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An ensemble theory of ideal quantum measurement processes

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Abstract A minimalist theory of ideal quantum measurements is presented. The tested system S and the apparatus A are treated as a single closed system, governed by standard quantum statistical mechanics, which relaxes towards generalised thermodynamic equilibrium. Our results are supported by detailed dynamic calculations, but they can be also derived at the level of thermodynamics, with a qualitative account of relaxation mechanisms. A quantum formalism without interpretation is first used, where the density operator encodes knowledge about properties of a statistical ensemble ~~or of its subensembles???~~ . In first step, the desired form for the final state of S+A associated with a large set of runs is thus derived, under some specified conditions to be fulfilled by the Hamiltonian. However, ~~due to a quantum ambiguity~~ *I propose here and also below*: due to the occurrence of spooky terms/decompositions, this is not sufficient to account for the occurrence of a well defined outcome for each individual run of the ensemble. Therefore, in a second step, a stronger result is established, concerning all possible subensembles of runs. Their associated density operators are shown to relax towards the required structure owing to a specific dynamical mechanism that acts near the end of the process. The equations thus formally obtained are interpreted in the last step by means of postulates which relate macrophysics to microphysics and pertain more to A than to S. The properties currently attributed to ideal measurements are thereby recovered most economically, and the status of Born's rule is ~~re-examined~~ redefined.
Khrennikov speaks about pre-probabilities, we about q-probabilities. Persist with that??

1 Introduction: the measurement problem

> *If one wants to be clear about what is meant by "position of an object," for example of an electron..., then one has to specify definite experiments by which the "position of an electron" can be measured; otherwise this term has no meaning at all.*
>
> Werner Heisenberg [1]

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After decades of discussion, one of the main foundational challenges of quantum theory is still the so-called measurement problem, which is closely related to the interpretation of quantum mechanics: Why can we make well-defined statements about an individual quantum system at the end of an ideal measurement in spite of the irreducibly probabilistic nature of quantum theory, which, as we shall stress, can be viewed as dealing with statistical ensembles? Or does measurement theory require a specific principle or interpretation of quantum mechanics? In particular, what is the status of Born's rule? What is the role of the apparatus? How can repeated measurements issued from the same initial state provide different individual outcomes? Already raised by the founding fathers, this crucial question, related to the connection between microscopic and macroscopic concepts has witnessed a revival [2–7,9]. Many alternative interpretations of quantum mechanics have been proposed, many theories going beyond it have been developed and many models have been explored, but they have not led to a consensus.

Here we analyse ideal measurements as dynamical processes undergone by a compound system S+A constituted by the tested system S coupled to the measuring apparatus A and governed by standard quantum statistical mechanics. Our scope is double, first *technical* then *conceptual*. In both respects, we wish to understand, in the most economic fashion (though completely), the various features of ideal measurements. Recognising first that the process amounts to relaxation towards thermodynamic equilibrium (Sec. 4), we describe this evolution within a purely formal Hamiltonian approach, postponing any interpretation (Secs. 5-7). Afterwards, we will introduce (Secs.8-10), the smallest possible number of principles (or postulates) needed to tackle the measurement problem, that is, to provide an interpretation in terms of physical individual runs to the formal results previously obtained.

As the quantum equations of motion refer to a large set of runs and not to individual runs, we will introduce, as an intermediate step, not only this large ensemble but also its *arbitrary subensembles*. (The ensemble formulation of quantum mechanics is recalled in Sec. 3.) They will be shown to undergo a specific mechanism of relaxation [7] which will help us to tackle the measurement problem in the end. An essential feature is the *macroscopic size* of the apparatus, which forces us to rely on non-equilibrium quantum statistical mechanics. We include in A a possible thermal bath or environment, so that S+A is an isolated quantum system governed by a Hamiltonian. Though large, the apparatus is treated as a finite object so as to keep control of the dynamics and of time scales.

We deal with ideal, non demolishing measurements. Their purpose is to test a single observable $\hat{s} = \sum_i s_i \hat{\pi}_i$ of S characterised by its (discrete) eigenvalues s_i and the associated eigenprojectors $\hat{\pi}_i$, while perturbing S minimally. Although ideal measurements are not currently performed in real experiments, their theoretical elucidation is a necessary step towards a full understanding of actual measurement processes, since any general quantum measurement (POVM) can be represented as a partial trace over an ideal measurement [13]. We denote by $\hat{\mathcal{D}}(t)$ the joint density operator of S+A for a large ensemble $\mathcal{E}$ of runs, and by $\hat{r}(t) = \mathrm{tr}_{\mathrm{A}}\hat{\mathcal{D}}(t)$ and $\hat{\mathcal{R}}(t) = \mathrm{tr}_{\mathrm{S}}\hat{\mathcal{D}}(t)$ the marginal density operators of S and A, respectively. At the initial time $t = 0$, S and A are uncorrelated, S lies in some state $\hat{r}(0)$ to be tested and A in a *metastable state*[1] $\hat{\mathcal{R}}(0)$.

The evolution of $\hat{\mathcal{D}}(t)$ is governed by a Hamiltonian including a suitable interaction $\hat{H}_{\mathrm{SA}}$. Ideality involves separation of S from A after the measurement is achieved, so that $\hat{H}_{\mathrm{SA}}$ will be switched off at the very last stage of the process. Then, initially triggered by this coupling, the apparatus will eventually relax towards one or another among its *stable states* $\hat{\mathcal{R}}_i$, which should have equal entropies and energies so as to avoid measurement bias. These states can be distinguished from one another through observation or registration of the pointer variable A_i, associated with some collective macroscopic observable $\hat{A}$ of A and identified as its expectation value $A_i = \mathrm{tr}_{\mathrm{A}}\hat{\mathcal{R}}_i\hat{A}$. The spectrum of $\hat{A}$ is dense. If Δ denotes the width of the distribution $\hat{\mathcal{R}}_i$ as regards $\hat{A}$, many eigenvalues of $\hat{A}$ lie in the interval $(A_i - \Delta, A_i + \Delta)$ while the condition $\Delta \ll |A_i - A_j|$ for $i \neq j$ ensures a neat distinction between the possible outcomes of A. We wish the final indication A_i of the pointer to be fully correlated with the eigenvalue s_i of the tested observable $\hat{s}$ and with the production of the final state $\hat{r}_i$ for S.

[1] This initial metastable state of A is often called "ready state"

The analysis of the process issued from the state $\hat{\mathcal{D}}(0)$ should therefore explain why we can assign to S+A at a final time t_f, *for each run of the measurement, one among the states* [14]

$$\hat{\mathcal{D}}_i = \hat{r}_i \otimes \hat{\mathcal{R}}_i, \qquad \hat{r}_i = \frac{1}{p_i}\hat{\pi}_i \hat{r}(0) \hat{\pi}_i. \tag{1}$$

The probability $\mathrm{tr}\ \hat{\mathcal{D}}(t_\mathrm{f})\hat{\Pi}_i$ of occurrence of the pointer outcome A_i should be given by Born's rule $p_i = \mathrm{tr}\ \hat{\mathcal{D}}(t_\mathrm{f})\hat{\pi}_i = \mathrm{tr}_\mathrm{S}\, \hat{r}(0)\hat{\pi}_i$. In particular, the measurement problem consists in understanding theoretically why an individual run provides one among the outcomes (1), whereas $\hat{\mathcal{D}}(0) = \hat{r}(0) \otimes \hat{\mathcal{R}}(0)$ is a probabilistic description of the initial state of S+A for a *large ensemble* $\mathcal{E}$ of runs.

2 Outline: ideal measurements in a subensemble based approach

We wish to uncover the most economic hypotheses and principles needed to explain the above expected features of ideal measurements. We will first describe the dynamics of S+A by using standard methods of quantum (statistical) mechanics (Secs. 5-7). We will thus exhibit the features of the Hamiltonian necessary to yield final density operators in agreement with the desired result (1). Such a formal framework, compatible with arbitrary interpretations, applies only to statistical ensembles. Extrapolation to individual runs requires principles that afford a physical interpretation to the mathematical results thus obtained. We will therefore introduce (Sec. 8) the weakest possible interpretative principles or postulates that solve the measurement problem. They are more generally relevant for interpretation of quantum mechanics. We will proceed in three steps, involving successively (*i*) the full ensemble of runs, (*ii*) its subensembles and (*iii*) the individual runs.

(*i*) *Full ensemble.* Most of the literature about solutions of measurement models is restricted to this step. Here it is sufficient to use a minimalist formulation of quantum theory (Sec. 3) without prejudging any specific interpretation. The isolated compound system S+A is regarded as a sample extracted from a large statistical ensemble $\mathcal{E}$, and its density operator $\hat{\mathcal{D}}(t)$ encodes the probabilistic properties of this ensemble at the time t. All elements of $\mathcal{E}$ are identically prepared in the state $\hat{\mathcal{D}}(0)$ and the dynamics of $\hat{\mathcal{D}}(t)$ is governed by the Liouville–von Neumann equation. Working out current methods of quantum statistical mechanics and assuming some suitable properties of the Hamiltonian of S+A, it can then be shown (Sec. 5) that the density matrix $\hat{\mathcal{D}}(t)$ (in a basis where $\hat{s}$ is diagonal) is subject to two independent relaxation processes, with different timescales, the decay of the off-diagonal blocks of $\hat{\mathcal{D}}(t)$ ("truncation") and the establishment of correlations between S and A in the final diagonal blocks $\hat{\mathcal{D}}_i$ ("registration"). At the final time t_f, the density operator that describes the collection of systems S+A belonging to $\mathcal{E}$ thus reaches

$$\hat{\mathcal{D}}(t_\mathrm{f}) = \sum_i p_i \hat{\mathcal{D}}_i, \qquad \hat{\mathcal{D}}_i = \hat{r}_i \otimes \hat{\mathcal{R}}_i. \tag{2}$$

We will see (Sec. 4) that $\hat{\mathcal{D}}(t_\mathrm{f})$ is a *generalised Gibbs state*, so that the evolution from $\hat{\mathcal{D}}(0)$ to $\hat{\mathcal{D}}(t_\mathrm{f})$ can be regarded as a relaxation towards a *thermodynamic equilibrium state*.

The result (2) thus derived also arises as a consequence of the two properties that we wish to explain, the expected outcome (1) of individual runs and Born's rule. However, the converse is not true for quantum reasons. If we were dealing with an ensemble of classical objects rather than a quantum ensemble, we would be entitled to interpret Eq. (2) as meaning that the ensemble $\mathcal{E}$ described by $\hat{\mathcal{D}}(t_\mathrm{f})$ at the end of the process might be split into subensembles $\mathcal{E}_i$, each of which being described by $\hat{\mathcal{D}}_i$ and containing a proportion p_i of runs. We shall see below why this naive interpretation is unfounded, so that having established (2) is *not sufficient* to allow the identification of the operators $\hat{\mathcal{D}}_i$ with quantum states and of the coefficients p_i with ordinary probabilities. Other results complementing (2) are needed.

(*ii*) *Subensembles.* To this aim, we will make a second step from $\mathcal{E}$ towards individual runs, while remaining within the ensemble formulation of quantum theory. Inspired by the frequency theory of classical probabilities [25] based on the structure of the set of all possible subensembles of events, we

note that different density operators may simultaneously be ascribed to the same system, depending on the information available about it, that is, on the ensemble in which it is embedded. This is standard in probability theory: when a dice is repeatedly thrown, the probability of the outcome "3" is $\frac{1}{6}$ for the full set of runs; it is $\frac{1}{3}$ when only the odd outcomes ("1", "3", "5") are selected and the even ones discarded; it is $\frac{1}{2}$ for a selection of the middle ones ("3" or "4"), and 1 for the subensemble containing only the outcome "3". Likewise, here (Sec. 3), the ensemble $\mathcal{E}$ is a large collection of individual systems S+A identically prepared and described by $\hat{\mathcal{D}}(t)$, while a subensemble $\mathcal{E}^{(k)}_{\rm sub}$ is a (still large) collection of elements extracted from $\mathcal{E}$ by means of some criterion, and the state $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t)$ describing $\mathcal{E}^{(k)}_{\rm sub}$ may well differ from $\hat{\mathcal{D}}(t)$. However, a specifically quantum difficulty arises. The construction of classical subensembles relies on the possibility of distinguishing individual events so as to select a part of them. Here, due to a quantum ambiguity (see below and in Sec. 6), the probabilistic description afforded by $\hat{\mathcal{D}}(t)$ for the full ensemble $\mathcal{E}$ does not allow such a theoretical recognition of the individual elements of $\mathcal{E}$ (although they are evidently distinguished in repeated experiments). In measurements, this turns out to be feasible only at the macroscopic scale and after achievement of the process, as the indications A_i of the pointer yield a criterion for selection. A subensemble $\mathcal{E}^{(k)}_{\rm sub}$ will thus be characterised after the final time $t_{\rm f}$ by the relative frequencies $q_i^{(k)}$ of runs having yielded the macroscopic outcome A_i and thus expected to have produced the state $\hat{\mathcal{D}}_i$ expressed by (1).

Hence any subensemble $\mathcal{E}^{(k)}_{\rm sub}$ of runs extracted from $\mathcal{E}$ should end up in a state of S+A of the form

$$\hat{\mathcal{D}}^{(k)}_{\rm sub}(t_{\rm f}) = \sum_i q_i^{(k)} \hat{\mathcal{D}}_i, \qquad \hat{\mathcal{D}}_i = \hat{r}_i \otimes \hat{\mathcal{R}}_i. \tag{3}$$

Here again as for (2), this desired form (3) is in thermodynamic equilibrium (Sec. 4). We need therefore to derive this result, and we foresee that the minimalist ensemble (or subensemble) formulation of quantum mechanics may suffice although it does not deal with individual systems.

