representation, and is an adaptation of the zero-temperature proofs from [BOO00, Joh01]—see Romik’s book [Rom15] for a pedagogical exposition. The second proof is based on the contour integral representation. Let us now give the analogous statement, implying Theorem 1.2, for the high temperature regime.

Proposition 5.2. *For $t = 1$ and in the edge high temperature regime*

$$\begin{cases} u = e^{-r} \\ r \rightarrow 0^+ \\ L = o(r^{-3}) \end{cases} \quad \begin{cases} a = \lfloor M + xr^{-1} \rfloor \\ b = \lfloor M + yr^{-1} \rfloor \\ m = \lfloor M + sr^{-1} \rfloor \end{cases}, \quad M := r^{-1} \ln \frac{I_0(2Lr)}{r}, \quad (5.12)$$

we have

$$r^{-1}K(a, b) \rightarrow \begin{cases} e^{-x} & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases} \quad (5.13)$$

and

$$\sum_{i \geq m} K(i, i) \rightarrow e^{-s}. \quad (5.14)$$

The proposition says essentially that, near the edge in the high temperature regime, the shift-mixed determinantal process degenerates to a Poisson point process with exponentially decreasing intensity. See also the discussion in [Joh07]. The proof is given in Appendix D and is based a slightly nonstandard analysis of the contour integral representation for K , which may have an interest on its own.

Proof that Proposition 5.1 implies Theorem 1.1. Observe that, to establish the convergence in distribution (1.5), we may replace λ_1 by the maximum of shift-mixed point configuration at $t = 1$, $\lambda_1 + c - 1/2$. Indeed, the difference between the two rescaled random variables is $(c - 1/2)/L^{1/3}$ which, by Lemma 2.1, converges to zero in probability for $t = 1$. (And taking $t \neq 1$ fixed simply amounts to a deterministic shift.)

Since the shift-mixed point process is determinantal, we are left with the task of proving the convergence of Fredholm determinants

$$\det(I - K)_{\ell^2(\{m, m+1, \dots\})} \rightarrow \det(I - M_\alpha)_{L^2(s, \infty)} = F_\alpha(s) \quad (5.15)$$

in the scaling regime (5.9). This is done by standard arguments, which we recall for convenience. Set $\tilde{K}(x, y) := L^{1/3}K(a, b)$ with a, b as in (5.9): by (5.10), $\tilde{K}(x, y)$ converges pointwise to $M_\alpha(x, y)$, while we have

$$\det(I - K)_{\ell^2(\{m, m+1, \dots\})} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(s', \infty)^n} \det_{1 \leq i, j \leq n} \tilde{K}(x_i, x_j) dx_1 \cdots dx_n. \quad (5.16)$$

with $s' = (m - 2L)/L^{1/3} \rightarrow s$. The integrand converges pointwise to $\det_{1 \leq i, j \leq n} M_\alpha(x_i, x_j)$, and Hadamard's inequality states that its modulus is bounded by $\prod_{i=1}^n \tilde{K}(x_i, x_i)$ (recall that K is positive semi-definite), so we may conclude using dominated convergence and the convergence of traces (5.11) that

$$\det(I - K)_{\ell^2(\{m, m+1, \dots\})} \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{(s, \infty)^n} \det_{1 \leq i, j \leq n} M_\alpha(x_i, x_j) dx_1 \cdots dx_n = \det(I - M_\alpha)_{L^2(s, \infty)} \quad (5.17)$$

as desired. \square

Remark 5.3. An extension of this argument shows that the joint law of the k first parts of λ converges to that of the k rightmost particles in the finite-temperature Airy process. See [BOO00, Section 4.2] for the zero-temperature case.

Proof that Proposition 5.2 implies Theorem 1.2. The argument of the previous proof applies mutatis mutandis. Moreover, we are now able to explicitly evaluate the limiting Fredholm determinant:

$$\det(I - K)_{\ell^2(\{m, m+1, \dots\})} \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{-ns} = e^{-e^{-s}} \quad (5.18)$$

since $\det_{1 \leq i, j \leq n} \tilde{K}(x_i, x_j) \rightarrow e^{-(x_1 + \cdots + x_n)}$ almost everywhere. Note that, in the regime (5.12), we have $2Lr = 2\gamma + \gamma r + o(1)$ so the deterministic leading-order term M is essentially the same as that in Theorem 1.2. \square

Proof of Proposition 5.1 via the Bessel representation. We shall use Nicholson's approximation (found in [Wat22, Section 8.43] but see [BOO00, Lemma 4.4] for a recent elementary proof), as stated in [Rom15, Theorem 2.27]:

- for $x \in \mathbb{R}$, uniformly on compact sets,

$$L^{1/3} J_{2L+xL^{1/3}}(2L) \rightarrow \text{Ai}(x), \quad L \rightarrow \infty; \quad (5.19)$$

- there exist constants c_1, c_2, L_0 such that for $L > L_0$ and $x > 0$,

$$L^{1/3} |J_{2L+xL^{1/3}}(2L)| < c_1 \exp(-c_2 x). \quad (5.20)$$

In the Bessel representation (5.3), we split the sum into three parts

$$L^{1/3} \sum_{\ell \in \mathbb{Z}'} = L^{1/3} \left(\sum_{\ell=-\infty}^{-TL^{1/3}-1} + \sum_{\ell=-TL^{1/3}}^{TL^{1/3}} + \sum_{\ell=TL^{1/3}+1}^{\infty} \right) =: \Sigma_1 + \Sigma_2 + \Sigma_3. \quad (5.21)$$

for some large enough T yet to be chosen.

The main asymptotic contribution comes from Σ_2 which, using Nicholson's approximation (5.19), becomes a Riemann sum for the finite-temperature Airy integral:

$$\begin{aligned} \Sigma_2 &\approx L^{-1/3} \sum_{\ell=-TL^{1/3}}^{TL^{1/3}} \left(L^{1/3} J_{2L+xL^{1/3}+\ell}(2L) \right) \left(L^{1/3} J_{2L+yL^{1/3}+\ell}(2L) \right) \frac{e^{\alpha L^{-1/3} \ell}}{1 + e^{\alpha L^{-1/3} \ell}} \\ &\approx L^{-1/3} \sum_{\ell=-TL^{1/3}}^{TL^{1/3}} \text{Ai}(x + \ell L^{-1/3}) \text{Ai}(y + \ell L^{-1/3}) \frac{e^{\alpha L^{-1/3} \ell}}{1 + e^{\alpha L^{-1/3} \ell}} \\ &\approx \int_{-T}^T \text{Ai}(x + v) \text{Ai}(y + v) \frac{e^{\alpha v}}{1 + e^{\alpha v}} dv \end{aligned} \quad (5.22)$$

and the latter can be made arbitrarily close to the result by choosing T large enough. It remains to show the contributions $|\Sigma_1|, |\Sigma_3| \rightarrow 0$. We begin with Σ_1 . By using the same approximation and recognizing a similar Riemann sum, we have

$$\begin{aligned}
|\Sigma_1| &\approx L^{-1/3} \left| \sum_{\ell=-\infty}^{-TL^{1/3}-1} \left(L^{1/3} J_{2L+xL^{1/3}+\ell}(2L) \right) \left(L^{1/3} J_{2L+yL^{1/3}+\ell}(2L) \right) \frac{e^{\alpha L^{-1/3}\ell}}{1+e^{\alpha L^{-1/3}\ell}} \right| \\
&\approx L^{-1/3} \left| \sum_{\ell=-\infty}^{-TL^{1/3}-1} \text{Ai}(x+\ell L^{-1/3}) \text{Ai}(y+\ell L^{-1/3}) \frac{e^{\alpha L^{-1/3}\ell}}{1+e^{\alpha L^{-1/3}\ell}} \right| \\
&\approx \left| \int_{-\infty}^{-T} \text{Ai}(x+v) \text{Ai}(y+v) \frac{e^{\alpha v}}{1+e^{\alpha v}} dv \right|
\end{aligned} \tag{5.23}$$

and the latter integral can be made arbitrarily small for appropriately large T . For instance, as the Airy function Ai is bounded, the Airy integral above is bounded by some positive constant times the integral $\int_{-\infty}^{-T} \frac{e^{\alpha v}}{1+e^{\alpha v}} dv = \log(1+\exp(-\alpha T))/\alpha$ which is arbitrarily small if one chooses T large enough.

We finally come to Σ_3 and for a change we shall use the asymptotics in (5.20):

$$\begin{aligned}
|\Sigma_3| &\approx L^{-1/3} \left| \sum_{\ell=TL^{1/3}+1}^{\infty} \left(L^{1/3} J_{2L+xL^{1/3}+\ell}(2L) \right) \left(L^{1/3} J_{2L+yL^{1/3}+\ell}(2L) \right) \frac{e^{\alpha L^{-1/3}\ell}}{1+e^{\alpha L^{-1/3}\ell}} \right| \\
&< L^{-1/3} \left| \sum_{\ell=TL^{1/3}+1}^{\infty} \left(L^{1/3} J_{2L+xL^{1/3}+\ell}(2L) \right) \left(L^{1/3} J_{2L+yL^{1/3}+\ell}(2L) \right) \right| \\
&< c_1^2 L^{-1/3} \sum_{m=1}^{\infty} e^{-c_2(x+y+2T+2mL^{-1/3})} = c_1^2 e^{-c_2(x+y+2T)} L^{-1/3} \sum_{m=1}^{\infty} q^m
\end{aligned} \tag{5.24}$$

where $q = \exp(-2c_2L^{-1/3})$; for the first inequality we have used the crude bound $e^v/(1+e^v) < 1$; for the second we applied (5.20) (note we need at least $T \geq \max(-x, -y)$ for this); and for the last equality we performed the change of variables $\ell - TL^{1/3} = m$. The infinite series evaluates to $q/(1-q) \approx L^{1/3}/(2c_2)$ and hence $|\Sigma_3| < c_3 \exp(-(x+y+2T))$ for an appropriate positive constant c_3 , and this latter bound can be made arbitrarily small by choosing an appropriately large T . This concludes the proof of (5.10), and the proof of (5.11) is similar mutatis-mutandis. \square

Remark 5.4. The above argument adapts easily to the case $\alpha = \infty$ or more precisely to the case $\alpha L^{-1/3} \rightarrow \infty$ as $L \rightarrow \infty$. In equation (5.22), the term $\frac{e^{\alpha L^{-1/3}\ell}}{1+e^{\alpha L^{-1/3}\ell}}$ becomes the indicator of $\{\ell > 0\}$ and by dominated convergence Σ_2 tends to the usual (here truncated) Airy kernel $(x, y) \mapsto \int_0^T \text{Ai}(x+v) \text{Ai}(y+v) dv$ while of course we still have $\Sigma_1, \Sigma_3 \rightarrow 0$. See also Remark D.4 for the case of the bulk limit.

Proof of Proposition 5.1 via the contour integral representation. We assume here that $\alpha < \infty$, see Remark 5.7 below for the case $\alpha = \infty$. We start with a useful lemma regarding the behavior of the propagator $\kappa(z, w)$.

Lemma 5.5. *Let $u = e^{-r}$ and $z/w = e^{r/2+i\theta}$ with $\theta \in [-\pi, \pi]$. Then we have*

$$\kappa(z, w) = \frac{\pi}{r \cosh \frac{\pi\theta}{r}} + O(e^{-\pi^2/r}/r), \quad r \rightarrow 0^+ \tag{5.25}$$

where the big O is uniform over θ .

Proof. This immediately follows from the Poisson summation formula

$$\kappa(z, w) = \sum_{m \in \mathbb{Z}'} \frac{e^{i\theta m}}{2 \cosh \frac{rm}{2}} = \sum_{k \in \mathbb{Z}} (-1)^k \int_{-\infty}^{\infty} \frac{e^{i\theta x - 2i\pi k x}}{2 \cosh \frac{rx}{2}} dx = \sum_{k \in \mathbb{Z}} (-1)^k \frac{\pi}{r \cosh \frac{\pi(\theta - 2\pi k)}{r}}. \tag{5.26}$$

\square

Remark 5.6. Lemma 5.5 is very similar to [Bor07, Proposition 3.2], which was proved using the elliptic form (3.14) and the imaginary Jacobi transform, which is another consequence of the Poisson summation formula. It seems that our proof is much simpler.

