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Results in perturbative quantum supergravity from string theory.

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présentée par Piotr Tourkine

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David Skinner	Rapporteur
Dimitrios Tsimpis	Rapporteur
Pierre Vanhove	Directeur de thèse

A Anne-Marie, Marie-Hélène et Audrey,
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Contents

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[1] Ou heureuse, selon que l'on travaille dans le BTP.

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[2] Je suspecte qu'Alexandre n'y est pas pour rien

[3] Eus-je eu le courage de procéder ainsi, j'aurais probablement ajouté à ceci une petite section appelée "Vannes Gratuites", à l'attention du Dr. Lazarescu.

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Personnal publications & Nota

This manuscript is based on the first four publications and preprints [PT1, PT2, PT3, PT4] below. The content of [PT5] has not been covered in this text.

[PT1] P. Tourkine and P. Vanhove, "An R^4 non-renormalisation theorem in $\mathcal{N} = 4$ supergravity," Class. Quant. Grav. **29**, 115006 (2012) [arXiv:1202.3692 [hep-th]].

[PT2] P. Tourkine and P. Vanhove, "One-loop four-graviton amplitudes in $\mathcal{N} = 4$ supergravity models," Phys. Rev. D **87**, no. 4, 045001 (2013) [arXiv:1208.1255 [hep-th]].

[PT3] P. Tourkine, "Tropical Amplitudes," [arXiv:1309.3551 [hep-th]].

[PT4] A. Ochirov and P. Tourkine, "BCJ duality and double copy in the closed string sector," JHEP **1405**, 136 (2014) [arXiv:1312.1326 [hep-th]].

[PT5] N. E. J. Bjerrum-Bohr, P. H. Damgaard, P. Tourkine and P. Vanhove, "Scattering Equations and String Theory Amplitudes," arXiv:1403.4553 [hep-th].

Nota. *The manuscripts of these works have not been included in this printed version, which is intended to be as self-contained as possible, and their most recent versions are accessible on the arXiv server.*

Chapter 1

Introduction

Throughout this text we use the standard system of notations where $\hbar = c = 1$ and our space-time signature is $(-, +, \ldots, +)$. Our kinematic Mandelstam invariants for two-to-two scattering are defined by $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_4)^2$ and $u = -(k_1 + k_3)^2$ where all particles are incoming particles with light-like momentum k_i. The bosonic sector of the heterotic string will always be the left-moving (anti-holomorphic) sector.

1.1 The UV question in quantum gravity.

Quantum gravity is one of the most challenging conundrums in modern physics. Conceptually, this theory is the missing link between quantum field theories that describe particles physics and Einstein's General Relativity that describes the dynamics of space-time. Einstein's equations relate the two realms as

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{space-time}} = \underbrace{16\pi G_N T_{\mu\nu}}_{\text{matter, energy}}\,, \tag{1.1.1}$$

then how could one be quantum and not the other ? The issue is that a naive quantization process leads quickly to inconsistencies, as we expose below.

The quantum nature of space-time is supposed to manifest itself at the Planck energy mass-scale, $M_{\text{Pl}} = 10^{19}$ GeV. Needless to say, this energy scale is far away from the reach of modern colliders. Quantum gravity effects are more likely to be detected in primordial cosmology experiments in the following decades, as the Big-Bang offers a more direct observational window to high energies.

Violation of unitarity One of the basic issues with a naive quantization of gravity is that it causes unitarity violations, already at the classical-level, as a consequence of the structure of gravitational interactions. These are described by the Einstein-Hilbert action coupled to matter

$$S_{\text{EH+matter}} = \frac{1}{2\kappa_D^2} \int \mathrm{d}^D x \sqrt{-g} (R + \mathcal{L}_{\text{matt}}(\phi, \psi, A_\mu))\,, \tag{1.1.2}$$

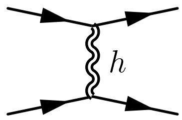


Figure 1.1: Graviton exchange violates unitary at tree-level.

where R is the scalar Ricci curvature, D is the space-time dimension and $\mathcal{L}_{\rm matt}$ is a given matter Lagrangian. By expanding this action around a flat background $g^{\mu\nu} = \eta^{\mu\nu} + \kappa_D h^{\mu\nu}$, where $h^{\mu\nu}$ is the spin-2 graviton field, standard manipulations [1] yield the Lagrangian of matter coupled to gravitons:

$$S_{EH} = \int \mathrm{d}^D x \left(\frac{1}{2}\partial h\, \partial h + \kappa_D c_0 h \partial h\, \partial h + O(\kappa_D h \partial h \partial h) + \dots \right. \\ \left. + \frac{1}{2\kappa_D^2}\mathcal{L}_{\rm mat}(\phi, \psi, A_\mu) + \frac{h}{2\kappa_D}\mathcal{L}_{\rm mat}(\phi, \psi, A_\mu) + \dots \right) . \tag{1.1.3}$$

