

Dynamics of Quantum Many Body Systems Far From Thermal Equilibrium – DRAFT

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The concept of thermodynamic equilibrium is one of the building blocks of our modern understanding of interacting many-particle systems. Together with quantum mechanics it provides a powerful framework to describe a large variety of natural phenomena. However, recent years have seen enormous experimental progress in preparing, controlling and probing strongly interacting quantum systems in different non-equilibrium regimes. These developments have brought fresh new interest around a set of fundamental questions concerning dynamics, dissipation, transport and the approach to thermal equilibrium in isolated and open quantum many-body systems. The aim of this course is to present the motivations and the main theoretical questions behind this research, and to discuss the recent progress in the general understanding of strongly interacting quantum systems far from equilibrium, with a special emphasis on their dynamics.

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I. INTRODUCTION, MOTIVATIONS AND PHYSICAL EXAMPLES

These lectures focus on the physics of quantum systems made of many interacting degrees of freedom, either fermions, bosons or quantum spins, **in regimes far away from thermal equilibrium**. This last statement is both crucial, since it will force us to look at things in a very different way from the traditional focus of condensed

matter physics, and intentionally vague, reflecting the intrinsic generality of out of equilibrium physics. This indeed requires to specify not only information on the system but also on the *protocol* used to bring it away from the thermal equilibrium state. The aim of this first lecture is to put this research in a broad general context, to give some introduction to the subject and to provide few physical examples that serve as motivation.

A. Falling Gently Out of Equilibrium

We start the discussion from a very general and qualitative definition of what thermal equilibrium is. We could say that a macroscopic system is said to be in thermal equilibrium when

- its state (physical properties) is defined in terms of a unique set of **intensive** and **extensive** variables which do not change with time (*stationarity*)
- no currents of charges associated to conserved quantities (particles, energy, ...) flow through it

Needless to say, such a concept is more an exception than a rule in every day life. Indeed, non equilibrium effects are extremely common in many different physical situations of the greatest simplicity, the flow of electric current through a metallic conductor being just a trivial example. This is even more true if we think that the very basic idea of *performing experiments* on materials and compounds amounts to act with some external field on an otherwise equilibrium system and to monitor its response to the applied perturbation. As a result one could be tempted to ask why, inspite of the restrictiveness of the above definition, the assumption of thermal equilibrium has been (and actually still is) so powerful and useful to describe physical properties of macroscopic systems. A possible answer to this question should be based on two observations.

1. From one side, as often happens in physics, what matters are the orders of magnitude of typical time scales a system need to reach a quasi-equilibrium state compared to the time scales on which observation takes place. As a consequence, with a good degree of approximation, a macroscopic system can be considered as in thermal equilibrium if all fast processes have taken place while the slowest one still have to occur. Clearly the distinction between *fast* and *slow* depends on the observation time that is considered¹⁻³. Example: in solids, the electronic relaxation times are of the order of *picoseconds* $10^{-12}s$ due to the coupling with lattice vibrations or other excitations, so essentially in most conventional experiments one can take the electronic liquid as in thermal equilibrium with its environment.
2. A second key observation comes from a basic result in the statistical theory of many particle systems which goes under the apparently innocuous name of linear response theory. It simply states that a small external perturbation can only probe small fluctuations around equilibrium. Hence, as long as applied fields are weak enough, the system can be considered as in thermal equilibrium for all practically purposes. This major result is of paramount importance in connecting theory to experiments, since it provides a way to compute experimental relevant quantities such as susceptibilities in terms of equilibrium correlation functions, which are the natural quantities in terms of which the many body problem is formulated.

Before moving into the realm of nonequilibrium quantum physics it might be therefore useful to recall some basic textbook fact about equilibrium quantum statistical physics that will be useful throughout the course to better appreciate the challenges associated with out of equilibrium quantum physics.

Let's consider for the sake of simplicity a system with Hamiltonian H in canonical thermal equilibrium at temperature T . Its properties can be described by a density matrix of Boltzmann-Gibbs form

$$\rho_{eq}(H) = \frac{e^{-\beta H}}{Z} \quad Z = \text{Tr} e^{-\beta H} \quad (1)$$

and from this quantity one can compute thermal averages,

$$\langle A \rangle_{eq} = \text{Tr} (\rho_{eq} A) \quad (2)$$

from which one can derive all thermodynamics, as well as dynamical correlation functions

$$C_{AB}(t, t') = \langle A(t)B(t') \rangle_{eq} = \text{Tr} (\rho_{eq} A(t)B(t')) \quad (3)$$

where operators are evolved in the Heisenberg picture as $A(t) = e^{iHt} A e^{-iHt}$. An important property of the equilibrium state is that its density matrix is directly known once the Hamiltonian of the system is, i.e.

$$[\rho_{eq}, H] = 0 \quad (4)$$

This has few important consequences: first, the system is time-translational invariant (a property that can equally be taken as a definition of stationarity) i.e. all single-time observable are time independent while multipoint functions only depend on time-differences, for example

$$C_{AB}(t, t') = \text{Tr}(\rho_{eq} A(t)B(t')) = \text{Tr}(\rho_{eq} A(t-t')B(0)) = C_{AB}(t-t') \quad (5)$$

Secondly, one can take advantage of Eq.(4) to derive a set of very general results linking these correlation functions, or to be more precise the so called "fluctuations"

$$F_{AB}(t) = \frac{1}{2}\langle\{A(t), B(0)\}\rangle = \frac{1}{2}(C_{AB}(t) + C_{BA}(-t)) \quad (6)$$

to experimentally relevant quantities such as the so called linear response functions, encoding the response of the system in thermal equilibrium to a small external perturbation, the most simple and gentle example of out of equilibrium quantum systems.

To see this let us assume first to perturb the system with a time-dependent field $f_{ext}(t)$ that couples to some observable B , i.e. $H' = H + V_{ext}$ with $V_{ext} = f_{ext}(t)B$, and then to compute an observable A in presence of such a small perturbation. To linear order in f_{ext} we get

$$\delta\langle A(t)\rangle \equiv \langle A(t)\rangle - \langle A\rangle_0 = \int dt' \chi_{AB}(t-t')f(t') \quad (7)$$

where the linear response or retarded susceptibility is defined as

$$\chi_{AB}(t) = -\frac{i}{\hbar}\theta(t)\langle[A(t), B(0)]\rangle \quad (8)$$

We can then define the dissipative response function as

$$\chi''_{AB}(t) = \frac{1}{2}\langle[A(t), B(0)]\rangle_{eq} = \frac{1}{2}(C_{AB}(t) - C_{BA}(-t)) \quad (9)$$

and use the result

$$C_{BA}(-\omega) = e^{-\beta\omega}C_{AB}(\omega) \quad (10)$$

to obtain the so called Fluctuation-Dissipation theorem

$$F_{AB}(\omega) = \coth\left(\frac{\beta\omega}{2}\right)\chi''_{AB}(\omega) \quad (11)$$

The above is the bosonic version, which is valid for physical observables. For fermionic fields, such as those entering the single particle Green's function, one defines for example the retarded and Keldysh components as

$$G^R(t, t') = -i\theta(t-t')\langle\{\psi(t), \bar{\psi}(t')\}\rangle \quad (12)$$

$$G^K(t; t') = i\langle[\psi(t), \bar{\psi}(t')]\rangle \quad (13)$$

and obtains the fermionic version of FDT relation

$$G^K(\omega) = \tanh\left(\frac{\beta\omega}{2}\right)2i\text{Im}G^R(\omega) \quad (14)$$

The basic fact encoded in Eq. (4), namely that the density matrix is an exponential function of the Hamiltonian, suggests the idea of performing a Wick rotation from real to imaginary time evolution and consider correlation functions of the form

$$C_{AB}(\tau) = -\langle T_\tau A(\tau)B(0)\rangle \quad A(\tau) = e^{H\tau} A e^{-H\tau}$$

The resulting Matsubara technique, which forms the basic of equilibrium quantum many body physics, allows to obtain real-frequency retarded correlation functions from imaginary time evolution through analytic continuation, and therefore, as a consequence, crucially removes "time" from the picture from the very beginning. We will see that an important consequence of going beyond linear response is that time evolution will come back at play and in a pretty major role.

Another interesting example of basic equilibrium assumptions that will need to be revised in the more general out of equilibrium context concerns the way quantum many body theory is usually built at zero temperature. Here one assumes to start from a non-interacting Hamiltonian H_0 in the infinite distant past and slowly switching on the interaction

$$H(t) = H_0 + e^{-|\varepsilon|t} H_{int} \quad (15)$$

Then one typically focus on multi-point correlations functions of the form

$$G(t_1, \dots, t_n) = \langle \Psi | T(A(t_1) \dots A(t_n)) | \Psi \rangle \quad (16)$$

where $|\Psi\rangle$ is the interacting ground state in the infinite future. A crucial result is that this can be written as

$$G(t_1, \dots, t_n) = \langle +\infty | S[+\infty, -\infty] T(A(t_1) \dots A(t_n)) | -\infty \rangle \quad (17)$$

where we have defined the S-matrix $S[+\infty, -\infty] = T \exp\left(-i \int_0^t dt V(t)\right)$ and where $|\pm\infty\rangle$ are the ground-states in the infinite past and infinite future, in the interaction representation. The crucial result⁴ that follows from adiabaticity is that these two states are essentially the same, up to a phase, since the process of switching on and off slowly the interaction does not affect the system, hence

$$|+\infty\rangle = e^{i\delta} |-\infty\rangle \quad e^{i\delta} = \langle -\infty | +\infty \rangle \quad (18)$$

so that we obtain

$$G(t_1, \dots, t_n) = \frac{\langle -\infty | S[+\infty, -\infty] T(A(t_1) \dots A(t_n)) | -\infty \rangle}{\langle -\infty | +\infty \rangle} \quad (19)$$

This is a crucial result to build the field theory approach to zero temperature quantum many body systems which however will need to be revised when dealing with arbitrary time dependent switching protocols which do not guarantee adiabaticity. The proper field theory approach goes under the name of Keldysh techniques (also due to Schwinger, Kadanoff, Baym, see Refs. 5–7). We will come back to it later in more detail, here we just mention the main idea. In a generic nonequilibrium condition the state of the system at infinite time is not known a-priori, but **it is itself an unknown of the problem** together with the energy spectrum, which can be strongly modified by the nonequilibrium perturbation. Then the basic is to design an approach where only the initial condition enters: this amounts to fold the real-time contour around the time axis and give rise to the celebrated Keldysh contour.

Equilibrium Quantum Statistical Physics is well developed and very solid theoretical framework to describe the properties of quantum many body systems (mostly inspired from solid state physics). From the basic assumptions reviewed above many powerful concepts have been developed, mainly inspired by the concept of equilibrium universality and by the fact that at low temperature the physics is **dominated by the groundstate plus low lying excitations**. Within this low energy approach one can attack the quantum many body problem using a broad array of concepts and tools, such as field theory techniques, renormalization group approaches, various sorts of mean field theories plus fluctuations, or powerful numerical methods (different flavours of Quantum Monte Carlo algorithms, DMRG and Matrix Product States algorithms and their extensions,..). Obviously this does not mean that everything that is equilibrium is simple and easy, for example strongly correlated quantum many body systems are a continuous challenge to theory and experiments for their unusual and exotic properties. Still it is good to appreciate the conceptual challenges associated with building a theory of quantum nonequilibrium systems.

B. Why Quantum Nonequilibrium?

A natural question that can arise at this point is therefore why should we care about quantum non-equilibrium problems. The reasons are twofold

- **Fundamental Interest:** the condition of thermal equilibrium is not only an idealized situation but also a very special one. Relaxing this constraint we open to questions about fundamental aspects of quantum mechanics in unconventional regimes. At the same time it is clear this cannot be enough, since many of these questions were around since the early days of quantum mechanics (Von Neumann 1929, for example) and the theoretical framework was developed in the mid sixties (Keldysh,..).

- **Experimental Relevance:** Recent years have seen tremendous progress in experimentally realizing quantum systems which can be brought out of equilibrium in a **controlled way**. Quantum Control is the key here and in particular *quantum control of light-matter interactions*. These progresses are happening at a new interface, between disciplines that did not use to cross each other that much in the past, such as Condensed Matter, Atomic Physics and Quantum Optics.

Below (see associated slides of Lecture 1) we will give a quick introduction to the relevant experimental systems, although unfortunately we won't have the time to cover in details the physics behind these platforms. Rather will present the main ideas that serve as a motivation for theoretical research. Luckily there are already few excellent **reviews that cover the latest experimental progresses** in this rapidly evolving field. For example, a classic and general review on Ultracold Atoms in Optical Lattices is Ref. 8, while Refs. 9–11 have more focus on nonequilibrium aspects. For what concerns strongly interacting light-matter systems we can mention few references: a recent and general review of collective quantum phenomena with light is Ref. 12, with Ref. 13 more focused on exciton-polariton condensates; for cavity/circuit QED arrays we can mention few reviews, such as Refs 14–16, while for Ultracold Atoms in Optical Cavities a classic reference is Ref. 17.

C. Outline of the course

These lectures will be divided in three main parts, covering the dynamics of quantum many body systems under different nonequilibrium regimes. Clearly the subject is extremely broad both in terms of nonequilibrium settings and model systems considered and we will have to make a selection.

In the first part we will focus on the **dynamics of thermally isolated systems** after rapid change in time of some parameters, the so called quantum quenches, motivated by experiments with ultracold gases and pump-probe spectroscopies. This will allow us to touch fundamental questions related to time evolution in quantum mechanics, the approach to a stationary regime and to the state of thermal equilibrium, the possibility of non trivial phenomenon in the quantum evolution, the existence of general mechanisms which can prevent the system from reaching equilibrium.

The second main subject of these lectures will be the **dynamics of open driven and dissipative quantum systems**, i.e. systems which are under the continuous effect of an external driving field and of a coupling to an environment. This part will be motivated by experiments with quantum optics platforms. We will discuss the theoretical framework of open quantum systems (master equations, Lindblad evolution,..) and connect it with the more conventional (in condensed matter) Keldysh field theory approach. We will address questions on the nature of the steady state in driven dissipative systems, the possibility of dissipative quantum phase transitions and their relation with standard (quantum) critical phenomena, the emergence of universality out of equilibrium.

In between will briefly review the problem of closed quantum many body systems under periodic driving, the so called Floquet quantum mechanics now in the context of a many body problem. This is emerging as a new exciting direction in nonequilibrium quantum physics, with several theoretical open questions and experimental platforms and it will be a natural crossover between the two worlds.

D. General Readings (to be completed)

The literature on systems far away from thermal equilibrium is very broad and there are a number of articles and reviews of general interest on the topic, covering a variety of subjects both in classical and quantum physics. Among these, I could mention in particular a recent Nature Physics Insight⁵⁷, together with the references already mentioned for the experimental platforms for quantum nonequilibrium physics (see previous page).

In addition there are excellent reviews focused on quantum many body systems far from equilibrium with some (or more than some) overlap with the material that will be covered in the lectures. We list here a few general references divided for topic and refer the reader to the extended bibliography at the end of the lecture notes, where we quote various works relevant to our discussion.