We now rewrite the contour integral representation (5.2) in the form

$$L^{1/3}K(a, b) = \frac{1}{(2i\pi)^2} \oint\!\!\!\oint e^{L(S(z)-S(w))} \kappa(z, w) \frac{L^{1/3} dz dw}{z^{a'+1} w^{-b'+1}} \quad (5.27)$$

with $S(z) := z - z^{-1} - 2\ln z$, $a' := a - 2L$, $b' := b - 2L$, and z and w are integrated over the circles of respective radii $e^{r/4}$ and $e^{-r/4}$, where we set again $u = e^{-r}$ with $r \rightarrow 0^+$. Note that, in the scaling regime (5.9), we have $rL^{1/3} \rightarrow \alpha$, $ra' \rightarrow \alpha x$ and $rb' \rightarrow \alpha y$.

Let us first record some useful properties of the ‘‘action’’ S : writing $z = e^{r/4+i\varphi}$ we have

$$\Re S(z) = 2 \sinh\left(\frac{r}{4}\right) \cos \varphi - \frac{r}{2} \quad (5.28)$$

which is maximal when $\varphi = 0$, i.e. when z is on the positive real axis. By the relation $S(w) = -S(w^{-1})$, $-\Re S(w)$ is also maximal on the positive real axis, and since $S(e^{r/4}) \sim r^3/196$, the exponential factor $e^{L(S(z)-S(w))}$ remains uniformly bounded.

Fixing an $\epsilon \in (0, 1/4)$, we split the double integral (5.27) in two: let I be the contribution from the z and w whose arguments are both smaller than $r^{1-\epsilon}$ in absolute value, and let I^c be the complementary contribution. We shall show that $I \rightarrow M_\alpha(x, y)$ while $I^c \rightarrow 0$:

- I^c tends to 0: we bound the integrand as follows. Since either z or w has an argument larger than $r^{1-\epsilon}$ in absolute value, we have by (5.28)

$$\Re S(z) - \Re S(w) - 2S(e^{r/4}) \leq 2 \sinh\left(\frac{r}{4}\right) (\cos r^{1-\epsilon} - 1) \leq -\frac{1}{4} \sinh\left(\frac{r}{4}\right) r^{2-2\epsilon} \quad (5.29)$$

from which we deduce that $|e^{L(S(z)-S(w))}| \leq C e^{-\alpha^3 r^{-2\epsilon}/8}$ for some $C > 0$. By Lemma 5.5, we have $\kappa(z, w) = O(r^{-1})$ uniformly, while $|z^{a'} w^{-b'}| = e^{r(a'+b')/4}$ remains bounded away from 0, so the integrand tends uniformly to 0 as wanted.

- I converges: we perform the change of variables

$$z = e^{r\zeta/\alpha}, \quad w = e^{r\omega/\alpha} \quad (5.30)$$

where $\Re(\zeta) = -\Re(\omega) = \alpha/4$ while $\Im(\zeta), \Im(\omega)$ are both smaller than $\alpha r^{-\epsilon}$ in absolute value. By a Taylor expansion of $S(z)$ around the ‘‘monkey saddle’’ $z = 1$, we find that

$$e^{L(S(z)-S(w))} = e^{\frac{\zeta^3}{3} - \frac{\omega^3}{3} + O(r^{1-4\epsilon})} \quad (5.31)$$

where the big O is uniform. By Lemma 5.5, we have

$$\kappa(z, w) \sim \frac{\pi}{r \cosh \frac{\pi \Im(\zeta - \omega)}{\alpha}} = \frac{\pi}{r \sin \frac{\pi(\zeta - \omega)}{\alpha}} \quad (5.32)$$

with a uniform exponentially small error term. Finally, $z^{a'} w^{-b'} \rightarrow e^{x\zeta - y\omega}$, which allows to conclude by dominated convergence that

$$I \rightarrow \frac{1}{(2i\pi)^2} \int_{\alpha/4+i\mathbb{R}} d\zeta \int_{-\alpha/4+i\mathbb{R}} d\omega \exp\left(\frac{\zeta^3}{3} - \frac{\omega^3}{3} - x\zeta + y\omega\right) \frac{\pi}{\alpha \sin \frac{\pi(\zeta - \omega)}{\alpha}} \quad (5.33)$$

(note that the $L^{1/3}$ of (5.27) and the $1/r$ of (5.32) are absorbed by the Jacobian of the change of variables $z \rightarrow \zeta, w \rightarrow \omega$).

It is straightforward to identify the right-hand side of (5.33) as $M_\alpha(x, y)$, using the integral representations

$$\frac{\pi}{\alpha \sin \frac{\pi(\zeta - \omega)}{\alpha}} = \int_{-\infty}^{\infty} \frac{e^{(\alpha + \omega - \zeta)v} dv}{1 + e^{\alpha v}}, \quad \text{Ai}(x + v) = \int_{\alpha/4 + i\mathbb{R}} e^{\frac{\zeta^3}{3} - (x+v)\zeta} \frac{d\zeta}{2i\pi} \quad (5.34)$$

and similarly for $\text{Ai}(y + v)$ which is given by the integral over ω . This concludes the proof of (5.10).

The proof of the convergence of traces (5.11) is entirely similar: in (5.27) the rightmost fraction should simply be replaced by $\frac{dzdw}{(z-w)z^m w^{-m+1}}$ (the factor $\frac{1}{z-w}$ is of order $r^{-1} \sim L^{1/3}$, and produces after the change of variables a factor $\frac{1}{\zeta - \omega}$ which we “decouple” by the usual integral representation $\frac{1}{\zeta - \omega} = \int_0^\infty e^{(\omega - \zeta)v'} dv'$). \square

Remark 5.7. In the case $\alpha = \infty$ the above proof may be adapted straightforwardly by integrating z and w over the circles of radii $e^{\pm L^{-1/3}}$ instead of $e^{\pm r/4}$. The change of variable (5.30) is simply replaced by $z = e^{L^{-1/3}\zeta}$ and $w = e^{L^{1/3}\omega}$ with $\Re(\zeta) = -\Re(\omega) = 1$ and, using the Poisson summation formula as for Lemma 5.5, we see that $\kappa(z, w)$ may now be approximated by $\frac{L^{1/3}}{\zeta - \omega}$ over the integration contour, instead of (5.32). We recover the usual zero-temperature Airy kernel as the limit.

Remark 5.8. As observed in several occasions—see e.g. [Oko02, OR03]—the asymptotic analysis of contour integral representations is particularly robust and explains why the limiting behavior should be universal. In our analysis, we simply use the fact that the action $S(z)$ has a double critical point on the real axis, a property that is characteristic of the “edge” regime. Our proof should therefore be easily adaptable to other instances of periodic Schur processes. For example, we expect that the finite-temperature Airy kernel will be observed in cylindric partitions as the scaling limit around any generic edge point (the action now involving the dilogarithm function [OR03, Bor07]).

6 The stationary cylindric Plancherel process

In this section we define a continuous-time periodic extension of the cylindric Plancherel measure. This process has the remarkable property of being stationary, i.e. invariant under time translation. It may be identified as the periodic analogue of the stationary process of Borodin and Olshanski [BO06b], whose fixed-time marginal is the ordinary poissonized Plancherel measure. As we constructed our process through Fock space considerations, our presentation is somewhat different from [BO06b].

Definition and basic properties. For β, ϑ two nonnegative parameters, let $T^{(\vartheta)}(\beta)$ be the “transfer matrix” acting on the set \mathcal{P} of integer partitions by

$$T^{(\vartheta)}(\beta)_{\lambda, \nu} := e^{\vartheta^2(e^{-\beta} - 1)} \sum_{\mu \in \mathcal{P}} e^{-\beta|\mu|} s_{\lambda/\mu}(\text{ex}_{\vartheta(1-e^{-\beta})}) s_{\nu/\mu}(\text{ex}_{\vartheta(1-e^{-\beta})}), \quad \lambda, \nu \in \mathcal{P} \quad (6.1)$$

where we recall that ex denotes the exponential specialization (3.4)—we may therefore rewrite $T^{(\vartheta)}(\beta)_{\lambda, \nu}$ in terms of the numbers of standard Young tableaux $\dim(\lambda/\mu)$ and $\dim(\nu/\mu)$. What is remarkable about our definition is that the family $(T^{(\vartheta)}(\beta))_{\beta \geq 0}$ forms a semigroup.

Proposition 6.1. *For any β, β', ϑ we have*

$$T^{(\vartheta)}(\beta)T^{(\vartheta)}(\beta') = T^{(\vartheta)}(\beta + \beta'). \quad (6.2)$$

Furthermore for any ϑ we have $\sum_{\lambda \in \mathcal{P}} T^{(\vartheta)}(\beta)_{\lambda, \lambda} = Z_\beta$, where $Z_\beta := \prod_{k \geq 1} (1 - e^{-\beta k})^{-1}$.

Proof. The transfer matrix is represented in fermionic Fock space as

$$T^{(\vartheta)}(\beta) = e^{\vartheta^2(e^{-\beta} - 1)} \Gamma_-(\text{ex}_{\vartheta(1-e^{-\beta})}) e^{-\beta H} \Gamma_+(\text{ex}_{\vartheta(1-e^{-\beta})}) \quad (6.3)$$

(we keep the same letter by a slight abuse of notation). We then leave to the reader the pleasure of checking that the commutation relations (B.8) imply the desired relation. The trace formula for $Z_\beta = \text{tr}(\Pi_0 T^{(\vartheta)}(\beta))$ may be checked using the Γ -elimination method of Section 4. \square

Remark 6.2. In fact, we have $T^{(\vartheta)}(\beta) = e^{-\beta H_\vartheta}$ where $H_\vartheta := H - \vartheta(\alpha_1 + \alpha_{-1}) + \vartheta^2$ with $\alpha_{\pm 1}$ being the bosonic operators defined in (B.6). The expression (6.3) corresponds to the normal ordered form.

Definition 6.3. Fix β, ϑ two nonnegative real parameters. The (stationary) *cylindric Plancherel process* of period β and intensity ϑ is the random partition-valued continuous-time process $\lambda(\cdot) : \mathbb{R} \rightarrow \mathcal{P}$ which is β -periodic and whose finite-dimensional distributions are given by

$$\text{Prob}(\lambda(0), \lambda(b_1), \dots, \lambda(b_n)) = \frac{1}{Z_\beta} \prod_{i=0}^n T^{(\vartheta)}(b_{i+1} - b_i)_{\lambda(b_i), \lambda(b_{i+1})} \quad (6.4)$$

where $0 = b_0 < b_1 < \dots < b_n < b_{n+1} = \beta$.

By Proposition 6.1 it is clear that (6.4) defines a consistent family of finite-dimensional distributions, so the process is well-defined by the Kolmogorov extension theorem. We may see that $\lambda(\cdot)$ can be realized as a partition-valued jump process (with finitely many jumps almost surely)—see again Figure 2 for an illustration—but we shall not detail these considerations here as we are chiefly interested in the properties of the finite-dimensional distributions themselves. Note that the operator $-H_\vartheta$ mentioned in Remark 6.2, while having positive off-diagonal elements in the canonical basis, is not an intensity matrix so does not directly generate a Markov process. However, H_ϑ is self-adjoint and nonnegative, hence our process might be thought as a quantum evolution in imaginary time.

The law of $\lambda(0)$ is the λ -marginal of the cylindric Plancherel measure (1.3) of parameters $u = e^{-\beta}$ and $\gamma = \vartheta(1 - e^{-\beta})$. This is also true for $\lambda(b)$ with any $b \in \mathbb{R}$ as, from the definition and from the semigroup property, one may check easily that the process is stationary. Moreover, from (6.1) we may check that, for any times $0 \leq b_1 < \dots < b_N \leq \beta$, the tuple $(\lambda(b_1), \dots, \lambda(b_N))$ is the λ -marginal of a periodic Schur process as defined in the introduction. Note that this requires to rewrite the weight (6.4) in the form (1.2): one sees that u is always $e^{-\beta}$ and that the specializations ρ_k^\pm will all be exponential, but their parameters have a slightly unnatural expression—which we omit here—since (1.2) breaks translation invariance due to the weight $u^{|\mu^{(0)}|}$.

Remark 6.4 (Infinite period/zero-temperature limit). In Fock space representation we have

$$\lim_{\beta \rightarrow \infty} T^{(\vartheta)}(\beta) = e^{-\vartheta^2} \Gamma_-(\text{ex}_\vartheta) |\emptyset\rangle \langle \emptyset| \Gamma_+(\text{ex}_\vartheta). \quad (6.5)$$

In other words $T^{(\vartheta)}(\infty)$ has rank one and we have the factorization $T^{(\vartheta)}(\infty)_{\lambda, \nu} = e^{-\vartheta^2} s_\lambda(\text{ex}_\vartheta) s_\nu(\text{ex}_\vartheta)$. From this observation, we see that the cylindric Plancherel process has well-defined limit as $\beta \rightarrow \infty$, which is an ordinary (nonperiodic) continuous-time Schur process whose marginal at any given time is the poissonized Plancherel measure of parameter ϑ . We may identify it with the stationary process of Borodin and Olshanski [BO06b]: this is not obvious from the definition, but it will be from the expression of the correlation functions.