The structure of this action indicates that gravitons couple to (massless) fields with a two-derivative interaction. Consequently, a single graviton exchange between massless fields such as the one depicted in figure 1.1 has an amplitude proportional to the dimensionless ratio E^2/κ_D where E is the energy of the process. Eventually, unitarity is violated for processes such that $E \gg \kappa_D^2$. At loop level, this classical breakdown of unitarity transfers directly to ultraviolet divergences of the theory. For instance, the amplitude of the double graviton-exchange depicted in fig. 1.2 contains an intermediate sum over states which induces a divergent behavior in the UV as

$$\frac{E^2}{\kappa_D^2} \int^\Lambda \mathrm{d}\tilde{E}\, \tilde{E} \sim \frac{E^2 \Lambda^2}{\kappa_D^2} \tag{1.1.4}$$

where Λ is a UV momentum cut-off. Alternatively these issues can be seen as the consequence of the positive mass dimension $\kappa_D = M_{\rm Pl}^{D-2}$ of the gravity coupling constant, which makes the theory non-renormalizable for $D > 2$. The first divergence of pure Einstein gravity occurs at the two-loop order [2, 3] and is followed by an infinite series of divergences which should be removed, thereby introducing an infinite amount of arbitrariness in the choice of the counterterms and making the quantum theory un-predictive or ill-defined.

Although this manuscript exclusively deals with the perturbative regime of quantum gravity and string theory, it is worth mentioning here that quantum gravity also seems to violate unitarity in black holes physics. On the one hand, the no-hair theorem states that classical black holes should be solely characterized by their mass, spin, and charge. Therefore, Hawking radiation

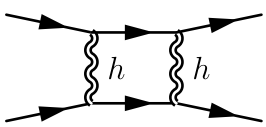


Figure 1.2: One-loop double graviton exchange diverge in the UV.

has to be thermal and can not radiate away the information that fells in the black hole. On the other hand, when the black hole has completely evaporated, this information is “lost”, which is impossible in a unitary quantum mechanical evolution. This goes under the name of the “information paradox”.

There exists nowadays two main paradigms to remedy these issues. The first possibility is to assume that gravity is truly a fundamental theory that should be quantized non-pertubatively and that the previously mentioned issues are artifacts of the perturbative quantization process. This is the point of view adopted by the Loop Quantum Gravity theory and by a somehow related Asymptotic Safety program.

The other option, that we follow and on which are based supergravity and string theory, postulates that general relativity is a low energy effective field theory of a more fundamental theory. Therefore, possibly drastic modifications of the law of physics at the quantum gravity scale are expected to happen.

UV divergences and effective field theories. The UV behavior of quantum gravity is of central importance in the effective field theory paradigm, where the presence of UV divergences signals a wrong identification of the microscopic degrees of freedom of the theory at high energy. In the language of effective actions, UV divergences correspond to local operators, called counterterms, that should be added to the effective action. They parametrize the ignorance about the high energy theory. These operators have to obey the symmetries of the theory, which in gravity include in particular diffeomorphism invariance. This constrains the counterterms to be expressed as tensorial combinations of the Riemann tensor $R_{\mu\nu\alpha\beta}$ or derivatives thereof. For a n-graviton scattering, they are of the form

$$\nabla^m R^n\,, \quad m = 0, 1, 2, \dots \tag{1.1.5}$$

where ∇ is the covariant derivative. These have mass dimension

$$[\nabla^m R^n] = M^{m+2n}\,. \tag{1.1.6}$$

In order to understand more precisely what kind of divergences these counterterm may cancel, let us look back at the structure of the action (1.1.3). The n-graviton vertex always carries at least two derivatives, therefore the most divergent graph at leading order in the gravity coupling constant is made of 3-valent vertices which bring two powers of loop momentum each. Using Euler's relation for a connected graph with L loops, V vertices and I internal edges legs:

$$V - I + L = 1 \tag{1.1.7}$$

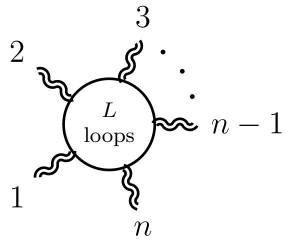
we obtain the *naive* superficial UV behavior of a L-loop n-graviton amplitude

$$\mathcal{M}_n^{L-\text{loop}} = \int \mathrm{d}^D\ell \frac{\ell^{2V}}{\ell^{2I}} \sim \int^\Lambda \frac{\mathrm{d}\ell}{\ell}\, \ell^{L(D-2)+2} \sim \Lambda^{L(D-2)+2}\,. \tag{1.1.8}$$

