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Integrability: old and new tools

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Abstract

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1 Introduction

Over the last few decades, integrability has become a tool of choice for the theoretical physicists, finding many application in condensed matter, cold atoms, stochastic processes, architecture, etc. Although it was believed to apply only to one-dimensional quantum systems (two space-time dimensions), it the last fifteen years it was also shown to apply to higher dimensional systems - provided we can reduce the problem to a two-dimensional one[1].

In the last fifteen years, surprisingly, the methods of integrability have found applications in the description of supersymmetric gauge theories in higher dimension, as well as for superstring theories in curved background. The most fascinating application, which will be the ultimate focus of this course, is to study the AdS/CFT correspondence. The first and most studied example is the exact equivalence of the (planar limit of the) maximally supersymmetric gauge theory in four dimensions ($\mathcal{N} = 4$ SYM) and (free) superstring theory in the curved background $AdS_5 \times S^5$. The two theories depend on a parameter, the 't Hooft coupling constant $\lambda = N g_{\rm YM}^2$ on the gauge side, and the string tension $\sqrt{\lambda}$ on the string side. The correspondence is a weak-strong correspondence, meaning that the one weakly coupled theory is mapped to a strongly coupled one. The obvious interest of this kind of equivalence stems in the quest of a descriprion of the strongly coupled gauge theories, as QCD, in terms of strings. This implies a non-trivial redefinition of degrees of freedom of the two theories, that we hope we can understand in terms of collective degrees of freedom usually handled by the integrable methods.

The discovery of integrabiltiy in the context of the planar $\mathcal{N} = 4$ SYM is based on the mapping of the dilatation operator of the theory, which generates scale transformations, to the action of the Hamiltonian of an integrable spin, albeit a rather complicated one. On the side of string theory, the description is possible in terms of a non-linear sigma model. According to the AdS/CFT correspondence, these two descriptions should exactly coincide, state by state. This statement goes beyond the one usually employed in condensed matter physics, that the low energy behaviour of spin chains is describable in terms of sigma models.

The first part of the lectures is devoted to the introduction of some prototypical examples of integrable systems which are used in the applications of integrability to the AdS/CFT correspondence. The first lecture is about the (coordinate and algebraic) Bethe Ansatz solution for the Heisenberg model and the second lecture is about the integrability of the $su(2)$ principal chiral field model, which is one of the simplest example of two-dimensional integrable field theory with $su(2)$ symmetry. The classical integrability, Lax connection and finite gap method will be explained. The second lecture will end with a comparison between spin chains and sigma models. The third and last lecture of the first part will introduce the integrable model associated with the $\mathcal{N} = 4$ SYM theory. Part of the lecture will be devoted to explaining the oscillator representation for the symmetry group of the SYM theory, $PSU(2, 2|4)$.

[1]Some three-dimensional integrable models were also studied.

2 Bethe ansatz solutions for the Heisenberg model

The Heisenberg model was proposed in the twenties to describe magnetism of metal, modelling the magnetic interaction of electrons in solid. The one-dimensional system, or spin chain, describes the nearest-neighbor interaction of magnetic momenta of electrons situated on a periodic lattice with L sites

$$H_2 = J \sum_{l=1}^{L} \sigma_l^a \sigma_{l+1}^a \,, \tag{2.1}$$

where σ_l^a are copies the Pauli matrices associated to the site $l = 1, \ldots, L$ with the commutation relations

$$[\sigma_l^a, \sigma_k^b] = i\epsilon^{abc}\sigma_l^c \; \delta_{lk} \,. \tag{2.2}$$

At each site the spin can take two values, $|\uparrow\rangle_l$ and $|\downarrow\rangle_l$. The space of states has dimension 2^L. If $J < 0$ the ground state is ferromagnetic (F), *i.e.* the spins tend to align in the same direction in the spin space, while for $J > 0$ it is antiferromagnetic (AF), *i.e.* the spins try to make singlets on adjacent sites $|\uparrow\rangle_l \otimes |\downarrow\rangle_{l+1} - |\downarrow\rangle_l \otimes |\uparrow\rangle_{l+1}$. Since this cannot be realised at every pair of adjacent sites, the resulting state is highly intricate.

Exercise: Show that H_2 commute with the generators of the total spin, $S^a = \frac{1}{2}\sum_{l=1}^{L} \sigma_l$. Show that the ferromagnetic state $|\Omega\rangle = |\uparrow \ldots \uparrow\rangle$ is degenerate and compute its degeneracy.

We will be ultimately interested in states not far from the ferromagnetic state so we choose the coupling constant to get, up to a constant

$$H_2 = \sum_{l=1}^{L} (1 - P_{l,l+1}) \,, \tag{2.3}$$

where $P_{l,l+1}$ permutes the spins at the sites l and $l+1$. In terms of the Pauli matrices we have $P_{l,l+1} = \frac{1}{2}\left(\sigma_l^a \sigma_{l+1}^a + 1\right)$.