- Dynamics of Closed Quantum Many Body Systems
 1. Polkovnikov, Sengupta, Silva, Vengalattore, *Review of Modern Physics*, **83**, 863 (2011)
 2. Eisert, Friesdorf, Gogolin, *Nature Physics*, **11**, 124 (2015) (see also arXiv:1503.07538)
- Dynamics of Open Driven and Dissipative Quantum Many Body Systems
 1. Sieberer, Buchhold, Diehl, arXiv:1512.00637

There are also few books that cover what is by now the more standard material, such as for example the Keldysh formalism. Among these we could mention

- Rammer, *Quantum Field Theory of Nonequilibrium States*, Cambridge University Press, 2007
- Kamenev, *Field Theory of Non-Equilibrium Systems*, Cambridge University Press, 2011
- Altland and Simons, *Condensed Matter Field Theory*, Cambridge University Press, 2010

Rammer's book has an introduction to Keldysh technique which is formal, abstract and field theory inspired (diagrammatics, generating functionals, Effective Action, ...) but lacks a bit on the side of applications and examples, which are few and pretty old fashioned. On the contrary Kamenev's book has more a focus on applications than formalism, although Keldysh technique is explained in detail (through several examples) mostly in the language of path integrals. Altland and Simons book is a truly excellent condensed matter theory book, which covers nonequilibrium physics (classical and quantum) among many other different subjects, but which is worth to be included in the list since the presentation is generally clear and slightly complementary to the previous two books. Overall the three books are good for what concerns general motivations and basic formalism, however they are all a bit off in terms of topics and subjects with respect to what is currently in the focus of the research on nonequilibrium quantum many body systems for which I would certainly recommend the reviews above and the articles mentioned in the lectures.

II. DYNAMICS OF CLOSED QUANTUM SYSTEMS: INTRO

Here we could discuss some general fact about dynamics of closed quantum systems evolving unitarily, motivated by experiments with ultracold atoms and (to a certain extent) pump and probe spectroscopies.

A. Quantum Dynamics, Generalities

We consider a closed quantum many-body system initially prepared at time $t_0 = 0$ in some initial state $|\Psi(0)\rangle$ and evolving unitarily under the action of its Hamiltonian $H(t)$, possibly time dependent. The state of the system at time t satisfies the time-dependent Schroedinger equation

$$i\partial_t|\Psi(t)\rangle = H(t)|\Psi(t)\rangle \quad (20)$$

whose general solution reads

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle = T \exp\left(-i \int_0^t dt' H(t')\right) |\Psi(0)\rangle \quad (21)$$

Let us recall some basic and general fact about unitary evolution, before start specifying more in detail the problem.

- First, the evolution is linear, that is the superposition principle holds, i.e. if $|\Psi_{1,2}(t)\rangle$ are two solutions of Eq. (20) with initial conditions $|\Psi_{1,2}(0)\rangle$ then one can construct the state

$$|\Psi(t)\rangle = c_1|\Psi_1(t)\rangle + c_2|\Psi_2(t)\rangle \quad (22)$$

which is again solution of Eq.(20) with initial condition $|\Psi(0)\rangle = c_1|\Psi_1(0)\rangle + c_2|\Psi_2(0)\rangle$.

- If we construct the density matrix $\rho(0) = |\Psi(0)\rangle\langle\Psi(0)|$ whose time evolution reads

$$\rho(t) = U(t)\rho(0)U^\dagger(t) \quad \partial_t\rho(t) = -i[H(t), \rho(t)] \quad (23)$$

we can see that the purity is conserved

$$\text{Tr}\rho^2(t) = \text{Tr}U(t)\rho(0)U^\dagger(t)U(t)\rho(0)U^\dagger(t) = \text{Tr}\rho^2(0) = 1 \quad (24)$$

i.e. in other words pure states remain pure.

- We can see this also by introducing the Von Neuman entropy associated to the density matrix $\rho(t)$, i.e.

$$S(t) = -\text{Tr} \rho(t) \log \rho(t) \quad (25)$$

It is then easy to show that the entropy remains constant for unitary evolution. Indeed using the definition of the logarithm of an operator, we have

$$\log \rho(t) = \log(U(t)\rho_0U^\dagger(t)) = U(t) \log \rho_0 U^\dagger(t), \quad (26)$$

from which we get

$$S(t) = -\text{Tr} \rho(t) \log \rho(t) = \text{Tr} \rho(0) \log \rho(0) = 0 \quad (27)$$

the last step being valid only for pure states. Alternatively we can compute the time derivative

$$\frac{d}{dt}S = -\frac{d}{dt}\text{Tr}\rho(t) \log \rho(t) = -\text{Tr}(\dot{\rho} \log \rho) - \text{Tr}(\rho \rho^{-1} \dot{\rho}) = i\text{Tr}([H, \rho] \log \rho) = 0 \quad (28)$$

- The average energy reads

$$E(t) = \langle\Psi(t)|H(t)|\Psi(t)\rangle = \langle\Psi(0)|U^\dagger(t) H(t) U(t)|\Psi(0)\rangle \quad (29)$$

is instead not constant in time and its time-derivative satisfies

$$\frac{d}{dt}E(t) = \langle\Psi(t)|\partial_t H(t)|\Psi(t)\rangle \quad (30)$$

which can be easily obtained using the fact that $i\partial_t U = H(t)U$.

These results emphasize the linearity of quantum evolution and the fact that for closed systems the quantum information is never really lost.

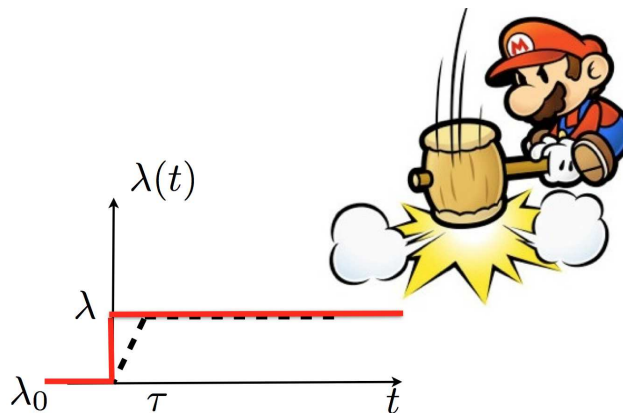


Figure 1: A Quantum Quench is among the simplest nonequilibrium protocol to study the unitary dynamics of closed quantum systems. The Hamiltonian of the system is assumed to depend on time through some classical parameter $\lambda(t)$ which is varied in time according to some protocol. In figure we show a linear ramp of $\lambda(t)$ between λ_0 and λ with a ramp time τ . Two limits have been particularly studied, the slow/adiabatic limit ($\tau \rightarrow \infty$) and the sudden one ($\tau \rightarrow 0^+$). In the main text we will focus on the latter.

1. The simplest nonequilibrium protocol: a "Quantum Quench"

So far we haven't said much about the explicit form of the time dependent Hamiltonian, and now we would like to be more specific about this and introduce a very neat and popular way of studying quantum dynamics, through a non-equilibrium protocol known as *quantum quench*¹⁸. This will give a bit more of physical understanding of the resulting nonequilibrium process.

Let us imagine that the system Hamiltonian depends on time through some global (i.e. equal everywhere in the system) control parameter $\lambda(t)$, i.e. $H[\lambda(t)]$. At time $t = 0$, when $\lambda(0) = \lambda_0$ the system is prepared in the ground-state of the Hamiltonian $H[\lambda_0]$ and then the parameter is varied between two values λ_0 at time $t = 0$ and λ at time $t = \tau$ according to some protocol (see figure). This is called *quantum quench*, in analogy with classical thermal quenches where the temperature of the bath is changed in time.

The limit $\tau \rightarrow 0^+$, a so called sudden quantum quench, corresponds to the case in which the Hamiltonian changes abruptly but the state has no time to adapt so, effectively, this corresponds to prepare a certain initial state and let it evolve. This is the opposite regime of the adiabatic switching that we have discussed in the introduction.

In the language of the quench it is useful to discuss the energetics once more. Let us assume to have an initial Hamiltonian H_0^i corresponding to the value λ_0 (where the superscript i reminds that this is the initial pre-quench) whose spectrum reads

$$H_0^i |\phi_n^i\rangle = E_n^i |\phi_n^i\rangle \quad (31)$$

and let assume to prepare the system in the groundstate $|\phi_0^i\rangle$ before performing the quench $\lambda_0 \rightarrow \lambda$. We can think of this quench process as a nonequilibrium transformation in the space of couplings $\lambda_0 \rightarrow \lambda_f$ and we can characterize the energetics by looking at the statistics of the work done after the quench¹⁹. In thermodynamics, the work is not a state function (such as the internal energy) and therefore, quantum-mechanically, it cannot be described in terms of an observable^{20,21}. Rather one can define its probability distribution, obtained by summing the conditional probabilities of starting from state $|\phi_0^i\rangle$ and ending in the state $|n\rangle$, such that $E_n - E_0^i = W$, i.e.

$$P(W) = \sum_n \delta(W - E_n + E_0^i) |\langle n | \phi_0^i \rangle|^2 \quad (32)$$

We notice that $P(W)$ has a threshold, i.e.

$$P(W) = 0, \quad W < \delta E = E_0 - E_0^i$$

and that the spectral weight of the threshold peak is given by $\langle 0 | \phi_0^i \rangle^2$ which is nothing but the overlap between the initial and final ground state. The average work done during the quench reads

$$\langle W \rangle = E_Q - E_0^i = E_0 - E_0^i + (E_Q - E_0) = \delta E + W_{exc} \quad (33)$$

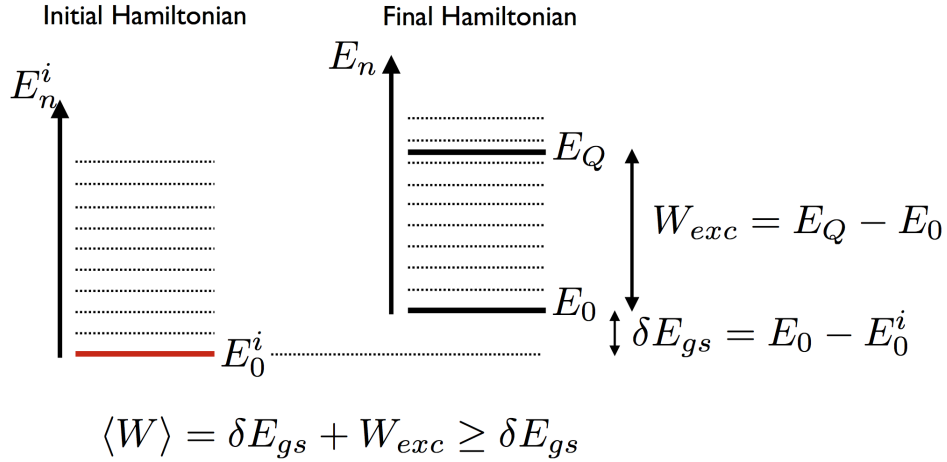


Figure 2: Energetics of a Quantum Quench: After the sudden change of parameters, the system evolves with a constant in time and extensive (i.e. proportional to the volume) energy E_Q which is in principle far above its groundstate. The average work performed during this change can be written as the sum of a purely "equilibrium" part - δE_{gs} the difference in ground state energy between initial and final Hamiltonians, which would correspond to an infinitely slow quench or a quasi-static transformation - and a non-equilibrium contribution that accounts for excitation due to the finite velocity of the quench, W_{exc} . Notice that while δE_{gs} has not a definite sign (could be positive or negative depending on the specifics), the excitation energy is positive and vanishes in the limit of $\tau \rightarrow \infty$.

where the excitation energy $W_{exc} = E_Q - E_0$ is in general positive and vanishes in the limit of infinitely slow process. Finally, in the general case of an initial state at finite temperature we would get for the work statistics

$$P(W) = \sum_{n,m} \rho_m^i \delta(W - E_n + E_m^i) |\langle n | \phi_m^i \rangle|^2 \quad (34)$$

with $\rho_m^i = e^{-\beta E_m^i} / Z$. Quite interestingly, one can show that the work statistics $P(W)$ satisfies very general relations, known as fluctuation theorems out of equilibrium^{22,23}, valid for both classical and quantum systems (for a review in the quantum case see Refs. 20,21). In particular, the Crooks relation relates the probability distribution of the work for two protocols (F, forward and B, backward) which are related by time reversal symmetry (so in the case of a quantum quench this would be $\lambda_0 \rightarrow \lambda$ and $\lambda \rightarrow \lambda_0$) and reads

$$\frac{P_F(W)}{P_B(-W)} = e^{\beta(W - \Delta F)} \quad (35)$$

where ΔF is the equilibrium free energy difference between states with λ_0, λ . From this relation it immediately follows the Jarzinsky equality, by integrating both sides of the equation

$$\int dW P_F(W) e^{-\beta W} = e^{-\beta \Delta F} \quad (36)$$

These are truly remarkable results since relate completely out of equilibrium quantities, the statistics of the work done during an arbitrary nonequilibrium process between two states, to the free energy difference between those equilibrium states. Their applications are far reaching, from biophysics to quantum transport, see for example the Nature Physics Insight on Nonequilibrium Physics, which I mentioned among general readings, for a recent review. In spite of their generality, such relations are extremely easy to prove in the case of quantum isolated systems. To this extent it is useful to introduce the generating function $G(s)$ for forward and backward protocol

$$G_F(s) = \frac{1}{Z_0} \text{Tr} (e^{-\beta H_0} e^{iH_0 s} e^{-iHs}) \quad G_B(s) = \frac{1}{Z} \text{Tr} (e^{-\beta H} e^{iHs} e^{-iH_0 s}) \quad (37)$$

Then it is easy using cycling property of the trace to show

$$Z_0 G_F(-s - i\beta) = Z G_B(s) \quad (38)$$

if we Fourier transform

$$P_B(W) = \int ds e^{iWs} G_B(s) = \frac{Z_0}{Z} \int ds e^{iWs} G_F(-s - i\beta) = \frac{Z_0}{Z} e^{\beta W} \int ds' G_F(s') e^{-iWs'} = \frac{Z_0}{Z} e^{\beta W} P_F(-W) \quad (39)$$

B. Time-Independent Hamiltonian, Isolated Systems

Let us now focus on the case of isolated and closed quantum systems, namely systems whose Hamiltonian is time-independent, the state of the system at time t simply reads

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle \quad (40)$$

and the energy is conserved

$$E(t) = \langle \Psi(0) | H | \Psi(0) \rangle \equiv E_Q \quad (41)$$

In this case there is a natural basis in which the quantum evolution becomes simple which corresponds to the eigenstates of the Hamiltonian H , i.e. the states such that

$$H |\Phi_n\rangle = E_n |\Phi_n\rangle \quad (42)$$

Then it is useful to decompose the initial state in this basis to obtain

$$|\Psi(0)\rangle = \sum_n c_n(0) |\Phi_n\rangle \quad (43)$$

with $c_n(0) = \langle \Phi_n | \Psi(0) \rangle$, such that the time evolution of the state simply becomes

$$|\Psi(t)\rangle = \sum_n c_n(t) |\Phi_n\rangle \quad c_n(t) = c_n(0) e^{iE_n t} \quad (44)$$

Then we can write the (conserved) energy

$$E_Q = \langle \Psi(0) | H | \Psi(0) \rangle = \sum_n |c_n(0)|^2 E_n \quad (45)$$

as well as the energy fluctuation

$$(\delta E_Q)^2 = \sum_n |c_n(0)|^2 (E_n - E_Q)^2 \quad (46)$$

It is useful to introduce the distribution function of the ensemble generated by the weights $|c_n(0)|^2$, so called *diagonal ensemble*, which reads

$$P_{de}(E) = \sum_n |c_n(0)|^2 \delta(E - E_n) \quad (47)$$

We notice that quite generically one can show that the diagonal ensemble is thermodynamically well behaved, i.e. the fluctuations of the energy go to zero in the $L \rightarrow \infty$, namely the distribution of $|C_n|^2$ narrows around the value E_Q .
HERE!!!! LOCAL HAMILTONIAN