However, we know that a divergence should be canceled by a counterterm of the form $\nabla^m R^n$. In other words, diffeomorphism invariance of the theory implies that divergent integrals have to factor out a term which can be canceled by such operators, and if $\mathcal{M}_n^{L-\text{loop}}$ in (1.1.8) diverges, the power-counting (1.1.6) indicate that there exists a m a d a n such that

$$\mathcal{M}_n^{L-\text{loop}} = \nabla^m R^n \int^\Lambda \frac{\mathrm{d}\ell}{\ell} \, \ell^{L(D-2)+2-m-2n} \,. \tag{1.1.9}$$

From this we read the actual superficial degree of divergence of such an amplitude, as depicted in eq. (1.1.10). A priori, all of these operators appear in the



$$\sim \quad \nabla^m R^n \times \Lambda^{L(D-2)+2-m-2n} \tag{1.1.10}$$

effective action, where they allow for an infinite number of UV divergences.

1.2 Supergravities

Supergravity theories are proposals for modifications of gravity at high energy with a enhanced UV behavior. These theories are conceptually close to Kaluza-Klein theories, where geometry in higher dimension provides matter and various interactions in lower dimensions. They differ from these in that the “higher” dimensions of supergravity theories include fermionic dimensions. Hence, the four-dimensional space-time is immersed in a higher-dimensional *superspace.*

From the field-theory viewpoint, the geometry of this superspace obeys a new local symmetry, called supersymmetry. This symmetry is characterized by a certain number of real anti-commuting charges called *supercharges*, from 4 in four dimensions to 32 for the maximal extension defined in any dimension up to $D = 11$ [4–6].[4] For definiteness, we shall refer to the number of four-dimensional supercharges $\mathcal{N}$ of the theory only when we talk about the four-dimensional theory. Consequently, $\mathcal{N} = 1$ supergravity is the minimal supergravity in four dimensions and has 4 real supercharges, while $\mathcal{N} = 8$ is maximal supergravity in four dimensions. Half-maximal supergravity or $\mathcal{N} = 4$ in four dimensions is the subject a full chapter of this manuscript, chap. 3. There, we distinguish between $(2, 2)$ and $(4, 0)$ type of constructions, depending on what string theory model the theory arises from.

These theories, some of which are listed in tab. 1.1, have much a richer spectrum than Einstein gravity and seemingly look more complicated. However, at the level of scattering amplitudes, the number of symmetries help to reduce the

[4]Adding more supercharges to a gravity theory forces to include an infinite tower of higher spin excitations

			$s=2$	$s=3/2$	$s=1$	$s=1/2$	$s=0$
$\mathcal{N}=0$	$s_{\max}=2$		1				
	$s_{\max}=2$	($\mathcal{N}=0$, YM)			1		
$\mathcal{N}=1$	$s_{\max}=2$		1	1			
	$s_{\max}=3/2$			1	1		
	$s_{\max}=1$				1	1	
	$s_{\max}=1/2$					1	1
$\mathcal{N}=2$	$s_{\max}=2$		1	1			
	$s_{\max}=3/2$			1	1		
	$s_{\max}=1$	($\mathcal{N}=2$, vect.)			1	1	
	$s_{\max}=1/2$	($\mathcal{N}=2$, hyper.)				1	1
$\mathcal{N}=4$	$s_{\max}=2$	($\mathcal{N}=4$, grav.)	1	4	6	4	1+1
	$s_{\max}=3/2$			1	4	6	4+4
	$s_{\max}=1$	($\mathcal{N}=4$, matt.)			1	4	6
$\mathcal{N}=6$	$s_{\max}=2$		1	6	15+1	20+6	15+15
	$s_{\max}=3/2$	($\mathcal{N}=6$, matt.)		1	6	15	20
$\mathcal{N}=8$	$s_{\max}=2$	($\mathcal{N}=8$)	1	8	28	56	70

Table 1.1: Partly reproduced after the textbook on supergravity theories [7]. Spin content of massless supersymmetry representations with maximal spin $s_{\max} \leq 2$ in four dimensions. The first line with $\mathcal{N}=0$ corresponds to pure Einstein gravity. The supermultiplet denominations within the parentheses correspond to notations used throughout the text.

complexity of the theories. Part of the discussion in this manuscript is focused on understanding these simplifications from the string theory perspective.

Among these extended supergravity theories, maximal supergravity have held a favorite position as the most promising candidate for a four-dimensional UV complete point-like theory of quantum gravity. It was however understood that an R^4 counterterm did respect the *linearized* maximal supersymmetry, very likely indicating a 3-loop divergence [8–12] in four dimensions. Despite this belief, curious similarities between the UV behavior of maximal super-Yang-Mills (SYM) and maximal supergravity were observed in particular in [13, 14]. Since maximal SYM is UV finite in four dimensions [15], this suggested that $\mathcal{N}=8$ might indeed be a UV finite theory. We recall that L-loop amplitudes in maximal SYM are UV finite in dimensions D such that [13, 16, 17]

$$D < D_c = 4 + 6/L\,, \tag{1.2.1}$$

where D_c is called the critical UV dimension and is defined here to be the smallest space-time dimension for which the theory diverges logarithmically at L loops.