The solution for the Hamiltonian (2.3) can be found by the so-called coordinate Bethe ansatz, which was originally employed by Bethe [**?**]. One starts with the reference state $|\Omega\rangle$, where all the spins are in the "up"state $|\uparrow\rangle$,

$$|\Omega\rangle = |\uparrow\uparrow\uparrow\uparrow \ldots \uparrow\rangle \tag{2.4}$$

Starting with this state, one can generate the whole Hilbert space by reversing an arbitrary number of spins. A reversed spin

$$|\downarrow\rangle = \sigma^- |\uparrow\rangle$$

will be called a *magnon*. Since the Hamiltonian (2.3) conserves the number of reversed spins, it can be diagonalized on a space with fixed number of magnons. Consider first the one-magnon states

$$\Psi_p = \sum_{k=1}^{L} e^{ipk} \sigma_k^- |\Omega\rangle \tag{2.5}$$

which are eigenstates of (2.3). The states are well defined if they are periodic, *i.e.*they are invariant under $k \to k + L$. This condition is satisfied if $e^{ipL} = 1$, so $p = 2\pi n/L$, $n = 0, \ldots, L-1$. The associated eigenvalue is

$$E(p) = (2 - e^{ip} - e^{-ip}) = 4 \sin^2 p/2 \ . \tag{2.6}$$

When p is small, the one-magnon wave function is slowly varying in space, while for $p = 0$ it is constant. The state with $p = 0$ is degenerate with the vacuum, and the magnons can be considered as the Goldstone bosons of the broken $su(2)$ symmetry, interpolating between different ground states.

2.1 Coordinate Bethe Ansatz

For a state with two magnons, the eigenfunctions are found with the following *ansatz*

$$\begin{aligned} \Psi_{p_1,p_2} &= \sum_{1\le k_1<k_2\le L} \left[A(p_1,p_2)e^{i(p_1k_1+p_2k_2)} + A(p_2,p_1)e^{i(p_2k_1+p_1k_2)}\right] \sigma^-_{k_1}\sigma^-_{k_2}|\Omega\rangle \\ &= \sum_{1\le k_2<k_1\le L} \left[A(p_1,p_2)e^{i(p_1k_2+p_2k_1)} + A(p_2,p_1)e^{i(p_2k_2+p_1k_1)}\right] \sigma^-_{k_1}\sigma^-_{k_2}|\Omega\rangle \ . \end{aligned} \tag{2.7}$$

This ansatz is clearly valid when $k_1 \ll k_2$, since the action of the hamiltonian (2.3) couples only two nearby sites. The condition that it is valid everywhere determines the ratio of the two coefficients $A(p_2, p_1)$ and $A(p_1, p_2)$ which defines the scattering matrix of two magnons

$$S(p_2,p_1) \equiv \frac{A(p_2,p_1)}{A(p_1,p_2)} = -\frac{e^{ip_1+ip_2} - 2e^{ip_2} + 1}{e^{ip_1+ip_2} - 2e^{ip_1} + 1} \tag{2.8}$$

while the periodicity under $k_{1,2} \to k_{1,2} + L$ imposes that

$$e^{ip_2L} = S(p_2,p_1) \ , \quad e^{ip_1L} = S(p_1,p_2) \ . \tag{2.9}$$

Exercise: Derive the scattering matrix from the condition that Ψ_{p_1,p_2} is an eigenfunction of H_2. Hint: check the terms with $k_1 = k_2 \pm 1$ in (2.7).

The equations (2.9) for the two-magnon wave function are valid for any spin model with local interaction, for some expression of $S(p_1, p_2)$. What is remarkable about the model (2.3) is that an ansatz of the type (2.7) is valid for any number M of magnons,

$$\Psi_{p_1,\ldots,p_M} = \sum_{1\le k_1<\ldots<k_M\le L} \sum_P \prod_{i<j} A(p_{P(i)},p_{P(j)}) e^{i(k_1p_{P(1)}+\ldots k_Mp_{P(M)})}|k_1,\ldots,k_M\rangle \tag{2.10}$$

The wave function in a different chamber $k_{P^{-1}(1)} < k_{P^{-1}(2)} < \ldots < k_{P^{-1}(M)}$ can be obtained from the one in $k_1 < k_2 < \ldots < k_M$ just by the permutation of the momenta $p_1, p_2, \ldots p_M \to p_{P(1)}, p_{P^{-1}(2)}, \ldots p_{P^{-1}(M)}$. Since any permutation P can be written as a product of transpositions, the elementary object is the two magnon scattering matrix (2.8). The magnon momenta are determined by the equations

$$e^{ip_n L} = \prod_{m \neq n=1}^{M} S(p_n, p_m) \ , \quad n = 1, \dots M \tag{2.11}$$

The property of factorized scattering is related, at least for the systems with a finite number of degrees of freedom, to the existence of a number of conserved quantities equal to the number of degrees of freedom, that is, to integrability.

2.2 Algebraic Bethe Ansatz

A different way to obtain the Bethe ansatz equations (2.11), as well as the conserved quantities, is via the algebraic Bethe ansatz. For a pedagogical review of different aspects of the algebraic Bethe ansatz, see [?]. One starts with a matrix $R(u)$ acting on $V \otimes V$, with $V = \mathbb{C}^2$ which satisfies the Yang-Baxter equation

$$R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v) \tag{2.12}$$

where the subscript indicates the space where the matrix act. In the $su(2)$ case, such a solution is provided by

$$R_{12}(u) = u + iP_{12} = u + \frac{i}{2}(\sigma_1^a \sigma_2^a + 1) \ , \tag{2.13}$$

where P_{12} is the permutation operator. The Yang-Baxter equation can be proven using the properties of the permutations, $P_{ij}^2 = 1$ and $P_{12}P_{13}P_{23} = P_{23}P_{13}P_{12}$. For $u = \pm i$ the matrix $R(u)$ is proportional to a projector. Starting with the R matrix, one can construct the monodromy matrix $T_0(u)$ by

$$T_0(u) = \mathrm{L}_1(u)\mathrm{L}_2(u)\dots\mathrm{L}_L(u) \ , \quad \text{with} \quad \mathrm{L}_n(u) \equiv R_{0n}(u - \theta_n - i/2) \tag{2.14}$$

which satisfies the Yang-Baxter equation in the form

$$R_{00'}(u-v)\, T_0(u)\, T_{0'}(v) = T_{0'}(v)\, T_0(u)\, R_{00'}(u-v). \tag{2.15}$$