In terms of the eigenstates and eigenenergies of the Hamiltonian the dynamics of any observable or correlation function becomes "trivial", a series of oscillations at different frequencies, for example

$$\langle O \rangle_t = \langle \Psi(t) | O | \Psi(t) \rangle = \sum_{nm} e^{-i(E_n - E_m)t} c_m^*(0) c_n(0) O_{mn} = \sum_n |c_n(0)|^2 O_{nn} + \sum_{n \neq m} e^{-i(E_n - E_m)t} c_m^*(0) c_n(0) O_{mn} \quad (48)$$

What can we say in general about this evolution? First and trivially not all operators will have a non trivial dynamics. Simple exceptions are, for example, the conserved quantities, i.e. any operator I that commutes with the Hamiltonian $[H, I] = 0$ will remain constant in time

$$\langle I \rangle_t = \langle \Psi(t) | I | \Psi(t) \rangle = I_0 \quad (49)$$

In the following, we will distinguish cases in which the system admits few global conserved quantities (total energy, particle number, total spin, etc) associated to obvious symmetries from cases of systems with an extensive set of (non trivial) conserved quantities, so called integrable systems.⁵⁸

Beside conserved charges what can we say about *generic* operators? Again we can cook up operators which have trivial evolution, i.e. operators which project on a given eigenstate of the Hamiltonian, $\Lambda_{nm} = |n\rangle\langle m|$ cannot relax but just oscillate in time

$$\Lambda_{nm}(t) = e^{i(E_n - E_m)t} \Lambda_{nm} \quad (50)$$

These however are very special cases which do not correspond to "reasonable" operators, since a many body eigenstate is a highly non-local object and so the projector built out of it. Instead we will focus our attention to generic (and physically motivated) *few-body, local* operators, for which we can expect in the thermodynamic limit, the dynamics to reach a stationary (time-independent) state. This can be seen going back to the expression for the time-dependent average and write it as

$$\langle O \rangle_t = \langle O \rangle_{DE} + \int d\omega e^{i\omega t} F_O(\omega) \quad (51)$$

where we have introduced the diagonal ensemble average (we will come back to it very soon)

$$\langle O \rangle_{DE} = \sum_n |c_n(0)|^2 O_{nn} \quad (52)$$

as well as the spectral density

$$F_O(\omega) = \sum_{n \neq m} c_m^*(0) c_n(0) O_{mn} \delta(\omega - (E_n - E_m)) \quad (53)$$

From this result we can understand why the process of relaxation depends crucially on taking the thermodynamic limit first (and of course on the nature of the operator considered). In this limit the many body spectrum becomes dense and the right-hand side vanishes at long time as a result of destructive interference of many very close frequencies, a process called *dephasing*.

For finite size systems, characterized by a discrete spectrum, the oscillations in Eq. (REF) will revive at some point in time. It is interesting to compute the temporal average as well as the temporal fluctuations of the *classical* variable $\langle O \rangle_t$. To this extent we define the time average of a time-dependent quantity as

$$\overline{f(t)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt f(t) \quad (54)$$

Then we have

$$\overline{\langle O \rangle_t} = \langle O \rangle_{DE} + \lim_{T \rightarrow \infty} i \sum_{n \neq m} c_m^*(0) c_n(0) O_{mn} \frac{e^{-i(E_n - E_m)T} - 1}{T(E_n - E_m)} = \langle O \rangle_{DE} \quad (55)$$

Similarly we can compute the the fluctuations around this time average, namely

$$\sigma_O^2 = \overline{\langle O \rangle_t^2} - \left(\overline{\langle O \rangle_t} \right)^2 \quad (56)$$

We note that these are classical fluctuations, and should not be confused with the fluctuations of the quantum average, (see later). We can compute σ_O^2 by direct substitution to get

$$\sigma_O^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \left(\overline{\langle O \rangle_t} + \sum_{n \neq m} e^{-i(E_n - E_m)t} c_m^*(0) c_n(0) O_{mn} \right)^2 - \left(\overline{\langle O \rangle_t} \right)^2 = \quad (57)$$

$$= 2 \overline{\langle O \rangle_t} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \sum_{n \neq m} e^{-i(E_n - E_m)t} c_m^*(0) c_n(0) O_{mn} + \quad (58)$$

$$+ \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \sum_{n \neq m; n' \neq m'} c_m^*(0) c_n(0) c_{m'}^*(0) c_{n'}(0) e^{-i(E_n - E_m + E_{n'} - E_{m'})t} \quad (59)$$

The first term vanishes while for the second term we get a finite contribution from the pairing $n = m'$, $m = n'$, i.e.

$$\sigma_O^2 = \sum_{n \neq m} |c_m(0)|^2 |c_n(0)|^2 |O_{nm}|^2 \leq \max |O_{nm}|^2 \quad (60)$$

In other words, the off diagonal matrix elements of an operator control the amplitude of temporal fluctuations around the long-time average. These (classical) temporal fluctuations should not be confused with the quantum fluctuations of the dynamics.

C. Quantum Thermalization, Eigenstate Thermalization Hypothesis

A fundamental question in statistical mechanics is whether and under which condition isolated systems approach thermal equilibrium under the evolution generated by their own Hamiltonian. Loosely with this statement we mean that the dynamics goes to a stationary state and that this stationary state is thermal. More precisely we want that

- In the long-time limit the system reaches a stationary state, which can be described in terms of few parameters, corresponding to the global conserved quantities of the dynamics (average energy, particle number,...).
- This should happen independently of the details of the initial condition (i.e. initial condition with the same value of the global conserved quantities should approach the same long time limit), so first of all thermalization requires losing memory of the initial data.
- This steady state can be described by the appropriate statistical ensemble consistent with the values of the global conserved quantities. Since we are considering the thermodynamic limit where the ensemble equivalence generally holds (we consider system with short-range interactions) we could equally require thermalization to the canonical ensemble at finite temperature T .

We are now in the position to start appreciating the apparent **paradox of quantum thermalization**. First, due to the intrinsic linearity of unitary evolution, at the level of quantum states in the Hilbert space there is no loss of memory, i.e. neither the wave function or the density matrix could ever *thermalize*. Therefore to begin with, a meaningful sense for thermalization can be only given for the average value of local operators (or for the density matrix of subsystems of finite size embedded in our thermodynamically large system). A different way of seeing why we need to focus on averages of operators is that in the Heisenberg picture they satisfy a dynamics, $\partial_t \langle O \rangle_t = i \langle [H, O] \rangle_t$ which is in general highly non-linear (and possibly chaotic in the sense of classical mechanics). The reason for "locality" is instead the intuition that as long as we observe a closed system locally we are essentially tracing out all the degrees of freedom outside the region of interest, thus in a certain sense as if the rest of the system was playing the role of a bath for a smaller part of itself.

Still, for an operator approaching a stationary value, this would require according to Eq. (48)

$$\langle O \rangle_{DE} = \sum_n |C_n(0)|^2 O_{nn} = \langle O \rangle_{MC}(E_Q) \quad (61)$$

where we have introduced the equilibrium micro-canonical (MC) ensemble average

$$\langle O \rangle_{MC}(E) = \frac{1}{N_{E, \Delta E}} \sum_{n, |E_n - E| < \Delta E} O_{nn} \quad (62)$$

with the energy set by the initial condition $E_Q = \langle \Psi_0 | \mathcal{H} | \Psi_0 \rangle$. At first sight the equality seems impossible, since on the right hand side the information on the initial condition is only encoded in the total energy (plus the fluctuation ΔE) while on the left hand side clearly we have that the initial condition enters the distribution of $C_n(0) = \langle n(0) | \Psi_0 \rangle$. Clearly, thermalization is hidden in the structure of the eigenstates and their projection on the initial state.

The Eigenstate Thermalization Hypothesis²⁴⁻²⁶ (ETH) sets a way out of this thermalization paradox by postulating, in its essence, that thermalization is encoded in each eigenstate of a generic (non-integrable) Hamiltonian, which are thermal.

The essential assumption is that for diagonal matrix element $O_{nn} = \langle O \rangle_{MC}(E_n)$, i.e. eigenstates are thermal. This can be understood a posteriori with naive two arguments: first, if thermalization holds for any initial condition it should also hold if we pick as initial condition an exact eigenstate of the system, in which case the time evolution is trivially $\langle n | O(t) | n \rangle = O_{nn}$. Hence in order this to be thermal one has to require that the eigenstate in the first place are thermal. Another way of understanding ETH ansatz is to consider this as an extreme limit of the microcanonical

ensemble, where the energy shell contains only a single eigenstate, so by equivalence of ensemble one should require O_{nn} to be thermal. As the name suggest, ETH is an *hypothesis*, which has been to certain extent tested in numerical calculations on (rather small) lattice quantum systems (see for example Refs. 27–29). We will come back to ETH at the beginning of the second lecture and discuss more in detail its consequences, the numerical evidence in its support and also discuss cases where it is known to fail.

1. More on ETH: Origin and Consequences

To be more precise ETH can be written as an ansatz on the matrix element of a "local" "few-body" operator in the eigenbasis

$$O_{mn} = \begin{cases} \langle O \rangle_{MC}(\bar{E}) & m = n \\ R_{mn} f_O(\bar{E}, \omega) e^{-S(\bar{E})/2} & m \neq n \end{cases}$$

where $O_{mc}(\bar{E})$ is the microcanonical equilibrium average of the operator at the energy $\bar{E} = (E_n + E_m)/2$, $f_O(\bar{E}, \omega)$ is a smooth function of the total energy \bar{E} and of the energy difference $\omega = E_n - E_m$, $S(E)$ is the thermodynamic (microcanonical) entropy while on the off diagonal R_{mn} is a random (real/complex) matrix, such that $\langle R_{nm} \rangle_{dis} = 0$, $\langle R_{nm}^2 \rangle_{dis} = 1$. Depending on whether the system satisfies time-reversal symmetry, i.e. has real or complex operators, one obtain different properties such as

$$R_{nm} = R_{mn} \rightarrow f_O(\bar{E}, -\omega) = f_O(\bar{E}, \omega) \quad (63)$$

$$R_{nm}^* = R_{mn} \rightarrow f_O^*(\bar{E}, -\omega) = f_O(\bar{E}, \omega) \quad (64)$$

In order to grasp the origin of the ETH ansatz it is useful to recall the limit of an Hamiltonian described by Random Matrix Theory. Let's assume eigenvectors are random and let's decompose an operator in its own eigenstates

$$O = \sum_{\alpha} O_{\alpha} |\alpha\rangle \langle \alpha| \quad (65)$$

Then for the matrix element

$$O_{mn} = \sum_{\alpha} O_{\alpha} (\psi_{\alpha}^m)^* \psi_{\alpha}^n \quad (66)$$

Let's average and let's assume

$$\langle \bar{\psi}_{\alpha}^m \psi_{\beta}^n \rangle = \frac{1}{D} \delta_{mn} \delta_{\alpha\beta} \quad (67)$$

we get

$$\langle O_{mm} \rangle = \frac{1}{D} \sum_{\alpha} O_{\alpha} \equiv \bar{O} \quad (68)$$

$$\langle O_{mn} \rangle = 0 \quad (69)$$

Fluctuations are

$$\langle O_{mm}^2 \rangle - \langle O_{mm} \rangle^2 = \frac{1}{D^2} \sum_{\alpha} O_{\alpha}^2 \equiv \frac{1}{D} \bar{O}^2 \langle O_{nn}^2 \rangle - \langle O_{mn} \rangle^2 = \frac{1}{D} \bar{O}^2 \quad (70)$$

therefore we can say

$$O_{mn} = \bar{O} \delta_{mn} + R_{mn} \frac{\bar{O}^2}{D} \quad (71)$$

where R_{mn} is some zero average unit variance random matrix.

a. *Thermalization from ETH* As we see ETH puts a lot of thermodynamic information into the matrix elements of an operator. Let's start discussing the first consequence of ETH, concerning the paradox of quantum evolution. Let's plug in the ETH ansatz in Eq. REF. We get

$$\langle O \rangle_{DE} = \sum_n |C_n(0)|^2 \langle O \rangle_{MC}(E_n) = \int dE P_{DE}(E) \langle O \rangle_{MC}(E) \simeq \langle O \rangle_{MC}(E_Q) \quad (72)$$

where we have used the fact that in the thermodynamic limit the distribution $P_{de}(E)$ is peaked around the value of the conserved energy E_Q . What about finite size fluctuations? We can expand $O_{eq}(E)$ around $E = E_Q$ and plug it in to get

$$\langle O \rangle_{de} = O_{eq}(E_Q) + \frac{1}{2}(\delta E)^2 O''(E_Q) \quad (73)$$

i.e. the DE average match the MC average up to corrections related to the energy fluctuations of the DE,

$$(\delta E)^2 = \int dE P_{DE}(E) (E - E_Q)^2 \quad (74)$$

which vanish in the thermodynamic limit. Is that the end of the story? Not really!

While ETH is believed to be true in the thermodynamic limit for generic non-integrable systems, there are cases in which we know for sure that ETH does not apply. The simplest example that we have already mentioned is the case of integrable systems when one does not expect thermalization to occur, due to the extensive set of conserved quantities which highly constraint the dynamics. A less obvious case which has been recently recognized is the case of strongly disordered quantum systems due to Anderson Localization. Finally, even for non-integrable systems we expect deviation from ETH due to finite size effects and indeed there are several numerical studies which indicate lack of thermalization after a quantum quench. In all these case it is interesting to understand why and in which sense ETH could fail.

A key point raised by Ref²⁸ concerns the effect of rare states which can strongly bias the diagonal ensemble distribution. In order to appreciate this point it is crucial to study, as the size of the system is increased, the distribution of the matrix element O_{nn} and eigenvalues E_n , namely

$$\mathcal{N}(e, o) = \sum_n \delta(e - E_n/L) \delta(o - O_{nn}) \sim e^{S(e, O)} \quad (75)$$

Quite generically one can show²⁸ that the width of such distribution shrinks in thermodynamic limit as $1/\sqrt{L}$ around the value predicted by ETH, i.e. the microcanonical average. However this is not enough to ensure thermalization since the crucial point is the behavior of the *support* of such distribution, namely if it remains finite or goes to zero as well. In the former case there will remain rare states different from the ETH one, and therefore if the distribution of $|c_n(0)|^2$ is strongly biased toward these states thermalization can be avoided. Indeed the diagonal ensemble average involves the weighted distribution

$$Z(e, o) = \sum_n |c_n(0)|^2 \delta(e - E_n/L) \delta(o - O_{nn}) \sim e^{F(e, O)} \quad (76)$$

which can be in principle different from (75), unless the weights $|c_n(0)|^2$ sample uniform the different eigenstates. As opposite, if the support of the distribution $\mathcal{N}(e, o)$ also shrinks, namely if in the thermodynamic limit all states satisfy ETH (also known as strong ETH), then thermalization is ensured. The results of Ref.²⁸ show that such rare states are indeed responsible for the absence of thermalization of integrable systems and might be at play as well in finite size non-integrable models.