In ref. [18], Green *et al.* predicted, using non-renormalization theorems in string theory, that the UV behavior of maximal supergravity might be softer than previously expected and that the three-loop divergence at could actually vanish. This issue was definitely settled by the explicit computation of Bern *et al.* in [19, 20], that was followed by a similar result at four loops [21]. Nowadays, the most elaborate predictions based on string theory non-renormalization theorems [18, 22, 23], supersymmetry [24–26] and duality symmetry analysis [27–32] predict that the critical behavior should change abruptly at five loops due to an allowed $\nabla^8 R^4$ counterterm, according to

$$\begin{aligned} D < D_c &= 4 + 6/L\,, \quad L < 5\,, \\ D < D_c &= 2 + 14/L\,, \quad L \geq 5\,, \end{aligned} \tag{1.2.2}$$

This critical behavior predicts that maximal supergravity may diverge at seven loops in four dimensions.

Despite the important progress made in the last decade in the field of scattering amplitude computations (see chapter 4 for a short review), the 7-loop order is still out of reach. Nonetheless, already the analysis of the UV behavior at five loops may indicate if the current string theory understanding is correct or needs to be deepened. If the critical dimension of $\mathcal{N} = 8$ is

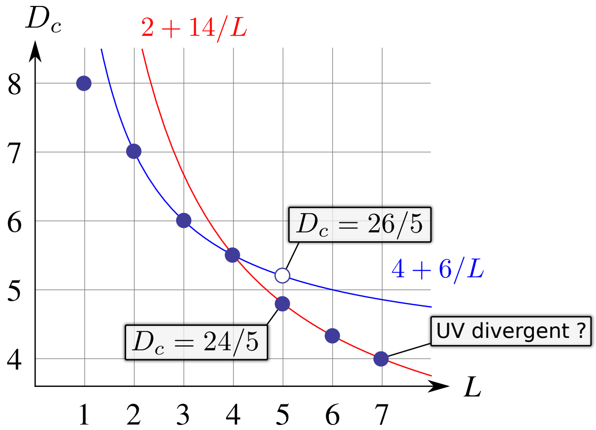


Figure 1.3: Critical UV behavior of maximal supergravity. ●: UV divergences predicted by string theory. ○ : 5-loop possible UV behavior indicating that $\mathcal{N} = 8$ might be UV finite.

strictly the same as the one of $\mathcal{N} = 4$ SYM, the five-loop divergence should occur in $D = 26/5$, corresponding to a $\nabla^{10} R^4$ counterterm. On the contrary, according to the previous predictions, the $\nabla^8 R^4$ counterterm is expected to cause a divergence in $D_c = 24/5$ at five loops.

The importance of the five-loop explicit computation is therefore crucial. As a matter of fact, this computation has been started for several years by the group of [21]. The first approach to the computation relied on the use of the "Bern-Carrasco-Johansson" duality [33] applied to the so-called double-copy construction of gravity amplitudes [34]. Despite important successes at three and four loops [34, 35], the prescription seems to work less efficiently at five-loop and for the moment has not been implemented [36]. In addition to the intrinsic interest of a first principle explanation of this BCJ duality, the five-loop problem acted as a motivation for the analysis of [PT4] which we describe in chap. 4, on first steps towards a string theoretic understanding of the BCJ duality at one-loop.

Another way to test the accuracy of string theory predictions is to study theories with reduced supersymmetry. This allows to trigger more divergent theories, thereby more accessible by explicit computations. In that perspective, half-maximal supergravity is the most natural candidate whose UV behavior should be investigated. This theory has a richer structure than maximal supergravity and can be realized in the low energy limit of various kind of string models, some of which we describe in chapter 3. The theory admits couplings to maximally SYM matter multiplets [37], which render the theory UV divergent already at one-loop for amplitudes with external matter fields [38]. The first section, sec. 3.1 of chap. 3 is dedicated to a review of the analysis given in [PT2] of graviton amplitudes at one-loop in several type of string theory models providing $\mathcal{N}=4$ supergravity, in heterotic string and orbifolds of type II string.

The following section, sec. 3.2 deals directly with the UV behavior of pure half-maximal supergravity. It was shown in [39–41] that R^4 is a half-BPS one-loop exact operator in heterotic string toroidal compactifications, and confirmed later in [42] by using the explicit two-loop computation of [43–49]. We review the analysis of [PT1] based on the use of the "Chaudhuri-Hockney-Lykken" [50–52] orbifolds of the heterotic string to show a non-renormalization theorem for the R^4 counterterm in pure half-maximal supergravity. This analysis provides a worldsheet supersymmetry argument for the origin of the vanishing of the R^4 3-loop logarithmic divergence in pure half-maximal supergravity originally observed in [53]. However, an additional element enters the analysis in this theory, due to the presence of a $U(1)$ anomaly [54] whose implication in the UV behavior of the theory is still unclear.