Once the expression (2) of $\hat{\mathcal{D}}(t_{\rm f})$ for $\mathcal{E}$ has been derived, its structure might suggest that it implies the form (3) of $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t_{\rm f})$ for $\mathcal{E}^{(k)}_{\rm sub}$. Such a deduction, which would be straightforward for classical distributions, is prohibited in quantum mechanics for the following reason. Let us associate with the subensemble $\mathcal{E}^{(k)}_{\rm sub}$ of $\mathcal{E}$, with relative size λ, its complement $\mathcal{E}^{(k')}_{\rm sub}$. The states describing $\mathcal{E}$, $\mathcal{E}^{(k)}_{\rm sub}$ and $\mathcal{E}^{(k')}_{\rm sub}$, respectively, satisfy $\hat{\mathcal{D}} = \lambda \hat{\mathcal{D}}^{(k)}_{\rm sub} + (1-\lambda)\hat{\mathcal{D}}^{(k')}_{\rm sub}$. Conversely, if only $\hat{\mathcal{D}}$ is given, one can mathematically decompose it, in an infinity of ways, as $\hat{\mathcal{D}} = \mu \hat{\mathcal{D}}_{\rm dec} + (1-\mu)\hat{\mathcal{D}}'_{\rm dec}$, where $\hat{\mathcal{D}}_{\rm dec}$ and $\hat{\mathcal{D}}'_{\rm dec}$ have the properties of quantum states, in particular positivity.[2] In classical statistical mechanics, these two components would describe probabilistically two complementary subsets $\mathcal{E}^{(k)}_{\rm sub}$ and $\mathcal{E}^{(k')}_{\rm sub}$ of individual systems. Such a seemingly natural property is violated here by a *quantum ambiguity* due to the matrix nature of $\hat{\mathcal{D}}_{\rm dec}$ and $\hat{\mathcal{D}}'_{\rm dec}$. As discussed in Sec. 6, physically nonsensical contradictions may occur, forbidding an individual quantum system that would be described by a separate term $\hat{\mathcal{D}}_{\rm dec}$ of most above mathematical decompositions to belong to some physical subensemble of $\mathcal{E}$. Hence, only some specific decompositions of $\hat{\mathcal{D}}$ are physically meaningful. Whereas a state $\hat{\mathcal{D}}^{(k)}_{\rm sub}$ associated with a physical subensemble $\mathcal{E}^{(k)}_{\rm sub}$ is an operator issued from a decomposition of $\mathcal{E}$, we cannot conversely construct physical states by analysing the decompositions of $\hat{\mathcal{D}}$, which yield in general mathematical objects $\hat{\mathcal{D}}_{\rm dec}$ without any physical relevance.

It is thus important to prove Eq. (3), which is not a consequence of (2), and we shall achieve this through a *dynamical mechanism* (Sec. 7). We first postulate that the subensemble $\mathcal{E}^{(k)}_{\rm sub}$, identified after the time $t_{\rm f}$ by the proportions $q_i^{(k)}$ of outcomes A_i, can already be described, at times $t > t'_{\rm f}$, where $t'_{\rm f}$ is *slightly earlier than* $t_{\rm f}$, by a state $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t)$ obeying the standard rules of quantum mechanics. (In its

[2] A multitude of mathematical decompositions is also known when in a quantum problem one is allowed to choose any basis of eigenfunctions.
Roger: Suppress. What you write is misleading. I do not understand and disagree: a change of basis is not a decompositions of a mixed state. Theo: I try to do good, I don't understand my guilt.

minimalist formulation, quantum mechanics does not allow us to make theoretical statements about the subensembles $\mathcal{E}_{\rm sub}^{(k)}$ at early times.) In particular, with the same physics involved, the evolution of $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t)$ between the times $t'_{\rm f}$ and $t_{\rm f}$ is governed by the same Hamiltonian as for $\mathcal{E}$, but the initial condition $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t'_{\rm f})$ at $t = t'_{\rm f}$ is not given. However, we know at least that $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t'_{\rm f})$ results from some decomposition of $\hat{\mathcal{D}}(t'_{\rm f})$. The latter feature imposes on $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t'_{\rm f})$ a constraint, if $t'_{\rm f}$ is chosen sufficiently late so that during the lapse $t'_{\rm f} < t < t_{\rm f}$ the state $\hat{\mathcal{D}}(t)$ of the full ensemble has already reached the form (2) after the interaction between A and S has been switched off.

We shall see (Sec. 7) that these properties are sufficient to prove the desired result (3). We rely on a specific mechanism, the *subensemble relaxation* [7], which is an extension of the known microcanonical relaxation [17-20] and which *involves only the (large) apparatus*. It entails rapid relaxation, on a delay shorter than $t_{\rm f} - t'_{\rm f}$, of both the diagonal and off-diagonal blocks of the density matrix $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t)$, so that the states associated with all subensembles reach at the time $t_{\rm f}$ the anticipated form (3), with the same desired building blocks $\hat{\mathcal{D}}_i$. The weights $q_i^{(k)}$ entering (3) are not known a priori. Owing to the form of the states $\hat{\mathcal{R}}_i$, they are expressed as $q_i^{(k)} = {\rm tr}\, \hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})\hat{\Pi}_i$, where $\hat{\Pi}_i$ denotes the projector onto the eigenspace of $\hat{A}$ associated with the sole eigenvalues entering $\hat{\mathcal{R}}_i$, namely, those lying in a range Δ around A_i. [3] The numbers $q_i^{(k)}$ thus appear as "quantum probabilities" for the indications A_i of the pointer, but cannot yet be interpreted as ordinary probabilities, although they satisfy, when two disjoint subensembles merge, the same additivity property of the quantities $q_i^{(k)}\mathcal{N}^{(k)}$ as ordinary probabilities, see eq. (13) below. Thus, for all possible subensembles, the various final states $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})$ satisfy a *hierarchic structure* characterised by the form (3) and the additivity of the coefficients $q_i^{(k)}$. Hence *the quantum ambiguity will be removed by a dynamical mechanism.*

(*iii*) *Individual runs.* The result (3), much stronger than (2), is the *most detailed property* that conventional quantum theory can afford. It is a necessary condition for ideal measurements, since it is a consequence of the required final states (1) to be assigned to individual runs; but conversely it is still not sufficient to entail (1), because the formal equations of quantum mechanics used to derive (3) neither provide an interpretation for the weights $q_i^{(k)}$ nor ensure that their range extends from 0 to 1, a property necessary to get (1) as a special case of (3)), namely to consider a subensemble $\mathcal{E}_i$ gathering the systems S+A that lie all in the same state $\hat{\mathcal{D}}_i$.

In order to extrapolate (3) towards individual runs, which lie beyond the realm of the ensemble formulation of quantum mechanics, and thus to be able to make statements about experimental facts, we need to supplement the abstract ensemble formulation of Sec. 3 with interpretative principles, presented and discussed in Sec.8. The main one will concern *only macroscopic variables*, here the pointer variables represented microscopically by the projectors $\hat{\Pi}_i$. We will show that the latter observables, when used within the states $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})$ *at the end of the measurement, effectively commute* with the whole algebra, so that they loose their specifically quantum features. The additivity property (13), the same as in the frequency theory of probabilities [25,26], will finally lead us (Sec. 8) to interpret, for any subensemble $\mathcal{E}_{\rm sub}^{(k)}$, each coefficient $q_i^{(k)}$, originally a formal object, as the *proportion of runs of* $\mathcal{E}_{\rm sub}^{(k)}$ *that provide the indication* A_i *of the pointer.*

In spite of its macroscopic nature, this principle will enable us (Sec. 9), owing to the full correlation between A_i and $\hat{r}_i$ created by the dynamics, to draw, at the final time, conclusions about the microscopic system S for individual runs of an ideal measurement, and to acknowledge the emergence of some classical probabilistic concepts for this quantum system S. Only then will $p_i = {\rm tr}\, \hat{\mathcal{D}}(t_{\rm f})\hat{\Pi}_i = {\rm tr}_{\rm S}\hat{r}(0)\hat{\pi}_i$ be interpreted[3] as Born's probability of occurrence of the eigenvalue s_i, and $\hat{r}_i$ as the final state of S in an individual run tagged by A_i. This assignment of von Neumann's *reduced state* $\hat{r}_i$ to S will appear as an *updating of information.*

For specific models, some parts of the above programme have been achieved in the literature [2–7,9]. We consider below arbitrary ideal measurements. Due to the generality of this scope, we will skip the technical, model-dependent aspects of the solution, but will demonstrate its feasibility

[3] It is essential to distinguish the projector $\hat{\pi}_i$ for the system S, associated with the eigenvalue s_i of $\hat{s}$, from the projector $\hat{\Pi}_i$ for the apparatus A, associated with the eigenvalues of $\hat{A}$ located in the range $(A_i - \Delta, A_i + \Delta)$.

by recalling in footnotes the outcomes of the detailed dynamical solution [6, 12] of the Curie–Weiss (CW) model[4].

3 Ensemble formulation of quantum mechanics

We tackle the measurement problem within a statistical ensemble formulation[5] of quantum mechanics. It is a *minimalist*, formal description suited to both microscopic and macroscopic systems. Its spirit is the same as in the C^*-algebraic approach [8–10], although we deal with finite non relativistic systems. It does not prejudge any specific interpretation of quantum oddities [13]. Physical interpretations should emerge at the macroscopic scale, in experimental contexts.

Physical quantities pertaining to a system are described as "observables" represented by Hermitean matrices in a Hilbert space. Observables behave as random objects, but, unlike ordinary random variables, their randomness arises from their non-commutative nature and is inescapable.

In the present formal scope, we regard a "quantum state", whether pure or not, merely as a theoretical tool for making probabilistic statements or predictions about the various observables. It is characterised by a correspondence associating with any observable $\hat{O}$ a real number and implemented as $\hat{O} \mapsto \langle \hat{O} \rangle = \mathrm{tr}\, \hat{\mathcal{D}}\hat{O}$ by means of a Hermitean, normalised and nonnegative density operator $\hat{\mathcal{D}}$. Such a definition[6] looks analogous to that of a state in classical statistical mechanics, encoded in a density in phase space which produces the expectation value of any physical quantity. However, "quantum expectation values" $\langle \hat{O} \rangle$ or "quantum information" (*q-information*, generalising q-bits) encoded in a density operator refer to a veiled reality. They may become physically available solely in special circumstances and solely in part, because our knowledge is limited not only by some ignorance about the values of physical quantities, but by their very nature of operators. An interpretation is then needed to extract from the q-information embedded in $\hat{\mathcal{D}}$ some ordinary probabilistic information affording predictions about real events, as we will see for ideal measurements in Secs. 8 and 9. For the time being, quantities such as a q-expectation $\langle \hat{s} \rangle$, a q-probability $\langle \hat{\pi}_i \rangle$ or a q-correlation $\langle \hat{s}\hat{A} \rangle$ are treated as formal objects. In particular, we will discuss (Secs. 8 and 9) which meaning should be given to the numbers $p_i = \langle \hat{\pi}_i \rangle = \mathrm{tr}_{\mathrm{S}} \hat{r}(0)\hat{\pi}_i$ since we do not postulate Born's rule, and in which sense the measurement process will nevertheless allow us to interpret these formal objects as probabilities[7].

Obviously, in case the q-variance of a macroscopic (or microscopic) observable $\hat{O}$ is negligible in relative value (or vanishes), we can readily interpret in terms of real events its formal q-expectation $\langle \hat{O} \rangle$, by identifying it with the actual physical value of $\hat{O}$. Controlling such variables that take a single value is a means for preparing a quantum state [13], for instance the initial state $\hat{\mathcal{R}}(0)$ of A or

[4] In the CW model [6, sect. 3], S is a spin $\frac{1}{2}$, the measured observable being its z-component $\hat{s}_z$, with outcomes $i = \uparrow$ or $\downarrow$. The apparatus simulates a magnetic dot, including $\mathcal{N} \gg 1$ spins $\hat{\sigma}^{(n)}$, which interact through the Ising coupling J, and a phonon thermal bath at temperature $T < J$; these spins and the phonons are coupled through a dimensionless weak coupling γ. Initially prepared in its metastable paramagnetic state, A may switch to one or the other stable ferromagnetic state. The pointer observable $\hat{A} = N\hat{m} = \sum_{n=1}^{N} \hat{\sigma}_z^{(n)}$ is the total magnetisation in the z-direction of the N Ising spins. The coupling between S and A is $\hat{H}_{\mathrm{SA}} = -\sum_{n=1}^{N} g\hat{s}_z\hat{\sigma}_z^{(n)}$, while $\hat{H}_{\mathrm{S}} = 0$.

[5] The term "statistical ensemble formulation" is intended to recall that "statistical ensembles and subensembles" will be essential in our approach and that quantum mechanics is irreducibly probabilistic. We do not speak here of "statistical interpretation" [2], a term which depends on the authors, but simply of "formulation" because we wish to introduce interpretations only in the end (Sec. 8). We shall abbreviate throughout by "ensemble" the expression "statistical ensemble", which refers to a real or virtual collection of systems characterised by some available information accounted for by a density operator. Increasing this information about some systems implies that they are embedded in a subensemble.