Remark 6.5. In view of the characterization (3.3) for nonnegative specializations, the exponential specialization is the only one having the “infinite divisibility property” needed to define a stationary continuous-time Schur process. Therefore the cylindric Plancherel process seems to be one of a kind. In [BO06a], Borodin and Olshanski construct Markov processes preserving z -measures but, as they point out, they do not seem to belong to the class of Schur process.

Remark 6.6. The cylindric Plancherel process appears as a “poissonian” limit of a measure on cylindric partitions in the following sense: let us consider the instance of the periodic Schur process of length N given by

$$\text{Prob}(\vec{\lambda}, \vec{\mu}) \propto \prod_{k=1}^N \left(s_{\lambda^{(k)}/\mu^{(k-1)}}(\vartheta\epsilon) e^{-\epsilon^{|\mu^{(k)}|}} s_{\lambda^{(k)}/\mu^{(k)}}(\vartheta\epsilon) \right). \quad (6.6)$$

This weight may be rewritten in the form (1.2) but here translation invariance is manifest. Since the specializations consist of single variables, the weight is supported on sequences $(\vec{\lambda}, \vec{\mu})$ corresponding to cylindric partitions, see again Figure 1. Then, the cylindric Plancherel measure of period β and intensity ϑ appears as the limit $N \rightarrow \infty$, $\epsilon \rightarrow 0^+$ with $N\epsilon \rightarrow \beta$. This is because the “elementary transfer matrix” reads here $\Gamma_-(\vartheta\epsilon) e^{-\epsilon H} \Gamma_+(\vartheta\epsilon) = 1 - \epsilon(H_\vartheta - \vartheta^2) + O(\epsilon^2)$ in the notation of Remark 6.2.

Correlation functions. We now turn to correlation functions, which characterize the point process

$$\mathfrak{S}(\lambda) := \left\{ \left(b, \lambda(b)_i - i + \frac{1}{2} \right), b \in [0, \beta], i \geq 1 \right\} \subset [0, \beta] \times \mathbb{Z}'. \quad (6.7)$$

Following the discussion of Section 3, we shall rather consider the shift-mixed process $\mathfrak{S}(\lambda) + (0, c)$, where $c \in \mathbb{Z}$ is distributed as (2.5) with t an arbitrary positive parameter.

Proposition 6.7. *The shift-mixed process $\mathfrak{S}(\lambda) + (0, c)$ is determinantal with correlation kernel given by*

$$K(b, k; b', k') = \frac{1}{(2i\pi)^2} \oint_{|z|=1^+} \frac{dz}{z^{k+1}} \oint_{|w|=1^-} \frac{dw}{w^{-k'+1}} \frac{e^{\vartheta(z-z^{-1})}}{e^{\vartheta(w-w^{-1})}} \kappa(ze^{-b}, we^{-b'}) \quad (6.8)$$

where: $b, b' \in [0, \beta]$, $k, k' \in \mathbb{Z}'$; κ is as in (3.13) with $u = e^{-\beta}$; and the notation 1^\pm means that we should take $|z|$ a bit larger than $|w|$ in the case $b = b'$, to circumvent the pole of κ at $z = w$.

Proof. That the shift-mixed process is determinantal is a consequence of Theorem 3.1. We may also derive the expression of $K(b, k; b', k')$ by converting the weight (6.4) in the form (1.2) and doing further manipulations, but we found it more instructive and less error-prone to rederive it directly from the free fermion formalism. From the discussion of Section 4, the generating function

$$\mathcal{K}(b, z; b', w) := \sum_{k \in \mathbb{Z}'} \sum_{k' \in \mathbb{Z}'} K(b, k; b', k') z^k w^{-k'} \quad (6.9)$$

admits the Fock space representation

$$\mathcal{K}(b, z; b', w) = \mathcal{Z}_{\beta, t}^{-1} \times \begin{cases} \text{tr} \left(t^C T^{(\vartheta)}(b) \psi(z) T^{(\vartheta)}(b' - b) \psi^*(w) T^{(\vartheta)}(\beta - b') \right) & \text{if } b \leq b', \\ -\text{tr} \left(t^C T^{(\vartheta)}(b') \psi^*(w) T^{(\vartheta)}(b - b') \psi(z) T^{(\vartheta)}(\beta - b) \right) & \text{if } b > b'. \end{cases} \quad (6.10)$$

where $\mathcal{Z}_{\beta, t} = \text{tr}(t^C T^{(\vartheta)}(\beta)) = \text{tr}(t^C e^{-\beta H})$. We now plug in (6.3) and perform Γ -elimination, which yields

$$\mathcal{K}(b, z; b', w) = \mathcal{Z}_{\beta, t}^{-1} \frac{e^{\vartheta(z-z^{-1})}}{e^{\vartheta(w-w^{-1})}} \times \begin{cases} \text{tr} \left(t^C e^{-bH} \psi(z) e^{-(b'-b)H} \psi^*(w) e^{-(\beta-b')H} \right) & \text{if } b \leq b', \\ -\text{tr} \left(t^C e^{-b'H} \psi^*(w) e^{-(b-b')H} \psi(z) e^{-(\beta-b)H} \right) & \text{if } b > b'. \end{cases} \quad (6.11)$$

Here, the somewhat surprising fact that the prefactor is independent of b and b' can be explained as follows: observe that, when performing Γ -elimination, the factor produced by the quasi-commutations with a given fermionic operator, say $\psi(z)$, does not depend on the position of the other fermionic operator(s) in the product. Therefore this factor should be the same as for $b = b'$, and by using translation invariance we may reduce to the case $b = b' = 0$, from which we may identify the factor with the one present in (5.2) for the cylindric Plancherel measure (note that $L = \vartheta$ in our current notations).

Finally, from the relations $e^{-bH} \psi(z) e^{bH} = \psi(ze^{-b})$ and $e^{-b'H} \psi^*(w) e^{b'H} = \psi^*(we^{-b'})$ (which result from the fact that ψ_k and ψ_{-k}^* “increase the energy” by k) and from the fermionic expression (4.9) of κ , we evaluate the remaining traces in (6.11) and get

$$\mathcal{K}(b, z; b', w) = \frac{e^{\vartheta(z-z^{-1})}}{e^{\vartheta(w-w^{-1})}} \kappa(ze^{-b}, we^{-b'}). \quad (6.12)$$

The ratio of the arguments of κ should be in the annulus $(1, e^\beta)$ for $b \leq b'$ and in $(e^{-\beta}, 1)$ for $b > b'$, and this condition is ensured by taking $|z| = 1^+$ and $|w| = 1^-$. The coefficient $K(b, k; b', k')$ may then be extracted by a double contour integral. \square

Remark 6.8. The Bessel representation (5.3) of the correlation kernel for the cylindric Plancherel measure may be generalized as

$$K(b, k; b', k') = \begin{cases} \sum_{\ell \in \mathbb{Z}'} J_{k+\ell}(2\vartheta) J_{k'+\ell}(2\vartheta) \frac{e^{(b-b')\ell}}{1+t^{-1}e^{\beta\ell}} & \text{if } b \leq b', \\ -\sum_{\ell \in \mathbb{Z}'} J_{k+\ell}(2\vartheta) J_{k'+\ell}(2\vartheta) \frac{e^{(b-b')\ell}}{1+te^{-\beta\ell}} & \text{if } b > b'. \end{cases} \quad (6.13)$$

In the limit $\beta \rightarrow \infty$, we recover the stationary version of the discrete extended Bessel kernel of [BO06b, Theorem 3.3].

Edge crossover behavior. We now study the edge crossover asymptotics of the kernel $K(b, k; b', k')$, which we find to be described by the *finite-temperature extended Airy kernel*, as defined in [LDMS17, Equation (99)]. Proposition 5.1 admits the following extension. (Note that the convergence of traces need not be extended.)

Proposition 6.9. *For $t = 1$ and in the edge crossover regime*

$$\begin{cases} \vartheta \rightarrow \infty \\ \beta \rightarrow 0^+ \\ \vartheta^{1/3}\beta \rightarrow \alpha \end{cases} \quad \begin{cases} b = \beta\tau/\alpha \\ k = \lfloor 2\vartheta + x\vartheta^{1/3} \rfloor \\ b' = \beta\tau'/\alpha \\ k' = \lfloor 2\vartheta + y\vartheta^{1/3} \rfloor \end{cases} \quad (6.14)$$

where we assume $\tau, \tau' \in [0, \alpha)$ without loss of generality, we have

$$\vartheta^{1/3}K(b, k; b', k') \rightarrow \begin{cases} \int_{-\infty}^{\infty} \frac{e^{(\tau-\tau')v}}{1+e^{-\alpha v}} \text{Ai}(x+v) \text{Ai}(y+v) dv & \text{if } \tau \leq \tau', \\ -\int_{-\infty}^{\infty} \frac{e^{(\tau-\tau')v}}{1+e^{\alpha v}} \text{Ai}(x+v) \text{Ai}(y+v) dv & \text{if } \tau > \tau'. \end{cases} \quad (6.15)$$

Proof. We simply adapt the contour integral proof of Proposition 5.1. Let us highlight the few changes. First, obviously, the role of r is now played by β , and that of L by ϑ . An immediate generalization of Lemma 5.5 shows that $\kappa = O(\beta^{-1})$ uniformly on the integration contours, and we have the estimate

$$\kappa(ze^{-b}, we^{-b'}) \sim \frac{\pi}{\beta \sin \frac{\pi(\zeta - \omega + \tau' - \tau)}{\alpha}} \quad (6.16)$$

in the vicinity of the saddle-point, where we perform the change of variables $z = e^{\beta\zeta/\alpha}$, $w = e^{\beta\omega/\alpha}$. Here, ζ and ω shall be integrated over lines parallel to the imaginary axis, with $\Re(\zeta) > 0 > \Re(\omega)$ and with $\Re(\zeta - \omega + \tau' - \tau) > 0$ if $\tau \leq \tau'$, < 0 otherwise. In view of (6.8), all the dependence in b, b' being in the κ factor, we conclude that the limiting kernel is given by (5.33) where we replace the last factor by the right-hand side of (6.16). The last step is to plug in the integral representations of the cosecant and Airy functions, but for the cosecant we should be careful that it depends on the sign of $\tau' - \tau$, namely

$$\frac{\pi}{\alpha \sin \frac{\pi(\zeta - \omega + \tau' - \tau)}{\alpha}} = \begin{cases} \int_{-\infty}^{\infty} \frac{e^{-(\zeta - \omega + \tau' - \tau)v} dv}{1+e^{-\alpha v}} & \text{if } \tau \leq \tau', \\ -\int_{-\infty}^{\infty} \frac{e^{-(\zeta - \omega + \tau' - \tau)v} dv}{1+e^{\alpha v}} & \text{if } \tau > \tau'. \end{cases} \quad (6.17)$$

This leads to the desired expression (6.15). \square

Of course, some extra steps of analysis—which we omit—are needed to deduce the convergence of the rescaled process from that of the kernel. Note that Lemma 2.1 still ensures that shift-mixing is irrelevant in the considered scaling limit. We have not investigated the edge high temperature regime, namely whether Proposition 5.2 may be extended to the cylindrical Plancherel process.

7 The periodic strict Schur process and neutral fermions

Vuletić [Vul07] defined the so-called shifted Schur process, which is a measure on sequences of *strict* partitions (as defined in Section 2), and whose definition involves Schur's P and Q functions instead of the ordinary Schur functions. In this section we explore the periodic variant of the shifted Schur process which we prefer to call the *strict Schur process* to emphasize that it deals with strict partitions.

Reminders on Schur's P and Q functions. (See [Mac95, Section III.8] for more general background.) Let $\text{SSym} \subset \text{Sym}$ be the subalgebra of symmetric functions generated by the p_n 's with n odd. This subalgebra contains the q_n 's which are defined via the generating series

$$Q(z) := \sum_{n \geq 0} q_n(x_1, x_2, \dots) z^n = \prod_i \frac{1 + zx_i}{1 - zx_i}. \quad (7.1)$$

This results from the relation $Q(z) = \exp(\sum_{n \geq 1, \text{ odd}} 2p_n z^n / n)$ and, in fact, the q_n 's with n odd form another algebraically independent family generating SSym .