There are two lessons to draw from the previous discussion. First, it appears that string theory is a good tool to understand the UV behavior of supergravity theories. Second, supergravities do not seem to be drastic enough modifications of gravity to ensure a proper quantum behavior. Therefore, the same reason for which string theory is an efficient tool also indicates it as an empirical necessary UV completion for supergravity theories.

1.3 String theory

String theory has an even richer history than maximal supergravity, which we do not intend to recapitulate completely here.[5] It was born almost half-a-century ago as a model to describe strong interactions with the Veneziano amplitude [56], that was soon after supplemented by a proposal from Virasoro [57], which we reproduce here:

$$M^{\mathrm{Vir}}(s,t,u)=\frac{\Gamma(-1-\alpha'\frac{s}{4})\Gamma(-1-\alpha'\frac{t}{4})\Gamma(-1-\alpha'\frac{u}{4})}{\Gamma(-2-\alpha'\frac{s}{4}-\alpha'\frac{t}{4})\Gamma(-2-\alpha'\frac{t}{4}-\alpha'\frac{u}{4})\Gamma(-2-\alpha'\frac{u}{4}-\alpha'\frac{s}{4})}, \tag{1.3.1}$$

The variables s and t and u are the usual kinematic Mandelstam invariants, respectively defined by $-(k_1+k_2)^2$, $-(k_1+k_4)^2$ and $-(k_1+k_3)^2$ and α' was called the Regge slope. Later it was understood that these amplitudes describe

[5] See [55] for a detailed historical perspective and list of references.

the interactions and scattering of open and closed relativistic bosonic strings of size $\ell_s = \sqrt{\alpha'}$ and tension $T = (2\pi\alpha')^{-1}$. Quantization and Lorentz invariance imposed that they propagate in a target 26-dimensional space-time and that their spectrum contains an infinite tower higher spin excitations with quantized masses given by

$$m^2_{\text{closed}} = \frac{4n}{\alpha'}, \quad m^2_{\text{open}} = \frac{n}{\alpha'}, \qquad n = -1, 0, 1, \ldots, +\infty \tag{1.3.2}$$

and maximal spins $J_{\max} = \alpha' m^2 + 1$. Both theories contained a tachyonic state (at $n = -1$) and the massless excitations of the closed string always contained a graviton. Later, the theory was extended to a theory of supersymmetric strings living in a 10-dimensional target space-time, where the tachyon was automatically projected out via the so-called "Gliozzi-Sherck-Olive" (GSO) projection [58]. This theory was shown to possess maximal supergravity in its massless spectrum in [59], making it a UV completion thereof.

Let us try to give a flavor of how string theory cures the structural problems of perturbative quantum gravity, namely unitarity violation and UV incompleteness. Firstly, the amplitude (1.3.1) has a high energy behavior now compatible with unitarity. In particular, in the hard scattering limit ($s, t \to +\infty$, fixed angle), this amplitude exhibits an exponentially soft behavior:

$$M^{\text{Vir}}(s,t) \sim \exp\left(-\frac{\alpha'}{2}(s \ln s + t \ln t + u \ln u)\right) \tag{1.3.3}$$

which can be seen as a restoration of the unitarity due to the infinite tower of massive states smoothing the interaction.

In order to comment on UV divergences, we need first to say a word on the quantization of string theory. String theory scattering amplitudes are computed in a first quantized formalism, as Feynman path integrals over the trajectories of the string in space-time. These trajectories draw a worldsheet, and the quantization process reduce the sum over trajectories to a finite dimensional integral over the moduli space of Riemann surfaces. The genus thereof, denoted by the letter g in this text, is related to the number of times the strings have split or joined during their evolution, and indicate the loop order of the interaction. One of the most notable features of string theory first-

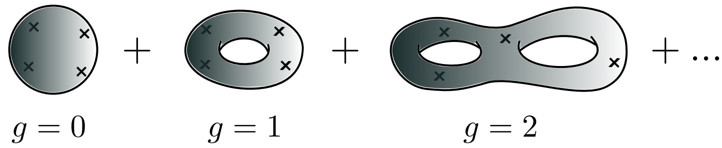


Figure 1.4: Perturbative expansion of string theory (four-point scattering exmaple).

quantized amplitudes is the compactness of the expressions. This is firstly a consequence of the fact that there is a single string graph at each order in perturbation theory; this is considerably simpler than the sum of Feynman graphs in quantum field theory. In addition, the computations of the string theory integrands are based on powerful conformal field theory (CFT) techniques which also simplify drastically the computations and give rise to a superior organization of the amplitude, in particular making manifest some cancellations not easily visible by other means. On the other hand, the integral over the moduli space of Riemann surfaces is most of the time impossible to carry, and despite the compactness of final answers, intermediate steps of computation can be fastidious.