[6] In this algebraic approach, the representation of a state by a density matrix is implemented from the correspondence $\hat{O} \mapsto \langle \hat{O} \rangle$ by taking the set of dyadics $|\eta\rangle\langle\eta'|$ as basis in the vector space of observables $\hat{O}$. The matrix element $\langle \eta'|\hat{\mathcal{D}}|\eta \rangle$ of $\hat{\mathcal{D}}$ is then defined as the q-expectation value of $|\eta\rangle\langle\eta'|$. Other so-called Liouville representations of states, such as the Wigner representation for a particle or the polarisation representation for a spin $\frac{1}{2}$, are defined through other choices of bases (the basis of Pauli operators for the spin).

[7] I degrade to footnote, because it was off the record: Likewise, in quantum computation, q-bits can be manipulated, but only blindly; the q-information that they carry is not available, and extracting from it real information requires a measurement process which destroys part of it.

the one $\hat{r}(0)$ of S in a measurement. Such an interpretation will be further discussed in Sec. 8 and extended to some macroscopic observables that may take several values.

As a standard probability distribution, a quantum state gathers q-information and refers, implicitly or not, to a statistical ensemble $\mathcal{E}$: We should regard an individual system as embedded in a large ensemble $\mathcal{E}$ of systems produced under the same conditions and characterised by the same available information. So a state does not "belong to a system", it is not an intrinsic property. The ensemble $\mathcal{E}$ may be a collection of real systems, as in a repeated experiment; it may also be virtual, containing the real studied system and mental copies.

Moreover, as noted in Sec. 2 with the classical example of the dice, gaining knowledge about an individual system leads to regard it as member of a different ensemble, and to modify its probabilistic description: It should be regarded as an element of some subensemble $\mathcal{E}_{\rm sub}^{(k)}$ of $\mathcal{E}$ accounting for this knowledge, and a new state is thus attributed to it. As a quantum example, take $\mathcal{E}$ for a spin $\frac{1}{2}$ as the union of two equal sized subsets $\mathcal{E}_{\rm sub}^{(k)}$ and $\mathcal{E}_{\rm sub}^{(k')}$, produced in the $+z$ and $-z$ directions, respectively. If we know that a given system is an element of $\mathcal{E}_{\rm sub}^{(k)}$, we should assign to it the state $\hat{\mathcal{D}}_{\rm sub}^{(k)}$ (which is $|z\rangle\langle z|$ in the example) that characterises the q-information associated with $\mathcal{E}_{\rm sub}^{(k)}$. If however we ignore its membership of $\mathcal{E}_{\rm sub}^{(k)}$ and consider it just as an element of $\mathcal{E}$, we can take into account only the (weaker) information associated with $\mathcal{E}$, and should assign to it for any prediction the state $\hat{\mathcal{D}}$ (the unpolarised state $\frac{1}{2}\hat{I}$ for a spin picked at random within $\mathcal{E}$ in the example). In each case, von Neumanns entropy $S(\hat{\mathcal{D}}) = -{\rm tr}\hat{\mathcal{D}}\ln\hat{\mathcal{D}}$ measures the amount of q-information missing when only $\hat{\mathcal{D}}$ is known. Such an occurrence of different probability distributions for the same system, *depending on the amount of information* retained about it, which is trivial in the dice example, may look odd for quantum states, but it is employed in practice when selecting measurement outcomes, hence it must enter theoretical treatments; indeed, it will be crucial in Sec. 7. Once the existence of different subensembles is granted, the evolutions in parallel of the corresponding states, governed for an isolated system by the same Hamiltonian, should be regarded for each considered subensemble as a transfer of q-information between the various observables.

The consideration of subensembles, which underlies the frequency approach to the classical probability theory [25], will be an essential ingredient of the present approach to quantum measurements.[8] Here, the density operator (2) of the compound system S+A encompasses q-information about the final state of a large set $\mathcal{E}$ of runs, whereas (3) accounts for the more detailed q-information associated with the subensemble $\mathcal{E}_{\rm sub}^{(k)}$. Assigning the final state $\hat{\mathcal{D}}_i$ given by (1) to a run belonging to a subensemble $\mathcal{E}_i$ relies on the possibility of reading the outcome A_i of the macroscopic pointer. Switching from $\hat{\mathcal{D}}(t_{\rm f})$ to $\hat{\mathcal{D}}_i$ will thus appear (Sec. 10) as an updating of information, similar to an updating of ordinary probabilities.

4 System plus apparatus in thermodynamic equilibrium

Let us first characterise the most general form of a Hamiltonian which may produce an ideal quantum measurement process. We suppose that the compound system S+A is isolated, including in A a

[8] In classical probability theory, the selection of the elements of a subensemble $\mathcal{E}_{\rm sub}^{(k)}$ of $\mathcal{E}$ is mathematically implemented [25,26] by numbering the events of $\mathcal{E}$ with an index n and introducing a function $f^{(k)}(n)$ that may take two values, 0 if the element n is discarded, 1 if it is selected. **Armen, aren't there conditions on the f??** Use of this probability concept requires distinguishing theoretically the individual events, which is of course experimentally performed in quantum measurements, but which is allowed in quantum theory only in special cases, such as at the issue of a measurement. This will be discussed in Sec. 7 and 8. Then, the subensembles $\mathcal{E}_{\rm sub}^{(k)}$ of theoretical interest for the present argument will be those for which the coefficients $q_i^{(k)}$ in (3) differ from p_i. However, the selections of all such subensembles within the full ensemble $\mathcal{E}$ have zero measure (in the sense of Lebesgue measure in the space of selections [26] when the number of elements of $\mathcal{E}$ becomes infinite). Nearly all subsets of $\mathcal{E}$, in particular those obtained by extracting systems at random from $\mathcal{E}$, would be described by the same state $\hat{\mathcal{D}}$ as the full set $\mathcal{E}$. It will therefore be essential for our purpose to consider *all subensembles* $\mathcal{E}_{\rm sub}^{(k)}$ of $\mathcal{E}$. Note also that, if $\hat{r}_i$ is a mixed state, the runs described by (3) are picked up at random within $\mathcal{E}_i$. Note finally that, if $\hat{\mathcal{D}}$ is a pure state $|\psi\rangle\langle\psi|$, this state $\hat{\mathcal{D}}$ should be assigned to any subensemble and to any individual system of $\mathcal{E}$. (The latter will not apply to measurements, since A has to start in a mixed state.)

thermal bath or an environment if present. The properties of S+A are governed by the Hamiltonian $\hat{H} = \hat{H}_{\rm S} + \hat{H}_{\rm A} + \hat{H}_{\rm SA}$, which must have specific features. If A is decoupled from S, the part $\hat{H}_{\rm A}$ governing the macroscopic apparatus should produce an initial *metastable* state $\hat{\mathcal{R}}(0)$, with lifetime longer than the duration of the measurement, and several thermodynamic equilibrium states $\hat{\mathcal{R}}_i$, the possible final states. A typical example[4] is given by spontaneously broken discrete invariance, the pointer variable A_i being the order parameter.

As we wish to deal with ideal measurements, the process should perturb S as little as possible: any observable compatible with $\hat{s}$, i.e., commuting with its eigenprojectors $\hat{\pi}_i$, should remain unaffected. The conservation of all these observables [14] is expressed by the fact that $\hat{H}$ depends on S only through the projectors $\hat{\pi}_i$. In particular *its coupling with* A *must have the form* $\hat{H}_{\rm SA} = \sum_i \hat{\pi}_i \otimes \hat{h}_i$, where $\hat{h}_i$ are operators of A. Moreover, if $\hat{s}$ takes the value s_i, that is, $\hat{\pi}_i$ the value 1, the apparatus A should end up in its stable state $\hat{\mathcal{R}}_i$, the pointer variable being close to A_i and also $\hat{\Pi}_i$ taking the value 1. This can be achieved if each $\hat{h}_i$ behaves as a source that breaks explicitly the invariance[9] by energetically favouring A_i.

Before analysing the dynamics of the measurement process (Sections 5 and 7), we determine for orientation the general form $\hat{\mathcal{D}}_{\rm eq}$ of the thermodynamic equilibrium states associated with the Hamiltonian $\hat{H}$ of S+A. We rely on the maximum von Neumann entropy criterion for assigning a state to a system in equilibrium [16,17]. We thus maximise $-\mathrm{tr}\,\hat{\mathcal{D}}\ln\hat{\mathcal{D}}$, which measures the amount of randomness of $\hat{\mathcal{D}}$, under constraints accounting for given values of the conserved quantities. As usual for isolated macroscopic systems, thermal equilibrium is implemented by specifying the average energy, identified with the q-expectation value $\langle\hat{H}\rangle$. Further constraints account for the other constants of the motion, to wit, the q-expectation values of all observables of S that commute with the projections $\hat{\pi}_i$. The Lagrange multipliers are β, for the Hamiltonian, and the coefficients multiplying the operators that commute with the $\hat{\pi}_i$.

This yields for the equilibrium states a generalised Gibbsian form where the exponent involves an arbitrary linear combination of all the conserved observables, i.e., $\hat{\mathcal{D}}_{\rm eq} \propto \exp(-\beta\hat{H} + \sum_i \hat{y}_i)$ where $\hat{y}_i$ is any operator of S acting inside the diagonal block i (so that $\hat{y}_i = \hat{\pi}_i\hat{y}_i\hat{\pi}_i$). For any possible equilibrium state of S+A, $\hat{y}_i$ is arbitrary; if this state is the result of relaxation, it depends on the initial conditions as specified below. Noting that the exponent, which commutes with the projections $\hat{\pi}_i$, has a block diagonal structure in a basis where $\hat{s}$ is diagonal, we find for these equilibrium states the related block diagonal structure

$$\hat{\mathcal{D}}_{\rm eq} = \sum_i q_i\hat{x}_i \otimes \hat{\mathcal{R}}_i^h, \qquad \sum_i q_i = 1. \tag{4}$$

Each factor $q_i\hat{x}_i$, which arises from $\exp(-\beta\hat{H}_{\rm S} + \hat{y}_i)$, is an arbitrary block diagonal operator of S, where $\hat{x}_i = \hat{\pi}_i\hat{x}_i\hat{\pi}_i$, $\mathrm{tr}_{\rm S}\hat{x}_i = 1$ behaves as a density operator of S. (If the eigenvalue s_i is non degenerate, $\hat{x}_i$ reduces to $\hat{\pi}_i$.) Each factor $\hat{\mathcal{R}}_i^h \propto \exp[-\beta(\hat{H}_{\rm A} + \hat{h}_i)]$ in (4) can be interpreted as a canonical equilibrium density operator in the space of A. Here the invariance is *explicitly broken* by adding to the Hamiltonian of A the source term $\hat{h}_i$ that arises from $\hat{H}_{\rm SA} = \sum_i \hat{\pi}_i \otimes \hat{h}_i$.

This term $\hat{h}_i$ should be sufficiently large so that the distribution $\mathrm{tr}_{\rm A}\hat{\mathcal{R}}_i^h\delta(A-\hat{A})$ of $\hat{A}$ has a single narrow peak, and sufficiently small so that this peak lies close to A_i, so as to let $\hat{A}$ behave as a pointer

[9] The interaction Hamiltonian $\hat{H}_{\rm SA}$ allows to describe not only ideal measurements involving well separated eigenvalues s_i of $\hat{s}$, but also more general measurements for which the projectors $\hat{\pi}_i$, still associated through $\hat{h}_i$ with the pointer indications A_i, are no longer in one-to-one correspondence with the eigenvalues of $\hat{s}$. For instance, if some $\hat{\pi}_i$ encompasses the eigenspaces of several different neighbouring eigenvalues, selecting the outcome A_i will not discriminate them, and the final state $\hat{r}_i = \hat{\pi}_i\hat{r}(0)\hat{\pi}_i/p_i$ of S will not be associated with a single eigenvalue of $\hat{s}$ as in an ideal measurement. As another example, consider two orthogonal rank-one projectors $\hat{\pi}_1$ and $\hat{\pi}_2$, coupled with sources $\hat{h}_1$ and $\hat{h}_2$ that produce different outcomes A_1 and A_2, and assume that $\hat{\pi}_1 + \hat{\pi}_2$ spans the two-dimensional eigenspace associated with a degenerate eigenvalue of $\hat{s}$; reading the outcome A_1 (or A_2) then provides more information than this eigenvalue.

observable. These properties are easy to satisfy for a macroscopic apparatus[10]. Thermodynamic equilibrium (4) thus entails a complete correlation between the eigenvalue s_i of $\hat{s}$ and the macroscopic value A_i of the pointer variable, implying the equality of the q-probabilities $\langle\hat{\pi}_i\rangle$ and $\langle\hat{\Pi}_i\rangle$. The moderate (neither too small nor too large) size of $\hat{h}_i$ also ensures that the state $\hat{\mathcal{R}}_i^h$ lies in the basin of attraction of the state $\hat{\mathcal{R}}_i$ with broken symmetry, so that $\hat{\mathcal{R}}_i^h$ will relax smoothly to $\hat{\mathcal{R}}_i$ at the end of the measurement process when the coupling $\hat{H}_{\rm SA}$ is switched off[11].

Let us return to measurement processes. We wish to explain how a final state $\hat{\mathcal{D}}_i$ of the form (1) should be assigned to each run with the probability $p_i = \langle\hat{\pi}_i\rangle = \langle\hat{\Pi}_i\rangle$. It is thus necessary (but, as stressed, not sufficient) to prove, by studying the dynamics of a large statistical ensemble $\mathcal{E}$ of runs issued from the initial state $\hat{\mathcal{D}}(0) = \hat{r}(0)\otimes\hat{\mathcal{R}}(0)$, that it ends up in the state $\hat{\mathcal{D}}(t_{\rm f})$ expressed by (2). We can identify (2) with a generalised thermodynamic equilibrium state (4), for which relaxation of $\hat{\mathcal{R}}_i^h$ to $\hat{\mathcal{R}}_i$ has taken place after switching off $\hat{H}_{\rm SA}$. Dynamics and conservation laws will determine the free parameters of $\hat{\mathcal{D}}_{\rm eq}$ from the initial state $\hat{\mathcal{D}}(0)$ by the conditions $q_i\hat{x}_i = \hat{\pi}_i\hat{r}(0)\hat{\pi}_i \equiv p_i\hat{r}_i$, thus indeed as $q_i = p_i$ and $\hat{x}_i = \hat{r}_i$.