Another way of saying: there are two scenarios

- Weak ETH. $S(e, o), F(e, o)$ peaked but the support doesn't shrink in the thermodynamic limit. Then one has to be careful because they are in principle different functions (due to $|c_n(0)|^2$) with different saddle points: $o_{MC}(e)$ such that $\partial_o S(e, o) = 0$ and $o_{DE}(e)$ such that $\partial_o F(e, o) = 0$. Thermalization emerges then only if $|c_n(0)|^2$ samples states uniformly and result in an overall constant, which does not affect the maximum, i.e. $o_{MC}(e) = o_{DE}(e)$. Otherwise, if the initial condition bias the distribution one can avoid thermalization (example of integrable systems).
- Strong ETH. $S(e, o), F(e, o)$ peaked and the support shrinks in the thermodynamic limit. then the ansatz is something like $Z(e, o) \sim \theta(o - o^* - \delta o) - \theta(o - o^* + \delta o) \sim e^{F(e, O)}$ with $\delta o \rightarrow 0$ in the thermodynamic limit. Then, all the states satisfy ETH and putative rare states do not play any role.

b. FDT from ETH Finally let's show how from the ETH ansatz we can obtain the Fluctuation Dissipation Relation²⁶. To this extent we introduce a dynamical correlation function of an operator O evaluated on a single eigenstate $|n\rangle$, i.e.

$$C_{On}(t) = \langle n|O(t)O(0)|n\rangle - \langle n|O(t)|n\rangle\langle n|O(0)|n\rangle \quad (77)$$

in terms of which we can write both the single-eigenstate retarded response function

$$\chi_{On}(\omega) = i \int_0^\infty dt e^{i\omega t} \langle [O(t), O(0)] \rangle = i \int_0^\infty dt e^{i\omega t} [C_{On}(t) - C_{On}(-t)] \quad (78)$$

as well as the symmetric and antisymmetric (fluctuation and dissipation) correlation functions

$$S_n^+(\omega) = \frac{1}{2} \int dt e^{i\omega t} \langle n| \{O(t), O(0)\} |n\rangle_c = \int dt e^{i\omega t} (C_{On}(t) + C_{On}(-t)) \quad (79)$$

$$S_n^-(\omega) = \frac{1}{2} \int dt e^{i\omega t} \langle n| [O(t), O(0)] |n\rangle_c = \int dt e^{i\omega t} (C_{On}(t) - C_{On}(-t)) \quad (80)$$

Using ETH we now derive the following result for $C_{On}(t)$, namely

$$C_{On}(t) = \int d\omega e^{\beta\omega/2} e^{-i\omega t} |f_O(E_n + \omega/2, \omega)|^2 \quad (81)$$

Let's start from the definition of $C_{On}(t)$ and plug in a set of eigenstates to get

$$C_{On}(t) = \sum_{m \neq n} e^{i(E_n - E_m)t} |O_{nm}|^2 \quad (82)$$

Using ETH ansatz for O_{nm} we get

$$C_{On}(t) = \sum_{m \neq n} e^{-S(\bar{E})} e^{-i\omega t} |f_O(\bar{E}, \omega)|^2 |R_{nm}|^2 \quad (83)$$

where we have defined

$$\omega = E_m - E_n \quad \bar{E} = \frac{E_n + E_m}{2} = E_n + \frac{\omega}{2} \quad (84)$$

Let us now assume that summing over m average out the random contribution from the matrix element R_{mn} , using $\langle |R_{mn}|^2 \rangle = 1$ and that we can cast this summation over an integral in the continuum limit, using $\int d\Omega \delta(\Omega + E_n - E_m) = 1$ to get

$$C_{On}(t) = \sum_{m \neq n} \int d\Omega \delta(\Omega + E_n - E_m) e^{-S(E_n + \omega/2)} e^{-i\omega t} |f_O(E_n + \omega/2, \omega)|^2 \quad (85)$$

We now use the definition of microcanonical entropy as logarithm of number of available eigenstates namely $S(E) = \log \Gamma(E)$, with

$$\Gamma(E) = \sum_n (\theta(E_n - E - \Delta E) - \theta(E_n - E)) \simeq \Delta E \sum_n \delta(E - E_n) = \Delta E \mathcal{N}(E) \quad (86)$$

where $\mathcal{N}(E)$ is the many body density of states. From this we can write

$$\sum_m \delta(\Omega + E_n - E_m) = \Delta E \exp S(E_n + \Omega)$$

and therefore

$$C_{On}(t) = \int \frac{d\Omega}{\Delta E} e^{S(E_n + \Omega) - S(E_n + \Omega/2)} e^{-i\Omega t} |f_O(E_n + \Omega/2, \Omega)|^2 \quad (87)$$

Finally we can expand the integrand for small Ω , i.e.

$$S(E_n + \Omega) - S(E_n + \Omega/2) = \frac{\beta\Omega}{2} \quad (88)$$

$$(89)$$

with $\beta = \partial S / \partial E$ to get the result quoted in Eq (REF).

We can now plug this result into the definition of both $S_{O_n}^\pm(\omega)$ and $\chi_{O_n}(\omega)$ to obtain for the former

$$S^\pm(\omega) = \frac{1}{2} \int dt e^{i\omega t} \int d\Omega e^{\beta\Omega/2} |f_O(E_n + \Omega/2, \Omega)|^2 (e^{i\Omega t} \pm e^{-i\Omega t}) = \quad (90)$$

$$= \pi \left(e^{\beta\omega/2} |f_O(E_n + \omega/2, \omega)|^2 \pm e^{-\beta\omega/2} |f_O(E_n - \omega/2, -\omega)|^2 \right) = \quad (91)$$

$$= \pi \cosh \beta\omega (|f_O(E_n + \omega/2, \omega)|^2 \pm |f_O(E_n - \omega/2, \omega)|^2) + \quad (92)$$

$$+ \pi \sinh \beta\omega (|f_O(E_n + \omega/2, \omega)|^2 \mp |f_O(E_n - \omega/2, \omega)|^2) \quad (93)$$

where we have used $\int dt e^{i\omega t} = 2\pi\delta(\omega)$ and the symmetry properties of $f_O(E, \omega)$. In particular we get for $S_n^+(\omega)$

$$S_n^+(\omega) = \pi \cosh \beta\omega (|f_O(E_n + \omega/2, \omega)|^2 + |f_O(E_n - \omega/2, \omega)|^2) + \quad (94)$$

$$+ \pi \sinh \beta\omega (|f_O(E_n + \omega/2, \omega)|^2 - |f_O(E_n - \omega/2, \omega)|^2) \quad (95)$$

Similarly we can obtain, using

$$i \int_0^\infty e^{i(\omega+i0^+)t} = i\pi\delta(\omega) - P\left(\frac{1}{\omega}\right) \quad (96)$$

the following result for the imaginary part of $\chi_{O_n}(\omega)$

$$\text{Im}\chi_{O_n}(\omega) = \pi \sinh \beta\omega (|f_O(E_n + \omega/2, \omega)|^2 + |f_O(E_n - \omega/2, \omega)|^2) + \quad (97)$$

$$+ \pi \cosh \beta\omega (|f_O(E_n + \omega/2, \omega)|^2 - |f_O(E_n - \omega/2, \omega)|^2) \quad (98)$$

We now notice that the functions in parenthesis are smooth and can be expanded for small ω to obtain

$$(|f_O(E_n + \omega/2, \omega)|^2 + |f_O(E_n - \omega/2, \omega)|^2) \simeq 2|f_O(E_n, \omega)|^2 \quad (99)$$

$$(|f_O(E_n + \omega/2, \omega)|^2 - |f_O(E_n - \omega/2, \omega)|^2) \simeq \omega \partial_E |f_O(E_n, \omega)|^2 \quad (100)$$

Then we find

$$S_n^+(\omega) = 2\pi \cosh \beta\omega/2 |f_O(E_n, \omega)|^2 \left(1 + \tanh \beta\omega/2 \frac{\omega \partial_E |f_O(E_n, \omega)|^2}{2|f_O(E_n, \omega)|^2} \right) \quad (101)$$

and similarly

$$\text{Im}\chi_O(\omega) = 2\pi \sinh \beta\omega/2 |f_O(E_n, \omega)|^2 \left(1 + \coth \beta\omega/2 \frac{\omega \partial_E |f_O(E_n, \omega)|^2}{2|f_O(E_n, \omega)|^2} \right) \quad (102)$$

From this we can get an effective distribution function, i.e.

$$\frac{S_n^+(\omega)}{\text{Im}\chi_O(\omega)} = F_{dist}^*(\omega) \quad (103)$$

where the effective distribution function reads

$$F_{dist}^*(\omega) = \coth(\beta\omega/2) \frac{\left(1 + \tanh \beta\omega/2 \frac{\omega \partial_E |f_O(E_n, \omega)|^2}{2|f_O(E_n, \omega)|^2} \right)}{\left(1 + \coth \beta\omega/2 \frac{\omega \partial_E |f_O(E_n, \omega)|^2}{2|f_O(E_n, \omega)|^2} \right)} \quad (104)$$

Assuming that the energy dependence of $|f_O(E, \omega)|$ is weak such that we can disregard the second term we finally get

$$F_{dist}^*(\omega) \simeq \coth(\beta\omega/2) \quad (105)$$

D. Dephasing after a Sudden Quench: The example of Quantum Ising Chain

We now turn the previous discussion, which was done in full generality, to a concrete example, and compute the dynamics after a quantum quench in a simple model, namely the Transverse Field Ising Chain (TFIC), which is characterized by the following Hamiltonian

$$H_0 = -J \sum_i^L \sigma_i^x \sigma_{i+1}^x - \Gamma_0 \sum_i^L \sigma_i^z \quad (106)$$

where L is the number of the spins in the chain and σ_i^α ($\alpha=x,y,z$) are the Pauli matrices relative to the i -th spin.

This model represents a paradigm solvable example of a quantum phase transition and it has been therefore the subject of a large literature³⁰. In equilibrium at zero temperature and in the thermodynamic limit it exhibits ferromagnetic ($\Gamma_0 < 1$) and paramagnetic ($\Gamma_0 > 1$) phases, separated by a quantum critical point at $\Gamma_c = 1$ (for choice of $J = 1$). For $\Gamma_0 < 1$ and $L \rightarrow \infty$ there are two degenerate ground states related by the Z_2 symmetry. Spontaneous symmetry breaking selects unique ground state, in which spins align along the x -direction. On the other hand, for fields $\Gamma > 1$ the ground state is non-degenerate and as the magnetic field Γ is increased, spins align more and more along the z -direction. The order parameter for the quantum phase transition is the ground state expectation value $\langle \sigma_i^x \rangle$. Furthermore we refer to the ferromagnetic phase as the ordered phase, and to the paramagnetic one as the disordered phase.

As discussed earlier,

Let's discuss the quench protocol we are going to focus in the following. We consider the system initially prepared at time $t_0 = 0$ in the ground state $|\psi_0\rangle$ of Eq. (106). We then suddenly change the value of the transverse field, $\Gamma_0 \rightarrow \Gamma$ (*global quench*), so that for $t > 0$ the system evolves with the new Hamiltonian

$$H = -J \sum_i^L \sigma_i^x \sigma_{i+1}^x - \Gamma \sum_i^L \sigma_i^z \quad (107)$$

The dynamics of the system after a sudden change of the transverse field, from Γ_0 to Γ can be obtained exactly using a Jordan-Wigner transformation and a time-dependent Bogolubov transformation, as we are going to show in the next sections. As a result the calculation of simple observables, such as for example the transverse magnetization, can be easily done analytically and we are going to present it in detail. The calculation of correlation functions is a more challenging task for which recent developments have been obtained³¹⁻³⁶. In the next section we first review briefly some basic equilibrium property of the TFIC and discuss its diagonalization which will be useful to discuss the nonequilibrium dynamics. The reader which is familiar with this material can skip this section and move on directly to XXX..

1. Diagonalization by Jordan-Wigner Transformation

The idea of mapping quantum spin 1/2 into (spinless) fermionic degrees of freedom has a long history in quantum mechanics. It can be seen as a very natural one from the point of view of local Hilbert space, since the two up/down states (eigenstates of σ^z) can be described in terms of presence/absence of a fermionic particle (eigenstates of the local charge $n = c^\dagger c$) and analogously the spin flip operators σ^\pm as creation/annihilation of such a particle, i.e. $\sigma^+ = f^\dagger$, $\sigma^- = f$ and $\sigma^z = 1 - 2n$. The problem with this naive identification comes from the statistics, namely from the fact that spins on different sites are expected to commutes

$$[\sigma_i^\alpha, \sigma_j^\beta] = 2i\varepsilon^{\alpha\beta\gamma} \delta_{ij} \sigma_j^\gamma \quad (108)$$

while fermions to anticommute

$$\{c_i, c_j^\dagger\} = \delta_{ij} \quad (109)$$

In one dimension there is a way to overcome such a difficulty, namely to have a fermionic representation of the spin which is consistent with the statistics, through the so called Jordan-Wigner transformation. The idea is to introduce a fermionic representation of the quantum spin at each lattice site j of the form

$$\sigma_j^x = K_j (c_j + c_j^\dagger) \quad (110)$$

$$\sigma_j^z = 1 - 2c_j^\dagger c_j \quad (111)$$

where the so called *string* operator K_j , defined as

$$K_j = \left[\prod_{l < j} (1 - 2c_l^\dagger c_l) \right], \quad (112)$$

sets the proper sign by counting for the number of fermions preceding the j -th site considered by σ_j^x . We notice that K_j is an hermitian operator, such that $[K_i, K_j] = 0$ and $K_i^2 = 1$. One can see that these definitions are consistent with fermionic anti-commutation relation/quantum spin algebra. In addition one can see that the definition of σ_i^z is consistent with this and does not involve the string operator, i.e.

$$\sigma_j^z = 1 - 2\sigma_j^+ \sigma_j^- = 1 - 2c_j^\dagger c_j \quad (113)$$

where we used the fact that $K_i^2 = 1$. Under these transformations the TFIC Hamiltonian becomes in terms of fermions

$$H = -J \sum_{i=1}^{L-1} (c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger + hc) + 2\Gamma \sum_{i=1}^L c_i^\dagger c_i + (-1)^{N_F} (c_L^\dagger c_1 + c_L^\dagger c_1^\dagger + hc) \quad (114)$$

where $N_F = \sum_{i=1}^L c_i^\dagger c_i$ is the number of fermions in the chain. The last term can be accounted for by choosing either periodic (PBC)/antiperiodic (ABC) boundary conditions for the fermions

$$c_{L+1} \equiv (-1)^{N_F+1} c_1 \quad (115)$$

depending on whether their total number N_F is odd/even. We have to focus to the fact that the Hamiltonian conserves the parity of fermions and we restrict to the even sector which contains the ground state. Note that restricting to one of the two sectors is justified only when one considers expectation values of operators which are defined in terms of products of an even number of fermionic operators, i.e., such that they do not change the parity of the state that they act on. We then perform a Fourier transform

$$c_j = \frac{1}{\sqrt{L}} \sum_k e^{i k R_j} c_k \quad (116)$$

where the set of momenta k is given by

$$k = \pm(2\pi/L) n \quad n = 0, 1, \dots, L/2 \quad (PBC) \quad (117)$$

$$k = \pm(\pi/L) (2n + 1) \quad n = 0, 1, \dots, L/2 - 1 \quad (ABC) \quad (118)$$

$$(119)$$

as one can immediately see by setting $c_L = \pm c_1$ in Eq. (116). (Since we have chosen N_F *even* we take in consideration k from ABC). The resulting Hamiltonian becomes

$$H = \sum_k \varepsilon_k c_k^\dagger c_k - J \sum_k (c_k^\dagger c_{-k}^\dagger e^{ika} + c_{-k} c_k e^{-ika}) \quad (120)$$

with $\varepsilon_k = 2\Gamma - 2J \cos k$. We can simplify further the second term by noticing that

$$\begin{aligned} \sum_k (c_k^\dagger c_{-k}^\dagger e^{ika} + c_{-k} c_k e^{-ika}) &= \sum_k \cos k (c_k^\dagger c_{-k}^\dagger + c_{-k} c_k) + \\ + i \sum_k \sin k (c_k^\dagger c_{-k}^\dagger - c_{-k} c_k) &= +i \sum_k \sin k (c_k^\dagger c_{-k}^\dagger - c_{-k} c_k) \end{aligned} \quad (121)$$

where in the last step we have used the fact that

$$\begin{aligned} \cos(-k) &= \cos(k) \\ (c_{-k}^\dagger c_k^\dagger + c_k c_{-k}) &= - (c_k^\dagger c_{-k}^\dagger + c_{-k} c_k) \end{aligned} \quad (122)$$

so the first summation in Eq. (121) cancels out due to the contribution of positive and negative momenta. Finally we have

$$H = \sum_k \varepsilon_k c_k^\dagger c_k - J \sum_k i \sin k (c_k^\dagger c_{-k}^\dagger - c_{-k} c_k) \quad (123)$$