In the equation (2.14) we have introduces an independent shift of the rapidities at each site, θ_n. The Lax matrix for a chain with spin s is obtained from the one in (2.14) by replacing the Pauli matrices with the spin generators in the spin s representation, $\sigma_n^a \to 2S_n^a$. $T_0(u)$ is a 2×2 dimensional matrix in the auxiliary space 0,

$$T_0(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} . \tag{2.16}$$

The Yang-Baxter equation (2.15) can be translated as commutation rules of the components, *e.g.*

$$\begin{aligned} A(u)B(v) &= \frac{u-v-i}{u-v} B(v)A(u) + \frac{i}{u-v} B(u)A(v) \ , \\ D(u)B(v) &= \frac{u-v+i}{u-v} B(v)D(u) - \frac{i}{u-v} B(u)D(v) \ . \end{aligned} \tag{2.17}$$

A consequence of the equation (2.15) is that the trace of the monodromy matrix, called the transfer matrix, commutes with itself for different values of the spectral parameter u

$$[t(u), t(v)] = 0 \ , \qquad t(u) \equiv \mathrm{Tr}_{\,0}\, T_0(u) \ , \tag{2.18}$$

such that the transfer matrix $t(u)$ generates the conserved charges. The commutation of the traces can be proven as follows

$$\begin{aligned} t(u)t(v) &= \mathrm{Tr}_{\,0}\mathrm{Tr}_{\,0'}\ R^{-1}_{00'}(u-v)\ R_{00'}(u-v)\ T_0(u)\ T_{0'}(v) \\ &= \mathrm{Tr}_{\,0}\mathrm{Tr}_{\,0'}\ T_{0'}(v)\ T_0(u)\ R_{00'}(u-v)\ R^{-1}_{00'}(u-v) = t(v)t(u) \ , \end{aligned} \tag{2.19}$$

where we have used the properties of the trace and the Yang-Baxter equation (2.15).

A special point is $u = i/2$, where the the conserved charges generated by the logarithm of transfer matrix are local. We consider for simplicity the homogeneous case when $\theta_1, \ldots, \theta_L = 0$. The first integral of motion is the shift operator,

$$U \equiv e^{iP} = i^{-L} t(0) = \mathrm{Tr}_{\,0} P_{01} \ldots P_{0L} = P_{LL-1} \ldots P_{21} \ . \tag{2.20}$$

The hamiltonian is the next conserved quantity

$$(L - H_2) = i \frac{d}{du} \ln t(u)|_{u=i/2} = \sum_{n=1}^{L} U^{-1} \mathrm{Tr}_{\,0} P_{01} \ldots \check{P}_{0n} \ldots P_{0L} = \sum_{n=1}^{L} P_{n,n+1} \ , \tag{2.21}$$

where we have used the notation $\check{P}_{0n}$ to denote the absence of that particular factor from the product. The next conserved quantities have an increasing range, *e.g.*

$$H_3 = \sum_{n=1}^{L} [P_{n-1,n} P_{n,n+1}] \ . \tag{2.22}$$

Another special point of the monodromy matrix is $u = \infty$. Let us expand $T_0(u)$ around this point,

$$\begin{aligned} T_0(u) &\equiv u^L \left(1 + \sum_{k=0}^{\infty} \frac{i^{k+1}}{u^{k+1}} J^a_k \sigma^a_0 \right) \\ &= u^L \left(1 + \frac{i}{u} \sum_{n=1}^{L} P_{0n} - \frac{1}{u^2} \sum_{n<m} P_{0n} P_{0m} + \ldots \right) \ . \end{aligned} \tag{2.23}$$

This shows that the large-rapidity limit of the matrix elements of the monodromy matrix become the global spin operators J^a_0 , *e.g.*

$$\frac{B(u)}{iu^{L-1}} \sim J^-_0 = \sum_n \sigma^-_n \ , \qquad \frac{C(u)}{iu^{L-1}} \sim J^+_0 = \sum_n \sigma^+_n \ . \tag{2.24}$$

The Yang-Baxter relations (2.15) can be translated in term of the commutation relations of the modes J^a_k defined in (2.23). The generators J^a_k define an infinite-dimensional algebra named the Yangian. Only the first two modes $k = 0, 1$ are independent, the remaining ones can be obtained from them by using the commutation relations. The Yangian generators J_1^a commute with the conserved quantities up to boundary terms. This means that the Yanginan *is not* a symmetry for the Heisenberg Hamiltonian for finite size, but it become close to an exact symmetry when L goes to infinity. An example of spin chain where the Yangian is an exact symmetry for any size L is the Haldane-Shastry Hamiltonian. The consequence of the existence of this symmetry is a huge degeneracy of the spectrum.

Exercise: Derive the expression of the generators J_1^a in terms of σ_n^a using the definition in (2.23) and find the commutation relations with the hamiltonian (difficult).