Physically, the mathematical reduction from all trajectories to Riemann surfaces is a consequence of string theory not being simply a theory of an infinite tower of interacting states; the latter wouldn't be a UV complete theory, as recalled in [60, sec. 7.3]. String theory has an additional, crucial, physical feature: it gives a minimal length to space-time phenomena, the string length $\sqrt{\alpha'}$. In loop amplitudes, this implies that the ultraviolet region is simply absent from the phase space of string theory ! As a consequence, string theory is a UV complete theory.

In contrast, the theory has an infrared (IR) region, which is precisely the one of interest for us in this text, as it describes the regime in which the strings effectively behave as particles. We shall alternatively refer to this regime as the $\alpha' \to 0$ limit[6], low energy limit or the field theory limit of string theory. One of the objectives of this text is to discuss some of the techniques known in the literature concerning this limit in the context of string theory amplitudes. It is not surprising that if the advantages of string theory amplitudes motivate the use of such procedures, its drawbacks should be encountered along the way. There are basically two classes of physical objects that can be extracted out of string theory amplitudes; field theory amplitudes – with their UV divergences – and low energy effective actions. The present text mostly describes the first type of computations.

In chap. 2 we discuss the general procedure to extract field theory amplitudes from string theory amplitudes. These techniques were pioneered by Green and Schwarz in [59], and their use culminated at one-loop with the development by Bern and Kosower of a set of rules to write one-loop gauge theory n-gluon amplitudes in [61–64], as the $\alpha' \to 0$ limits of string theory amplitudes. The reason why such a procedure is efficient is because of the compactness of string amplitudes. The technical difficulties that are faced in general ($g \geq 2$) involve firstly the geometry of the moduli space of Riemann surfaces, which should be described correctly in order to reproduce the various graph topologies in the low energy. Another class of difficulties come precisely from the degenerations of the CFT on higher genus Riemann surfaces. In [PT3], we argued that tropical geometry, a somewhat recent branch of mathematics, helps to solve these issues.

Another remarkable feature of string theory is that it provides a framework where it is possible to carry the exact computation of some coefficient of the operators in low energy effective action. Let us here simply mention that the

[6]In strict rigor, it is rather defined as a limit where the energy E of the scattered states is much smaller than the string scale $\alpha' E \ll 1$.

automorphic form program [22, 23, 65–71] led to the *exact non-perturbative* computation of R^4, $\nabla^4 R^4$ and $\nabla^6 R^4$ couplings in the type II string effective action in various dimensions. These exhibit directly non-renormalization properties, since they receive only a finite number of loop corrections. For instance, the essence of the previous prediction on the critical UV behavior of maximal supergravity follows from the fact that R^4 is not perturbatively renormalized beyond one-loop, $\nabla^4 R^4$ beyond two loops, $\nabla^6 R^4$ beyond three loops. The coupling corresponding to $\nabla^8 R^4$ has not been computed yet, but it is expected to receive contributions through all loop orders by different type of arguments mentioned above [24–32]. The counterpart of these computations in string theory amplitudes corresponds to factorization of derivatives in the pure spinor formalism [17]. In [PT1] we presented an explication for the vanishing of the three-loop divergence of $\mathcal{N} = 4$ pure supergravity [53] due to a non-renormalization theorem in heterotic string orbifold models for the R^4 term at two-loops. The computation is based on the explicit factorization of two derivatives in the computation of D'Hoker and Phong at two loops [43–49].

Structure of the manuscript.

Below is a quick summary of the organization of this manuscript.

In chap. 2 we review the analysis of [PT3] on the low energy limit of string theory amplitudes and the connexion with tropical geometry. We discuss applications in sec. 2.3, where we provide a novel analysis on the low energy limit and in particular the graph structure of the four-graviton three-loop amplitude computed in [72].

In chap. 3 we cover the content of [PT1, PT2] on half-maximal supergravity amplitudes at one-loop and the UV divergences of this theory at higher loops. We provide novel piece of analysis on the worldline structure of these amplitudes at two-loop.

In chap. 4, we describe the arguments presented in [PT4] towards a string theoretic understanding of the BCJ duality at one-loop.

The final chapter contains open questions and future directions of research.

Chapter 2

The field theory limit of closed string amplitudes and tropical geometry

In the introduction, we motivated the study of string theory amplitudes as an efficient way to access field theory amplitudes. Physically, there is no doubt that the perturbative expansion of string theory reproduces the Feynman graph expansion of the low energy effective field theory in the point-like limit $\alpha' \to 0$ of the string. However, this procedure has not been applied beyond one-loop[7] and a lot of technical tools are missing at higher genus. In [PT3], the author initiated a program to develop these tools by using a previously unnoticed connexion between the $\alpha' \to 0$ limit of string theory amplitudes and a recent field of mathematics called tropical geometry.[8] This chapter is a review of this work. We also present in the last section some elements of a novel three-loop analysis.