We can write this expression in terms of the positive momenta only as

$$H = \sum_{k>0} \varepsilon_k \left(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger \right) - 2J \sum_{k>0} i \sin k \left(c_k^\dagger c_{-k}^\dagger - c_{-k} c_k \right) \quad (124)$$

To proceed with the diagonalization let's introduce the Nambu-Gorkov spinor notation

$$\Psi_k = \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

to write the Ising Hamiltonian in the final form

$$H = \sum_{k>0} \bar{\Psi}_k \hat{h}_k \Psi_k \quad (125)$$

where \hat{h}_k is a 2×2 matrix which reads

$$\hat{h}_k = \begin{pmatrix} \varepsilon_k & -i\gamma_k \\ i\gamma_k & -\varepsilon_k \end{pmatrix}$$

where we have defined $\gamma_k = 2J \sin k$. This formalism is particularly convenient to diagonalize the problem. Indeed we can introduce the Pauli Matrices $\tau_{i=x,y,z}$ (to not be confused with the physical spin 1/2 operators!) and write

$$\hat{h}_k = \varepsilon_k \tau_z + \gamma_k \tau_y = E_k \tau_z \left(\frac{\varepsilon_k}{E_k} - i \tau_x \frac{\gamma_k}{E_k} \right) = E_k \tau_z e^{-2i\theta_k \tau_x} = E_k e^{-i\theta_k \tau_x} \tau_z e^{i\theta_k \tau_x} \quad (126)$$

Here we have defined the angle θ_k such as

$$\cos 2\theta_k = \frac{\varepsilon_k}{E_k} \quad \sin 2\theta_k = -\frac{\gamma_k}{E_k} \quad (127)$$

where $E_k = \sqrt{\varepsilon_k^2 + \gamma_k^2}$ is the quasiparticle spectrum which reads

$$E_k = 2\sqrt{J^2 + \Gamma^2 - 2J\Gamma \cos k} \quad (128)$$

From this result we see that quite generically the spectrum is gapped, i.e. the minimum amount of excitation energy (obtained for $k=0$) is finite and given by

$$\Delta_g = 2\sqrt{J^2 + \Gamma^2 - 2J\Gamma} \quad (129)$$

but as a function of Γ/J there is a special point $\Gamma_c = J$ where the gap vanishes continuously

$$\Delta_g = 2J |1 - \Gamma/\Gamma_c| \quad (130)$$

the nature of the spectrum changes and the system becomes gapless: the dispersion relation becomes linear at low momenta.

$$E_k = 4\sqrt{2}J \sin k/2 \sim vk \quad (131)$$

In order to complete the diagonalization process, we can now introduce a new set of fermionic fields

$$\Phi_k \equiv \begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix} = e^{i\theta_k \tau_x} \Psi_k = \begin{pmatrix} \cos \theta_k & i \sin \theta_k \\ i \sin \theta_k & \cos \theta_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

or equivalently

$$\Psi_k = e^{-i\theta_k \tau_x} \Phi_k$$

such that the so called Bogolubov transformation reads

$$c_k = \cos \theta_k \eta_k - i \sin \theta_k \eta_{-k}^\dagger \quad (132)$$

$$c_{-k}^\dagger = -i \sin \theta_k \eta_k + \cos \theta_k \eta_{-k}^\dagger \quad (133)$$

which diagonalize the Hamiltonian

$$H = \sum_{k>0} E_k \bar{\Phi}_k \tau_z \Phi_k = \sum_{k>0} E_k \left(\eta_k^\dagger \eta_k - \eta_{-k} \eta_{-k}^\dagger \right) \quad (134)$$

Alternatively, we could have obtained the same result by starting from the Hamiltonian (123), introducing a new set of fermionic fields through the Bogolubov rotation of Eq.(132) and then substitute them into the Hamiltonian to finally impose all the anomalous terms of the form $\eta^\dagger \eta^\dagger$ or $\eta \eta$ to be absent in the quasiparticle Hamiltonian.

E. Equilibrium Transverse Magnetization and Total Energy

Here we use the mapping to fermions and the Bogolubov transformation to compute simple observables in thermal equilibrium, both at zero temperature (corresponding to our initial state) and at finite temperature, which can be relevant for comparison with the long-time steady state limit of the dynamics.

Transverse magnetization Let's start with the definition and its expression in the fermionic language

$$m^z = \frac{1}{L} \sum_i \langle \sigma_i^z \rangle = 1 - \frac{2}{L} \sum_k \langle n_k \rangle \quad (135)$$

Then, we use the Bogolubov rotation (132) to write

$$\langle c_k^\dagger c_k \rangle = \cos^2 \theta_k \langle \eta_k^\dagger \eta_k \rangle + \sin^2 \theta_k \langle \eta_{-k} \eta_{-k}^\dagger \rangle = \cos 2\theta_k \langle \eta_k^\dagger \eta_k \rangle + \sin^2 \theta_k \quad (136)$$

and obtain

$$m^z = 1 - \frac{2}{L} \sum_k \sin^2 \theta_k - \frac{2}{L} \sum_k f(E_k) \cos 2\theta_k \quad (137)$$

where we have used the fact that the quasiparticle excitations are free fermions for which we have

$$\langle \eta_k^\dagger \eta_{k'} \rangle = \delta_{k,k'} f(E_k) \quad (138)$$

Using the bisection identities we can write the result above also as

$$m^z = \frac{1}{L} \sum_k \cos 2\theta_k (1 - 2f(E_k)) = \frac{1}{L} \sum_k \frac{\varepsilon_k}{E_k} \tanh \frac{\beta E_k}{2} \quad (139)$$

At zero temperature we obtain the result

$$m^z(T=0) = \frac{1}{L} \sum_k \frac{\varepsilon_k}{E_k} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{\Gamma - J \cos k}{\sqrt{\Gamma^2 + J^2 - 2J\Gamma \cos k}} \quad (140)$$

We can easily check that this recovers the trivial limits, i.e. $m^z = 0$ for $\Gamma = 0$ and $m^z = 1$ for $J = 0$. The above result could be re-derived also within the Nambu-Gorkov formalism.

Average Energy Here we compute the average energy at finite temperature T , within the Nambu-Gorkov formalism.

$$E(T) = \langle H \rangle = \sum_{k>0} \varepsilon_k \langle (c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) \rangle - 2J \sum_{k>0} i \sin k \langle (c_k^\dagger c_{-k}^\dagger - c_{-k} c_k) \rangle = \sum_{k>0} \langle \bar{\Psi}_k \hat{h}_k \Psi_k \rangle \quad (141)$$

The simplest way to proceed is to notice that an average is not affected by a unitary transformation. Therefore we can insert in Eq. (141) the operator that diagonalize the Hamiltonian and obtain

$$E(T) = \sum_{k>0} E_k \langle \bar{\Phi}_k \tau_z \Phi_k \rangle = - \sum_{k>0} E_k \tanh \frac{\beta E_k}{2} \quad (142)$$

1. Quantum Dynamics in the TFIC

We now consider the situation of interest, namely the case in which the system is taken out of equilibrium by a quantum quench of the transverse field. We want to study averages of temporal dependent operators, whose dynamic is governed by quenched Hamiltonian. Using Heisenberg notation:

$$\langle \hat{O}(t) \rangle = \langle \psi_0 | e^{iHt} \hat{O} e^{-iHt} | \psi_0 \rangle \quad (143)$$

Where the Hamiltonian H used for the dynamic of the operator O can not act directly on the ground state, because it doesn't commute with the equilibrium hamiltonian H_0 . A convenient way to calculate the associated dynamics correlations after the quench, consist in expressing the (time dependent) operators $\{c_k(t), c_{-k}^\dagger(t)\}$, in terms of

the quasi-particle operators $\{\gamma_k, \gamma_{-k}^\dagger\}$ which diagonalize the original Hamiltonian $H(\Gamma_0)$ ³⁴. The merit of this procedure is evident when calculating expectation values over $|\psi_0\rangle$, because γ_k operators, act trivially on their vacuum ψ_0 .

We start from the initial and final Hamiltonian written in the fermionic language

$$H_0 = \sum_{k>0} \varepsilon_{k0} \left(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger \right) - \sum_{k>0} i\gamma_k \left(c_k^\dagger c_{-k}^\dagger - c_{-k} c_k \right) \quad (144)$$

and

$$H = \sum_{k>0} \varepsilon_k \left(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger \right) - \sum_{k>0} i\gamma_k \left(c_k^\dagger c_{-k}^\dagger - c_{-k} c_k \right) \quad (145)$$

where we have defined:

$$\varepsilon_{k0} = \Gamma_0 - 2J \cos k \quad \gamma_k = J \sin k \quad (146)$$

and similarly for ε_k with $\Gamma_0 \rightarrow \Gamma$. The two quadratic Hamiltonians can be diagonalized in terms of two sets of fermionic quasiparticles

$$H_0 = \sum_{k>0} E_{k0} \eta_k^\dagger \eta_k \quad H = \sum_{k>0} E_k \gamma_k^\dagger \gamma_k \quad (147)$$

with energies

$$E_{k0} = \sqrt{\varepsilon_{k0}^2 + \gamma_k^2} \quad E_k = \sqrt{\varepsilon_k^2 + \gamma_k^2} \quad (148)$$

A sudden change of the transverse field corresponds therefore to a sudden change of the gap.

It is useful to relate the quasi-particle operators η_k, η_k^\dagger of the initial hamiltonian $H(\Gamma_0)$ to quasi-particle operators $\gamma_k, \gamma_k^\dagger$ of the final hamiltonian $H(\Gamma)$. To this extent we write the first Bogolubov rotation

$$\begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \cos \theta_k^0 & -i \sin \theta_k^0 \\ -i \sin \theta_k^0 & \cos \theta_k^0 \end{pmatrix} \begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix} = e^{-i\theta_k^0 \tau_x} \begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix}$$

as well as the second one

$$\begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \cos \theta_k & -i \sin \theta_k \\ -i \sin \theta_k & \cos \theta_k \end{pmatrix} \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix} = e^{-i\theta_k \tau_x} \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix}$$

Then, by combining the two we finally get

$$\begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix} = e^{i(\theta_k - \theta_k^0) \tau_x} \begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix}$$

which can be written as

$$\begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \cos \delta\theta_k & i \sin \delta\theta_k \\ i \sin \delta\theta_k & \cos \delta\theta_k \end{pmatrix} \begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix}$$

where $\delta\theta_k = \theta_k - \theta_k^0$ and the Bogolubov angles θ_{k0}, θ_k are defined respectively as

$$\cos 2\theta_{k0} = \frac{\varepsilon_{k0}}{E_{k0}} \quad \sin 2\theta_{k0} = -\frac{\gamma_k}{E_{k0}} \quad (149)$$

$$\cos 2\theta_k = \frac{\varepsilon_k}{E_k} \quad \sin 2\theta_k = -\frac{\gamma_k}{E_k} \quad (150)$$

Then we compute the time evolution of the fermionic operators

Using this result we can obtain the full time-dependence of the fermionic operators which is needed to evaluate dynamical averages

$$\begin{pmatrix} c_k(t) \\ c_{-k}^\dagger(t) \end{pmatrix} = \mathcal{M}_k(t) \begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix}$$

where the dynamical matrix reads in compact form

$$\mathcal{M}_k(t) = \cos \theta_{k0} \cos E_k t - i \cos(\theta_k + \delta\theta_k) \sin E_k t \tau^z + \\ -i \cos E_k t \sin \theta_{k0} \tau^x - i \sin(\theta_k + \delta\theta_k) \sin E_k t \tau^y$$

with $\tau_{\alpha=x,y,z}$ the Pauli matrices. In order for the anti commutation relation for $\{c_{k_1}^\dagger, c_{k_2}\} = \delta_{k_1, k_2}$ to be conserved, we could easily check that the matrix obtained for the dynamical transformation, has $Det = 1$. Thus we have found the unitary transformation, which allows us to compute average quantities, starting from an out of equilibrium Hamiltonian.

$$\mathbf{M}(\Gamma, \Gamma_0) = \begin{pmatrix} \cos \theta_k \cos \Delta\theta_k e^{-iE_k t} + \sin \theta_k \sin \Delta\theta_k e^{iE_k t} & i \cos \theta_k \sin \Delta\theta_k e^{-iE_k t} - i \sin \theta_k \cos \Delta\theta_k e^{iE_k t} \\ -i \sin \theta_k \cos \Delta\theta_k e^{-iE_k t} + i \cos \theta_k \sin \Delta\theta_k e^{iE_k t} & \sin \theta_k \sin \Delta\theta_k e^{-iE_k t} + \cos \theta_k \cos \Delta\theta_k e^{iE_k t} \end{pmatrix}$$

namely

$$\mathbf{M}(\Gamma, \Gamma_0) = \begin{pmatrix} \cos \theta_{k0} \cos E_k t - i \cos(\theta_k + \Delta\theta_k) \sin E_k t & -i \cos E_k t \sin \theta_{k0} + \sin(\theta_k + \Delta\theta_k) \sin E_k t \\ -i \cos E_k t \sin \theta_{k0} - \sin(\theta_k + \Delta\theta_k) \sin E_k t & \cos \theta_{k0} \cos E_k t + i \cos(\theta_k + \Delta\theta_k) \sin E_k t \end{pmatrix}$$

F. Results for the TFIC Quench Dynamics

Out of Equilibrium Transverse Magnetization As we have done in thermal equilibrium, we now focus our attention on the transverse magnetization and compute its time evolution after the quench using the time-dependent Bogolubov transformation we have obtained. We notice that while this is a particularly straightforward observable to evaluate, this is not true for the order parameter, $\langle \sigma_i^x(t) \rangle$ which mixes the two parity sectors of the Hilbert space and its calculation requires a more sophisticated analysis. We could refer to a large literature where this quantity was widely studied, In particular^{31,32,34} where quantities such that one-point $\langle \sigma_i^x(t) \rangle$ and two-point $\langle \sigma_i^x(t) \sigma_{i+l}^x(t) \rangle$ correlation functions of the order parameter, were studied.