The common eigenvectors of the conserved quantities can be constructed from the reference vector $|\Omega\rangle$ as follows. The conserved quantities are generated by

$$t(u) = A(u) + D(u) \tag{2.25}$$

while the eigenvectors are build as

$$|u_1, u_2, \dots u_M\rangle = B(u_1)B(u_2)\dots B(u_M)|\Omega\rangle \ . \tag{2.26}$$

Let us start with an arbitrary Bethe state written as above, and and act with the transfer matrix (2.25) on it. We will use that

$$\begin{aligned} A(u)|\Omega\rangle &= a(u)|\Omega\rangle \ , \qquad D(u)|\Omega\rangle = d(u)|\Omega\rangle \ , \\ a(u) &\equiv \prod_{n=1}^{L}(u+\theta_n+i/2) \ , \qquad d(u) \equiv \prod_{n=1}^{L}(u+\theta_n-i/2) \end{aligned} \tag{2.27}$$

and the commutation relations (2.17) to push the operators $A(u)$ and $D(u)$ to the right. It is instructive to note that in (2.17) the first term in the r.h.s. just exchanges the order of the operators (these are "wanted" terms), while the second term also exchanges the value of the rapidities ("unwanted" terms). From the "wanted" terms we obtain

$$t(u)|u_1, u_2, \dots u_M\rangle = \left(a(u)\prod_{m=1}^{M}\frac{u-u_m-i}{u-u_m} + d(u)\prod_{m=1}^{M}\frac{u-u_m+i}{u-u_m} \right)|u_1, u_2, \dots u_M\rangle \ . \tag{2.28}$$

Let us now analyse the "unwanted" terms and try to set them to zero. The kind of unwanted term we obtain are the following (we use the index of the rapidity as an index for the operator)

$$(A+D)B_1B_2\dots B_M \to \frac{i}{u-u_1}BA_1B_2\dots B_M - \frac{i}{u-u_1}BD_1B_2\dots B_M \tag{2.29}$$

We are looking for the condition which will set to zero the coefficient of $BB_2\dots B_M$ [2]. Further commutation of A_1 and D_1 with the B_m operators cannot produce an object of

[2] This argument was suggested to us by V. Pasquier.

this type through extra unwanted terms, so the other commutations should be of wanted type. We deduce that we have to require

$$a(u_1)\frac{i}{u-u_1}\prod_{m\neq 1}\frac{u_1-u_m-i}{u_1-u_m}-d(u_1)\frac{i}{u-u_1}\prod_{m\neq 1}\frac{u_1-u_m+i}{u_1-u_m}=0\;, \tag{2.30}$$

condition which is, surprisigly, independent of u. Given that the operators B_n commute with each other, the same argument can be repeated similarly with the other unwanted terms. We conclude that the vectors (2.26) are eigenvectors of the operators (2.25) if the rapidities $u_1, u_2, \ldots u_M$ satisfy the equations

$$\frac{a(u_k)}{d(u_k)}=\prod_{m\neq k=1}^{M}\frac{u_k-u_m+i}{u_k-u_m-i}\;,\qquad k=1,\ldots,M\;, \tag{2.31}$$

which are identical with the set of equations (2.11) upon the identification (with the inhomogeneities θ set to 0)

$$u=\frac{1}{2}\cot\frac{p}{2}\;. \tag{2.32}$$

The energy is given, in these variables, by

$$E_2=\sum_{k=1}^{M}\frac{1}{u_k^2+1/4}\;. \tag{2.33}$$

It is clear that the magnons with infinite rapidity, that is, zero momentum, have also zero energy. This is in agreement with the finding in (2.24) that the magnons at infinite rapidity are equal to a generator of the global symmetry. Let us emphasise that the equations (2.31) allow trivially that some of the rapidites are situated at infinity. The low-energy excitations have

$$u\simeq\frac{1}{2}\cot\frac{\pi n}{L}\sim\frac{\pi n}{L}\sim\frac{1}{L}\;. \tag{2.34}$$

Baxter equation and Baxter polynomial. The Bethe equations (2.31) could be also obtained by introducing an operator $Q(u)$ which can be diagonalised simultaneously with the transfer matrix $t(u)$,

$$t(u)=a(u)\frac{Q(u-i)}{Q(u)}+d(u)\frac{Q(u+i)}{Q(u)}\;. \tag{2.35}$$

Taken as a relation between the eigenvalues, the Baxter equation above can be solved as follows: for the spin chain of length L, $t(u)$ is a polynomial of degree L in u, as are $a(u)$ and $d(u)$. Therefore, the equation can be solved by a polynomial $Q(u)=\prod_{m=1}^{M}(u-u_m)$. The conditions of regularity of $t(u)$ when $u=u_k$, with u_k the roots of the Baxter polynomial $Q(u)$, are exactly the Bethe Ansatz equations (2.31). The Baxter equation can be also written as second-order difference equation, discretisation of a Schrödinger-like equation,

$$t(u)Q(u)=a(u)Q(u-i)+d(u)Q(u+i)\;. \tag{2.36}$$

Under this quantum-mechanical analogy, $Q(u)$ is the equivalent of the wave-function. The positions of the Bethe roots u_k are the zeroes of the wave-function.