For $a < b$ distinct nonnegative integers let $Q_{(a,b)} := qaqb + 2 \sum_{i=1}^b (-1)^i q_{a+i} q_{b-i}$ and set $Q_{(b,a)} = -Q_{(a,b)}$. For strict partitions μ, λ , write them such that $\lambda_1 > \dots > \lambda_m > 0$ and $\mu_1 > \dots > \mu_n \geq 0$ so we can assume without loss of generality that $m+n$ is even. Then the skew Schur's Q function $Q_{\lambda/\mu}$ is defined by $Q_{\lambda/\mu} = \text{pf } M_{\lambda,\mu}$ where $M_{\lambda,\mu}$ is the $(m+n) \times (m+n)$ antisymmetric matrix block-defined by

$$M_{\lambda,\mu} = \begin{pmatrix} M_\lambda & N_{\lambda,\mu} \\ -N_{\mu,\lambda} & 0 \end{pmatrix} \quad (7.2)$$

with $(M_\lambda)_{i,j} = (Q_{(\lambda_i,\lambda_j)})$ and $N_{\lambda,\mu}$ the $m \times n$ matrix $(q_{\lambda_i - \mu_j})$. This should be viewed as an analogue of the Jacobi–Trudi formula for Schur's Q functions. We define the companion Schur's P -functions by $Q_{\lambda/\mu} = 2^{\ell(\lambda) - \ell(\mu)} P_{\lambda/\mu}$. We note that $Q_{\lambda/\mu} = 0$ unless $\mu \subset \lambda$ and likewise for P .

A *strict specialization* ρ is an algebra homomorphism from SSym to the complex numbers, and is completely determined by the generating function

$$Q(\rho; z) := \sum_{n \geq 0} q_n(\rho) z^n = \exp \left(\sum_{n \geq 1, \text{ odd}} \frac{2p_n(\rho) z^n}{n} \right). \quad (7.3)$$

It is said to be nonnegative if $Q_{\lambda/\mu}(\rho) \geq 0$ for all λ, μ . A necessary and sufficient condition for ρ to be nonnegative is [Naz90, Naz92] that its generating function is of the form

$$Q(\rho; z) = e^{\gamma z} \prod_{i \geq 1} \frac{1 + \alpha_i z}{1 - \alpha_i z} \quad (7.4)$$

for a summable sequence of nonnegative real numbers $\gamma, \alpha_1, \alpha_2, \dots$. For two strict specializations ρ, ρ' , we define the specialization $\rho \cup \rho'$ by

$$Q(\rho \cup \rho'; z) := Q(\rho; z) Q(\rho'; z). \quad (7.5)$$

Recalling that \mathcal{SP} denotes the set of strict partitions, we can finally define $Q(\rho; \rho')$ by

$$Q(\rho; \rho') := \sum_{\lambda \in \mathcal{SP}} Q_\lambda(\rho) P_\lambda(\rho') = \sum_{\lambda \in \mathcal{SP}} 2^{\ell(\lambda)} P_\lambda(\rho) P_\lambda(\rho'). \quad (7.6)$$

This is associated to the Cauchy identity for Schur's P and Q functions [Mac95, Section III.8], which amounts to

$$Q(\rho; \rho') = \exp \left(\sum_{n \geq 1, \text{ odd}} \frac{2p_n(\rho) p_n(\rho')}{n} \right). \quad (7.7)$$

Definition of the process and the partition function. Fix N a nonnegative integer, u a nonnegative real number smaller than 1, and let ρ_i^\pm , $1 \leq i \leq N$ be $2N$ strict specializations such that $z \mapsto Q(\rho_i^\pm; z)$ is analytic in a disk of radius $R > 1$. The *periodic strict Schur process* is a measure on sequences of *strict* partitions of the form (1.1), which to a given sequence assigns the weight

$$\mathcal{W}_s(\vec{\lambda}, \vec{\mu}) := u^{|\mu^{(0)}|} \prod_{k=1}^N (Q_{\lambda^{(k)}/\mu^{(k-1)}}(\rho_k^+) P_{\lambda^{(k)}/\mu^{(k)}}(\rho_k^-)). \quad (7.8)$$

The *partition function* $Z_s \equiv Z_s(u, \rho_1^+, \rho_1^-, \dots, \rho_N^+, \rho_N^-)$ is the sum of weights of all sequences.

Proposition 7.1. *The partition function of the periodic strict Schur process reads*

$$Z_s = \prod_{1 \leq k \leq \ell \leq N} Q(\rho_k^+; \rho_\ell^-) \prod_{n \geq 1} Q(u^n \rho^+; \rho^-) (1 + u^n) \quad (7.9)$$

where

$$\rho^\pm := \rho_1^\pm \cup \rho_2^\pm \cup \dots \cup \rho_N^\pm. \quad (7.10)$$

Remark 7.2. For $u = 0$ we recover the partition function of the shifted Schur process [Vul07].

Correlation functions. The *point configuration* associated with a sample $(\vec{\lambda}, \vec{\mu})$ of the periodic strict Schur process is the set

$$\mathfrak{S}^s(\vec{\lambda}) := \left\{ \left(i, \lambda_j^{(i)} \right), 1 \leq i \leq N, 1 \leq j \leq \ell(\lambda^{(i)}) \right\} \subset \mathbb{N} \times \mathbb{N}^*. \quad (7.11)$$

Note that, since the $\lambda^{(i)}$'s are strict, we need not shift the parts to obtain a simple point process and, as before, considering only the partitions $\lambda^{(1)}, \dots, \lambda^{(N)}$ causes no loss of generality.

Fixing an auxiliary real parameter t , let c be a variable which can take the values 0 and 1, and define the *shift-mixed* periodic strict Schur process as the probability measure

$$\text{Prob}(c, \vec{\lambda}, \vec{\mu}) = \frac{1}{(1+t)Z_s} \cdot t^c \cdot \mathcal{W}_s(\vec{\lambda}, \vec{\mu}). \quad (7.12)$$

In other words, c is Bernoulli random variable of parameter $t/(1+t)$ independent of $(\vec{\lambda}, \vec{\mu})$. The *shift-mixed* point configuration $\mathfrak{S}_t^s(\vec{\lambda})$ is the same as the point configuration, except viewed in the larger space $\{0, 1\} \times \mathbb{N} \times \mathbb{N}^*$.³ In this regard, t is a dummy parameter here more so than it was in the case of the shift-mixed periodic (non strict) Schur process, where the dependence of t and u was intertwined in the theta function $\theta_3(t; u)$. We nevertheless choose to keep it for keeping our exposition parallel to that of the case of the non strict process. We now state our main result.

Theorem 7.3. *Let $c \in \{0, 1\}$ be a Bernoulli random variable of parameter t and $U = \{(i_1, k_1), \dots, (i_n, k_n)\} \subset \{1, \dots, N\} \times \mathbb{N}^*$ with $i_1 \leq \dots \leq i_n$. The n -point shift-mixed correlation function $\tilde{\varrho}^s(U) := \text{Prob}(c \times U \subset \mathfrak{S}_t^s(\vec{\lambda}))$ is a pfaffian:*

$$\tilde{\varrho}^s(U) = \text{pf } K \quad (7.13)$$

where K is the $2n \times 2n$ antisymmetric matrix given by

$$K_{\gamma, \delta} = \begin{cases} 2^{-1} [z^{k_\gamma} w^{k_\delta}] F^s(i_\gamma, z) F^s(i_\delta, w) \kappa^s(z, w), & 1 \leq \gamma < \delta \leq n, \\ 2^{-1} [z^{k_\gamma} w^{-k_{2n+1-\delta}}] (-1)^{k_{2n+1-\delta}} F^s(i_\gamma, z) F^s(i_\delta, w) \kappa^s(z, w), & 1 \leq \gamma \leq n < \delta \leq 2n, \\ 2^{-1} [z^{-k_{2n+1-\gamma}} w^{-k_{2n+1-\delta}}] (-1)^{k_{2n+1-\gamma} + k_{2n+1-\delta}} F^s(i_\gamma, z) F^s(i_\delta, w) \kappa^s(z, w), & n < \gamma < \delta \leq 2n \end{cases} \quad (7.14)$$

and

$$F^s(i, z) = \frac{\prod_{1 \leq \ell \leq i} Q(\rho_\ell^+; z)}{\prod_{i \leq \ell \leq N} Q(\rho_\ell^-; z^{-1})} \prod_{n \geq 1} \prod_{1 \leq \ell \leq N} \frac{Q(u^n \rho_\ell^+; z)}{Q(u^n \rho_\ell^-; z^{-1})}, \quad (7.15)$$

$$\kappa^s(z, w) = \frac{\theta_u\left(\frac{w}{z}\right)}{\theta_u\left(-\frac{w}{z}\right)} \cdot (1+t)(u; u)_\infty^2 (-u; u)_\infty^{-1}, \quad u^{1/2} < |w| < |z| < u^{-1/2}.$$

Remark 7.4. If $u = 0$ (in which case one can take $t = 0$) we recover the pfaffian correlations of the shifted Schur process [Vul07]. See also [Mat05] for the case of the shifted Schur measure.

Remark 7.5. The coefficient extractions in the kernel K can be replaced by double contour integrals similar to those in Theorem 3.1.

Proof of Theorem 7.3. We again use the formalism of free fermions, but this time in their “neutral” flavor. See Appendix C for our conventions and notations. Let us first sketch the proof of Proposition 7.1. We write

$$Z_s = \text{tr}_0 \left(u^{H_s} \Gamma_+^s(\rho_1^+) \Gamma_-^s(\rho_1^-) \dots \Gamma_+^s(\rho_N^+) \Gamma_-^s(\rho_N^-) \right) \quad (7.16)$$

where tr_0 stands for trace over the subspace \mathcal{NF}_0 of even grading. We then proceed as in Section 4 to eliminate the Γ -operators, with the following two minor modifications. First, commuting the Γ^s operators will yield $Q(u^n \rho_i^+; \rho_j^-)$ factors instead of the H factors. Second, at the end one is left with $\text{tr}_0(u^{H_s}) = \prod_{i \geq 1} (1+u^i)$. For the shift-mixed cylindric strict Schur process, the partition function is computed similarly except one traces over all of $\mathcal{NF} = \mathcal{NF}_0 \oplus \mathcal{NF}_1$, the result being $(1+t)Z_s$.

³The fiber $c \times \mathfrak{S}^s(\vec{\lambda})$ — $c \in \{0, 1\}$ —corresponds to states $|\lambda, c\rangle$ in neutral Fock space \mathcal{NF} in the notation of Appendix C.

We now turn to the proof of Theorem 7.3 itself, which is very similar to that of Theorem 3.1. First we notice that

$$\begin{aligned} \tilde{\varrho}^s(U) &= \frac{1}{(1+t)Z_s} \operatorname{tr} \left(u^{H_s} t^{C_s} \Gamma_+^s(\rho_1^+) \Gamma_-^s(\rho_1^-) \cdots \Gamma_+^s(\rho_{i_1}^+) \left(\frac{1}{2} \phi_{k_1} \phi_{k_1}^* \right) \Gamma_-^s(\rho_{i_1}^-) \cdots \right. \\ &\quad \left. \Gamma_+^s(\rho_{i_n}^+) \left(\frac{1}{2} \phi_{k_n} \phi_{k_n}^* \right) \Gamma_-^s(\rho_{i_n}^-) \cdots \Gamma_+^s(\rho_N^+) \Gamma_-^s(\rho_N^-) \right) \end{aligned} \quad (7.17)$$

where the operators $\frac{1}{2} \phi_{k_\gamma} \phi_{k_\gamma}^*$ measure whether $\lambda^{(i_\gamma)}$ has a part of size k_γ and we recall that $\phi_k^* = (-1)^k \phi_{-k}$ for $k > 0$. By commuting out the Γ -operators, we can rewrite the correlation function as

$$\tilde{\varrho}^s(U) = 2^{-n} \frac{\operatorname{tr} \left(u^{H_s} t^{C_s} \Phi_{k_1}(i_1) \Phi_{k_1}^*(i_1) \cdots \Phi_{k_n}(i_n) \Phi_{k_n}^*(i_n) \right)}{\operatorname{tr}(u^{H_s} t^{C_s})}. \quad (7.18)$$

with

$$\Phi_k(i) := \operatorname{Ad} \left(\Gamma_+^s(\rho_i^{\rightarrow}) \Gamma_-^s(\rho_i^{\leftarrow})^{-1} \right) \cdot \phi_k, \quad \Phi_k^*(i) := \operatorname{Ad} \left(\Gamma_+^s(\rho_i^{\rightarrow}) \Gamma_-^s(\rho_i^{\leftarrow})^{-1} \right) \cdot \phi_k^* \quad (7.19)$$

where Ad denotes the Lie group adjoint action $\operatorname{Ad}(A) \cdot B := ABA^{-1}$ and where

$$\rho_i^{\rightarrow} := \bigcup_{\ell=1}^i \rho_\ell^+ \cup \bigcup_{n \geq 1} u^n \rho^+, \quad \rho_i^{\leftarrow} := \bigcup_{\ell=i}^N \rho_\ell^- \cup \bigcup_{n \geq 1} u^n \rho^-. \quad (7.20)$$