We also need to prove a stronger result. For a subset $\mathcal{E}_{\rm sub}^{(k)}$ having yielded a proportion $q_i^{(k)}$ of runs with outcomes A_i, the final corresponding state $\hat{\mathcal{D}}_{\rm sub}^{(k)}$ should have the form (3). This final state is again a generalised thermodynamic equilibrium state (4) with $\hat{x}_i = \hat{r}_i$, and the same property $\hat{x}_i = \hat{r}_i$ holds for the special subensembles $\mathcal{E}_i$.

Thus, an *ideal measurement process* appears as a mere *relaxation of* S+A *to generalised thermodynamic equilibrium*, for the full ensemble $\mathcal{E}$ of runs as well as for all[8] its subensembles $\mathcal{E}_{\rm sub}^{(k)}$. In quantum mechanics, relaxation of $\hat{\mathcal{D}}(t)$ and $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t)$ towards Gibbsian generalised thermodynamic equilibrium states (2) and (3) is not granted [18]. We must therefore justify these properties within the quantum statistical *dynamics* framework. We sketch the main steps of such a proof in Sec. 5 and Sec. 7, as a prerequisite to the consideration of individual runs.

5 Dynamics of system and apparatus for the full set of runs

Our first task [11] consists in deriving the final state (2) by solving the Liouville–von Neumann equation $i\hbar{\rm d}\hat{\mathcal{D}}(t)/{\rm d}t = [\hat{H},\hat{\mathcal{D}}(t)]$ with initial condition $\hat{\mathcal{D}}(0) = \hat{r}(0)\otimes\hat{\mathcal{R}}(0)$. Taking into account the above form of $\hat{H}$ and the commutation $[\hat{H}_{\rm S},\hat{r}(0)] = 0$ which ensures that the marginal state $\hat{r}(t)$ of S is perturbed only by the interaction $\hat{H}_{\rm SA}$ during the process, we check that $\hat{\mathcal{D}}(t)$ can be parameterised as

$$\hat{\mathcal{D}}(t) = \sum_{i,j}\hat{\pi}_i\hat{r}(0)\hat{\pi}_j\otimes\hat{\mathcal{R}}_{ij}(t) \tag{5}$$

in terms of a set $\hat{\mathcal{R}}_{ij}(t) = \hat{\mathcal{R}}_{ji}^\dagger(t)$ of operators in the Hilbert space of A, to be determined by the equations

$$i\hbar\frac{{\rm d}\hat{\mathcal{R}}_{ij}(t)}{{\rm d}t} = (\hat{H}_{\rm A}+\hat{h}_i)\hat{\mathcal{R}}_{ij}(t) - \hat{\mathcal{R}}_{ij}(t)(\hat{H}_{\rm A}+\hat{h}_j), \tag{6}$$

with the initial conditions $\hat{\mathcal{R}}_{ij}(0) = \hat{\mathcal{R}}(0)$. The dynamics thus involves *solely the apparatus*, irrespective of the tested system (involved in (6) only through $\hat{h}_i$ and $\hat{h}_j$), a specific property of ideal measurements. Moreover, *for each block* $\hat{\mathcal{R}}_{ij}$ of the density matrix $\hat{\mathcal{D}}$, whether $i = j$ of $i > j$, *the dynamics of is decoupled from the one in other blocks*, while the $i < j$ blocks follow by Hermiticity.

[10] In the CW model[4], the factors $\hat{h}_\downarrow = -\hat{h}_\uparrow = \sum_{n=1}^N g\hat{\sigma}_z^{(n)}$ that occur in the coupling $\hat{H}_{\rm SA}$ behave as a magnetic field applied to A. The conditions for $\hat{h}_i$ are satisfied if $N \gg T/g$ (which lets the probability of the states with $m < 0$ vanish for $s_z = 1$), and $g < T$ [6, sect. 9.4].

[11] In the CW model $g < T$ ensures this relaxation [6, sect. 7.2].

Though macroscopic, A should be treated as a finite system so as to ensure a better control of the dynamics and to discuss under which conditions the process can be regarded as an ideal measurement. We must then explain how the expected irreversible relaxation from $\hat{\mathcal{D}}(0)$ to $\hat{\mathcal{D}}(t_\mathrm{f})$ can be governed by the reversible equations (6), so that we run into the old paradox of irreversibility. As usual in statistical mechanics, it is legitimate for finite but large systems to disregard events having an extremely small probability, to forget about recurrences that would occur after large, unattainable times, and to neglect physically irrelevant correlations between a macroscopic number of degrees of freedom. Such approximations, although not exact mathematically, are fully justified when their outcome is physically indistinguishable from the exact solution. A large apparatus, and a suitable choice of parameters in $\hat{H}_\mathrm{A}$ and $\hat{H}_\mathrm{SA}$ will therefore be needed, for each model, to explain the required relaxations and to estimate their time scales, as illustrated by the CW model treated extensively in [7]. In decoherence approaches, which focus on the disappearance of the off-diagonal blocks $\hat{\mathcal{R}}_{ij}$ for $i \neq j$, irreversibility is ensured by the large size of an external environment (that we regard here as a bath included in A). The resulting equations should anyhow be supplemented as below in Sec. 8 by interpretative principles.

Two types of relaxation arise independently from the dynamical equations (6).

(*i*) *"Truncation":* For $i \neq j$, the coherent contributions $\hat{\mathcal{R}}_{ij}(t)$ decay for all practical purposes owing to the difference between $\hat{h}_i$ and $\hat{h}_j$, and eventually vanish. The off-diagonal blocks of the density matrix $\hat{\mathcal{D}}(t)$ are thus truncated as regards the physically attainable observables.[12] Depending on the model, this decay may be governed by different mechanisms.[13]

(*ii*) *"Registration":* For $i = j$, the evolution of $\hat{\mathcal{R}}_{ii}(t)$ governed by (6) is a mere relaxation from the metastable state $\hat{\mathcal{R}}(0)$ to the equilibrium state $\hat{\mathcal{R}}_i^h$ in the presence of the source $\hat{h}_i$, and then to $\hat{\mathcal{R}}_i$ after $\hat{H}_\mathrm{SA}$ is switched off. The correlation between s_i and A_i needed to register the outcome is thereby established[14]. Since registration requires a dumping of free energy into the bath, it its typically slower than truncation.

Thus, microscopic dynamics confirms the generalised thermodynamic equilibrium surmise (2) for the final state of S+A in the ensemble $\mathcal{E}$. The coupling between S and A can already be switched off at an appropriate moment before this state is reached, and the remainder of our discussion will involve *only the apparatus*.

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[12] The matrix elements of $\hat{\mathcal{R}}_{ij}(t)$ with $i \neq j$ contain rapidly oscillating phase factors. As for any irreversible process, physical quantities involve sums over very many of them, which cancel out for times less than the huge recurrence time. So for all practical purposes they can be omitted after the relaxation time owing to the macroscopic size of the apparatus, in spite of the constant value of the sum $\mathrm{tr}_\mathrm{A}\hat{\mathcal{R}}_{ij}(t)\hat{\mathcal{R}}_{ij}^\dagger(t)$ of the modulus square of the matrix elements of $\hat{\mathcal{R}}_{ij}(t)$. However, would one wish to calculate mathematical objects, for instance to check that the fine-grained entropy remains constant, they would definitely be needed.

[13] Truncation is the most studied phenomenon in the literature on measurements, decoherence being currently ensured by an external environment. Several processes are involved in the CW model[4]. Over the short time scale $\hbar/g\sqrt{N}$, truncation results [6, sect. 5] from the dephasing between the oscillations yielded by the factor $\exp 2it\hbar^{-1}\sum_{n=1}^{N} g\hat{\sigma}_z^{(n)}$ entering $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$, which have different frequencies (due to the randomness of $\sigma_z^{(n)}$ in the initial paramagnetic state of A). Information is thereby lost through a cascade of correlations of higher and higher order, less and less accessible, between the spins of A, so that $\hat{\mathcal{R}}_{\uparrow\downarrow}(t)$ practically tends to zero as regards the accessible observables. Recurrences are wiped out [6, sect. 6], either by the coupling γ with the phonon bath (provided $T/J \gg \gamma \gg g/NT$), or by a spread δg in the couplings g of $\hat{H}_\mathrm{SA}$ (provided $\delta g \gg g/\sqrt{N}$).

[14] Authors do not always give the same meaning to the various words used. We term as *truncation* the disappearance of the off-diagonal blocks of the density matrix of S+A under the effect of an arbitrary mechanism, and specialise *decoherence* to the production of this effect by interaction with an environment or a thermal bath. We term as *registration* the process which leads each diagonal block to the correlated state $\hat{r}_i \otimes \hat{\mathcal{R}}_i$, and as *reduction* the transition from $\hat{r}(0)$ to some $\hat{r}_i$ for an individual run. While much attention has been paid to the vanishing of the off-diagonal blocks, the relaxation of the diagonal blocks is too often disregarded, although it produces the correlations that ensure registration. In the CW model [6, sect. 7], this process is triggered by $\hat{h}_i$ which makes $\hat{\mathcal{R}}(0)$ unstable and should be sufficiently large to exclude false registrations ($g \gg J/\sqrt{N}$). Later on, the relaxation of $\hat{\mathcal{R}}_{ii}(t)$ to $\hat{\mathcal{R}}_i^h$, and finally to $\hat{\mathcal{R}}_i$ after $\hat{H}_\mathrm{SA}$ is switched off, is governed by the dumping of free energy from the magnet to the phonon bath; its characteristic duration is the registration time $\hbar/\gamma(J-T)$.

6 Through subensembles towards individual runs

We have thus shown how $\hat{\mathcal{D}}(t)$ evolves from $\hat{\mathcal{D}}(0)$ to $\hat{\mathcal{D}}(t_{\rm f}) = \sum_i p_i \hat{\mathcal{D}}_i$ for the large set $\mathcal{E}$ of all runs. It would be tempting to interpret this result as meaning that $\mathcal{E}$ contains a proportion p_i of individual runs having ended up in the state $\hat{\mathcal{D}}_i$. This conclusion would be justified if the density operator $\hat{\mathcal{D}}(t_f)$ did behave as an ordinary probability distribution or as a density in phase space of classical statistical mechanics. However, as indicated in Sec. 2, such a seemingly natural inference is here fallacious due to the quantum ambiguity of the decompositions [20]. Namely, there exist many mathematical decompositions $\hat{\mathcal{D}} = \sum_n \mu_n \hat{\mathcal{D}}^{(n)}$ of an arbitrary mixed state $\hat{\mathcal{D}}$ into a weighted sum of operators $\hat{\mathcal{D}}^{(n)}_{\rm dec}$ having the properties of density operators, and different decompositions can present incompatible features. Hence, we cannot interpret physically the operators $\hat{\mathcal{D}}^{(n)}_{\rm dec}$ issued from a decomposition of $\hat{\mathcal{D}}$.

We illustrate this general ambiguity by a simple, well known example, that of an unpolarised ensemble $\mathcal{E}$ of spins, described by the density operator $\frac{1}{2}\hat{I}$. One might naively infer from the decomposition $\frac{1}{2}\hat{I} = \frac{1}{2}|z\rangle\langle z| + \frac{1}{2}|-z\rangle\langle -z|$ that $\mathcal{E}$ would be the union of two equal sized subensembles of spins, polarised along the z-axis in either direction and described by the pure states $|z\rangle$ and $|-z\rangle$, respectively. However, considering the alternative decomposition $\frac{1}{2}\hat{I} = \frac{1}{2}|x\rangle\langle x| + \frac{1}{2}|-x\rangle\langle -x|$, the same interpretation would let one believe in the possibility of splitting $\mathcal{E}$ into two subensembles of spins polarised along the x-axis in either direction. If these two decompositions were physically meaningful, each individual spin would be polarised simultaneously in two orthogonal directions, but this is nonsensical. And many other decompositions exist[15]. To describe statistically an individual spin, the only issue is to assign to it the state $\frac{1}{2}\hat{I}$. Such a contradiction forbids us to give a physical meaning to the separate terms of a decomposition of the mixed state $\hat{\mathcal{D}}$ describing an ensemble $\mathcal{E}$ (unless information other than $\hat{\mathcal{D}}$ is available). For measurements, once the expression (2) of $\hat{D}(t_{\rm f})$ has been globally derived as in Sec. 4, the existence of decompositions of $\hat{D}(t_{\rm f})$ incompatible with the particular one $\hat{D}(t_{\rm f}) = \sum_i p_i \hat{\mathcal{D}}_i$ makes it illegitimate to postulate that each individual run ends up in one or another of the states $\hat{\mathcal{D}}_i$.