2.3 Solutions of the Bethe equations and the large L limit

In this section we analyse the solutions of the Bethe Ansatz equation in the limit where the length of the spin chain L and the number of magnons M are large. Before going to the limit, let us say a few words about the solutions of the Bethe equations in general. Due to the global $su(2)$ symmetry, each of the eigenstates is degenerate. Let us consider a solution with M non trivial magnons, *i.e.* the rapidities are all finite. From the commutation relations of the elements of the monodromy matrix one can show that

$$[S^+, B(u)] \sim A(u) - D(u) \ . \tag{2.37}$$

The constant of proportionality is not relevant here. Let us act with the raising operator S^+ on the Bethe state

$$S^+ B(u_1)B(u_2)\dots B(u_M)|\Omega\rangle = B_1 S^+ B_2 \dots |\Omega\rangle + A_1 B_2 B_3 \dots |\Omega\rangle - D_1 B_2 B_3 \dots |\Omega\rangle \ . \tag{2.38}$$

By the same argument we have used for (2.29), we can show that, if $u_1, u_2, \dots u_M$ obey the BAE, tehn

$$S^+ B(u_1)B(u_2)\dots B(u_M)|\Omega\rangle = 0 \ . \tag{2.39}$$

This means that the state $|u_1, u_2, \dots u_M\rangle$ is a highest state of a $su(2)$ multiplet with spin $L/2 - M$, therefore it is $L - 2M + 1$ times degenerate. The other states in the multiplet can be obtained by repeatedly acting with S^-, that is by creating magnons with infinite rapidity. When L is even, the state with $M = L/2$ is non-degenerate and unique, and it is the antiferromagnetic state. When the coupling constant J is positive, this state is the ground state of the Heisenberg Hamiltonian.

Bound state solutions. The BAE have generically solutions with real u, which means real momentum p. In terms of the wave-function (2.7), solutions with real momentum are associated with propagating plane waves. From (2.11), we see that a solution with complex imaginary part means that, for large L, the l.h.s. either grows exponentially or vanishes exponentially, which signals that the r.h.s should have a pole or a zero, respectively, so, in terms of rapidities, we should have another rapidity satisfying

$$u_1 = u_2 \pm i + \mathcal{O}(e^{-L}) \ . \tag{2.40}$$

Denote now

$$u^{(2)} = \frac{u_1 + u_2}{2} \ . \tag{2.41}$$

If the imaginary part of $u^{(2)}$ is zero, this means that $p^{(2)} = p_1 + p_2$ is real and we are back to the previous situation of having an excitation which propagates over large distances. This excitation is a bound state of two magnons (or two-string). If the imaginary part of $u^{(2)}$ is not zero, we continue the procedure until we get a bound state of k magnons such that $u^{(k)} = \frac{1}{k}\sum u_k$ is real. The scattering matrix for two bound states is given by the product of scattering matrices of the constituents.

Exercise: Show that if one replaces $S(p_1, p_2)$ in (2.11) with $S^{-1}(p_1, p_2)$, the resulting Bethe equations admit no bound state solution.

Sutherland solutions and semiclassical limit. A particularly interesting type of solutions of the Bethe Ansatz equations are those close to the ferromagnetic ground state. Suppose that we consider a string as above, with a number of constituents $M \sim L$. These solution were first discovered by Sutherland. In this case, the distance between two consecutive rapidities in the string is not exactly i, because the exponential corrections add up to give a finite contribution. If one takes a low-lying excitation mode of such macroscopic strings, the rapidities of all the constituents are of the order L as in (2.34). We have then a small parameter $\epsilon = 1/L$ at our disposal, which can be considered as the Planck constant. We are therefore going to call the limit $L \to \infty$ of the Sutherland solutions the semiclassical limit.

Let us consider the Bethe equations for one of the Sutherland solutions and define the quasi-momentum $\mathrm{p}(u)$ as

$$e^{2i\mathrm{p}(u)} = -\left(\frac{u+i/2}{u-i/2}\right)^L \prod_{k=1}^{M} \frac{u-u_k-i}{u-u_k+i} \ . \tag{2.42}$$

The quantisation conditions are given by

$$e^{2i\mathrm{p}(u_k)} = 1 \ , \qquad k = 1, \ldots, M \ , \tag{2.43}$$

or, in a logarithmic form,

$$\mathrm{p}(u_k) = \pi m \qquad k = 1, \ldots, M \ , \tag{2.44}$$

where m is the same for rapidities in the same string. This can be seen by analysing the determination of the branch of logarithm. It is worth to notice that the equation $e^{2i\mathrm{p}(u)} = 1$ has other solutions than $u_1, \ldots, u_M$. These L extra solutions are called *hole solutions* and they are nothing else than the roots of the polynomial $t(u)$ in (2.35).