Each $\Phi_k(i)$ and $\Phi_k^*(i)$ is a linear combination of the ϕ_k 's (recall that $\phi_k^* = (-1)^k \phi_{-k}$ is just a relabelling of ϕ_k added for convenience), a statement which follows easily if one passes to the generating field $\phi(z)$ and uses the commutation between Γ_\pm^s and $\phi(z)$ given in (C.18). By Wick's lemma C.1, the left-hand side of equation (7.18) is a $2n \times 2n$ pfaffian of a certain antisymmetric matrix K whose entries $K_{\gamma,\delta}$ for $\gamma < \delta$ are

$$\begin{cases} 2^{-1} \operatorname{tr} \left(u^{H_s} t^{C_s} \Phi_{k_\gamma}(i_\gamma) \Phi_{k_\delta}(i_\delta) \right) / \operatorname{tr}(u^{H_s} t^{C_s}), & 1 \leq \gamma < \delta \leq n, \\ 2^{-1} \operatorname{tr} \left(u^{H_s} t^{C_s} \Phi_{k_\gamma}(i_\gamma) \Phi_{k_{2n+1-\delta}}^*(i_{2n+1-\delta}) \right) / \operatorname{tr}(u^{H_s} t^{C_s}), & 1 \leq \gamma \leq n < \delta \leq 2n, \\ 2^{-1} \operatorname{tr} \left(u^{H_s} t^{C_s} \Phi_{k_{2n+1-\delta}}^*(i_{2n+1-\delta}) \Phi_{k_{2n+1-\gamma}}^*(i_{2n+1-\gamma}) \right) / \operatorname{tr}(u^{H_s} t^{C_s}), & n < \gamma < \delta \leq 2n. \end{cases} \quad (7.21)$$

Finally one checks that, after passing to the generating field $\phi(z)$ via coefficient extraction, one has

$$\begin{aligned} \operatorname{tr} \left(u^{H_s} t^{C_s} \Phi_{k_\gamma}(i_\gamma) \Phi_{k_\delta}(i_\delta) \right) &= \\ [z^{k_\gamma} w^{k_\delta}] \operatorname{tr} \left(u^{H_s} t^{C_s} \Gamma_+^s(\rho_1^+) \cdots \phi(z) \cdots \phi(w) \cdots \Gamma_-^s(\rho_N^-) \right), & 1 \leq \gamma < \delta \leq n, \\ \operatorname{tr} \left(u^{H_s} t^{C_s} \Phi_{k_\gamma}(i_\gamma) \Phi_{k_{2n+1-\delta}}^*(i_{2n+1-\delta}) \right) &= (-1)^{k_{2n+1-\delta}} \times \\ [z^{k_\gamma} w^{-k_{2n+1-\delta}}] \operatorname{tr} \left(u^{H_s} t^{C_s} \Gamma_+^s(\rho_1^+) \cdots \phi(z) \cdots \phi(w) \cdots \Gamma_-^s(\rho_N^-) \right), & 1 \leq \gamma \leq n < \delta \leq 2n, \\ \operatorname{tr} \left(u^{H_s} t^{C_s} \Phi_{k_{2n+1-\delta}}^*(i_{2n+1-\delta}) \Phi_{k_{2n+1-\gamma}}^*(i_{2n+1-\gamma}) \right) &= (-1)^{k_{2n+1-\gamma} + k_{2n+1-\delta}} \times \\ [z^{-k_{2n+1-\gamma}} w^{-k_{2n+1-\delta}}] \operatorname{tr} \left(u^{H_s} t^{C_s} \Gamma_+^s(\rho_1^+) \cdots \phi(z) \cdots \phi(w) \cdots \Gamma_+^s(\rho_N^+) \right), & n < \gamma < \delta \leq 2n. \end{aligned} \quad (7.22)$$

Note that the first equation above corresponds to $\phi\phi$ correlators, the second to $\phi\phi^*$ while the third to $\phi^*\phi^*$ correlators. This explains the presence of minus signs in the second and third equations as $\phi_k = (-1)^k \phi_k^*$. The kernel has been arranged in this way so as to make the $u = t = 0$ limit correspond to the kernels of Matsumoto and Vuletić [Mat05, Vul07]. Finally, each of the above quantities inside a coefficient extraction can be computed in exact form by Γ -elimination. At the end one is left with $\operatorname{tr}(u^{H_s} t^{C_s} \phi(z) \phi(w))$ which is given by Proposition C.3. Putting it all together, one arrives at the stated values for the entries of the correlation kernel. This concludes the proof of Theorem 7.3. \square

We can also derive a strict analogue of Proposition 3.2. Recall that $\langle O \rangle := \operatorname{tr}(u^{H_s} t^{C_s} O) / \operatorname{tr}(u^{H_s} t^{C_s})$. We

write the n -point correlation function as follows:

$$\begin{aligned}
\tilde{\varrho}^s(U) &= 2^{-n} \left[\prod_{\ell=1}^n z_\ell^{k_\ell} (-1)^{k_\ell} w_\ell^{-k_\ell} \right] \text{tr} \left(u^{H_s} t^{C_s} \Gamma_+^s(\rho_1^+) \Gamma_-^s(\rho_1^-) \cdots \Gamma_+^s(\rho_{i_1}^+) \phi(z_1) \phi(w_1) \Gamma_-^s(\rho_{i_1}^-) \cdots \right. \\
&\quad \left. \Gamma_+^s(\rho_{i_n}^+) \phi(z_n) \phi(w_n) \Gamma_-^s(\rho_{i_n}^-) \cdots \Gamma_+^s(\rho_N^+) \Gamma_-^s(\rho_N^-) \right) / \text{tr} \left(u^{H_s} t^{C_s} \right) \\
&= 2^{-n} \left[\prod_{\ell=1}^n z_\ell^{k_\ell} (-1)^{k_\ell} w_\ell^{-k_\ell} \right] \prod_{\ell=1}^n (F(i_\ell, z_\ell) F(i_\ell, w_\ell)) \cdot \langle \phi(z_1) \phi(w_1) \cdots \phi(z_n) \phi(w_n) \rangle
\end{aligned} \tag{7.23}$$

where the second equality follows from the first after cyclically moving the Γ^s operators out of the way and picking up the corresponding F factors and where brackets stand for coefficient extraction. Using the pfaffian evaluation from Proposition C.4 and replacing coefficient extraction with contour integration we arrive at the following result.

Proposition 7.6. *The n -point correlation has the form*

$$\begin{aligned}
\tilde{\varrho}^s(U) &= \frac{(1+t)^n (u; u)_{\infty}^{2n}}{2^n (-u; u)_{\infty}^n} \oint \cdots \oint \prod_{\ell=1}^n \left(\frac{dz_\ell dw_\ell F(i_\ell, z_\ell) F(i_\ell, w_\ell)}{(2\pi i)^2 (-1)^{k_\ell} z_\ell^{k_\ell+1} w_\ell^{-k_\ell+1}} \cdot \frac{\theta_u \left(\frac{z_\ell}{w_\ell} \right)}{\theta_u \left(-\frac{z_\ell}{w_\ell} \right)} \right) \times \\
&\quad \prod_{1 \leq \ell < m \leq n} \frac{\theta_u \left(\frac{z_\ell}{z_m} \right) \theta_u \left(\frac{w_\ell}{w_m} \right) \theta_u \left(\frac{z_\ell}{w_m} \right) \theta_u \left(\frac{w_\ell}{z_m} \right)}{\theta_u \left(-\frac{z_\ell}{z_m} \right) \theta_u \left(-\frac{w_\ell}{w_m} \right) \theta_u \left(-\frac{z_\ell}{w_m} \right) \theta_u \left(-\frac{w_\ell}{z_m} \right)}
\end{aligned} \tag{7.24}$$

where the conditions on the contours are the same as in Proposition 3.2.

8 Conclusion

We conclude with a few remarks about how this work fits in the bigger picture. Let us first mention that this paper is part of a project to analyze Schur processes with “nonstandard” boundary conditions. In another paper written in collaboration with Peter Nejjar and Mirjana Vuletić, we introduced the Schur process with free boundaries [BBNV18]. For the edge behavior of this process, we expect to encounter pfaffian variants of the finite-temperature Airy process [BBNV19]. Note that this is a priori unrelated to the pfaffian process considered in Section 7, which we discuss below.

As said in Remark 5.8, the finite-temperature Airy kernel should be the universal scaling limit of the periodic Schur process around a generic point at the edge of the limit shape. It is however known that, in the zero-temperature case, there exist other scaling limits (e.g., around nongeneric points such as cusps) and other specializations (e.g., those of z -measures) which lead to interesting objects like the Pearcey kernel [OR07], the cusp Airy kernel [OR06], the hypergeometric kernel [BO00], etc. It is natural to ask whether these kernels have finite-temperature variants.

We have defined in Section 6 the stationary cylindric Plancherel process through its finite-dimensional marginals, but we believe that it admits alternative nice combinatorial/probabilistic constructions via variants of the Robinson-Schensted correspondence (as used in [BO06b] for the zero-temperature limit), Hammersley’s process [AD95] or the polynuclear growth (PNG) model [PS02]. We might return to this subject in another publication.

On a related aspect, it would be nice to devise a sampling algorithm—for the stationary cylindric Plancherel process or for the periodic Schur process in general—as was done in [Bor11, BBB⁺18] for the zero-temperature case. In fact, we believe that the approach of [BBB⁺18] may be adapted straightforwardly (but we have not implemented it yet), building on ideas present in [Lan12, BCC17].

Our characterization in Section 7 of the correlation functions of the periodic strict Schur process raises the question whether they admit new (bulk or edge) scaling limits. For the edge, we expect to still get generically the finite-temperature Airy process (pfaffians reducing to determinants in the limit). Bulk limits might be more interesting and involve pfaffian variants of the kernels obtained in [Bor07]. See [Vul07] for the zero-temperature case.

As said in the introduction, M_α has first appeared in edge asymptotics of the Moshe–Neuberger–Shapiro (MNS) matrix model [Joh07], 1D free fermions in a harmonic trap [DLDMS15, DLDMS16, LW17] and nonintersecting Ornstein–Uhlenbeck processes on a cylinder [LDMS17]. All these models are different facets of the same continuous space-time finite-temperature free-fermionic model. Let us point out that the free fermions used here are slightly different from those considered in the above references. They correspond essentially to an effective theory when the number of fermions gets large.

Perhaps coincidentally, up to reparametrization, M_α has also appeared in non-free-fermionic models and at the level of the KPZ equation itself, as can be seen in the works of Amir–Corwin–Quastel and Sasamoto–Spohn (see [ACQ11, SS10], and references therein) where it appears in the scaling of the weakly asymmetric exclusion process, and in works of Calabrese–Le Doussal–Rosso, Borodin–Corwin–Ferrari and Imamura–Sasamoto [CDR10, BCF14, IS17] regarding fluctuations of the free energy of the O’Connell–Yor directed polymer. This coincidence was noticed in [DLDMS15] and further discussed in [DLDMS16, LW17, CG18] but, to the best of our knowledge, its fundamental origin is not understood.

Finally, we note that the approach of Baik, Deift and Johansson [BDJ99] for the edge behavior of Plancherel random partitions was based on Riemann–Hilbert techniques, while the approach that we follow here is much closer in spirit to that developed by Borodin, Okounkov and Olshanski [BOO00]. It seems an interesting technical challenge to adapt the Riemann–Hilbert approach to analyze the Fredholm determinant F_α , even though its asymptotics may be studied through other techniques—see [CG18] and references therein. We might return to this question in the future.