2.1 Closed string theory amplitudes and their field theory limit.

Our intention here is not to provide an exhaustive recapitulation of the material present in the standard textbooks [60, 77–79] on the perturbative quantization of string theory, but rather to recall some essential facts about string perturbation theory in order to introduce some important notions for the discussion of this chapter.

[7] Except the recent work at two loops of [73] in the Schottky parametrization, continuing the older works [74, 75]

[8] Let us take the opportunity here to mention that the construction of [76] is actually the first time where tropical geometry was used (even before its "official" birth !) in physics, in a different context though. There, tropical varieties, called "grid diagrams", were defined as configurations of branes in the five-dimensional decompactification limit of four-dimensional $\mathcal{N} = 2$ gauge theories.

2.1.1 Superstring theory amplitudes

Bosonic string path integral String theory scattering amplitudes or S-matrix elements are computed in a first quantized formalism. The coordinates X^μ of the string define an embedding of the two-dimensional manifold swept by the string – the worldsheet – in the target space-time in which it evolves. From this worldsheet viewpoint, the X^μ's are scalar fields, whose dynamics is governed by Polyakov action[9]

$$S_{\text{Polyakov}} = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma \mathrm{d}\tau \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \tag{2.1.1}$$

where σ and τ are the worldsheet coordinates, g^{ab} is the worldsheet quantum metric and $G_{\mu\nu}(X)$ is the target space-time metric. Lorentz invariance and mathematical consistency allow for only two kind of space-time interactions between strings (open or closed): splitting and joining.

The quantum mechanical amplitude for a process including propagation with or without interactions is given by a path integral over worldsheets that connect initial and final asymptotic states, weighted by the string action,

$$\int \frac{\mathcal{D}X\,\mathcal{D}g}{V_{\text{Diff}\times\text{Weyl}}} \exp(-S) \tag{2.1.2}$$

The factor $V_{\text{Diff}\times\text{Weyl}}$ is the volume of the diffeomorphisms and gauge freedom on the worldsheet required to counterbalance the over-counting of the path-integral. At the g-th order in perturbation theory, for an n-point scattering, standard BRST procedure fixes this gauge redundancy and reduces the integration to a finite dimensional space of dimension $3g-3+n$: the moduli space of genus-g n-pointed Riemann surfaces $\mathcal{M}_{g,n}$. The outcome of the quantization of the bosonic string is well known; the theory should live in 26 dimensions, has no fermions and has a tachyon.

Superstring path-integral Extending the bosonic formulation to a supersymmetric one projects out the bosonic string tachyon by introducing fermions. Conceptually, this gives a heuristic motivation for the existence of fermions, as a necessity to produce a sensible quantum theory of strings. A similar situation happens for supergravity, where fermions soften the bad UV behavior of Einstein gravity. Three formulations of superstring theory are known; the Green-Schwarz [77, 80] and Berkovits pure spinor [81, 82] space-time supersymmetric formalisms, and the Ramond-Neveu-Schwarz worldsheet supersymmetric formalism.

The advantage of the first two is to implement the very appealing physical idea that the superstring should move in a "super-spacetime". In this case the bosonic coordinates X^μ are supplemented with fermionic ones θ^α and space-time supersymmetry is guaranteed from the start. The drawback of the Green-Schwarz formalism is the difficulty to gauge the so-called κ-symmetry, which

[9] We skipped the traditional pedagogical introduction of the Nambu-Goto action with its square root that creates difficulties in the quantization.

the pure spinor formalism manages to do thanks to the introduction of a pure spinor ghost field. A lot of results have been obtained in this formalism, among which a recent three-loop four-graviton amplitude computation [72] which we discuss in this text. It should be noted that above genus five, the prescription to compute the pure spinor ghost path integral has to be changed [83], and that the impact of this change in explicit computations has not been cross-checked so far.

On the other hand, the Ramond-Neveu-Schwarz formulation has the advantage of mathematical robustness. The formulation is based on the extension of the usual worldsheet to a super-worldsheet, by supersymmetrizing the Polyakov action and adding superpartners to the X^μ scalars, the fermionic fields ψ^μ, and a superpartner to the metric field g, the gravitino field χ_α:

$$S_{RNS} = -\frac{1}{8\pi}\int_\Sigma \mathrm{d}\sigma\mathrm{d}\tau\sqrt{g}\left(\frac{2}{\alpha'}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\mu + 2i\psi^\mu\sigma^\alpha\partial_\alpha\psi^\mu \right.$$
$$\left. - i\psi^\mu\sigma^\beta\sigma^\alpha\chi_\beta\left(\sqrt{\frac{2}{\alpha'}}\partial_\alpha X^\mu - \frac{i}{4}(\chi_\alpha\psi^\mu)\right)\right) \tag{2.1.3}$$