Let us now look at the large L limit of the Bethe equations in the Sutherland limit, when the string solutions condense on some contours $\mathcal{C}_m$. We introduce a density of the (unknown) Bethe roots through

$$\rho(u) = \sum_{k=1}^{M} \delta(u - u_k) \ . \tag{2.45}$$

In the semiclassical limit, with $u \sim L$, the density will be supported on the contours $\mathcal{C}_m$. We have, taking the logarithm of (2.42) and expanding in the small shifts

$$\mathrm{p}(u) = \frac{L}{2u} - \int_{\mathcal{C}_m} \frac{\rho(v)dv}{u-v} \equiv \frac{L}{2u} - G(u) \ . \tag{2.46}$$

Obviously, the quasi-momentum is singular when u approaches the support $\mathcal{C}_m$ of the Bethe solution. Therefore, in order to write the continuum limit of the (2.43) one has to regularise the integral by excluding the point u itself in the integral (2.46). This can be done by the principal value prescription,

$$\mathrm{p}(u+i0) + \mathrm{p}(u-i0) = \frac{L}{u} - 2 \,⨍_{\mathcal{C}_m} \frac{\rho(v)dv}{u-v} = 2\pi m \ . \tag{2.47}$$

On the other hand, using

$$\frac{1}{u-v-i0} - \frac{1}{u-v+i0} = 2\pi i\, \delta(u-v) \tag{2.48}$$

one can show that $\rho(u)$ is related to the jump of $\mathrm{p}(u)$ across its cut situated on $\mathcal{C}_m$,

$$\mathrm{p}(u-i0) - \mathrm{p}(u+i0) = 2\pi i\, \rho(u)\ . \tag{2.49}$$

Equations like (2.47) can be solved by a square root Ansatz. Suppose for simplicity that the endpoints of $\mathcal{C}_m$ are situated at $\pm a$, with a to be determined. Then one can take

$$\mathrm{p}(u) = \sqrt{u^2-a^2} f(u) + \pi m \tag{2.50}$$

with $f(u)$ a function regular on the cut.

Exercise: Find the function $f(u)$ for a single-cut solution. Use the analytic structure in u implicit in (2.47), together with the normalisation of the density, $\int_{\mathcal{C}_m} \rho(u) du = M$.

3 Integrable field theories with $su(2)$ symmetry

In this section we are considering two examples of two-dimensional field theories with $su(2)$ symmetry in order to illustrate the interplay between the classical and quantum integrability. The first example is the Landau-Lifshits model, which gives a field-theoretical description of the Heisenberg model we have considered in the first lecture. We are going to re-derive the solutions in the Sutherland limit presented in the previous chapter from the classical integrability of the Landau-Lifshits model. The second model, the so-called principal chiral model, is a relativistic version of the first. One of the interests of the principal chiral model is that it can be promoted to a model of strings on $S^3 \times \mathbb{R}$ and as such it gives a baby-version of the string sigma model for strings in $AdS_5 \times S^5$.

3.1 The Landau-Lifshits model

The Landau-Lifshits model is a description of the low-energy sector of the Heisenberg chain (2.1), close to the ferromagnetic ground state. It action is given by

$$\mathcal{S} = s \int d\tau \int_0^{2\pi} d\sigma \int_0^1 ds\ \vec{n}(\partial_\tau \vec{n} \times \partial_s \vec{n}) - J s^2 \int d\tau \int_0^{2\pi} d\sigma \partial_\sigma\ \vec{n}\ \partial_\sigma \vec{n}\ , \tag{3.1}$$

where $\vec{n} = (n_1, n_2, n_3)$ is a unit vector $\vec{n}^2 = 1$ and we consider the spin $s = 1/2$ case. The first term in the action is the so-called Wess-Zumino term, insuring the quantum nature of the spin. The integral over the variable s is used to simplify the expression of the WZ term; $\vec{n}(\sigma, \tau) = \vec{n}(\sigma, \tau, s = 1)$. The second term comes from the interaction (2.1). The spatial coordinate $\sigma = 2\pi l/L$ is periodic. A derivation of the action (3.1) using the coherent states representation can be found in the book by E. Fradkin.

Exercise: Show that the equations of motion of the LL model are given by

$$\partial_\tau \vec{n} = J \vec{n} \times \partial_\sigma^2 \vec{n}\ . \tag{3.2}$$

From the equations of motion it is clear that the theory is non-relativistic, since $E(k) \sim |J| k^2$.

Classical integrability and the Lax connection. The equations of motion above can be interpreted as the flatness (no-curvature) condition for a connection defined by

$$J_\sigma = \frac{i\,\vec{n}\vec{\sigma}}{2u}\ , \qquad J_\tau = \frac{iJ\,\vec{n}\vec{\sigma}}{u^2} + \frac{iJ(\vec{n}\times\partial_\sigma\vec{n})\vec{\sigma}}{u}\ , \tag{3.3}$$

where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ is a vector whose components are the Pauli matrices.

Exercise: Show that the equations of motion of the LL model (3.2) are equivalent to the flatness condition

$$[\partial_\tau - J_\tau,\, \partial_\sigma - J_\sigma] = 0\ . \tag{3.4}$$

The flatness condition is the compatibility condition of the system of equations (auxiliary problem)

$$\begin{aligned} (\partial_\tau - J_\tau)\,\Psi &= 0 \\ (\partial_\sigma - J_\sigma)\,\Psi &= 0\ . \end{aligned} \tag{3.5}$$

The first equation above can be solved formally by

$$\Psi(\sigma,\tau,u) = P\exp\int_0^\sigma d\sigma' J_\sigma(\sigma',\tau,u)\ \Psi(0,\tau,u)\ , \tag{3.6}$$

with P standing for the path-ordering operator. The flatness condition insures that the integral in the exponential does not depend on the path of integration, but only the endpoints, as long as we replace $d\sigma' J_\sigma$ by $d\sigma' J_\sigma + d\tau' J_\tau$. Since the system is periodic in $\sigma \to \sigma + 2\pi$, after a period the solution for Ψ should be back to itself, possibly rotated,

$$\begin{aligned} \Psi(\sigma+2\pi,\tau,u) &= \Omega(\sigma,u)\Psi(\sigma,\tau,u)\ , \\ \Omega(\sigma,u) &= P\exp\int_\sigma^{\sigma+2\pi} d\sigma' J_\sigma(\sigma',\tau,u)\ . \end{aligned} \tag{3.7}$$