In order to face this impossibility of getting direct conclusions about individual events, we will pursue within the quantum formalism as far as possible, postponing interpretation. As noted in Sec. 3, quantum statistical mechanics applies not only to the full ensemble $\mathcal{E}$ but also to its subensembles $\mathcal{E}^{(k)}_{\rm sub}$. Our next task in Sec. 7 will therefore consist in proving that, for *all possible subensembles* [8,16] $\mathcal{E}^{(k)}_{\rm sub}$ of $\mathcal{E}$, S+A ends up in a state of the form (3), $\hat{\mathcal{D}}^{(k)}_{\rm sub} = \sum_i q_i^{(k)} \hat{\mathcal{D}}_i$. This property, a necessary ingredient to solve the measurement problem, does not follow from (2) but will be ensured by a new relaxation process pertaining to the apparatus and taking place near the end of the measurement [7].

Keeping aside for a while the measurement problem, we remind (Sec. 3) that a given individual system can statistically be described by different quantum states, depending on our information about the physical subensemble in which it is embedded (as in the dice example of Sec. 2 (ii)). These states, usually mixed, are related to one another. When an ensemble $\mathcal{E}$ (with $\mathcal{N}$ elements described by $\hat{\mathcal{D}}$) gathers a subensemble $\mathcal{E}^{(l)}_{\rm sub}$ (with $0 < \mathcal{N}^{(l)}_{\rm sub} < \mathcal{N}$ elements described by $\hat{\mathcal{D}}^{(l)}_{\rm sub}$) and its complement $\mathcal{E}^{(l')}_{\rm sub}$ (with $\mathcal{N}^{(l')}_{\rm sub} = \mathcal{N} - \mathcal{N}^{(l)}_{\rm sub}$ elements described by $\hat{\mathcal{D}}^{(l')}_{\rm sub}$), the q-expectation values $\langle \hat{O} \rangle$ for $\mathcal{E}$, $\mathcal{E}^{(l)}_{\rm sub}$ and $\mathcal{E}^{(l')}_{\rm sub}$ defined by the correspondence $\hat{O} \mapsto \langle \hat{O} \rangle$ have the same additivity property as ordinary averages. This is expressed at each time by

$$\hat{\mathcal{D}}(t) = \lambda \hat{\mathcal{D}}^{(l)}_{\rm sub}(t) + (1-\lambda)\hat{\mathcal{D}}^{(l')}_{\rm sub}(t), \tag{7}$$

with the weight $\lambda = \mathcal{N}^{(l)}_{\rm sub}/\mathcal{N}$. All three states $\hat{\mathcal{D}}(t)$, $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t)$ and $\hat{\mathcal{D}}^{(k')}_{\rm sub}(t)$ are governed by the same dynamical equations, including the Hamiltonian that characterises the considered system. Consider

[15] An arbitrary mixed state can be decomposed in an infinity of ways as a weighted sum of projectors onto pure states, which cannot be decomposed further and are not necessarily orthogonal.

[16] If $\mathcal{E}$ has $\mathcal{N}$ elements, its number of subensembles is $2^{\mathcal{N}} - 1$ since each run may or may not be selected and since we exclude the empty set.

more generally an ensemble $\mathcal{E}$ that encompasses different populations of similar systems, and the *whole collection of its physical subensembles* $\mathcal{E}_{\rm sub}^{(l)}$. As $\mathcal{E}$ is the merger of some "elementary" subensembles $\mathcal{E}_j$ (with relative sizes p_j) which gather individual systems described by the same state $\hat{\mathcal{D}}_j$, the state that describes $\mathcal{E}$ is expressed by $\hat{\mathcal{D}} = \sum_j p_j \hat{\mathcal{D}}_j$ according to Eq. (7). An arbitrary subensemble $\mathcal{E}_{\rm sub}^{(l)}$ is characterised by the relative numbers $q_j^{(l)}$ of individual systems extracted from each $\mathcal{E}_j$, so that the state associated with $\mathcal{E}_{\rm sub}^{(l)}$ is expressed as $\hat{\mathcal{D}}_{\rm sub}^{(l)} = \sum_j q_j^{(l)} \hat{\mathcal{D}}_j$; the coefficients $q_j^{(l)}$ satisfy the additivity property expressed by Eq. (13) below. We will term as *hierarchic structure* this general form for the set of physical states $\hat{\mathcal{D}}_{\rm sub}^{(l)}$. The hierarchic property, characterised by the building blocks $\hat{\mathcal{D}}_j$ and the additive coefficients $q_j^{(l)}$, ensures the consistency of the various density operators $\hat{\mathcal{D}}_{\rm sub}^{(l)}$ assigned to a given individual system, depending on the subensemble $\mathcal{E}_{\rm sub}^{(l)}$ to which it is regarded to belong.

Conversely, whereas one can readily infer from a classical probability distribution a *unique set of subensembles* $\mathcal{E}_{\rm sub}^{(l)}$ and their associated distributions $\hat{\mathcal{D}}_{\rm sub}^{(l)}$, the quantum ambiguity *makes this impossible* for a mixed density operator $\hat{\mathcal{D}}$. Due to the matrix nature of $\hat{\mathcal{D}}$, one cannot deduce from it the states associated with physical subensembles: There exist *many sets* of operators $\hat{\mathcal{D}}_{\rm dec}$ issued from decompositions of $\hat{\mathcal{D}}$, which produce different hierarchical structures (with different building blocks $\hat{\mathcal{D}}_j$), or which are even incompatible with the hierarchic structure needed for decompositions associated with physical subensembles. Thus, if nothing else than $\hat{\mathcal{D}}$ is known, this *logical incompatibility between different mathematical decompositions prevents us from giving them a physical meaning*, and their elements do not describe physical subensembles.

In the case of measurements, the final mixed state $\hat{\mathcal{D}}(t_{\rm f})$ of S+A possesses, besides the decomposition $\hat{\mathcal{D}}(t_{\rm f}) = \sum_i p_i \hat{\mathcal{D}}_i$ expected to be physical, many other mathematical decompositions $\hat{\mathcal{D}}(t_{\rm f}) = \sum_n \mu_n \hat{\mathcal{D}}_{\rm dec}^{(n)}$ that cannot be associated with physical subensembles. This forbids us to interpret separately the terms $p_i \hat{\mathcal{D}}_i$ of $\hat{\mathcal{D}}(t_{\rm f})$ unless extra information is found. Neither can we infer from the sole global knowledge of $\hat{\mathcal{D}}(t_{\rm f})$ whether a term $\hat{\mathcal{D}}_{\rm dec}^{(1)}$ issued from some of its decompositions is or is not a density operator describing a real subset of runs. Although suggestive, the form (2) of $\hat{\mathcal{D}}(t_{\rm f})$ alone is not sufficient to give a meaning to its ingredients p_i and $\hat{\mathcal{D}}_i$ and to imply the existence of subensembles of runs that would be described by $\hat{\mathcal{D}}_i$. Nevertheless, we will present below a dynamical argument which will help us to reach the desired conclusion.

Armen, your opinion is needed: Don't we already modify the q-formalism if we say that we can evolve unphysical E's according to our taste? Namely, in that case we say that the q-formalism leaves a choice there, but in my opinion the q-formalism is unique and leaves us no choice. The same point comes back below, see "Armen".

7 Subensemble relaxation

In order to overcome this quantum ambiguity while deriving results in agreement with the desired features of ideal measurements, we need at least to prove that all the states assigned to the *physical* subensembles $\mathcal{E}_{\rm sub}^{(k)}$ relax towards the form $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f}) = \sum_i q_i^{(k)} \hat{\mathcal{D}}_i$. This property might be regarded as intuitive, since this is just a relaxation towards a generalised thermodynamic equilibrium state, often supposed to be ensured by an environment. However, even when the probability distribution associated with an *ensemble* $\mathcal{E}$ has reached its equilibrium form, the distributions associated with *subensembles* may still be off equilibrium: The equilibrium form (2) for $\hat{\mathcal{D}}(t_{\rm f})$ does not entail the equilibrium form (3) for $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})$, a very serious difficulty in quantum mechanics since we must deal with ensembles. A rigorous derivation is necessary to establish this latter property. To this aim, we will rely on a new relaxation mechanism suited to subensembles [7]. We adopt the following strategy.

Experimentally, the indication of the pointer can be read at the macroscopic scale after achievement of the process, and one can use this indication to tag the individual runs so as to select determined subensembles $\mathcal{E}_{\rm sub}^{(k)}$. Acknowledging this fact, we are ascertained that a state can be assigned to S+A for each of these subensembles. At the final time, but before reading, the explicit

form of these states $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t_{\rm f})$ cannot be specified, though their existence is ensured. It is *natural to postulate* that, at least slightly earlier, during the very last stage $t'_{\rm f} < t < t_{\rm f}$ of the process, one can assign to each subensemble $\mathcal{E}^{(k)}_{\rm sub}$ a state $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t)$ governed by quantum statistical dynamics, with the same Hamiltonian as $\hat{\mathcal{D}}(t)$. This allows us to study the evolution of $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t)$ between $t'_{\rm f}$ and $t_{\rm f}$. However, at earlier times and especially at the beginning of the process, one cannot make any theoretical statement about subensembles of $\mathcal{E}$, even though individual runs are experimentally distinguished and followed during the whole process.

The new initial time $t'_{\rm f}$ after which we will describe the subensemble $\mathcal{E}^{(k)}_{\rm sub}$ can be chosen sufficiently late so that $\hat{H}_{\rm SA}$ has been switched off and that $\hat{\mathcal{D}}(t'_{\rm f})$ has reached the equilibrium form $\hat{\mathcal{D}}(t_{\rm f}) = \hat{\mathcal{D}}(t'_{\rm f}) = \sum_i p_i \hat{\mathcal{D}}_i$. Thus, $\hat{\mathcal{D}}(t'_{\rm f})$ does not lie in the full Hilbert space $\mathcal{H}$ of S+A but in a subspace $\mathcal{H}_{\rm corr}$, containing only the eigenspaces of $\hat{A}$ with eigenvalues close to a macroscopic equilibrium value A_i, and involving *correlations* between the tested system and the pointer. Although we do not know the state $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t'_{\rm f})$ to be assigned to a given subensemble $\mathcal{E}^{(k)}_{\rm sub}$, we can ascertain that it must be an element of some decomposition of $\hat{\mathcal{D}}(t'_{\rm f}) = \hat{\mathcal{D}}(t_{\rm f})$. This provides a constraint on the initial condition $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t'_{\rm f})$ for the ensuing dynamics. Each state $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t)$ will at the new initial time $t'_{\rm f}$ *lie in the subspace* $\mathcal{H}_{\rm corr}$ and will during the time lapse $t'_{\rm f} \leq t \leq t_{\rm f}$ be *governed by the Hamiltonian* $\hat{H}_{\rm A}$ of the sole apparatus, remaining in the space $\mathcal{H}_{\rm corr}$ because the transitions induced by $\hat{H}_{\rm A}$ within this space are much faster than the transitions towards the remainder of $\mathcal{H}$.

As $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t_{\rm f})$ is unknown, we will consider for $t'_{\rm f} \leq t \leq t_{\rm f}$ *all mathematically allowed decompositions* of $\hat{\mathcal{D}}$ which split it into a sum $\hat{\mathcal{D}} = \mu \hat{\mathcal{D}}^{(1)}_{\rm dec} + (1-\mu)\hat{\mathcal{D}}^{(2)}_{\rm dec}$ of two nonnegative Hermitean operators $(0 < \mu < 1)$. Nothing ensures that an operator $\hat{\mathcal{D}}_{\rm dec}$ issued from such a decompostion describes a subensemble of runs and is therefore physically meaningful; most often it will not be. In fact, $\hat{\mathcal{D}}_{\rm dec}$ is, in the space $\mathcal{H}_{\rm corr}$ an *arbitrary operator* (non-negative and normalised). However, since *all physical states* $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t)$ associated for $t'_{\rm f} \leq t \leq t_{\rm f}$ with real subensembles $\mathcal{E}^{(k)}_{\rm sub}$ of runs are issued from a decomposition (7), they obviously *belong to the class of operators* $\hat{\mathcal{D}}_{\rm dec}$. As we cannot distinguish the physical states $\hat{\mathcal{D}}^{(k)}_{\rm sub}$ within this class $\hat{\mathcal{D}}_{\rm dec}$, we will regard all possible operators $\hat{\mathcal{D}}_{\rm dec}(t)$ as candidates for physical density operators. As the physical states $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t)$ obey the same dynamics as $\hat{\mathcal{D}}(t)$, and as they belong to the set of operators $\hat{\mathcal{D}}_{\rm dec}(t)$ issued from arbitrary mathematical decompositions, we extend this dynamics to the set of operators $\hat{\mathcal{D}}_{\rm dec}(t)$. Our purpose will then be to show, by taking any allowed initial condition $\hat{\mathcal{D}}_{\rm dec}(t'_{\rm f})$ and solving for $t'_{\rm f} \leq t \leq t_{\rm f}$ the equations of motion for $\hat{\mathcal{D}}_{\rm dec}(t)$ governed by the Hamiltonian of S+A, that *any possible candidate* $\hat{\mathcal{D}}_{\rm dec}(t)$ *for a physical state ends up in the expected form* $\hat{\mathcal{D}}_{\rm dec}(t_{\rm f}) = \sum_i q_i \hat{\mathcal{D}}_i$. It will remain impossible to distinguish among the final operators $\hat{\mathcal{D}}_{\rm dec}(t_{\rm f})$ which are the physical ones $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t_{\rm f})$, but this does not matter since, as we demonstrate below, all of them reach the same form (3), so that we can forget in the end about the unphysical operators included in the class $\hat{\mathcal{D}}_{\rm dec}$. We shall thereby have proven the relaxation of $\hat{\mathcal{D}}^{(k)}_{\rm sub}(t)$ towards the required equilibrium form (3) at the time $t_{\rm f}$ for all physical subsets of runs.