We compute the transverse magnetization taking into account the dynamical matrix obtained in Eq.(??). Thus we transform the expression below

$$m^z(t) = \frac{1}{L} \sum_i \langle \sigma_i^z(t) \rangle = 1 - \frac{2}{L} \sum_k \langle c_k^\dagger(t) c_k(t) \rangle = -\frac{2}{L} \sum_{k>0} \left(\langle c_k^\dagger(t) c_k(t) \rangle - \langle c_{-k}(t) c_{-k}^\dagger(t) \rangle \right) \quad (151)$$

Using the matrix of the dynamical transformation defined above and taking the averages at $T = 0$, the resulting expression is:

$$m^z(t) = \frac{2}{L} \sum_{k>0} [\cos 2\theta_k \cos 2\Delta\theta_k + \sin 2\theta_k \sin 2\Delta\theta_k \cos(2E_k t)] \quad (152)$$

We can manipulate Eq. (152) with the transformation in Eq 127 and in the continuum limit it becomes

$$m^z(t) = 2 \int_0^\pi \frac{dk}{2\pi} \frac{1}{E_k^2 E_{k^0}} [\epsilon_k (\epsilon_k \epsilon_{k^0} + \gamma_k^2) + \gamma_k^2 (\epsilon_{k^0} - \epsilon_k) \cos(2E_k t)] \quad (153)$$

The quantity in Eq.(153), shows dephasing effect in the thermodynamic limit. Infact looking at Fig.(3), we could observe *quantum revivals* effect: increasing the size of the system, revivals are moved later in time. Taking the continuum limit in Eq.(153) (thus $L \rightarrow \infty$), quantum recurrence "disappears", and we could observe just *dephasing*. Dephasing effect is due to the locality of the global transverse magnetization in real space: infact if we consider the quantity $\langle n_k \rangle$, we could state that it doesn't dephase because it is defined for a fixed momentum k (just one wave function), thus using Fourier transformation, it is completely delocalized in real space. But if we integrate over all momenta $\frac{1}{L} \sum_k \langle n_k \rangle$, superposition of many oscillatory curves, each one of them having a slightly different frequency, with time progressing, they get out of phase and so their average value decays.

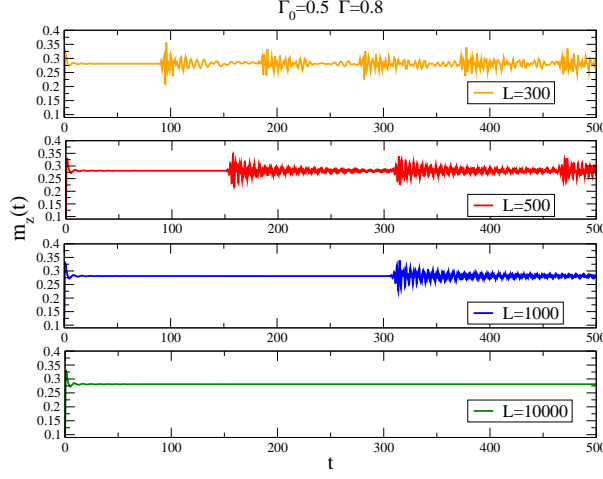


Figure 3: Plot of the global transverse magnetization $m_z(t)$ for fixed values of the quench parameter at $\Gamma_0 = 0.5$ and $\Gamma = 0.8$ (within the ferromagnetic case); for each of the four panels, we fixed the size L of the system increasing it from the top to the bottom.

Quench Energy In order to check energy conservation, let's compute the mean value of the Hamiltonian.

$$H = \sum_{k>0} \varepsilon_k \left(c_k^\dagger(t) c_k(t) - c_{-k}(t) c_{-k}^\dagger(t) \right) - i \sum_{k>0} \gamma_k \left(c_k^\dagger(t) c_{-k}^\dagger(t) - c_{-k}(t) c_k(t) \right) \quad (154)$$

Finally, the solution for the average energy could be written as the sum of two terms:

$$\langle \hat{H} \rangle = E_q + E(t) \quad (155)$$

The former is time independent and corresponds to the total amount of energy given by the quench while the latter is time dependent and is zero. Indeed

$$E(t) = \sum_{k>0} [\varepsilon_k (-\sin 2\theta_k \sin 2\Delta\theta_k \cos(2E_k t)) - \gamma_k (\sin 2\theta_k \cos 2\Delta\theta_k \cos(2E_k t))] \quad (156)$$

$$= \sum_{k>0} \cos(2E_k t) \sin 2\Delta\theta_k \underbrace{(-\varepsilon_k \sin 2\theta_k - \gamma_k \cos 2\theta_k)}_{=0 \text{ from } E_q \text{ (127)}} = 0 \quad (157)$$

and

$$E_q = \sum_{k>0} [\varepsilon_k (-\cos 2\theta_k \cos 2\Delta\theta_k) + \gamma_k \sin 2\theta_k \cos 2\Delta\theta_k] \quad (158)$$

$$= \sum_{k>0} \cos 2\Delta\theta_k (\gamma_k \sin 2\theta_k - \varepsilon_k \cos 2\theta_k) \quad (159)$$

$$= - \sum_{k>0} \frac{\varepsilon_k \varepsilon_k^0 + \gamma_k^2}{E_k E_k^0} \doteq E_q(\Gamma_0, \Gamma) \quad (160)$$

G. Generalized Gibbs Ensemble

As we have argued above average values of local operators in the stationary state after the quench in the TFIC are not described by the standard thermal equilibrium ensemble. The reason, as we have mentioned, is that the TFIC is an exactly solvable, i.e. integrable, model with an extensive set of local integral of motions. For this class of systems one can nonetheless describe the long-time stationary properties in terms of a generalized statistical ensemble, which take into account the extra conserved quantities. The idea is to construct an ensemble which maximizes the entropy under

the constraints imposed by the conserved quantities. Let's indicate with I_n a set of mutually commuting integrals of motion

$$[I_n, I_m] = 0 \quad [I_n, H] = 0 \quad (161)$$

Then let's introduce the functional of density matrix ρ

$$F[\rho] = \sum_n \lambda_n (\text{Tr} \rho I_n - I_{0n}) + \text{Tr} \rho \log \rho \quad (162)$$

where $I_{0n} = \langle \Psi_0 | I_n | \Psi_0 \rangle$ is the initial value of the integral of motion which is conserved, and let's require its stationarity with respect to ρ and λ_n to obtain

$$\rho^* \sim \exp \left(\sum_n \lambda_n I_n \right) \quad (163)$$

$$\text{Tr} \rho^* I_n = I_{0n} \quad (164)$$

This so called Generalized Gibbs ensemble (introduced firstly in the context of classical statistical mechanics by Jaynes in the late fities^{37,38} and then more recently in the context of quantum quenches³⁹) describes well the long-time properties of several integrable quantum systems, in particular those that can be mapped onto quadratic Hamiltonians. Its general validity is however matter of debate, that we won't have the time to discuss in detail, mainly because as we mentioned already, the definition of conserved charge is a bit tricky in quantum mechanics and the understanding of which conserved quantity has to be included in the GGE description to reproduce correctly the long-time steady state is a delicate question⁴⁰.

III. SHORT-TO-INTERMEDIATE TIME DYNAMICS: PRETHERMALIZATION AND DYNAMICAL TRANSITIONS

Here we discuss some non trivial aspects of the short-to-intermediate time dynamics of isolated interacting quantum systems. In certain case thermalization can be a fast process, while in other situations the approach to equilibrium can involve multiple time scales and proceed as a two-stage process, where on intermediate time scales a quasistationary state is reached which eventually decay away toward thermal equilibrium. Such a phenomenon has been dubbed *prethermalization*⁴¹ and appears ubiquitously in non-equilibrium dynamics of a variety of quantum systems and recently also observed experimentally^{42,43}.

Another interesting effect which has been discussed extensively recently is the possibility of some sort of **dynamical critical behavior of dynamical transition** in the time evolution of isolated quantum systems. Such an issue was raised first in the context of Fermi-Hubbard model solved with Dynamical Mean Field Theory^{44,45} but later was recognized to appear in many other models close to the mean field limit, including fermions within the time-dependent Gutzwiller approximation^{46,47}, fully connected Ising spins in transverse field and Bose-Hubbard models⁴⁸, interacting field theories⁴⁹⁻⁵¹ and Fermi-Hubbard models with antiferro-magnetic correlations^{52,53}.

A. Prethermalization in the Fermi Hubbard Model

To discuss prethermalization let us consider the Fermi Hubbard model describing electrons hopping on a lattice and interacting with an onsite Coulomb repulsion. Since we are interested in quench dynamics we will consider a time dependent interaction that we will suddenly switch on,

$$H = - \sum_{ij} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + U(t) \sum_i (n_i - 1)^2 \quad (165)$$

Let's start at $t = 0$ with a Fermi sea and let's ask the following basic and simple question: How does the distribution function evolves in time after a weak perturbation of this form? This problem was studied in 2008 by Kehrein and Moeckel⁵⁴ using a real-time RG-perturbative approach and later using Keldysh perturbation theory⁵⁵.

The result for the time dependent quasiparticle weight reads

$$Z_{neq}(t) = 1 - 4 \int d\omega \frac{\sin^2(\omega t/2)}{\omega^2} \Gamma_{k_F}(\omega) \quad (166)$$

where $\Gamma_k(\omega)$ is the interaction matrix element and has the Fermi-golden rule structure

$$J_k(\omega) = U^2 \sum \delta(k+p=k'+p') \delta(\omega+\varepsilon_p = \varepsilon_{k'} + \varepsilon_{p'}) [n_k n_p (1-n_{k'}) (1-n_{p'}) - n_{k'} n_{p'} (1-n_k) (1-n_p)] \quad (167)$$

which can also be related to the imaginary part of the self-energy for the fermions, giving their lifetime.

It turns out to be a much more general phenomenon, known under the name of *prethermalization*. How can we understand this trapping into a nonthermal state after a small quantum quench? First remark. since we are talking about weak perturbations breaking integrability one might be tempted to use real-time perturbation theory, for example in the Keldysh framework. In order to do so one has to be careful about how to proceed. In general non self-consistent perturbation theory might lead to singular terms diverging in time. Why: think about a perturbative expansion of some exponential decay $e^{-\gamma(U)t}$ with $\gamma \sim U^2$ for small U . Then if you expand this in power of U you would get $1 - U^2 t + \dots$. These are trivial singularity or secular terms which can be avoided using either self-consistent resummation schemes or as we are going to see unitary transformations⁵⁶.

Let's consider an Hamiltonian of the form

$$H = H_0 + gH_1 \quad (168)$$

where $H_0 = \sum_{\alpha} \varepsilon_{\alpha} I_{\alpha}$ is diagonal in terms of some set of integrals of motion

$$[I_{\alpha}, I_{\beta}] = 0 \quad [H, I_{\alpha}] = 0 \quad (169)$$

which are not conserved anymore at finite g . We would like to compute the dynamics of an observable O under the action of H , in particular we will focus on an observable that (i) we will assume to commute with all the conserved quantities (i.e. some linear combination of them) and that further we assume (ii) to satisfy $O|\psi(0)\rangle = 0$. In the example of the Hubbard model we see that this observable could be the momentum distribution (which is indeed conserved by H_0) minus the average over the initial Fermi sea (such that $A|\Psi(0)\rangle = 0$), i.e.

$$O = n_k - \langle n_k \rangle_0$$

In the limit of a weak perturbation, $g \ll 1$, we can compute things using a unitary transformation

Let's introduce a unitary transformation $\Omega = e^{-S}$ generated by an antiunitary operator $S^{\dagger} = -S$ that we expand formally in power of g

$$S = S_0 + gS_1 + \frac{g^2}{2}S_2 + o(g^3) \quad (170)$$

The effective Hamiltonian, $H_{eff} = \Omega^{\dagger} H \Omega = e^S H e^{-S}$ can be also expanded in series of g

$$H_{eff} = H_0 + g(H_1 + [S_1, H_0]) + g^2(\dots) \quad (171)$$

and we will require that, order by order, this Hamiltonian becomes diagonal in the same conserved quantities, i.e.

$$H_{eff} = H_0 + gH_{diag}^{(1)} + g^2H_{diag}^{(2)} + O(g^3) \quad (172)$$

with $H_{diag}^{(i)} = \sum_n E_n^{(i)} |n\rangle\langle n|$. Such a condition will translate into equation for the matrix elements of S_i as well as for the correction to the eigenvalues $E_n^{(i)}$. We get for the linear order

$$\langle n|S_1|m\rangle = (1 - \delta_{nm}) \frac{\langle n|H_1|m\rangle}{E_n - E_m} \quad (173)$$

$$E_n^{(1)} = \langle n|H_1|n\rangle \quad (174)$$

while for the term order g^2

$$\langle n|S_2|m\rangle = (1 - \delta_{nm}) \frac{\langle n|[S_1, H_0 + H_{diag}^{(1)}]|m\rangle}{E_n - E_m} \quad (175)$$

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle m|H_1|n\rangle|^2}{(E_n - E_m)} \quad (176)$$

Let's now compute the time-dependent average of an operator O

$$\langle O \rangle_t = \langle \Psi(0) | e^{iHt} O e^{-iHt} | \Psi(0) \rangle \quad (177)$$

and let's insert the identity $1 = e^{-S} e^S$ using the fact that $[H_{eff}, O] = 0$ up to $o(g^2)$, to obtain

$$\langle O \rangle_t = \langle \Psi(0) | e^{-S} e^{S(t)} O e^{-S(t)} e^S | \Psi(0) \rangle \quad (178)$$

where we have defined $e^{S(t)} = e^{iH_{eff}t} e^S e^{-iH_{eff}t}$. We now expand the inner and outer exponential and use the Baker Hausdorff formula

$$e^S X e^{-S} = X + [S, X] + \frac{1}{2}[S, [S, X]] + \dots \quad (179)$$

to obtain to second order

$$\langle O \rangle_t = \langle \Psi(0) | O + [S(t) - S, O] - \frac{1}{2}[S, [2S(t) - S, O]] + \frac{1}{2}[S(t), [S(t), O]] | \Psi(0) \rangle + \dots \quad (180)$$

$$\langle O \rangle_0 + \langle (S(t) - S)O - O(S(t) - S) \rangle_0 - \frac{1}{2} \langle [S, (2S(t) - S)O - O(2S(t) - S)] \rangle_0 + \frac{1}{2} \langle [S(t), (S(t)O - OS(t))] \rangle_0 + \dots$$

In evaluating these commutators we now use the fact that the operator O annihilates the initial state, i.e. $O|\Psi(0)\rangle = 0$ to simplify the result that becomes,

$$\langle O \rangle_t = \frac{1}{2} \langle (2S(t) - S)OS \rangle_0 + \frac{1}{2} \langle SO(2S(t) - S) \rangle_0 - \langle S(t)OS(t) \rangle_0 = \quad (181)$$

$$= \langle S(t)OS \rangle_0 + \langle SOS(t) \rangle_0 - 2 \langle SOS \rangle_0 = 2g^2 \text{Re} \langle S_1(t)OS_1 \rangle_0 - 2g^2 \langle S_1OS_1 \rangle_0 \quad (182)$$

where in the last step we used the fact that the result is second order in S , therefore to get $\langle O \rangle_t$ to order g^2 we can set $S = gS_1$. The first term can be written as

$$\langle S_1(t)OS_1 \rangle_0 = \sum_n e^{-i(E_n - E_0)t} \langle n|O|n \rangle \frac{|\langle 0|H_1|n \rangle|^2}{(E_n - E_0)^2} = - \int d\omega \frac{J(\omega)}{\omega^2} e^{-i\omega t} \quad (183)$$

where we have introduced the spectral density

$$J(\omega) = - \sum_n |\langle 0|H_1|n \rangle|^2 \langle n|O|n \rangle \delta(\omega - (E_n - E_0)) \quad (184)$$