The unimodular monodromy matrix $\Omega(\sigma,u)$ depends on the starting/endpoint as

$$\Omega(\sigma,u) = h(\sigma)\,\Omega(0,u)\,h^{-1}(\sigma) \tag{3.8}$$

so that its trace is invariant independent of the starting/endpoint σ,

$$\mathrm{Tr}\,\Omega(\sigma,u) = \mathrm{Tr}\,\Omega(0,u) = e^{i\mathrm{p}(u)} + e^{-i\mathrm{p}(u)}\ . \tag{3.9}$$

The invariance with respect the time translations suggests that $\mathrm{Tr}\,\Omega(0,u)$ generates the integrals of motion. Indeed, the monodromy matrix $\Omega(0,u)$ is the field-theoretical counterpart of the monodromy matrix $T_0(u)/u^L$ in (2.14),

$$\frac{T_0(u)}{u^L} = \left(1 + \frac{i\vec{\sigma}_0\vec{S}_1}{2u}\right)\ldots\left(1 + \frac{i\vec{\sigma}_0\vec{S}_L}{2u}\right) \simeq P\exp\int_0^{2\pi} d\sigma\,\frac{i\vec{\sigma}_0\vec{S}(\sigma)}{2u}\ . \tag{3.10}$$

This justifies the use of the notation $e^{i\mathrm{p}(u)}$ for the eigenvalues of the monodromy matrix, which can be identified in the semiclassical limit with the two terms in the Baxter equation

$$e^{i\mathrm{p}(u)} \simeq \left(\frac{u+i/2}{u}\right)^L \frac{Q(u-i)}{Q(u)}, \qquad e^{-i\mathrm{p}(u)} \simeq \left(\frac{u-i/2}{u}\right)^L \frac{Q(u+i)}{Q(u)}. \tag{3.11}$$

Let us now have a closer look at the structure of the quasi-momentum $\mathrm{p}(u)$ as implied by the structure of the monodromy equation. Generically $\mathrm{p}(u)$ is real. At the critical points u_*, the eigenvalues of the monodromy matrix become degenerate, which means

$$\mathrm{p}(u_*) = -i\mathrm{p}(u_*) + 2\pi m \; . \tag{3.12}$$

These equations are the same as (2.49) at the end of the distribution of Bethe roots. Consider a generic point for the monodromy matrix,

$$\Omega(u) = -\begin{pmatrix} a & b \\ c & d \end{pmatrix} \; . \tag{3.13}$$

Its eigenvalues will be, around the critical points u_*,

$$e^{i\mathrm{p}_{1,2}(u)} = \frac{1}{2}(a+d\pm\sqrt{(a-d)^2+4bc}) \simeq 1\pm\alpha\sqrt{u-u_*}+\ldots = e^{1\pm\alpha\sqrt{u-u_*}+\ldots} \; . \tag{3.14}$$

This means that the quasi-momentum can have a branch point at the critical point (or a double point), where the two eigenvalues are exchanged, and the two eigenvalues live on a double-sheeted Riemann surface with branch cuts. On the cuts we have

$$\mathrm{p}(u+i0) + \mathrm{p}(u-i0) = 2\pi m \; . \tag{3.15}$$

which is exactly the same condition as in (2.47). The object

$$Y(u) = \partial_u \log \Omega(u) \sim \begin{pmatrix} i\partial_u \mathrm{p}(u) & 0 \\ 0 & -i\partial_u \mathrm{p}(u) \end{pmatrix} \tag{3.16}$$

has only poles and branch cuts and therefore lives on an algebraic curve. This curve can be uniquely determined by the positon of the branch cuts and the conditions at infinity (values of the conserved charges). According to the relations (3.11), $\mathrm{p}(u)$ has a simple pole at $u=0$.

3.2 The $su(2)$ principal chiral model

Another example of integrable field theory with $su(2)$ symmetry is the principal chiral model (PCM). By adding an extra direction, it can be promoted to a string theory for strings which move on the sphere $S^3\times\mathbb{R}$. Let us consider the coordinates $\Phi_1, ..., \Phi_4$ with $\sum_i \Phi_i^2 = 1$ be the coordinates of the sphere S^3, which is the also the group manifold for $SU(2)$, and Φ_0 the timelike coordinate

$$\mathfrak{g} = \begin{pmatrix} \Phi_1+i\Phi_2 & \Phi_3+i\Phi_4 \\ -\Phi_3+i\Phi_4 & \Phi_1-i\Phi_2 \end{pmatrix} \equiv \begin{pmatrix} Z & X \\ -\bar{X} & \bar{Z} \end{pmatrix} \in SU(2) \tag{3.17}$$

The string action is given by

$$S_{S^3} = g\int_0^{2\pi} d\sigma \int d\tau \left[(\partial_\alpha \Phi_i)^2 - (\partial_\alpha \Phi_0)^2\right] \tag{3.18}$$

$$= -g\int_0^{2\pi} d\sigma \int d\tau \left[\frac{1}{2}\mathrm{Tr}\,(\mathfrak{g}^{-1}\partial_\alpha \mathfrak{g})^2 + (\partial_\alpha \Phi_0)^2\right] \tag{3.19}$$