We have first to characterise the initial operators $\hat{\mathcal{D}}_{\rm dec}(t'_{\rm f})$. These operators cannot a priori be determined, but their general form is constrained to arise from some decomposition of $\hat{\mathcal{D}}_{\rm dec}(t'_{\rm f})$, which implies that $\hat{\mathcal{D}}(t'_{\rm f}) - \mu\hat{\mathcal{D}}_{\rm dec}(t'_{\rm f})$ is nonnegative for some $0 < \mu < 1$. To simplify the discussion, we assume here the eigenvalues of $\hat{s}$ to be non degenerate[17] so that $\hat{r}_i = \hat{\pi}_i = |s_i\rangle\langle s_i|$. As A is macroscopic, the fluctuations of $\hat{H}_{\rm A}$ around $\langle \hat{H}_{\rm A} \rangle$ and of the pointer observable around the macroscopic value A_i are relatively small, and we can replace in $\hat{\mathcal{D}}(t'_{\rm f})$ the *canonical* equilibrium states $\hat{\mathcal{R}}_i$ by *microcanonical* ones, $\hat{\mathcal{R}}^{\mu}_i$. Within the Hilbert space of A, we denote as $|A_i, \eta\rangle$ a basis of kets characterised by a value

[17] For degenerate eigenvalues s_i, the only change in the forthcoming derivation, if the states $\hat{r}_i \equiv |i\rangle\langle i|$ are pure, is the replacement of $|s_i\rangle$ by the ket $|i\rangle$ in the eigenspace of $\hat{s}$ associated with s_i. If the density operator $\hat{r}_i$ is mixed, we note that this operator of S is not modified by the process, while remaining fully coupled with A_i for $t > t'_{\rm f}$. We should therefore preserve this property when we split $\hat{\mathcal{D}}$ so as to build the candidates $\hat{\mathcal{D}}_{\rm dec}$ for states $\hat{\mathcal{D}}^{(k)}_{\rm sub}$ of physical subensembles $\mathcal{E}^{(k)}_{\rm sub}$. The subensemble relaxation of A then produces again the final state (10), at least for all physical subensembles.

of $\hat{A}$ lying between $A_i - \Delta$ and $A_i + \Delta$, and by the value $\mathrm{tr}_\mathrm{A}\hat{\mathcal{R}}_i\hat{H}_\mathrm{A}$ of the macroscopic energy. As the spectrum is dense, the index η may take a very large number G_i of values, and we denote as $\hat{\Pi}_i$ the projector[3] over the Hilbert subspace for A spanned by the basis $|A_i, \eta\rangle$. The equivalence between the canonical and microcanonical states $\hat{\mathcal{R}}_i$ and $\hat{\mathcal{R}}_i^\mu$ is ensured ensured by choosing the width 2Δ, or equivalently the dimension G_i of the projector Π_i, in such a way that $\mathrm{tr}_\mathrm{A}\hat{\mathcal{R}}_i\hat{\Pi}_j \simeq \mathrm{tr}_\mathrm{A}\hat{\mathcal{R}}_i^\mu\hat{\Pi}_j = \delta_{ij}$[18]. The microcanonical equilibrium state of A is then

$$\hat{\mathcal{R}}_i^\mu = \frac{1}{G_i}\sum_\eta |A_i,\eta\rangle\langle A_i,\eta| = \frac{1}{G_i}\hat{\Pi}_i. \tag{8}$$

The state $\hat{\mathcal{D}}(t'_\mathrm{f}) = \hat{\mathcal{D}}(t_\mathrm{f}) \simeq \sum_i p_i\hat{r}_i \otimes \hat{\mathcal{R}}_i^\mu$ thus involves only diagonal matrix elements within the Hilbert subspace $\mathcal{H}_\mathrm{corr}$ of S+A spanned by the correlated kets $|s_i\rangle\,|A_i,\eta\rangle$. Hence, since $\hat{\mathcal{D}}(t'_\mathrm{f}) - \lambda\hat{\mathcal{D}}_\mathrm{dec}(t'_\mathrm{f})$ must be nonnegative, the matrix elements of any initial operator $\hat{\mathcal{D}}_\mathrm{dec}(t'_\mathrm{f})$ must lie within this Hilbert subspace $\mathcal{H}_\mathrm{corr}$, so that $\hat{\mathcal{D}}_\mathrm{dec}(t'_\mathrm{f})$ must have the form

$$\hat{\mathcal{D}}_\mathrm{dec}(t'_\mathrm{f}) = \sum_{i,j,\eta,\eta'} |s_i\rangle|A_i,\eta\rangle K(i,\eta;j,\eta';t'_\mathrm{f})\langle s_j|\langle A_j,\eta'|, \tag{9}$$

where K is a Hermitean, normalised and nonnegative matrix.

Since S and A have already been decoupled, the evolution for $t > t'_\mathrm{f}$ of the operator $\hat{\mathcal{D}}_\mathrm{dec}(t)$ issued from (9) is governed by $\hat{H}_\mathrm{S} + \hat{H}_\mathrm{A}$, and, as in Sec. 5, $\hat{H}_\mathrm{S}$ is ineffective, so any further effect has to come from the *sole Hamiltonian $\hat{H}_\mathrm{A}$ of the apparatus*. Only the part of $\hat{H}_\mathrm{A}$ that lives in the Hilbert subspace $\mathcal{H}_\mathrm{corr}$ is relevant for the evaluation of $\hat{\mathcal{D}}_\mathrm{dec}(t)$. This part is supposed to involve weak interactions inducing among the kets $|A_i,\eta\rangle$ rapid transitions[19] that modify η without affecting the macroscopic value of A_i. Such interactions are realistic for a macroscopic apparatus; they have little effect on the processes described in Sec. 5. The absence of transitions between different pointer values is needed to ensure the stability of the states $\hat{\mathcal{R}}_i$. This mechanism is a generalisation of the standard microcanonical relaxation [17-20] which leads, for a single microcanonical equilibrium, to a decay of the off-diagonal elements in the energy basis and to the equalisation of the populations of the microstates. Here, we deal with *several thermodynamic equilibrium states*, each one labelled by a value A_i, and practically uncoupled. Such a "quantum collisional process" is irreversible for a large apparatus. Acting here separately in each sector, on both sides $|A_i,\eta\rangle$ and $\langle A_j,\eta'|$ in (9), it produces over a single time scale τ_sub two different effects. (*i*) For $i \neq j$, all contributions to (9) fade out and eventually vanish, so that *the different sectors become uncorrelated*. (*ii*) For $i = j$, all terms such that $\eta \neq \eta'$ disappear, while the coefficients $K(i,\eta;i,\eta;t)$ of the various terms η all tend to one another, their sum remaining constant. Altogether, the coherences disappear and *the populations equalise within each sector*. The duration τ_sub of both relaxations is by definition shorter[19] than $t_\mathrm{f} - t'_\mathrm{f}$, and the mechanism is already effective before t'_f, so that the relaxation is likely to have already been effective at t'_f; otherwise $\hat{\mathcal{D}}_\mathrm{dec}$ anyway reaches at the final time $t_\mathrm{f} > t'_\mathrm{f} + \tau_\mathrm{sub}$ the "multi-microcanonical" equilibrium[20]

[18] Strictly speaking, this property holds only for the microcanonical states $\hat{\mathcal{R}}_i^\mu$. However, $\mathrm{tr}_\mathrm{A}\hat{\mathcal{R}}_i\hat{\Pi}_i$ is close to 1 if Δ is sufficiently large compared to the width of $\hat{\mathcal{R}}_i$, and $\mathrm{tr}_\mathrm{A}\hat{\mathcal{R}}_i\hat{\Pi}_j$ for $i \neq j$ is negligible if Δ is small compared to the distance between the possible outcomes A_i.

[19] Two different mechanisms achieving such a process have been fully worked out for the CW model [6, §11.2], and it has been shown that they produce the result (10). In the more realistic one [6, Appendices H and I], the transitions that modify η are produced by an interaction $\hat{V}$ between the magnet and the bath which has a variance $v^2 = \mathrm{tr}\,\hat{V}^2$; an average delay θ separates successive transitions. Microcanonical relaxation may take place even if $\hat{V}$ is not macroscopic, with a variance scaling as $v \propto N^a$ $(a < 1)$ for large N. For a short θ that scales as $\theta \propto 1/N^b$ $(a < b < 2a)$, the characteristic time $\tau_\mathrm{sub} = \hbar^2/v^2\theta$ scales as $1/N^c$ where $c = 2a - b$, $0 < c < a < 1$; it is short compared to the registration time, which dominates t_f because registration involves a macroscopic dumping of energy from the magnet to the bath, in contrast to the subensemble relaxation.

[20] Such a result may be obtained through decoherence by an external environment. However, the present process, contrary to the truncation of Sec. 5 (which may also result from decoherence), involves only the apparatus, and thus does not depend on the tested observable of S.

$$\hat{\mathcal{D}}_{\rm dec}(t_{\rm f}) = \sum_i q_i \hat{r}_i \otimes \hat{\mathcal{R}}_i^{\mu}, \quad q_i = \sum_{\eta} K(i,\eta;i,\eta;t_{\rm f}'). \tag{10}$$

Since the above derivation holds for arbitrary operators $\hat{\mathcal{D}}_{\rm dec}$ issued from a mathematical decomposition of $\hat{\mathcal{D}}$, it encompasses *the physical states* $\hat{\mathcal{D}}_{\rm sub}^{(k)}$ *associated with all the possible subsets* $\mathcal{E}_{\rm sub}^{(k)}$ of runs, which therefore end up in the required equilibrium form (3). We can from now on *forget about unphysical decompositions*, which were introduced only as a tool for establishing the expression (3) of $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})$. All physical subensembles $\mathcal{E}_{\rm sub}^{(k)}$ involve at the final time the *same building blocks* $\hat{\mathcal{D}}_i$, and *the quantum ambiguity has been removed.*

By using the form (3) for the state $\hat{\mathcal{D}}_{\rm sub}^{(k)}$, thus justified from (10), and the property[20] ${\rm tr}_{\rm A}\hat{\mathcal{R}}_i\hat{\Pi}_j = \delta_{ij}$, we get for any subensemble $\mathcal{E}_{\rm sub}^{(k)}$

$${\rm tr}\,\hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})\hat{\Pi}_i = q_i^{(k)}. \tag{11}$$

Each weight $q_i^{(k)}$ is therefore identified as the *q-probability* of occurrence for the pointer of the macroscopic value A_i in the subensemble $\mathcal{E}_{\rm sub}^{(k)}$ of runs of the measurement. The narrowness ($\Delta \ll |A_i - A_j|$) of the spectrum of the projectors $\hat{\Pi}_i$ entails that for any $\mathcal{E}_{\rm sub}^{(k)}$ the *q-distribution* ${\rm tr}\,\hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})\delta(\hat{A} - A)$ of $\hat{A}$ is strongly peaked around the values A_i, with the weights $q_i^{(k)}$. These quantum properties are still formal and call for an interpretation (Sec. 8).

We also note, by using the commutation $[\hat{\mathcal{R}}_i, \hat{\Pi}_j] = 0$, that the q-expectation values

$${\rm tr}\,\hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})[\hat{\Pi}_i, \hat{O}] = 0 \tag{12a}$$

vanish for arbitrary operators $\hat{O}$. More generally, if the typical dimension G of the projectors $\hat{\Pi}_i$ is large, and if $\hat{P}$ and $\hat{P}'$ denote arbitrary projection operators with finite dimension whereas $G \gg 1$, one readily shows, by expansion on the basis $|s_i\rangle|A_i,\eta\rangle$, that

$${\rm tr}\,\hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})\hat{P}'[\hat{\Pi}_i, \hat{P}]\hat{P}'' = \mathcal{O}\left(\frac{1}{G}\right) \tag{12b}$$

is small. Any operator of S+A containing as a factor a commutator $[\hat{\Pi}_i, \hat{O}]$ of a pointer observable with an arbitrary observable $\hat{O}$ (finite for large G) can be written as a weighted sum of terms (12b). Hence, Eqs. (12a-b) express that the *q-expectation value in the final state* of any operator depending on the projectors $\hat{\Pi}_i$ through *commutators* $[\hat{\Pi}_i, \hat{O}]$ *with arbitrary finite observables* $\hat{O}$*, is negligible* for a macroscopic pointer.

As discussed in Sec. 6, the states $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})$ associated with the various physical subensembles are related to one another by a *hierarchic structure*, embedded in the following additivity of the q-probabilites $q_i^{(k)}$: If two non overlapping subensembles $\mathcal{E}_{\rm sub}^{(k')}$ and $\mathcal{E}_{\rm sub}^{(k'')}$ of $\mathcal{E}$, described by $\hat{\mathcal{D}}_{\rm sub}^{(k')}$ and $\hat{\mathcal{D}}_{\rm sub}^{(k'')}$ and having $\mathcal{N}^{(k')}$ and $\mathcal{N}^{(k'')}$ elements, respectively, merge into a new subensemble $\mathcal{E}_{\rm sub}^{(k)}$, the above proof implies

$$q_i^{(k)} = \frac{\mathcal{N}^{(k')}q_i^{(k')} + \mathcal{N}^{(k'')}q_i^{(k'')}}{\mathcal{N}^{(k')} + \mathcal{N}^{(k'')}}, \tag{13}$$

as a consequence of Eq. (7) and of the expression (10) of the still formal q-probabilities q_i. This being just the addition rule for ordinary probabilities, we may suspect that we are beginning to land in *standard probability theory*, but the results proved above are still formal and call for physical interpretation.