Evaluating this result at $t = 0$ and combining it in Eq. REF we finally get

$$\langle O \rangle_t = -2g^2 \int d\omega \frac{J(\omega)}{\omega^2} (1 - \cos(\omega t)) = -4g^2 \int d\omega \frac{J(\omega)}{\omega^2} \sin^2(\omega t/2) \quad (185)$$

Back now to the Fermi Hubbard model

$$\langle N_k \rangle_t = \langle N_k \rangle_0 + 4g^2 \int d\omega \frac{J(\omega)}{\omega^2} \sin^2(\omega t/2) \quad (186)$$

In this case we can interpret this directly for the jump and we get

$$Z_{neq}(t) = \langle N_{k=k_F^-} \rangle_t - \langle N_{k=k_F^+} \rangle_t = 1 - 4g^2 \int d\omega \frac{J(\omega)}{\omega^2} \sin^2(\omega t/2) \quad (187)$$

the result that we get for the is then the same obtained by Moeckel and Kehrein where one can show after some algebra that the kernel $J(\omega)$ is given by nothing that the Fermi-Golden rule one. The long time limit of this quantity is controlled by the low frequency behavior of the phase space factor, which we now from Fermi Liquid theory goes as ω^2 at low frequency with some finite energy cutoff. If we use a form

$$J(\omega) = \theta(\Lambda - \omega) \omega^2 / \Lambda \quad (188)$$

we obtain

$$Z_{neq}(t) = 1 - 4g^2 \int_0^\Lambda d\omega \sin^2(\omega t/2) = 1 - 4g^2 \frac{1}{\Lambda t} \left(\Lambda t/2 - \frac{1}{2} \sin \Lambda t \right) \simeq 1 - 2g^2 + o(1/t) \quad (189)$$

Two remarks are in order here: first, the result is clearly different from what one would get at finite temperature, where the step would be vanishing. Quite interestingly this is not even related to zero temperature we would get for the renormalization factor (or wave function renormalization) mismatch at the final value of the interaction. It is instructive to compare this with the analog finite temperature expression obtained in equilibrium which would read

$$\langle O \rangle_{eq} = g^2 \sum_{nm} \rho_m^0 \frac{|\langle m | H_1 | n \rangle|^2}{(E_n - E_m)^2} \langle n | O | n \rangle \quad (190)$$

from which we would get at zero temperature

$$Z_{eq} = 1 - g^2 \int_0^\Lambda d\omega \frac{J(\omega)}{\omega^2} \quad (191)$$

Bottomline - the metastable fermi liquid is more correlated than its zero temperature counterpart and has still a sharp fermi surface! More recent works have investigate the properties of this metastable Fermi Liquid for what concerns for example its response functions (structure factor)? .

B. Dynamical Transitions in the Fully Connected Transverse Field Ising

Let's consider our favourite Ising model in a transverse field, but let's now consider the limit of a fully connected model, i.e. a model where the exchange interaction involves all pairs of spins

$$H = -\frac{J}{N} \sum_{ij} \sigma_i^x \sigma_j^x - \Gamma \sum_i \sigma_i^z \quad (192)$$

We can write this in terms of the components of the total spin $\mathbf{S} = (S^x, S^y, S^z)$ where we have defined

$$S^\alpha = \frac{1}{2N} \sum_i \sigma_i^\alpha \quad (193)$$

We get

$$H = -\frac{4J}{N} (S^x)^2 - 2\Gamma S^z \quad (194)$$

from which we immediately conclude that the total spin, $\mathbf{S} \cdot \mathbf{S} = (S^x)^2 + (S^y)^2 + (S^z)^2$ is conserved in this fully connected model

$$[H, \mathbf{S} \cdot \mathbf{S}] = 0 \quad (195)$$

Using the spin algebra

$$[S^\alpha, S^\beta] = i\varepsilon^{\alpha\beta\gamma} S^\gamma \quad (196)$$

we can obtain the equations of motion of the various spin components evolving under the action of H , i.e. $\dot{S}^\alpha = -i[S^\alpha, H]$, which read

$$\dot{S}^x = 2\Gamma S^y \quad (197)$$

$$\dot{S}^y = -\frac{4}{N} (S^z S^x + S^x S^z) \quad (198)$$

$$\dot{S}^z = -\frac{4}{N} (S^y S^x + S^x S^y) \quad (199)$$

We now take the average over the initial state and use the fact that, in the thermodynamic limit which also corresponds to the large spin limit, the spin behaves semiclassically which allows us to factorize the correlations

$$\langle S^\alpha S^\beta \rangle \simeq \langle S^\alpha \rangle \langle S^\beta \rangle$$

Corrections to REF can be shown to vanish in the thermodynamic limit (REF). This can be seen by noticing that if we define the reduced ("intensive") spin operators $m^\alpha = S^\alpha/N$, the commutation relations for these operators read

$$[m^\alpha, m^\beta] = \frac{i}{N} \varepsilon^{\alpha\beta\gamma} m^\gamma \quad (200)$$

,i.e. in the thermodynamic limit these objects behave as classical (commuting) variables. We then get the mean field equations

$$\langle \dot{S}^x \rangle = 2\Gamma \langle S^y \rangle \quad (201)$$

$$\langle \dot{S}^y \rangle = \frac{8}{N} \langle S^z \rangle \langle S^x \rangle - 2\Gamma \langle S^x \rangle \quad (202)$$

$$\langle \dot{S}^z \rangle = -\frac{8}{N} \langle S^y \rangle \langle S^x \rangle \quad (203)$$

which can be easily written as classical equations of motion using the parametrization

$$\langle \dot{S}^x \rangle = \frac{N}{2} \cos \theta \quad (204)$$

$$\langle \dot{S}^y \rangle = \frac{N}{2} \sin \theta \sin \phi \quad (205)$$

$$\langle \dot{S}^z \rangle = \frac{N}{2} \sin \theta \cos \phi \quad (206)$$

$$(207)$$

From the S^z equation we immediately get

$$\dot{\theta} = -2\Gamma \sin \phi \quad (208)$$

$$(209)$$

while from the equations for S^x, S^y , by multiplying for $\cos \phi$ and $\sin \phi$ and subtracting we get

$$\sin \theta \dot{\phi} = 4 \sin \theta \cos \theta - 2\Gamma \cos \theta \cos \phi \quad (210)$$

These equations conserve the total energy of the systems, which play the role of the classical effective Hamiltonian and reads

$$\varepsilon(\theta, \phi) = -\cos^2 \theta - \Gamma \sin \theta \cos \phi \quad (211)$$

Analogously one can show that this amounts to consider a coherent evolution of the form

$$|\theta, \phi\rangle = \exp\left(\frac{1}{2}\theta e^{i\varphi} S_- - \frac{1}{2}\theta e^{-i\varphi} S_+\right) |S, M\rangle \quad (212)$$

and making stationary the action

$$S = \int dt \langle \theta, \phi | i\partial_t - H | \theta, \phi \rangle \quad (213)$$

Using the conservation of energy we can write the problem in terms of an effective one dimensional classical dynamics. Indeed if we define $m = S^x/N = \cos \theta/2$ we get

$$\dot{m} = -\frac{1}{2} \sin \theta \dot{\theta} = \Gamma \sin \theta \sin \phi \quad (214)$$

and using the fact that

$$\sin \phi = \sqrt{1 - \cos^2 \phi} = \sqrt{1 - \left(\frac{\varepsilon + \cos^2 \theta}{\Gamma \sin \theta}\right)^2} \quad (215)$$

we finally get

$$\dot{m} = \pm \sqrt{f(m)} \quad (216)$$

with the effective potential

$$f(m) = \Gamma^2(1 - 4m^2) - (\varepsilon + 4m^2)^2 \quad (217)$$

We also notice that in terms of the variable m, ϕ the classical energy reads

$$E(m, \phi) = -4m^2 - \Gamma\sqrt{1 - 4m^2} \cos \phi \quad (218)$$

Before turning into the discussion of the resulting dynamics, let's conclude by presenting another (more rigorous) derivation of the semiclassical dynamics where corrections $1/N$ are explicitly present. Let's go back to the fully connected TFI Hamiltonian,

$$H = -\frac{4J}{N}(S^x)^2 - 2\Gamma S^z \quad (219)$$

and let's study the dynamics at finite N . We already mentioned that the total spin is conserved, $[H, \mathbf{S}^2] = 0$, so one can block diagonalize the Hamiltonian in sectors of the Hilbert space at fixed S . Similarly one can study the dynamics in these subspaces. To this extent let's decompose the wave function in the basis states at fixed S, M , where M is the eigenvalue of S^x

$$|\Psi(t)\rangle = \sum_{M=-S}^S \psi_M(t)|S, M\rangle \quad (220)$$

and apply the time dependent Schroedinger equation $i\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$ to get

$$\begin{aligned} \sum_M i\partial_t \psi_M(t)|S, M\rangle &= -\frac{4J}{N} \sum_M \psi_M(t)(S^x)^2|S, M\rangle - 2\Gamma \sum_M \psi_M(t)S^z|S, M\rangle = \\ &= -\frac{4J}{N} \sum_M M^2 \psi_M(t)|S, M\rangle - 2\Gamma \sum_M \psi_M(t) \left(\sqrt{(S-M)(S+M+1)}|S, M+1\rangle + \sqrt{(S+M)(S-M+1)}|S, M-1\rangle \right) \end{aligned} \quad (221)$$

If we now project back on the state $|S, M'\rangle$ we get an equation for the amplitudes

$$i\partial_t \psi_M(t) = -\frac{4JM^2}{N} \psi_M(t) - 2\Gamma \sum_{\alpha=\pm} \sqrt{S(S+1) - M(M+\alpha)} \psi_{M+\alpha}(t) \quad (222)$$

We now introduce the reduced magnetization $m = 2M/N$ and the reduced spin $S = N(1 - 2k)/2$ such that we can re-write this equation as

$$\left(\frac{1}{N}\right) i\partial_t \psi_m(t) = -Jm^2 \psi_m(t) - 2\Gamma \sqrt{(1-2k)^2 - m^2} \sum_{\alpha} \psi_{m+\alpha}(t) \quad (223)$$

We now can use the fact that $\psi_{m+a} = e^{a\partial_m} \psi_m$ and finally obtain

$$\left(\frac{1}{N}\right) i\partial_t \psi_m(t) = -Jm^2 \psi_m(t) - 2\Gamma \sqrt{(1-2k)^2 - m^2} \cos(\partial_m) \psi_m(t) = H_{eff} \psi_m(t) \quad (224)$$

which shows how the dynamics at finite N is quantum mechanical with an effective planck constant of order $1/N$. If one wants to study the limit of large N this corresponds to do a semiclassical analysis (WKB) of the quantum problem, i.e. solving a classical dynamics for the degrees of freedom m and its conjugate $\phi \equiv \partial_m$ with an Hamiltonian

$$H_{eff} = -Jm^2 - 2\Gamma \sqrt{(1-2k)^2 - m^2} \cos \phi \quad (225)$$

which matches the result we have obtained with the mean field decoupling.

1. Dynamical Restoration of the Symmetry

Let's now specify the quench and the initial condition. Let's assume to start from $\Gamma_0 = 0$, corresponding to an ideal classical Ising ferromagnet and to switch on a finite Γ . The initial values of the angles θ, ϕ therefore corresponds to

$\theta = 0$, with arbitrary ϕ , and the conserved energy reads $\varepsilon = -1$. Then one can easily show that the effective potential has, for $\Gamma < 1$ four roots

$$m_{eq} = \pm 1/2 \quad m_* = \pm \sqrt{\frac{1 - \Gamma^2}{2}} \quad (226)$$

where $m_{eq} = 1/2$ is the initial equilibrium condition which is always a stationary point of the dynamics. The motion of this one dimensional problem is confined in the regions which are classically allowed, corresponding to $f(m) > 0$ which are nothing but the regions $[m_-, 1/2]$. The motion is periodic and one can easily compute the period of these oscillations

$$T = \int_{m_-}^{1/2} \frac{dm}{\sqrt{16(1/4 - m^2)(m^2 - m_-^2)}} \quad (227)$$

As the transverse field approaches the value $\Gamma_c^{dyn} = 1$ the root $m_- \rightarrow 0$ and the dynamics changes qualitatively. Right at the critical point the order parameter relaxes to zero exponentially

$$dt = \int_0^{1/2} \frac{dm}{m\sqrt{1/4 - x^2}} \rightarrow m(t) \sim \exp(-\gamma t) \quad (228)$$

Above the threshold, for $\Gamma > 1$, the shape of the effective potential changes since now only two roots are present, namely $m_{eq} = \pm 1/2$ which always remain a stationary point. In other words in this regime the order parameter oscillates freely around zero, i.e. the symmetry is dynamically broken.

It is interesting to notice the relation between this dynamical critical point and the quantum phase transition in equilibrium. Indeed if one solves for the model at $T = 0$ using the same mean field decoupling one finds the self-consistent equation

$$m = \frac{2Jm}{\sqrt{16J^2m^2 + \Gamma^2}} \tanh \beta \sqrt{16J^2m^2 + \Gamma^2} \quad (229)$$

which has a quantum critical point at $\Gamma_c = 2J$. Therefore the dynamical transition, corresponding to a restoration of the symmetry, happens well within the ordered phase.

C. The Role of Dephasing in the Dynamical Transition: Quenches in the AFM Hubbard Model

We have seen that at the mean field level the dynamical transition is sharp but lacks for example some basic feature like the effect of damping. Now we discuss how one can improve this result by including some sort of fluctuations. To see this point it is useful to go back again to the Hubbard model

$$H = - \sum_{ij} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i (n_i - 1)^2 \quad (230)$$

While earlier we considered the Fermi Liquid regime, where the state remains fully symmetric, the HM has a wide range of broken symmetry phases. We focus on the more conventional one, the antiferromagnetic (AFM) pattern, which typically arises in hypercubic lattice (cubic lattice in d dimension). The critical temperature has the characteristic bell shape, starting small at weak coupling (BCS-like) and going as $J \sim t_{ij}^2/U$ at large interaction.

We are now going to study quenches in this weak coupling regime using time-dependent Hartree Fock. The idea is, as previously, to decouple the interaction with a time-dependent and spin dependent average density profile, i.e.

$$\langle n_{i\sigma} \rangle_t = \frac{1}{2} + m(t)\sigma(-1)^i \quad (231)$$

i.e. in other words the order parameter reads

$$\langle n_{i\uparrow} \rangle_t - \langle n_{i\downarrow} \rangle_t = (-1)^i m(t) \quad (232)$$

The resulting Hamiltonian reads

$$H_{eff} = - \sum_{ij} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} - m(t)U \sum_i (-1)^i (n_{i\uparrow} - n_{i\downarrow}) = \sum_{k\sigma} \varepsilon_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma} - Um(t) \sum_k \left(c_{k\uparrow}^\dagger c_{k+Q\uparrow} - c_{k\downarrow}^\dagger c_{k+Q\downarrow} \right) \quad (233)$$

where we have used the fact that $e^{iQRi} = (-1)^i$ and we have to satisfy the self-consistency condition

$$m(t) = \sum_i (-1)^i [\langle n_{i\uparrow} \rangle_t - \langle n_{i\downarrow} \rangle_t] \quad (234)$$

(to complete: derive equations of motion for the momentum distributions and show they admit an extensive set of conserved quantities, basically the length of a pseudospin).