A Eta and theta functions

Fix q a complex parameter of modulus less than 1. The q -Pochhammer symbol of argument z and length $n \in \mathbb{N} \cup \{\infty\}$, the *multiplicative* theta function and the Dedekind eta function are respectively defined as

$$(z; q)_n := \prod_{k=0}^{n-1} (1 - zq^k), \quad \theta_q(z) := (z; q)_\infty (q/z; q)_\infty, \quad \eta(q) := q^{\frac{1}{24}} (q; q)_\infty. \quad (\text{A.1})$$

We also introduce the (“additive”) Jacobi theta functions θ_3 and θ_1 :

$$\theta_3(z; q) := \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}} z^n, \quad \theta_1(z; q) := \frac{1}{i} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{(n+1/2)^2}{2}} z^{n+1/2} = \frac{q^{\frac{1}{8}} z^{\frac{1}{2}}}{i} \theta_3(-q^{1/2} z; q). \quad (\text{A.2})$$

These conventions differ by the change $q \rightarrow q^{1/2}$ from those in [EMOT55, p.355] and that for θ_1 is also slightly different from [Bor07, p.411] (the i factor makes $\theta_1(z; q)$ real for q real and $|z| = 1$). The Jacobi triple product identity can then be written in the two equivalent forms

$$\theta_3(z; q) = (q; q)_\infty \theta_q(-q^{1/2} z), \quad \theta_1(z; q) = iq^{\frac{1}{8}} z^{-\frac{1}{2}} (q; q)_\infty \theta_q(z) \quad (\text{A.3})$$

where in the last equation we have used the relation $\theta_q(qz) = \theta_q(1/z) = -(1/z)\theta_q(z)$.

B Basics of charged free fermions

We recall the definitions necessary for Section 4—see also [BBNV18, Section 3] and references therein. The *fermionic Fock space*, denoted \mathcal{F} , may be seen as the infinite dimensional Hilbert space with orthonormal basis indexed by Maya diagrams, as defined in Section 2. We use the bra–ket notation throughout, hence denote respectively by $\langle \underline{n} |$ and $|\underline{n} \rangle$ the bra and ket associated with the Maya diagram \underline{n} . The charge C and energy H , as defined in (2.1), are naturally promoted as diagonal operators on \mathcal{F} . We also defined the shift operator R , that shifts a Maya diagram one unit to the right.

Creation/annihilation operators. For $k \in \mathbb{Z}'$, the fermionic *creation and annihilation operators* ψ_k and ψ_k^* are defined by

$$\psi_k |\underline{n} \rangle := \begin{cases} 0 & \text{if } n_k = 1, \\ (-1)^{\sum_{j>k} n_j} |\underline{n}^{(k)} \rangle, & \text{if } n_k = 0, \end{cases} \quad \psi_k^* |\underline{n} \rangle := \begin{cases} (-1)^{\sum_{j>k} n_j} |\underline{n}^{(k)} \rangle, & \text{if } n_k = 1, \\ 0 & \text{if } n_k = 0, \end{cases} \quad (\text{B.1})$$

where $\underline{n}^{(k)}$ denotes the Maya diagram obtained from \underline{n} by inverting the value found at position k . Because of the signs, we have the *canonical anticommutation relations*

$$\{\psi_k, \psi_\ell^*\} = \delta_{k,\ell}, \quad \{\psi_k, \psi_\ell\} = \{\psi_k^*, \psi_\ell^*\} = 0, \quad k, \ell \in \mathbb{Z}' \quad (\text{B.2})$$

where $\{a, b\} := ab + ba$ denotes the anticommutator. We also define the generating series

$$\psi(z) := \sum_{k \in \mathbb{Z}'} \psi_k z^k, \quad \psi^*(w) := \sum_{k \in \mathbb{Z}'} \psi_k^* w^{-k}. \quad (\text{B.3})$$

Note that the charge and energy operators can be rewritten as bilinears in the creation/annihilation operators, namely

$$C = \sum_{k \in \mathbb{Z}'} : \psi_k \psi_k^* :, \quad H = \sum_{k \in \mathbb{Z}'} k : \psi_k \psi_k^* :, \quad : \psi_k \psi_k^* : := \begin{cases} \psi_k \psi_k^*, & \text{if } k > 0, \\ -\psi_k^* \psi_k, & \text{if } k < 0. \end{cases} \quad (\text{B.4})$$

Here it is convenient to introduce the normal ordering $:\cdot:$ with respect to the vacuum so as to have well-defined single sums. Such bilinear quantities are essential to free field theory. A manifestation of this is Wick's lemma at finite temperature given below.

Γ -operators. For ρ a specialization with generating function $H(\rho, \cdot)$ as defined in (3.2), we consider the sequence $(p_n(\rho))_{n \geq 1}$ of *power sums specialized at ρ* given by

$$\sum_{n \geq 1} \frac{p_n(\rho)}{n} z^n := \ln H(\rho; z). \quad (\text{B.5})$$

The *half-vertex operators* $\Gamma_\pm(\rho)$ are then defined by

$$\Gamma_\pm(\rho) := \exp \left(\sum_{n \geq 1} \frac{p_n(\rho) \alpha_{\pm n}}{n} \right), \quad \alpha_n := \sum_{k \in \mathbb{Z}'} \psi_{k-n} \psi_k^*. \quad (\text{B.6})$$

If x is a variable, we denote by $\Gamma_\pm(x)$ (respectively $\Gamma'_\pm(x)$) the half-vertex operators for the specialization in the single variable x (respectively its dual \bar{x}), for which $p_n(x) = x^n$ (respectively $p_n(\bar{x}) = (-1)^{n-1} x^n$). It is well-known that

$$\langle \underline{n} | \Gamma_+(\rho) | \underline{m} \rangle = \langle \underline{m} | \Gamma_-(\rho) | \underline{n} \rangle = \begin{cases} s_{\mu/\nu}(\rho) & \text{if } c = c', \\ 0 & \text{otherwise,} \end{cases} \quad (\text{B.7})$$

where (ν, c) and (μ, c') are the charged partitions associated with the Maya diagrams \underline{n} and \underline{m} respectively, via the correspondence of Section 2. The Γ -operators commute with the charge (C) and shift (R) operators and satisfy the following quasi-commutation relations

$$\begin{aligned} \Gamma_\pm(\rho) \psi(z) &= H(\rho; z^{\pm 1}) \psi(z) \Gamma_\pm(\rho), & \Gamma_\pm(\rho) \psi^*(w) &= H(\rho; w^{\pm 1})^{-1} \psi^*(w) \Gamma_\pm(\rho), \\ \Gamma_+(\rho) \Gamma_-(\rho') &= H(\rho; \rho') \Gamma_-(\rho') \Gamma_+(\rho), & H(\rho; \rho') &:= \exp \left(\sum_{n \geq 1} \frac{p_n(\rho) p_n(\rho')}{n} \right), \\ \Gamma_\pm(\rho) u^H &= u^H \Gamma_\pm(u^{\pm 1} \rho), & H(u^{\pm 1} \rho; z) &:= H(\rho; u^{\pm 1} z). \end{aligned} \quad (\text{B.8})$$

We may reconstruct the creation/annihilation operators from the Γ -operators as

$$\psi(z) = z^{C-\frac{1}{2}} R \Gamma_-(z) \Gamma'_+(-z^{-1}), \quad \psi^*(w) = R^{-1} w^{-C+\frac{1}{2}} \Gamma'_-(-w) \Gamma_+(w^{-1}), \quad (\text{B.9})$$

a result often referred to as the *boson-fermion correspondence*.

Wick’s lemma. For $u \in (0, 1)$ and $t > 0$, we denote by $\langle \cdot \rangle_{u,t}$ the grand canonical expectation value defined by $\langle O \rangle_{u,t} := \text{tr}(t^C u^H O) / \text{tr}(t^C u^H)$ for any operator O acting on Fock space. Then we have the following “finite-temperature” version of Wick’s lemma.

Lemma B.1. Let Ψ be the vector space spanned by (possibly infinite) linear combinations of the ψ_k and ψ_k^* , $k \in \mathbb{Z}'$. For $\varphi_1, \dots, \varphi_{2n} \in \Psi$, we have

$$\langle \varphi_1 \cdots \varphi_{2n} \rangle_{u,t} = \text{pf } A \quad (\text{B.10})$$

where A is the $2n \times 2n$ antisymmetric matrix defined by $A_{i,j} = \langle \varphi_i \varphi_j \rangle_{u,t}$ for $i < j$.

In particular, if φ_{2i-1} (respectively φ_{2i}) is a linear combination of the ψ_k ’s only (respectively the ψ_k^* ’s only) for all $i = 1, \dots, n$, we have

$$\langle \varphi_1 \cdots \varphi_{2n} \rangle_{u,t} = \det_{1 \leq i, j \leq n} \langle \mathcal{T}(\varphi_{2i-1}, \varphi_{2j}) \rangle_{u,t} \quad (\text{B.11})$$

where \mathcal{T} is the “time-ordered product”: $\mathcal{T}(\varphi_{2i-1}, \varphi_{2j}) = \varphi_{2i-1} \varphi_{2j}$ for $i \leq j$ and $\mathcal{T}(\varphi_{2i-1}, \varphi_{2j}) = -\varphi_{2j} \varphi_{2i-1}$ for $i > j$.

The more usual Wick’s lemma at zero temperature corresponds to the case $u = 0$, for which $\langle \cdot \rangle_{u,t}$ reduces to the vacuum expectation value $\langle \emptyset | \cdot | \emptyset \rangle$. For convenience we provide a proof of Lemma B.1, which seems basically due to Gaudin [Gau60].

Proof. Let us introduce the density matrix $D := t^C u^H / \text{tr}(t^C u^H)$ so that $\langle O \rangle_{u,t} = \text{tr}(DO)$ for any operator O . By direct computation we have

$$D\psi_k = tu^k \psi_k D, \quad D\psi_k^* = (tu^k)^{-1} \psi_k^* D. \quad (\text{B.12})$$

By multilinearity, it suffices to prove (B.10) when each φ_i is equal to either ψ_{k_i} or $\psi_{k_i}^*$ for some k_i . The left hand side of (B.10) can be telescopically rewritten as:

$$\begin{aligned} \text{tr}(D\varphi_1 \cdots \varphi_{2n}) &= \sum_{i=2}^{2n} (-1)^i \text{tr}(D\varphi_2 \cdots \varphi_{i-1} \{\varphi_1, \varphi_i\} \varphi_{i+1} \cdots \varphi_{2n}) - \text{tr}(D\varphi_2 \varphi_3 \cdots \varphi_{2n} \varphi_1) \\ &= \sum_{i=2}^{2n} (-1)^i \{\varphi_1, \varphi_i\} \text{tr}(D\varphi_2 \cdots \varphi_{i-1} \varphi_{i+1} \cdots \varphi_{2n}) - c_1 \text{tr}(D\varphi_1 \varphi_2 \varphi_3 \cdots \varphi_{2n}) \end{aligned} \quad (\text{B.13})$$

where $c_1 = (tu^{k_1})^{-1}$ (respectively $= tu^{k_1}$) if $\varphi_1 = \psi_{k_1}$ (respectively $= \psi_{k_1}^*$)—to pass to the second line we use the fact that, by the canonical anticommutation relations, $\{\varphi_1, \varphi_i\}$ is a scalar, and we use cyclicity and (B.12) to rewrite the rightmost trace. In particular, for $n = 1$, (B.13) yields

$$\{\varphi_1, \varphi_2\} = (1 + c_1) \langle \varphi_1 \varphi_2 \rangle_{u,t} \quad (\text{B.14})$$

which of course still holds when replacing φ_2 by φ_i for any i . We deduce the recursion relation

$$\langle \varphi_1 \cdots \varphi_{2n} \rangle_{u,t} = \sum_{i=2}^{2n} (-1)^i \langle \varphi_1 \varphi_i \rangle_{u,t} \langle \varphi_2 \cdots \varphi_{i-1} \varphi_{i+1} \cdots \varphi_{2n} \rangle_{u,t}. \quad (\text{B.15})$$

The proof of (B.10) is then done by induction: it is a tautology for $n = 1$, and assuming that it holds at rank $n - 1$, we have $\langle \varphi_2 \cdots \varphi_{i-1} \varphi_{i+1} \cdots \varphi_{2n} \rangle_{u,t} = \text{pf } A^{(1,i)}$ where $A^{(1,i)}$ is the $(2n - 2) \times (2n - 2)$ submatrix of A with the first and i -th rows and columns removed. We recognize in the right-hand side of (B.15) the expansion of the pfaffian of A with respect to the first row/column, and conclude that (B.10) holds at rank n .

In the case where φ_{2i-1} (respectively φ_{2i}) is a linear combination of the ψ_k ’s only (respectively the ψ_k^* ’s only) for all $i = 1, \dots, n$, $A_{i,j}$ vanishes whenever i and j have the same parity, hence $\text{pf } A = \det_{1 \leq i, j \leq n} A_{2i-1, 2j}$ which is equivalent to the stated form (B.11). \square

Remark B.2. The main ingredients of the proof are the facts that (i) the anti-commutator of any two elements of Ψ is a scalar, and (ii) the density matrix D is the exponential of a bilinear combination of elements of Ψ . Here we assume that D has a diagonal form, which simplifies the proof but is not necessary—see e.g. [BR86, Ch. 4 and P4.1]. The canonical density matrix $\Pi_0 u^H$ does not satisfy the property (ii), which is why we need to pass to the grand canonical ensemble to have determinantal correlations.