8 Interpretative principles and individual runs

In order to elucidate the measurement problem for ideal quantum measurements, it remains to show that the operators $\hat{\mathcal{D}}_i$ are not only the building blocks of the expressions (2) and (3) derived above for the density operators $\hat{\mathcal{D}}(t_\mathrm{f})$ and $\hat{\mathcal{D}}^{(k)}_\mathrm{sub}(t_\mathrm{f})$, but also that they are the possible final states (1) of individual runs. However, we have stressed (Sec. 2 and Sec. 3) that quantum mechanics, in its minimalist formulation, does not deal with individual systems but only provides q-information on statistical ensembles – possibly Gedanken, but physically consistent. In such a strictly formal framework free from any interpretation, we have shown how the initial state $\hat{\mathcal{D}}(0)$ of S+A for a run randomly extracted from the ensemble $\mathcal{E}$ relaxes to $\hat{\mathcal{D}}(t_\mathrm{f})$, then how the states associated with *all possible subensembles* $\mathcal{E}^{(k)}_\mathrm{sub}$ reach at the final time t_f the equilibrium structure $\hat{\mathcal{D}}^{(k)}_\mathrm{sub}(t_\mathrm{f}) = \sum_i q_i^{(k)} \hat{\mathcal{D}}_i$.

These results (2) and (3) are *the most detailed ones* about ideal measurements that can be derived through the mere formalism of quantum statistical mechanics. In order to relate them to physical events, and in particular to properties of individual runs, we ought to supplement the quantum rules of Sec. 3 with additional principles. These principles or postulates, which we gather below, will allow us to interpret the equations proved above. Natural and as economic as possible, they refer to macroscopic variables that may be observed. They lie *astride macrophysics and microphysics* and are consistent both with our macroscopic experience and with the quantum formalism.

1. We have introduced a first, trivial principle in Sec. 3 by noting that, if a macroscopic observable has a q-variance negligible in relative size, its q-expectation value can be interpreted as an *ordinary macroscopic value*. We have used this identification in Sec. 4, together with the maximum entropy criterion, to assign to the apparatus the Boltzmann-Gibbs equilibrium state $\hat{\mathcal{R}}^h_i$ (or $\hat{\mathcal{R}}_i$) for a given macroscopic energy including (or not) the source term $\hat{h}_i$. Likewise, the initial metastable quantum state $\hat{\mathcal{R}}(0)$ of A, the starting point of Sec. 5, is characterised by some physical macroscopic data. In case the initial state $\hat{r}(0)$ of S to be tested is specified by a controlled preparation, it is characterised by microscopic well-defined variables.

2. Our second principle was at the root of the subensemble approach of Sec. 7. It expresses that density operators which obey the probabilistic and dynamic rules of quantum mechanics may in certain circumstances be assigned not only to systems picked at random from a full statistical ensemble $\mathcal{E}$, but also to systems extracted from some of its subensembles. However, on account of the quantum ambiguity, this introduction of *several subensemble dependent density operators* is meaningful only if such subensembles can be characterised by some macroscopic property that ensures their physical reality. After the end of a quantum measurement process, this is granted by the possibility of observing and selecting the pointer indications, and then the relevant subensembles $\mathcal{E}^{(k)}_\mathrm{sub}$ are characterised by the proportions $q_i^{(k)}$ of runs that have eventually produced the outcome A_i. We inferred from such an evidence a postulate, namely that these physical subensembles can already be described, at times $t > t'_\mathrm{f}$ slightly earlier than the end t_f of the process, by the density operators $\hat{\mathcal{D}}^{(k)}_\mathrm{sub}(t)$. Nevertheless, subensembles cannot theoretically be identified at early stages of the process, although individual runs evidently exist experimentally at all times. The possibility of assigning the states $\hat{\mathcal{D}}^{(k)}_\mathrm{sub}(t)$ to the subensembles $\mathcal{E}^{(k)}_\mathrm{sub}$ emerges from the dynamics, since this may be achieved only after the state $\hat{\mathcal{D}}(t)$ for the full ensemble $\mathcal{E}$ has reached the equilibrium form (2).

Armen, your opinion is needed: Theo is not convinced that we have the right to read into the mess process. Look at $t = t'_\mathrm{f}$: there are still many difficult children (decompositions), but we don't throw them away. We see that when they get adult at $t = t_\mathrm{f}$, they are just OK. The subE relaxation is just christianising them. Roger's reply: *599 No!!! We precisely postulate here that we can read into the process, that is, make a retrodiction from t_f to t'_f so as to specify some properties of the unknown state that we wish to associate with real subensembles. We never christianise the unphysical decompositions but throw them away once we have proven (3). We only introduce them to recognise that the physical states (which are part of the set of decompositions) are OK. Isnt this clear from Sec 7? If not, please suggest changes.*

Theo responds: Technically, the subE relaxation makes any bad decomposition relax to a good one, namely to a micro canonical Gibbs state. Hence I see that their badness is a quantum oddity which is physically not serious, since finally they come out fine. But the situation would be dramatic if the Hamiltonian did not possess the weak random-matrix terms – it would be a horrible apparatus.

3. To achieve our task, we will introduce a last interpretative principle, which refers to a *macroscopic quantity* that may take *several discrete values*[21], here the different indications of the pointer. It will amount to the interpretation of the *formal q-probabilities* $q_i^{(k)}$ expressed by (11) *as physical probabilities, i. e., as relative frequencies of occurrence of the pointer outcomes* A_i in the subensemble $\mathcal{E}_{\rm sub}^{(k)}$ of runs. We justify below heuristically the introduction of this principle, then bring out its core by showing that *that at the final time the pointer observable has*, roughly speaking, *a "classical behaviour"* expressed by Eqs. (12a-b). The argument covers two properties that we now specify more precisely.

On the one hand, we may compare the present situation with the frequency approach to the theory of ordinary probabilities [21], which also relies on the consideration of subensembles. There, probabilities are introduced as numbers associated with a large ensemble and with its subensembles[8], which among other properties have the following ones: they are non negative and normalised, they are additive for disjoint subensembles, they may take any value between 0 and 1. Here, a similar structure arises for the q-probabilities $q_i^{(k)} = {\rm tr}\, \hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})\hat{\Pi}_i$, where $\hat{\Pi}_i$ is the projector associated with the macroscopic value A_i of the pointer. They satisfy the above properties of classical probabilities including the additivity (13), except for the last one, to take all values between 0 and 1. In fact, they came out in Eq.(10) as formal objects, and the quantum formalism is not sufficient to ensure that their *range extends from 0 to 1.* Our postulate makes up for this.

On the other hand, the essential feature that distinguishes quantum observables from classical ones is the non-commutative nature of their algebra. The projectors $\hat{\Pi}_i$ present in this respect a remarkable feature. Consider their commutators $[\hat{\Pi}_i, \hat{O}]$ with arbitrary observables $\hat{O}$ ($\hat{O}$ being bounded when the typical dimension G of the projectors becomes large). Eqs. (12a-b) imply that, in any state $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t_{\rm f})$ describing the outcome of a subensemble, all q-expectation values involving commutators $[\hat{\Pi}_i, \hat{O}]$ are negligible as $1/G$. The observables $\hat{\Pi}_i$ thus behave *at the final time* $t_{\rm f}$ *as if they commuted with the full algebra.* The quantum nature of these *macroscopic variables* has become concealed, so that they have a *classical behaviour at the final stage* of the process.

It then becomes natural to *postulate* that the q-probabilities $q_i^{(k)}$ can be interpreted as standard probabilities in the sense of frequencies, which implies that *their range extends down to* 0. In fact, this identification, not imposed by the quantum rules but in harmony with them, is imposed by macroscopic experience, since in our minimalist formulation of quantum mechanics without interpretation, we can ascertain that each individual run provides a well defined indication A_i for the pointer *only* if it is satisfied.

The interpretation of (2) and (3) now becomes obvious. Each $q_i^{(k)}$ can be understood as the proportion of individual runs tagged by A_i in the subensemble $\mathcal{E}_{\rm sub}^{(k)}$. As such coefficients may vary between 0 and 1 in (3), we theoretically acknowledge the (experimentally obvious) existence of subensembles $\mathcal{E}_i$ characterised by the relative frequencies $q_j = \delta_{ij}$, that is, by the specific value A_i of the pointer. An arbitrary *individual run* belonging to $\mathcal{E}_i$ is thus described by the state $\hat{\mathcal{D}}_i$ of S+A. The q-probabilities $p_i = {\rm tr}\, \hat{\mathcal{D}}(t_{\rm f})\hat{\Pi}_i$ are interpreted as the proportion of initial runs having ended up with the indication A_i. A formal quantum approach using some properties of the Hamiltonian of S+A, supplemented by the above interpretative principles, has therefore been sufficient to distinguish the *individual outcomes* of an ideal measurement process, and thus to provide *a solution to the measurement problem.*

[21] Introduced here for the pointer variable to solve the measurement problem, this interpretation of q-probabilities for macroscopic quantities can be used in other contexts, such as the quantum dynamics of phase transitions with spontaneously broken invariance. There, $\hat{\mathcal{D}}(t)$ denotes the state of a statistical ensemble $\mathcal{E}$ of systems, identically prepared at the macroscopic scale, and $\hat{\mathcal{D}}_i$ the equilibrium states characterised by discrete values of the macroscopic order parameter A_i. We assume that the initial state $\hat{\mathcal{D}}(0)$ and the Hamiltonian $\hat{H}$ are sufficiently symmetric so as to avoid favouring the occurrence at the final time of a single outcome A_i. We thus expect $\hat{\mathcal{D}}(t)$ to relax towards a state $\hat{\mathcal{D}}(t_{\rm f})$ of the form (2). Provided time scales are suitable, we also expect, as in Sec. 7, the states $\hat{\mathcal{D}}_{\rm sub}^{(k)}(t)$ associated with all subensembles to relax to the hierarchical structure (3). The present principle can then be used to explain within quantum mechanics why, in the considered circumstances, the order parameter takes a well-defined value in a single experiment (but not always the same value). The community has therefore rightfully not be bothered about this complication, but the question may remain open for not too large systems.

We stress that the above principle, which allows interpreting some q-probabilities as ordinary probabilities, has been founded on several stringent requirements and cannot be extended carelessly.

(*i*) The *effective classicality* of the projectors $\hat{\Pi}_i$ relies on the *macroscopic character* of these observables, since Eq. (12b) holds only for a macroscopic pointer ($G \gg 1$). More generally, we expect the same ideas to hold for macroscopic systems involving several equilibrium states distinguished through high-dimensional projectors, for instance systems with broken discrete invariance[21] (A_i is then replaced by the value of the order parameter). However, it is illegitimate, for microscopic observables, to identify q-expectation values with ordinary expectation values, as exemplified by the GHZ paradox [19], which cannot be understood within ordinary logics and probabilities: The experimental context has to be taken into account.

(*ii*) Moreover, the effective classicality of these projectors $\hat{\Pi}_i$ is ensured *only at the final time*, as Eqs. (12a-b) involve the final states $\hat{\mathcal{D}}_{\rm sub}^{(k)}$. During the process, the commutators of $\hat{\Pi}_i$ cannot be neglected since the pointer must evolve, and since this time-dependence of $\mathrm{tr}\,\hat{\mathcal{D}}(t)\hat{\Pi}_i$ requires that $\mathrm{tr}\,\hat{\mathcal{D}}(t)[\hat{\Pi}_i,\hat{H}]$ does not vanish. Note that, whereas the projector $\hat{\Pi}_i$ pertaining to A does not commute with $\hat{H}$, the projector $\hat{\pi}_i$ pertaining to S commutes with $\hat{H}$, so that $\mathrm{tr}\,\hat{\mathcal{D}}(t)\hat{\pi}_i$ remains constant. Below we shall explain why this q-probability equals the probability p_i related to the apparatus.

(*iii*) The consideration of *subensembles* has also been essential. If we wish individual runs to provide the outcomes $\hat{\mathcal{D}}_i$, the necessary conditions (3) must be fulfilled. Due to the existence of incompatible decompositions of $\hat{\mathcal{D}}(t_{\rm f})$ (Sec. 6), it is not justified to postulate directly, as generally done, that the coefficients p_i in $\hat{\mathcal{D}}(t_{\rm f}) = \sum_i p_i \hat{\mathcal{D}}_i$ might be interpreted as frequencies of the outcomes $\hat{\mathcal{D}}_i$ in the full ensemble $\mathcal{E}$. The dynamical elimination of this quantum ambiguity through subensemble relaxation (Sec. 7), which provides the expected structure for the states $\hat{\mathcal{D}}_{\rm sub}^{(k)}$, allowed us to escape this loophole.

It is therefore disputable to admit blindly Born's rule which identifies the q-probability $p_i = \mathrm{tr}_{\rm S}\hat{r}(0)\hat{\pi}_i$ with a genuine probability (although this rule will be recovered in Sec. 9 within some reservation), since one should neither keep the apparatus aside nor disregard the two conditions (*i*) and (*iii*) above. It is only the quantity p_i defined by $p_i = \mathrm{tr}\hat{\mathcal{D}}(t_{\rm f})\hat{\Pi}_i$ which comes out, interpreted as the probability of observing the value A_i of the pointer. Its interpretation as a probability of $\hat{s}$ to take the eigenvalue s_i when the system S lies in the initial state $\hat{r}(0)$ cannot be regarded as a principle of quantum mechanics.