IV. PERIODICALLY DRIVEN ISOLATED QUANTUM SYSTEMS

A. Intro to Floquet Theory - First formulation

Let's now discuss a first formulation of Floquet theory. Let us consider an isolated quantum system which is periodically driven, such that its Hamiltonian is periodic

$$H(t) = H(t + T)$$

According to the Floquet theorem the time-dependent Schroedinger solution admits a complete set of (quasi-)periodic solutions (Floquet Basis)

$$|\Psi_\alpha(t)\rangle = e^{-i\varepsilon_\alpha t} |\Phi_\alpha(t)\rangle \quad (235)$$

where the Floquet mode is periodic $|\Phi_\alpha(t+T)\rangle = |\Phi_\alpha(t)\rangle$ and ε_α are the so called Floquet quasi-energies. To see this we can apply the operator $(H(t) - i\partial_t)$ to a given Floquet mode to get

$$(H(t) - i\partial_t) |\Phi_\alpha(t)\rangle = \varepsilon_\alpha |\Phi_\alpha(t)\rangle + e^{i\varepsilon_\alpha t} (H - i\partial_t) |\Psi_\alpha\rangle \quad (236)$$

from which it immediately follows that $|\Psi_\alpha\rangle$ is solution provided we impose

$$(H(t) - i\partial_t) |\Phi_\alpha(t)\rangle = \varepsilon_\alpha |\Phi_\alpha(t)\rangle \quad (237)$$

Given a Floquet mode $|\Phi_\alpha(t)\rangle$ one can obtain another one by a phase transformation

$$|\tilde{\Phi}_\alpha(t)\rangle = e^{-in\Omega t} |\Phi_\alpha(t)\rangle \quad (238)$$

this new mode satisfy the same Schroedinger equation with quasi-energy

$$\varepsilon_\alpha - n\Omega \quad (239)$$

As a result the quasi-energies are defined modulo an integer multiple of Ω which means we can restrict ourselves to the interval $[-\Omega/2, \Omega/2]$. Such a restriction is completely similar to the Bloch theorem for quasi-periodic real-space wave functions.

Since the Floquet modes are periodic it is useful to expand them in harmonics

$$|\Phi_\alpha(t)\rangle = \sum_n e^{-in\Omega t} |\Phi_\alpha^n\rangle \quad (240)$$

and to write the eigenvalue equation for the quasi-energies in this basis to get firsts

$$\sum_n (H(t) - n\Omega) |\Phi_\alpha^n\rangle e^{-in\Omega t} = \varepsilon_\alpha \sum_n |\Phi_\alpha^n\rangle e^{-in\Omega t} \quad (241)$$

then if one introduces a proper scalar product and use the orthogonality condition

$$\frac{1}{T} \int_0^T dt \exp(i(m-n)t) = \delta_{m,n} \quad (242)$$

finally ends up with the result

$$\sum_n (H_{mn} - \delta_{mn} m\Omega) |\Phi_\alpha^n\rangle = \varepsilon_\alpha |\Phi_\alpha^n\rangle \quad (243)$$

with

$$H_{mn} = \frac{1}{T} \int_0^T dt e^{i(m-n)\Omega t} H(t) \quad (244)$$

The previous equation takes the form of a conventional eigenvalue equation in an enlarged Hilbert space that now contains the space of harmonics as well.

B. Floquet Theory, second formulation

We now further elaborate on the Floquet theory, considering it now in the Heisenberg rather than Schroedinger picture, where it will take the form of a specific ansatz on the evolution operator. For the sake of simplicity we assume to switch on the periodic driving at $t = 0$ and to consider the evolution operator up to time t which is of the form

$$U(t, 0) = T \exp \left(-i \int_0^t dt' H(t') \right) \quad (245)$$

and satisfies as well the SE, $i\partial_t U(t) = H(t)U(t)$ with initial condition $U(0, 0) = 1$. A general property of the periodic evolution operator is that if one evaluates it at times which are multiples of the period, $t = kT$, then one can easily prove

$$U(kT, 0) = (U(T, 0))^k \quad (246)$$

this result follows immediately from the definition above. Indeed we have

$$U(kT, 0) = T \exp \left(-i \int_0^{kT} dt' H(t') \right) = T \exp \left(-i \sum_{n=0}^{k-1} \int_{nT}^{(n+1)T} dt' H(t') \right) = \quad (247)$$

$$= T \exp \left(-i \sum_{n=0}^{k-1} \int_{nT}^{(n+1)T} dt' H(t') \right) = T \exp \left(-i \sum_{n=0}^{k-1} \int_0^T dt'' H(t'' + nT) \right) = \quad (248)$$

$$T \exp \left(-i \sum_{n=0}^{k-1} \int_0^T dt'' H(t'') \right) = T \prod_n \exp \left(-i \int_0^T dt'' H(t'') \right) = \prod_n U(T, 0) = (U(T, 0))^k \quad (249)$$

This result is very suggestive, since it essentially says that if one observe the system *stroboscopically*, that is at integer multiples k of the fundamental period, the evolution operator is equivalent to applying k times the same operator $U(T, 0)$ which acts therefore as it was a time independent Hamiltonian. In other words this suggests to define a time-independent Floquet Hamiltonian as

$$U(T, 0) \equiv e^{-iH_F T} \quad (250)$$

But there's more to the role of the evolution in a single cycle. If one takes the Schroedinger equation with a time-periodic Hamiltonian it is easy to see that if $|\Psi_\alpha(t)\rangle$ is a solution, then also $|\Psi_\alpha(t+T)\rangle$ it is a solution, since

$$i\partial_t |\Psi_\alpha(t+T)\rangle = H(t) |\Psi_\alpha(t+T)\rangle \rightarrow \quad (251)$$

$$\rightarrow i\partial_{t+T} |\Psi_\alpha(t+T)\rangle = H(t+T) |\Psi_\alpha(t+T)\rangle \quad (252)$$

Therefore there should be an operator which connects these two solutions. It is easy to see using the semigroup properties of the unitary operator that

$$|\Psi_\alpha(t+T)\rangle = U(t+T, 0) |\Psi_\alpha(0)\rangle = U(t+T, 0) U^\dagger(t, 0) U(t, 0) |\Psi_\alpha(0)\rangle = U(t+T, t) |\Psi_\alpha(t)\rangle \quad (253)$$

namely the two solutions at any time t are connected by the evolution operator during a cycle T . If we plug now the Floquet ansatz for the solution of the t.d. Schroedinger equation we have

$$U(t+T, t) |\Psi_\alpha(t)\rangle = U(t+T, t) |\Phi_\alpha(t)\rangle e^{-i\varepsilon_\alpha t} = \quad (254)$$

$$e^{-i\varepsilon_\alpha(t+T)} |\Phi_\alpha(t+T)\rangle = e^{-i\varepsilon_\alpha(t+T)} |\Phi_\alpha(t)\rangle \quad (255)$$

from which we conclude that the Floquet modes are the eigenstates of the evolution operators during a period T with the eigenvalues given by $e^{-i\varepsilon_\alpha T}$, i.e.

$$U(t+T, t) |\Phi_\alpha(t)\rangle = e^{-i\varepsilon_\alpha(t+T)} |\Phi_\alpha(t)\rangle \quad (256)$$

which suggests the decomposition

$$U(t+T, t) = \sum_\alpha e^{-i\varepsilon_\alpha T} |\Phi_\alpha(t)\rangle \langle \Phi_\alpha(t)| \quad (257)$$

which evaluated at time $t = 0$ gives

$$U(T, 0) = e^{-iH_F T} = \sum_{\alpha} e^{-i\varepsilon_{\alpha} T} |\Phi_{\alpha}(0)\rangle \langle \Phi_{\alpha}(0)| \quad (258)$$

In addition, one can obtain for the evolution operator at time t within the first cycle the result

$$U(t, 0) = \sum_{\alpha} e^{-i\varepsilon_{\alpha} t} |\Phi_{\alpha}(t)\rangle \langle \Phi_{\alpha}(0)| \quad (259)$$

which can be easily verified by inspection. Indeed if we take the time derivative of both sides we get

$$i\partial_t U(t, 0) = \sum_{\alpha} \varepsilon_{\alpha} e^{-i\varepsilon_{\alpha} t} |\Phi_{\alpha}(t)\rangle \langle \Phi_{\alpha}(0)| + \sum_{\alpha} e^{-i\varepsilon_{\alpha} t} i\partial_t |\Phi_{\alpha}(t)\rangle \langle \Phi_{\alpha}(0)| = H \sum_{\alpha} e^{-i\varepsilon_{\alpha} t} |\Phi_{\alpha}(t)\rangle \langle \Phi_{\alpha}(0)| \quad (260)$$

From this result we see that Floquet eigenmodes and quasi-energies play for H_F a very similar role to the standard eigenstates and eigenvalues of a static Hamiltonian in terms of a natural basis onto which one can expand any initial state and study easily the stroboscopic dynamics. The above results also suggest practical numerical approaches to compute solve the Floquet problem.

One possibility involves computing the evolution operator up to time T , solving the equation of motion in matrix form from $t = 0$ up to T and use Eq. (REF) to diagonalize it and find the Floquet modes at time $t = 0$ and the quasi-energies. Then one can just propagate these states to get them in the full first cycle. Otherwise one can solve for the evolution operator in the full first cycle $U(t, 0)$ and then diagonalize it (i.e. get right and left eigenvectors) to obtain directly the Floquet modes over the first cycle.

C. Floquet Approach and the Transformation to Rotating Frame

Let's start from some very general consideration about time-dependent unitary transformations. Let's assume to have a system satisfying the td Schroedinger equation

$$i\partial_t |\Psi(t)\rangle = H(t) |\Psi(t)\rangle \quad (261)$$

Let's introduce a time-dependent unitary transformation

$$|\Psi(t)\rangle = \Omega(t) |\Phi(t)\rangle \quad (262)$$

and we would like to ask what kind of equation of motion is satisfied by the state $|\Phi(t)\rangle$. This can be easily found by computing the time-derivative of the state $|\Phi(t)\rangle = \Omega^{\dagger}(t) |\Psi(t)\rangle$ to get

$$i\partial_t |\Phi(t)\rangle = (i\partial_t \Omega^{\dagger}) |\Psi(t)\rangle + \Omega^{\dagger} i\partial_t |\Psi(t)\rangle = ((i\partial_t \Omega^{\dagger}) \Omega + \Omega^{\dagger} H \Omega) |\Phi(t)\rangle \quad (263)$$

from which we get that the transformed Hamiltonian reads

$$\tilde{H} = (i\partial_t \Omega^{\dagger}) \Omega + \Omega^{\dagger} H \Omega \quad (264)$$

It is also interesting to define a transformed evolution operator, such that

$$\tilde{U}(t, 0) = \Omega^{\dagger}(t) U(t, 0) \Omega(0) \quad (265)$$

which can be easily shown to satisfy the equation

$$i\partial_t \tilde{U}(t, 0) = \tilde{H} \tilde{U}(t, 0) \quad (266)$$

as one can immediately see from inspection and using the definition of $\tilde{U}(t, 0)$.

Let's now consider the following periodic Hamiltonian

$$H = H_0(S^2, S_z) + \lambda(S_x \cos \omega t - S_y \sin \omega t) = H_0(S^2, S_z) + \frac{\lambda}{2}(S_+ e^{i\omega t} + S_- e^{-i\omega t}) \quad (267)$$

describing a quantum spin S , with components $S_{x,y,z}$ driven by circularly polarized light. Here H_0 includes the rotationally invariant part of the Hamiltonian as well as the part diagonal in S_z . Let's now consider a time-dependent unitary transformation generated by S_z of the form

$$\Omega = e^{iS_z \omega t} \quad (268)$$

Under the action of this operator we know that

$$\Omega^\dagger S_x \Omega = S_x \cos \omega t + S_y \sin \omega t \quad (269)$$

$$\Omega^\dagger S_y \Omega = -S_x \sin \omega t + S_y \cos \omega t \quad (270)$$

$$\Omega^\dagger S_z \Omega = S_z \quad (271)$$

as one can easily show using the spin algebra $[S_\alpha, S_\beta] = -i\varepsilon_{\alpha\beta\gamma}S_\gamma$. Clearly the rotation is such that the rotational invariant part, function of $S^2 = S_x^2 + S_y^2 + S_z^2$ remains constant. The rotated Hamiltonian becomes

$$\tilde{H} = H_0 + \lambda(\cos \omega t (S_x \cos \omega t + S_y \sin \omega t) - \sin \omega t (-S_x \sin \omega t + S_y \cos \omega t)) + \omega S_z = H_0 + \omega S_z + \lambda S_x \quad (272)$$

namely in the transformed frame the problem becomes time-dependent. The evolution operator in the rotated frame is therefore just $\tilde{U}(t) = e^{-i\tilde{H}t}$, and when transformed back we get

$$U(t, 0) = \Omega(t)e^{-i\tilde{H}t}\Omega^\dagger(0) = e^{iS_z\omega t}e^{-i\tilde{H}t} \quad (273)$$

which assumes the Floquet form of a periodic operator times an evolution with a static (time-independent) Hamiltonian.

We can cast the Floquet approach in this language, which is another angle to the Floquet problem. Let's consider a periodic Hamiltonian $H(t+T) = H(t)$ and let's look for a unitary "gauge" transformation

$$\Omega(t) = e^{iF(t)} \quad (274)$$

generated by some periodic hermitian operator $F(t+T) = F(t)$ such that in the new gauge the transformed Hamiltonian i.e.

$$\tilde{H} = (i\partial_t e^{-iF}) e^{iF} + e^{-iF} H e^{iF} \quad (275)$$

is time independent. If we assume this operator F exists and if therefore \tilde{H} is time-independent, it is immediate to conclude that the Floquet theorem holds. Indeed let's then call the eigenfunctions of the new Hamiltonian \tilde{H} with $|\Phi_\alpha(0)\rangle$, their dynamics induced by the effective Hamiltonian is just a trivial phase with an eigenvalue ε_α

$$e^{-i\tilde{H}t}|\Phi_\alpha(0)\rangle = e^{-i\varepsilon_\alpha t}|\Phi_\alpha(0)\rangle \quad (276)$$

while in the original frame the time evolution corresponds to the action of the operator e^{iF} , i.e. in other words

$$|\Psi_\alpha(t)\rangle = e^{iF(t)} e^{-i\varepsilon_\alpha t} |\Phi_\alpha(0)\rangle = e^{-i\varepsilon_\alpha t} |\Phi_\alpha(t)\rangle \quad (277)$$

where we can now identify the Floquet state with the periodic function $e^{iF(t)}|\Phi_\alpha(0)\rangle$. Interesting we have that the unitary operator in the original frame reads

$$U(t, 0) = e^{iF(t)} e^{-i\tilde{H}t} e^{-iF(0)} \quad (278)$$

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