C Basics of neutral free fermions

We recall the theory of neutral free fermions, useful for the study of the strict Schur process in [Vul07], following the conventions of [Whe11, Sections 1.4 and 3.8] (note they differ by a factor of 2 from the conventions of [Kac90, Exercise 14.13] and by a factor of 4 from those of [Mat05, Vul09]). We give a more detailed outline—neutral fermions being less often used in the literature—than the one in Appendix B on which we rely in our construction of neutral fermions.

Neutral Fock space and fermionic operators. We begin by constructing the *neutral fermionic Fock space* \mathcal{NF} in analogy with how we constructed \mathcal{F} . We start with the *neutral fermionic operators*, which we define in terms of the charged fermionic operators ψ_k, ψ_k^* as follows:

$$\phi_i = \psi_{i+\frac{1}{2}} + (-1)^i \psi_{-i+\frac{1}{2}}^*, \quad i \in \mathbb{Z} \quad (\text{C.1})$$

(notice how we have switched the indexing from \mathbb{Z}' to \mathbb{Z}). They form the *neutral fermionic field*

$$\phi(z) = \sum_{n \in \mathbb{Z}} \phi_n z^n \quad (\text{C.2})$$

and satisfy the following canonical anticommutation relations:

$$\{\phi_i, \phi_j\} = 2(-1)^j \delta_{i,-j} \quad (\text{C.3})$$

which in particular implies

$$\phi_i^2 = \delta_{i,0}. \quad (\text{C.4})$$

Let us further define

$$\phi_i^* = (-1)^i \phi_{-i}, \quad i > 0. \quad (\text{C.5})$$

Given an ordered subset $\{\mu_1 > \dots > \mu_r \geq 0\} \subset \mathbb{N}$, a basis for \mathcal{NF} (respectively its dual \mathcal{NF}^*) is given by states of the form

$$\phi_{\mu_1} \cdots \phi_{\mu_r} |\emptyset\rangle, \quad \langle \emptyset | \phi_{\mu_r}^* \cdots \phi_{\mu_1}^* \quad (\text{C.6})$$

which can be naturally identified with pairs $|\mu, c\rangle$ (respectively $\langle \mu, c|$) where μ is a *strict partition* (a partition with all its parts distinct) and $c = 0$ if r is even, and 1 otherwise. The number $c \in \mathbb{Z}_2$ can be seen as an even/odd grading $\mathcal{NF} = \mathcal{NF}_0 \oplus \mathcal{NF}_1$, based on whether a basis state comes from an even/odd number of fermions acting on the vacuum. For example, the vectors $\phi_3 \phi_2 \phi_0 |\emptyset\rangle =: |(3, 2), 1\rangle \in \mathcal{NF}_1, \phi_3 \phi_2 |\emptyset\rangle =: |(3, 2), 0\rangle \in \mathcal{NF}_0$ correspond to the subsets $\{3, 2, 0\}, \{3, 2\} \subset \mathbb{N}$ and both to the same strict partition $(3, 2)$. One can think of basis states in \mathcal{NF} as collections of finitely many particles sitting on the \mathbb{N} lattice.

Note ϕ_k^* for $k > 0$ is the adjoint of ϕ_k under the inner product $\langle \lambda, c | \mu, d \rangle = 2^{\ell(\lambda)} \delta_{\lambda, \mu} \delta_{c, d}$ for λ, μ strict partitions and $c, d \in \mathbb{Z}_2$. We have $\phi_{-n} |\emptyset\rangle = \phi_n^* |\emptyset\rangle = 0 = \langle \emptyset | \phi_n = \langle \emptyset | \phi_{-n}^* |\emptyset\rangle$ for all $n > 0$.

We denote $\langle \lambda^\vee, c | = 2^{-\ell(\lambda)} \langle \lambda, c |$ so that $\langle \lambda^\vee, c | \mu, d \rangle = \delta_{\lambda, \mu} \delta_{c, d}$.

The *energy operator* H_s and *grading operator* C_s (the analogue of the charge operator C) are given by (the subscript standing for *strict*)

$$u^{H_s} |\lambda, c\rangle := u^{|\lambda|} |\lambda, c\rangle, \quad t^{C_s} |\lambda, c\rangle := t^c |\lambda, c\rangle. \quad (\text{C.7})$$

For $k > 0$ we have

$$\frac{1}{2} \phi_k \phi_k^* |\lambda, c\rangle = \mathbb{1}_{\lambda \text{ has a part of size } k} |\lambda, c\rangle. \quad (\text{C.8})$$

We also have a (finite-temperature) Wick lemma for neutral fermions similar to Lemma B.1:

Lemma C.1. *Let Φ be the vector space spanned by (possibly infinite linear combinations of) the ϕ_k 's. For $\varphi_1, \dots, \varphi_{2n} \in \Phi$, we have*

$$\text{tr}(u^{H_s} t^{C_s} \varphi_1 \cdots \varphi_{2n}) = \text{pf } A \quad (\text{C.9})$$

where A is the antisymmetric matrix defined by $A_{i,j} = \text{tr}(u^{H_s} t^{C_s} \varphi_i \varphi_j) / \text{tr}(u^{H_s} t^{C_s})$ for $i < j$.

Remark C.2. We remark that above, the trace of an operator O over \mathcal{NF} is understood as

$$\mathrm{tr} O = \sum_{\lambda \in \mathcal{SP}, c \in \{0,1\}} \langle \lambda^\vee, c | O | \lambda, c \rangle \quad (\text{C.10})$$

and we have that $\mathrm{tr}(u^{H_s} t^{C_s}) = (1+t) \prod_{i \geq 1} (1+u^i)$.

Proof of Lemma C.1. The proof of Lemma B.1 applies mutatis-mutandis except we now use the canonical anticommutation relations for neutral fermions from equation (C.3). Note we still have the simple commutations

$$D\phi_k = u^k t \psi_k D, \quad D\phi_k^* = (u^k t)^{-1} \phi_k^* D \quad (\text{C.11})$$

for $k > 0$ where $D := u^{H_s} t^{C_s} / \mathrm{tr}(u^{H_s} t^{C_s})$. \square

Bosonic and half-vertex operators. We can define the so-called *bosonic operators* $\alpha_{\pm n}^s$ as follows. Fix n a positive *odd* integer. Then set

$$\alpha_n^s := \frac{1}{4} \sum_{k \in \mathbb{Z}} (-1)^k \phi_{-k-n} \phi_k. \quad (\text{C.12})$$

We have that α_{-n}^s is the adjoint of α_n^s , that $\alpha_n^s |\emptyset\rangle = 0$ for $n > 0$, and for $n, m \in 2\mathbb{Z} + 1$ that they satisfy the following commutation relations

$$[\alpha_n, \alpha_m] = \frac{n}{2} \delta_{n,-m}, \quad [\alpha_n, \phi(z)] = z^n \phi(z). \quad (\text{C.13})$$

For ρ a (strict) specialization of the algebra $\mathbb{S}\mathrm{Sym}$ we define the *half-vertex operators* $\Gamma_{\pm}^s(\rho)$ by

$$\Gamma_{\pm}^s(\rho) := \exp \left(\sum_{n \in 2\mathbb{N}+1} \frac{2p_n(\rho) \alpha_{\pm n}^s}{n} \right). \quad (\text{C.14})$$

When x is a variable, we denote by $\Gamma_{\pm}^s(x)$ the half-vertex operators for the specialization in the single variable x , for which $p_n(x) = x^n$. $\Gamma_{-}^s(\rho)$ is the adjoint of $\Gamma_{+}^s(\rho)$ for any real ρ , and

$$\Gamma_{+}^s(\rho) |\emptyset\rangle = |\emptyset\rangle, \quad \langle \emptyset | \Gamma_{-}^s(\rho) = \langle \emptyset|. \quad (\text{C.15})$$

Given two strict specializations ρ, ρ' , as $p_n(\rho \cup \rho') = p_n(\rho) + p_n(\rho')$, we have

$$\Gamma_{+}^s(\rho) \Gamma_{+}^s(\rho') = \Gamma_{+}^s(\rho \cup \rho') = \Gamma_{+}^s(\rho') \Gamma_{+}^s(\rho). \quad (\text{C.16})$$

The commutation relations (C.13) and the Cauchy identity (7.7) imply that

$$\Gamma_{+}^s(\rho) \Gamma_{-}^s(\rho') = Q(\rho; \rho') \Gamma_{-}^s(\rho') \Gamma_{+}^s(\rho) \quad (\text{C.17})$$

while

$$\Gamma_{\pm}^s(\rho) \phi(z) = Q(\rho; z^{\pm 1}) \phi(z) \Gamma_{\pm}^s(\rho). \quad (\text{C.18})$$

These latter relations always make sense at a formal level; at an analytic level they require that the parameter of $Q(\rho; \cdot)$ be within its disk of convergence. The crucial property of these half-vertex operators is that skew Schur's P and Q functions arise as their matrix elements (see [Mat05] or [Whe11, Sections 1.4.9 and 3.2.8]):

$$\langle \lambda^\vee, c | \Gamma_{+}^s(\rho) | \mu, d \rangle = Q_{\mu/\lambda}(\rho) \delta_{c,d}, \quad \langle \mu^\vee, c | \Gamma_{-}^s(\rho) | \lambda, d \rangle = P_{\mu/\lambda}(\rho) \delta_{c,d}. \quad (\text{C.19})$$

where λ, μ are strict partitions and $c, d \in \mathbb{Z}_2$. This results from (C.18), Wick's lemma C.1 at $u = 0$, and the Jacobi–Trudi-like identity for Schur's P and Q functions we took for their definition—see [Mat05], [Whe11, Sections 1.4.9 and 3.2.8] for elementary proofs.

The half-vertex operators Γ^s commute with the grading operator C_s and satisfy the following quasi-commutation with the energy operator H_s :

$$\Gamma_{\pm}^s(\rho) u^{H_s} = u^{H_s} \Gamma_{\pm}^s(u^{\pm 1} \rho). \quad (\text{C.20})$$

Finally, the boson–fermion correspondence in this setting reads [Kac90, Exercise 14.13]:

$$\phi(z) = R_s \Gamma_-^s(z) \Gamma_+^s(-z^{-1}), \quad (\text{C.21})$$

where R_s satisfies $R_s^2 = 1$, $R_s|\emptyset\rangle = \phi_0|\emptyset\rangle$ and $R_s\phi_i = \phi_i R_s$ for $i \neq 0$.

The following averages are useful for our purposes.

Proposition C.3. *We have*

$$\begin{aligned} \langle \emptyset | \phi(z) \phi(w) | \emptyset \rangle &= \frac{z-w}{z+w}, & |w| < |z|, \\ \text{tr}(u^{H_s} t^{C_s} \phi(z) \phi(w)) &= \frac{\theta_u\left(\frac{w}{z}\right)}{\theta_u\left(-\frac{w}{z}\right)} \cdot (1+t)(u; u)_\infty^2 (-u; u)_\infty^{-1}, & u^{1/2} < |w| < |z| < u^{-1/2}. \end{aligned} \quad (\text{C.22})$$

Proof. Notice that the first equality is the $u = t = 0$ case of the second. For the second we use the boson–fermion correspondence (C.21) and Γ -elimination. \square

Using neutral fermions, we can also derive the following pfaffian evaluation.

Proposition C.4. *We have*

$$\text{pf}_{1 \leq i < j \leq 2n} \frac{\theta_u\left(\frac{x_i}{x_j}\right)}{\theta_u\left(-\frac{x_i}{x_j}\right)} = \prod_{1 \leq i < j \leq 2n} \frac{\theta_u\left(\frac{x_i}{x_j}\right)}{\theta_u\left(-\frac{x_i}{x_j}\right)}. \quad (\text{C.23})$$

Proof. For an operator O acting on neutral Fock space \mathcal{NF} let us introduce, as usual, its expectation to be $\langle O \rangle = \text{tr}(u^{H_s} t^{C_s} O) / \text{tr}(u^{H_s} t^{C_s})$. By Wick’s lemma C.1 we have

$$\langle \phi(x_1) \phi(x_2) \cdots \phi(x_{2n-1}) \phi(x_{2n}) \rangle = \text{pf}_{1 \leq i < j \leq n} \langle \phi(x_i) \phi(x_j) \rangle. \quad (\text{C.24})$$

The entries of the pfaffian are given by Proposition C.3, and the left-hand side can also be written as a product of theta functions using the boson–fermion correspondence. The result follows upon cancelling all diagonal terms appearing on both sides. \square

Remark C.5. This pfaffian, while simple to write down, has appeared only recently in the (mathematical) literature in the works [Ros07, Remark 2.9] and [Ros08, Lemma 3.1] by Rosengren. Remarkably, using an algebraic geometric-type no-go argument, Rains (personal communication with the first author) has proved it was the most general pfaffian evaluation of the form $\text{pf}_{i < j} A_{i,j} = \prod_{i < j} A_{i,j}$. Taking $u \rightarrow 0$ leads to a famous pfaffian evaluation of Schur (see, e.g., [Mac95, Section III.8]) $\text{pf}_{i < j} \frac{x_i - x_j}{x_i + x_j} = \prod_{i < j} \frac{x_i - x_j}{x_i + x_j}$.