Indeed, since what is observed is the relative frequency p_i of occurrence of the macroscopic value A_i of the pointer of A, we are informed about the eigenvalue s_i of $\hat{s}$ only through the correlation between s_i and A_i produced by the dynamics. Thus, the usual Born rule will come out here (Sec. 9) as a consequence of these correlations and of above principle 3 about the macroscopic pointer variable. If the coupling $\hat{H}_{\rm SA}$ is too weak, the "registration" process considered in Sec. 4 may be imperfect, driving $\hat{\mathcal{R}}_{ii}(t)$ with some probability to a wrong equilibrium state $\hat{\mathcal{R}}_j$ with $j \neq i$ (see see ref. [7], sect. 8). The resulting incompleteness of the correlation between s_i and A_i then produces a violation of Born's rule, with an observed probability $\mathrm{tr}\,\hat{\mathcal{D}}(t_{\rm f})\hat{\Pi}_i$ different from $\mathrm{tr}_S\,\hat{r}(0)\hat{\pi}_i$.

9 Emergence of classical probabilistic features

Whether the pointer is observed or not, the quantum dynamics and the above principle ensure the uniqueness of the outcome of each run and the existence of subensembles $\mathcal{E}_i$ ending up in the state $\hat{\mathcal{D}}_i = \hat{r}_i \otimes \hat{\mathcal{R}}_i$. However, the latter subensembles can be identified explicitly only through reading or registering the macroscopic pointer indication. A specific subensemble $\mathcal{E}_i$ is selected by picking out the runs tagged by the value A_i. Two steps are thus necessary to go from the initial state $\hat{\mathcal{D}}(0)$ to $\hat{\mathcal{D}}_i$. First, the Hamiltonian evolution (irreversible since A is large) of the coupled system S+A leads to the state $\hat{\mathcal{D}}(t_{\rm f}) = \sum_i p_i \hat{\mathcal{D}}_i$ for the ensemble $\mathcal{E}$, and to the states $\hat{\mathcal{D}}_{\rm sub}^{(k)} = \sum_i q_i^{(k)} \hat{\mathcal{D}}_i$ with unknown coefficients $q_i^{(k)}$ for its subensembles. The second step, leading from the state $\hat{\mathcal{D}}(t_{\rm f})$ to one of its components $\hat{\mathcal{D}}_i$ is *not* a consequence of some *evolution*, but the result of *selecting* the particular outcome A_i (as in the dice example of Sec. 2). This change of state, associated with the extraction

from $\mathcal{E}$ of the subset $\mathcal{E}_i$, is merely an updating of information[22], in the same way as the change of an ordinary probability distribution resulting from some gain of information.

For the time being, we have dealt only with the macroscopic pointer. The complete correlation established by the process between the pointer indications A_i and the final states $\hat{\mathcal{D}}_i$ allows us to get access to some features of microscopic reality. By selecting the outcome A_i and disregarding the apparatus, we set an individual system S into the quantum state $\hat{r}_i$, for which the tested observable $\hat{s}$ has the well-defined value s_i. We thereby recover *von Neumann's reduction*, with the final state $\hat{r}_i = \hat{\pi}_i\hat{r}(0)\hat{\pi}_i/p_i$ in terms of the state $\hat{r}(0)$ initially assigned to S (or $r_i = \hat{\pi}_i$ if the eigenvalue s_i is non degenerate). From the state $\hat{r}_i$ thus prepared, we can predict the q-expectation values of all observables of S after this selection[22]. In particular, the q-expectation values $\mathrm{tr}_\mathrm{S}\hat{r}_i\hat{O}_\mathrm{off}$ of all off-diagonal observables $\hat{O}_\mathrm{off}$ of S (characterised by the conditions $\hat{\pi}_i\hat{O}_\mathrm{off}\hat{\pi}_i = 0$) vanish. The *loss of q-information* about these observables (which do not commute with $\hat{s}$) is a *price to pay for testing the observable* $\hat{s}$.

The correlated form of $\hat{\mathcal{D}}_i = \hat{r}_i \otimes \hat{\mathcal{R}}_i$ and the identity $\mathrm{tr}_\mathrm{A}\hat{\mathcal{R}}_i\hat{\Pi}_j = \mathrm{tr}_\mathrm{S}\hat{r}_i\hat{\pi}_j = \delta_{ij}$ allow us to express the probability $p_i = \mathrm{tr}\,\hat{\mathcal{D}}(t_\mathrm{f})\hat{\Pi}_i$ that the pointer gives the final indication A_i in the alternative form $p_i = \mathrm{tr}_\mathrm{S}\hat{r}(0)\hat{\pi}_i$ eliminating the apparatus (*Born's rule*), provided the process has produced the needed complete correlation between S and A. Hence, for an ideal measurement, we can safely interpret p_i as being, *at the end of the process, an ordinary probability* in the sense of frequency, that is, the relative number of runs for which $\hat{s}$ takes the value s_i. Accordingly, $\mathrm{tr}\,\hat{\mathcal{D}}(t_\mathrm{f})\hat{\pi}_j\hat{\Pi}_i/\mathrm{tr}\,\hat{\mathcal{D}}(t_\mathrm{f})\hat{\Pi}_i = \delta_{ij}$ is the standard conditional probability for $\hat{s}$ to equal s_j if the pointer takes the value A_i; likewise, $\langle\hat{s}\rangle = \mathrm{tr}_\mathrm{S}\hat{r}(0)\hat{s}$ is an ordinary expectation value, and $\langle\delta\hat{s}^2\rangle = \mathrm{tr}_\mathrm{S}\hat{r}(0)\hat{s}^2 - \langle\hat{s}\rangle^2$ an ordinary variance in the final state.

However, *in the initial state*, Born's expression $p_i = \mathrm{tr}_\mathrm{S}\hat{r}(0)\hat{\pi}_i$ is a only a q-probability; *it should not be interpreted as a genuine probability*, since $\hat{s}$ is not macroscopic and keeps its quantum nature in the initial state $\hat{r}(0)$. Even if $\hat{r}(0)$ contains no off-diagonal block, the repeatedly discussed quantum ambiguity forbids such an interpretation. Retrodiction from the outcomes of measurements allows us to determine the diagonal part $\sum_i p_i r_i$ of $\hat{r}(0)$, but not to interpret it with usual concepts. For instance, in an EPR setting, the measurement aims at inferring, from the outcomes of the two detectors, correlations between the spins of the tested systems S, all prepared similarly in a given initial state $\hat{r}(0)$. For each setting of A with a fixed pair of spin directions, it is legitimate to interpret the outcomes as *true correlations* between the two spins of S in the considered directions, but *only in the final state*. However, the corresponding *q-correlations in the initial state* cannot be regarded as standard correlations (they violate Bell's inequalities when the results of experiments involving different contexts – different directions of the detectors – are put together).

As usual in statistical mechanics, the irreversibility of the measurement process emerges from reversible microscopic dynamics owing to the macroscopic size of the apparatus. But moreover, this macroscopic size together with dynamics produce at our macroscopic scale other remarkable types of emergence of features qualitatively different from those of quantum theory.

On the one hand, we have relied on a subensemble analysis to explain the apparent "bifurcation" (or "multifurcation") which leads from the single initial state $\hat{\mathcal{D}}(0)$ to several final states $\hat{\mathcal{D}}_i$. According to our second principle of Sec. 8, it is only by the end of the measurement, at the time t'_f when $\hat{\mathcal{D}}(t)$ has reached the form $\sum_i p_i\hat{\mathcal{D}}_i$, that a theoretical treatment of subensembles is allowed (including the necessary relaxation towards (3)). One cannot recognise the state of any subensemble at earlier times, especially before the achievement of truncation. Thus, the possibility of identifying physical subensembles, in particular the sets $\mathcal{E}_i$, *emerges from the dynamics*. This property *requires the irreversibility* of the process, which remarkably is needed to let us gain information.

[22] Measurements involve both a physical process of interaction between S and A and a selection of outcomes for repeated experiments. If we do not select the indications of A, knowledge about S is updated by replacing $\hat{r}(0)$ by $\sum_i p_i\hat{r}_i$. If the tested observable is not fully specified, the least biased subsequent predictions should rely on a state obtained by averaging over all possible interaction processes. If for instance, one is aware that an ensemble of spins initially prepared in the state $\hat{r}(0)$ have been measured in some direction, but if one knows neither in which direction nor the results obtained, one should assign to the final state the density operator $\frac{1}{3}[\hat{1} + \hat{r}(0)]$ as being the best (but imperfect) description. As repeatedly stressed above, a quantum state should not be regarded as an intrinsic property of a physical system but it reflects our knowledge about the ensemble in which it is embedded.

On the other hand, as indicated above, *classical probabilities emerge* from quantum theory for some physical quantities of S selected by the measurement context, although these quantities are formally represented by operator-valued observables. It is the possibility of reading the macroscopic outcomes A_i of individual runs which reveals, for these observables, the values of their q-probabilities and transfers to them the status of ordinary probabilities in the frequency sense.

10 Conclusion: transfers of information

We used quantum mechanics according to Bohr's celebrated precept: "There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature." Thus, a "state" defined by the mapping $\hat{O} \mapsto \langle \hat{O} \rangle = \mathrm{tr}\hat{\mathcal{D}}\hat{O}$ has only a mathematical nature; the c-numbers $\langle \hat{O} \rangle$ present only a formal analogy with ordinary expectation values. The bridge between formal results and physical reality is provided by the third interpretative principle of Sec. 8, which allows us to identify q-probabilities with relative frequencies for the macroscopic pointer. In this prospect, we may regard an ideal measurement as a *processing of information*, involving in particular transformations between q-information (carried formally by quantum states) and ordinary information (accessible to experiment). At the risk of being tedious, we review below the informational aspects in the above treatment.

As regards the tested system, the state $\hat{r}(0)$, given or not, encodes the q-information characterising initially the ensemble $\mathcal{E}$ in which it is embedded. The part of the q-information that pertains to the observable $\hat{s}$ of interest will, in the end of the ideal measurement process, be disclosed as ordinary information, with standard probabilities for the eigenvalues s_i. We have seen above that fully reaching this information requires destroying the whole q-information about off-diagonal observables $\hat{O}_{\mathrm{off}}$. The subsequent assignment of the state $\hat{r}_i$ to an individual system tagged by the value A_i of the pointer will provide q-information in view of future experiments.

However, it is the apparatus which plays the major role in the above theoretical analysis, and the tested system is kept aside except for the truncation and the registration process of Sec. 5. The initial state of A, characterised by some macroscopic data, is described by the least biased (least informative) density operator $\hat{\mathcal{R}}(0)$ accounting for these data. The dynamics of the coupled system S+A have the effect of transferring q-information from some degrees of freedom to others, with possible downgrading. In fact, some q-information is lost due to irreversibility. Such a loss occurs in truncation (Sec. 5): Through a cascade of q-correlations between S and an increasingly large number of degrees of freedom of A, the q-information that flow towards inaccessibly complicated q-correlations (with a huge recurrence time) irremediably fade out. Downgrading also occurs in registration (Sec. 5): The increase of entropy of A when it relaxes to an equilibrium state $\hat{\mathcal{R}}_i$ is equivalent to a loss of q-information, partly compensated for by the creation of a complete q-correlation between s_i and A_i. In Sec. 7, some q-information about the subensembles $\mathcal{E}^{(k)}_{\mathrm{sub}}$, hidden in the state $\hat{\mathcal{D}}(t_{\mathrm{f}})$ of the full ensemble, was used to set constraints about the states $\hat{\mathcal{D}}^{(k)}_{\mathrm{sub}}(t'_{\mathrm{f}})$ to lie in the Hilbert subspace $\mathcal{H}_{\mathrm{corr}}$.

Within standard quantum mechanics, the dynamics thus transforms part of the initial q-information embedded in $\hat{\mathcal{D}}(0)$ into q-information embedded in the universal building blocks $\hat{\mathcal{D}}_i$, plus q-information embedded in the q-probabilities $q_i^{(k)}$ for the outcomes A_i of the macroscopic pointer. The latter q-probabilities can then be identified (Sec. 8) with ordinary probabilities. Finally, the correlation between s_i and A_i allows us (Sec. 9) to interpret the q-information about the microscopic observable $\hat{s}$ as ordinary information.

Thus all the features of ideal measurements, including the solution of the measurement problem and the classical aspects of the outcomes, have been explained by the sole use (i) of a *minimalist and abstract formulation* of quantum theory, applied to a large apparatus satisfying adequate dynamical properties, and (ii) of *natural interpretative principles* pertaining to macroscopic observables. Interpretation came out naturally, but only in the end. Since alternative interpretations involve unnecessary assumptions of one kind or another, the usage of the ensemble formulation in teaching is advocated. For tutorial purposes, one should distinguish q-probabilities from ordinary probabilities. However, the coefficients $p_i = \mathrm{tr}\,\hat{\mathcal{D}}(t_{\mathrm{f}})\hat{\Pi}_i$, though equal to Born's q-probabilities $p_i = \mathrm{tr}_{\mathrm{S}}\hat{r}(0)\hat{\pi}_i$, come out as relative frequencies of occurrence of the pointer indications in measurements, and should there-

fore be interpreted, not as intrinsic properties of quantum systems, but as partial information about them indirectly collected owing to the full correlation between the eigenvalues of $\hat{s}$ and the apparatus. The distinction between ordinary probabilities, governing macroscopic data, and q-probabilities, characterising quantum states, would then clarify the status of the quantum peculiarities that arise when one puts together results of experiments involving different settings.

11 Left overs from yesterday

In facto, we should anyhow mention in the conclusion the word "interpretation of QM".

Even if we don't, most others will consider what we do as an interpretation. We "interpret" the Q-rules as allowing to start in a metastable initial state etc. Pretty realistic though.

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