The global symmetry of the action is given by $SU(2)_L \otimes SU(2)_R \simeq SO(4)$. The two $su(2)$ symmetries correspond to the left and right multiplication respectively, with currents given by $l_\alpha = \partial_\alpha \mathfrak{g}\mathfrak{g}^{-1}$ and $j_\alpha = \mathfrak{g}^{-1}\partial_\alpha \mathfrak{g}$. Under the first group, $(Z, -\bar{X})$ and $(X, \bar{Z})$ transform as doublets, while under the second (Z, X) and $(-\bar{X}, \bar{Z})$ are doublets. We consider the action in the gauge $\Phi_0 = \kappa\tau$ and the energy of the string is given by

$$\Delta = 2g\int_0^{\infty} d\sigma \partial_\tau \Phi_0 = 4\pi g\kappa\ . \tag{3.20}$$

The equations of movement for the action (3.18) can be written, in the light cone coordinates $\sigma_\pm = \frac{1}{2}(\tau \pm \sigma)$, $\partial_\pm = \partial_\tau \pm \partial_\sigma$

$$\begin{gathered} \partial_+ j_- + \partial_- j_+ = 0\ ; \quad \partial_+\partial_-\Phi_0 = 0 \\ \partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0\ , \end{gathered} \tag{3.21}$$

where the first equation corresponds to the current conservation and the second line is the equation of motion. These equations are to be supplemented with the Virasoro condition, which express the vanishing of the stress-energy tensor,

$$\frac{1}{2}\mathrm{Tr}\ j_+^2 = \frac{1}{2}\mathrm{Tr}\ j_-^2 = -\kappa^2\ . \tag{3.22}$$

The equations (3.21) can be reformulated as the zero curvature condition [?]

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0 \tag{3.23}$$

for the connection

$$J_\pm = \frac{j_\pm}{1 \mp x}\ . \tag{3.24}$$

Of course, this connection is a particular case of the one presented in section (??). The monodromy matrix

$$\Omega(x) = P\exp \int_0^{2\pi} d\sigma \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right) \tag{3.25}$$

can be diagonalized in terms of the quasi-momentum $p(x)$ defined by

$$\mathrm{Tr}\,\Omega(x) = 2\cos p(x)\ . \tag{3.26}$$

Near the singular points $x = \pm 1$, the quasi-momentum behaves as

$$p(x) = -\frac{\pi\kappa}{x \pm 1} + \dots\ . \tag{3.27}$$

while the asymptotics at $x = 0, \infty$ are related to the conserved charges,

$$p(x) = -\frac{Q_L}{2gx} + \dots \quad x \to \infty \qquad (3.28)$$
$$p(x) = 2\pi n + \frac{Q_R}{2g} x + \dots \quad x \to 0 \, .$$

where $Q_L = \int d\sigma \, j_\tau^3$, $Q_R = \int d\sigma \, l_\tau^3$ are the charges associated with the two $\mathfrak{su}(2)$ currents. This behavior suggests that the transformation $x \to 1/x$ exchanges the two $su(2)$ components. In order to remove the poles at $x = \pm 1$, one introduces the resolvent

$$G(x) = p(x) + \frac{\pi\kappa}{x-1} + \frac{\pi\kappa}{x+1} \, . \qquad (3.29)$$

which is analytic on the physical sheet, with possible branch cuts. The resolvent can be represented in terms of a density $\rho(x)$ with support on the branch cuts of $G(x)$

$$G(x) = \int dy \, \frac{\rho(y)}{x-y} \qquad (3.30)$$

such that $2\pi i \rho(x) = G(x - i0) - G(x + i0)$. The asymptotic conditions for $p(x)$ at $x = 0$ and ∞ translate into normalisation conditions on $\rho(x)$

$$\int dx \; \rho(x) = \frac{\Delta + 2J - L}{2g} \, , \qquad (3.31)$$
$$\int dx \; \frac{\rho(x)}{x} = 2\pi m \, , \qquad \int dx \; \frac{\rho(x)}{x^2} = \frac{\Delta - L}{2g}$$

The continuous part of $p(x)$ on the cut is

$$p(x + i0) + p(x - i0) = 2\pi m \qquad (3.32)$$

where m is specific to the branch cut C, or, in terms of the resolvent

$$G(x + i0) + G(x - i0) = 2 \fint dy \frac{\rho(y)}{x-y} = \frac{4\pi\kappa x}{x^2 - 1} + 2\pi m \, . \qquad (3.33)$$

In conclusion, the analysis of the PCM is very similar to that of the LL model, with two differences: one is the appearance of the poles at $x = \pm 1$ for the PCM quasi-momentum, to be compared with the single pole at $u = 0$ for the LL model. The second difference is the symmetry, which is $su(2)_L \times su(2)_R$ in one case and $su(2)$ in the other case. It is interesting to notice that in the principal chiral model one has two different descriptions in terms of the left and right currents, which can be exchanged by (up to the mode number n)

$$\mathrm{p}_L(x) \leftrightarrow -\mathrm{p}_R(1/x) \, . \qquad (3.34)$$

This symmetry is related to the relativistic nature of the underlying field theory. In terms of energy/momenta of the excitations, the transformation $x \to 1/x$ amounts to going from particles to antiparticles.

A spin model with two $su(2)$ types of spin was proposed by Faddeev and Reshetikhin as a discretization of the principal chiral model .