D On the discrete finite-temperature Bessel kernel

We provide some further analysis of the discrete finite-temperature Bessel kernel of Section 5. Our approach relies on the contour integral representation (5.2) where we set $u = e^{-r}$ and $t = 1$.

A convenient change of variables. Set

$$z = e^{\frac{\zeta}{2} + i\phi}, \quad w = e^{-\frac{\zeta}{2} + i\phi} \quad (\text{D.1})$$

so that $z/w = e^\zeta$ and $\sqrt{zw} = e^{i\phi}$, where ϕ may be integrated over the interval $[-\pi, \pi]$ and ζ over $\zeta_0 + i[-\pi, \pi]$ with $\zeta_0 \in (0, r)$. We get

$$K(a, b) = \frac{1}{4i\pi^2} \int_{-\pi}^{\pi} d\phi e^{i(b-a)\phi} \int_{\zeta_0 - i\pi}^{\zeta_0 + i\pi} d\zeta \frac{\kappa(\zeta)}{e^{\frac{a+b}{2}\zeta}} e^{4L \cos \phi \sinh \frac{\zeta}{2}}. \quad (\text{D.2})$$

where $\kappa(\zeta) := \sum_{m \in \mathbb{Z}'} \frac{e^{\zeta m}}{1 + e^{r m}}$. Note that the modulus of the exponential factor $e^{4L \cos \phi \sinh \frac{\zeta}{2}}$ is equal to $e^{4L \cos \phi \sinh \frac{\zeta_0}{2} \cos \frac{\Im(\zeta)}{2}}$ which is maximal at $(\phi, \Im(\zeta)) = (0, 0)$, a property which is essential for doing saddle-point approximations.

Remark D.1. Applying the Poisson summation formula as in Lemma 5.5, we get

$$\kappa(\zeta) = \sum_{\ell \in \mathbb{Z}} (-1)^\ell \frac{\pi}{r \sin \pi \frac{\zeta - 2i\pi\ell}{r}}. \quad (\text{D.3})$$

from which it is manifest that κ is a meromorphic function of ζ in \mathbb{C} , having a simple pole at each point of the lattice $r\mathbb{Z} + 2i\pi\mathbb{Z}$, with residue $(-1)^{n+\ell}$ at $\zeta = nr + 2i\pi\ell$. If we assume that $\arg(\zeta) \in [-\pi, \pi]$, then keeping only the term $\ell = 0$ in (D.3) yields an approximation of $\kappa(\zeta)$ with uniform error $O(e^{-\pi^2/r}/r)$.

Remark D.2. By integrating over ϕ in (D.2), we get

$$K(a, b) = \frac{1}{2i\pi} \int_{\zeta_0 - i\pi}^{\zeta_0 + i\pi} d\zeta \frac{\kappa(\zeta)}{e^{\frac{a+b}{2}\zeta}} I_{b-a} \left(4L \sinh \frac{\zeta}{2} \right) \quad (\text{D.4})$$

where I_D is the modified Bessel function of the first kind and order D .

The bulk limit revisited. We now give a short rederivation of Borodin's bulk limiting kernel for the cylindric Plancherel measure.

Proposition D.3 (see [Bor07, Example 3.4]). *Consider the limit $u = e^{-r} \rightarrow 1^-$ with $Lr \rightarrow \gamma$ fixed, and assume that $ra, rb \rightarrow \tau \in \mathbb{R}$ with $D := b - a$ fixed. Then*

$$K(a, b) \rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{iD\phi} d\phi}{1 + e^{\tau - 2\gamma \cos \phi}} \quad (\text{D.5})$$

Proof. We use the integral representation (D.2) for $K(a, b)$, and perform the change of variable $\zeta = \frac{\tau}{2} + i r \nu$ with $\nu \in [-\pi/r, \pi/r]$. Then, using Lemma 5.5 and simple Taylor expansions, we get

$$\begin{aligned} K(a, b) &= \int_{-\pi}^{\pi} \frac{e^{iD\phi} d\phi}{4\pi^2} \int_{-\pi/r}^{\pi/r} d\nu e^{(\gamma \cos \phi - \tau/2)(1+2i\nu) + o(1)} \left(\frac{\pi}{\cosh \pi\nu} + O(e^{-\pi^2/r}) \right) \\ &\rightarrow \int_{-\pi}^{\pi} \frac{e^{iD\phi} d\phi}{4\pi} \int_{-\infty}^{\infty} d\nu \frac{e^{(\gamma \cos \phi - \tau/2)(1+2i\nu)}}{\cosh \pi\nu}, \end{aligned} \quad (\text{D.6})$$

where we use standard tails pruning/completion arguments to justify passing to the second line. By integrating over ν we obtain wanted expression (D.5). \square

We believe that this proof may be adapted—keeping the same change of variables $(z, w) \rightarrow (\phi, \nu)$ —to give a streamlined derivation of [Bor07, Theorem 3.1] regarding the bulk limiting kernels for more general periodic Schur processes.

Remark D.4. If we let $\gamma \rightarrow \infty$ with $\tau/\gamma = x$ fixed in the right-hand side of (D.5), we recover the discrete sine kernel as for the (noncylindric) poissonized Plancherel measure—see the remark at the end of [Bor07, Example 3.4]. In all rigor, Proposition D.3 does not apply to the situation where we let $u \rightarrow 1^-$ and $\gamma \rightarrow \infty$ jointly as in Theorem 1.1. But it is not difficult to adapt to the finite-temperature setting the proof given in [Oko02, Section 3.2] for the convergence of the discrete Bessel kernel to the sine kernel, the key fact being that $\kappa(z, w)$ has a residue 1 at $z = w$ —as mentioned in Section 4 this is a general consequence of the fermionic canonical anticommutations.

Proof of Proposition 5.2. The difficulty to circumvent is that, in the contour integral representation (5.2) of $K(a, b)$, the exponential factor $e^{L(z-z^{-1}-w+w^{-1})}$ is now subdominant. Instead, we may see that the asymptotics is governed by the pole of κ at $z/w = u^{-1}$. The strategy is to deform the integration contours into two circles such that $|z/w|$ is larger than u^{-1} (but smaller than u^{-2}). As we cross the pole, we pick a contribution from the residue which turns out to be dominant.

More precisely, let us first establish the pointwise convergence (5.13). We use the reparametrization (D.2) and change the integration contour of ζ into the segment $\frac{3r}{2} + i[-\pi, \pi]$ plus a small negatively oriented circle around the pole at r . Noting that the residue of κ at this pole is -1 and using the integral representation

$$I_D(G) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{iD\phi + G \cos \phi} \quad (\text{D.7})$$

for the modified Bessel function of the first kind and integer order D , we find that

$$r^{-1}K(a, b) = \underbrace{\frac{I_{b-a}(G)}{r e^{\frac{a+b}{2}r}}}_{=:T_1} + \underbrace{\frac{1}{4i\pi^2 r} \int_{-\pi}^{\pi} d\phi e^{i(b-a)\phi} \left(\int_{\frac{3r}{2}-i\pi}^{\frac{3r}{2}+i\pi} d\zeta \frac{\kappa(\zeta)}{e^{\frac{a+b}{2}\zeta}} e^{4L \cos \phi \sinh \frac{\zeta}{2}} \right)}_{=:T_2} \quad (\text{D.8})$$

where $G := 4L \sinh \frac{r}{2}$. We claim that, in the edge high temperature regime (5.12), the first term T_1 in (D.8) tends to $e^{-x} \delta_{x,y}$ (the Kronecker delta) while the second term T_2 tends to 0. Note that $G = 2Lr + o(1) \sim 2\gamma$ may or may not remain finite, and the forthcoming discussion is valid in both cases.

The fact that $T_1 \rightarrow e^{-x}$ for $x = y$ (i.e. $b - a = 0$) is a straightforward consequence of the scaling (5.12). Indeed we have

$$e^{\frac{a+b}{2}r} \sim \frac{I_0(2Lr)}{r} e^{\frac{x+y}{2}} \sim \frac{I_0(G)}{r} e^{\frac{x+y}{2}} \quad (\text{D.9})$$

where it is safe to replace the argument of I_0 even if $G \rightarrow \infty$ since $I_0(G) \sim \frac{e^G}{\sqrt{2\pi G}}$.

For $x \neq y$, the order $D := b - a$ of the Bessel function now tends to $\pm\infty$ as $(y - x)r^{-1}$, and showing that $T_1 \rightarrow 0$ amounts to showing that

$$I_D(G) = o(I_0(G)) = o\left(\frac{e^G}{\sqrt{G}}\right) \quad (\text{D.10})$$

which is done by the saddle-point method: in the integral representation (D.7), let us move the integration path to the segment $[-\pi, \pi] + i \operatorname{arsinh}(\frac{D}{G})$. On this segment we may bound the integrand as

$$|e^{iD\phi} e^{G \cos \phi}| = e^{-D \operatorname{arsinh}(\frac{D}{G}) + G \sqrt{1 + \frac{D^2}{G^2}} \cos \Re(\phi)} \leq e^{-Gf(\frac{D}{G}) - G \frac{\Re(\phi)^2}{8}} \quad (\text{D.11})$$

where $f(x) := x \operatorname{arsinh}(x) - \sqrt{1 + x^2}$. By integrating over ϕ we get $I_D(G) = O(G^{-1/2} \exp(-Gf(\frac{D}{G})))$. We now observe that $f'(x) = \operatorname{arsinh}(x)$ so f is convex and minimal at $x = 0$, with $f(x) \geq -1 + \frac{x^2}{2}$. Hence $e^{-Gf(\frac{D}{G})} = O(e^{G - \frac{D^2}{2G}}) = O\left(e^{G - \frac{(y-x)^2}{4Lr^3}}\right) = o(e^G)$, which establishes (D.10) hence that $T_1 \rightarrow 0$ for $x \neq y$.

We now show that $T_2 \rightarrow 0$: using (D.3) and proceeding as for Lemma 5.5, we see that $|\kappa(\zeta)|$ may be bounded over the ζ -integration contour by $\frac{\pi}{r \cosh \frac{\pi \Im(\zeta)}{r}}$ up to some uniform constant. We then bound the exponential factors as

$$\left| \frac{e^{4L \cos \phi \sinh \frac{\zeta}{2}}}{e^{\frac{a+b}{2}\zeta}} \right| = \frac{e^{4L \sinh \frac{3r}{2} \cos \phi \cos \Im(\zeta)}}{e^{\frac{3(a+b)r}{4}}} = O\left(\frac{e^{\frac{3G}{2}(1 - \frac{\phi^2}{8})}}{e^{\frac{3(a+b)r}{4}}}\right) = O\left((r\sqrt{G})^{3/2} e^{-\frac{3G\phi^2}{16}}\right) \quad (\text{D.12})$$

where we use again (D.9) and the fact that $I_0(G) = \Theta\left(\frac{e^G}{\sqrt{G}}\right)$. Upon integrating over ϕ and ζ , and multiplying by the prefactor r^{-1} , we deduce that $T_2 = O((r\sqrt{G})^{1/2}) \rightarrow 0$, which completes the proof of the pointwise convergence (5.13).

The proof of the trace convergence (5.14) is entirely similar: starting from (D.8), we find that

$$\sum_{i \geq m} K(i, i) = \frac{I_0(G)}{(1 - e^{-r})e^{mr}} + \frac{1}{4i\pi^2} \int_{-\pi}^{\pi} d\phi \left(\int_{\frac{3r}{2}-i\pi}^{\frac{3r}{2}+i\pi} \frac{d\zeta}{1 - e^{-\zeta}} \frac{\kappa(\zeta)}{e^{m\zeta}} e^{4L \cos \phi \sinh \frac{\zeta}{2}} \right) \quad (\text{D.13})$$

and all the above analysis of (D.8) may be adapted straightforwardly. \square

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