In this formalism, suitable gauging of the supergravity fields on the a genus-g n-pointed super-worldsheet induces an integration over $3g-3+n$ bosonic and $2g-2+n$ fermionic moduli which span the moduli space of genus-g n-pointed super-Riemann surfaces $\mathfrak{M}_{g,n}$ [84–86]. The amplitude is obtained by integrating a correlation function of vertex operators $V_1,\ldots,V_n$ corresponding to external scattered states as

$$\mathsf{A}_{\alpha'}^{(g,n)} = \int_{\mathfrak{M}_{g,n}} \mathrm{d}\mu_{SS}\,\langle V_1 ... V_n \mathcal{O}_1 ... \mathcal{O}_k\rangle \tag{2.1.4}$$

where $\mathrm{d}\mu_{SS}$ is the supermoduli space measure and $\mathcal{O}_1\ldots\mathcal{O}_k$ are a certain number of picture changing operators, required to saturate superghosts background charges. Until the recent series of papers [86–91], the procedure to compute such integrals was believed to rely on the existence of a global holomorphic section of $\mathfrak{M}_{g,n}$ [85, 92]. This would allow to integrate out the odd moduli first and reduce the integral to an integral over its bosonic base. Such a procedure is now known not to exist in the general case. In particular, for $g\geq 5$ it is known that $\mathfrak{M}_{g,0}$ is not holomorphically projected [93], while the question remains open for $g=3,4$.

Our case In [PT3], the author discussed the low energy limit of string amplitudes in the cases where they *can be* written as integrals over the ordinary bosonic moduli space $\mathcal{M}_{g,n}$. As a consequence of the non-projectedness issues, the discussion is restricted to genus $g\leq 4$ amplitudes. Note that in the Green-Schwarz and pure spinor formalisms, this "restricted" RNS set-up is the standard set-up and the question of the compatibility of the three formalisms in view of this discussion is open. In this context, the amplitudes take the generic form:

$$\mathsf{A}_{\alpha'}^{(g,n)} = \int_{\mathcal{M}_{g,n}} \mathrm{d}\mu_{\rm bos}\ \mathcal{W}_{g,n}\ \exp(\mathcal{Q}_{g,n})\,, \tag{2.1.5}$$

where $\mathrm{d}\mu_{\rm bos}$ is a $(3g-3+n)$-dimensional integration measure and the string integrand is written as a product of $\mathcal{W}_{g,n}$, which generically accounts for the kinematics of the scattering process, with $\exp(\mathcal{Q}_{g,n})$, the universal Koba-Nielsen factor. It comes from the plane-wave parts of the vertex operators[10]

$$\langle : e^{ik_1 X(z_1,\bar z_1)} : \cdots : e^{ik_n X(z_n,\bar z_n)} :\rangle = \exp\Big(\sum_{i<j} k_i\cdot k_j \langle X(z_i,\bar z_i)X(z_j,\bar z_j)\rangle\Big) \tag{2.1.6}$$

and writes explicitly

$$\mathcal{Q}_{g,n} = \sum_{i<j} k_i\cdot k_j\, \mathcal{G}(z_i-z_j,\bar z_i-\bar z_j) \tag{2.1.7}$$

in terms of the momenta k_i of the n scattered states and of the two-point function

$$\mathcal{G}(z-w,\bar z-\bar w) = \langle X(z,\bar z)X(w,\bar w)\rangle\,. \tag{2.1.8}$$

whose explicit expression given below in eq. (2.2.34) was determined in [85, 94]. We shall describe several type of $\mathcal{W}_{g,n}$, these are obtained from application of Wick's theorem and typically write as products of two-point correlators of the X and ψ fields, as well as of ghosts and superghosts fields.

2.1.2 The field theory limit.

How could one create a graph out of a closed Riemann surface ? The first thing one would have in mind is to stretch the surface to create very long and thin tubes. This actually does not produce graphs but degenerate Riemann surfaces with nodes. Nevertheless, it is a good start, and physically these stretched surfaces probe the IR region of string theory. To obtain a graph out of these tubes one still has to integrate out their cylindrical dependence. A good flavor of what is tropical geometry can be found in the survey [95], where the tropicalization procedure is presented as a way to "*forget the phases of complex numbers*". In the previous example, if σ and τ are respectively angular and longitudinal coordinates along the tube, $w=\tau+i\sigma$ can be conformally mapped to the plane via $w\to z=e^{iw}$, and we see that integrating out the cylindrical dependence of w amounts to integrating out the phase of z. Therefore tropical geometry describes how surfaces are turned into graphs by integrating out the phases of complex numbers.

The genus one bosonic string partition function is a handful example to illustrate the basic mechanism of the field theory limit in string theory, and point out where do come from the "phases" and "modulus" of complex numbers. It can be found in standard string theory textbooks mentioned before and writes

$$Z(\tau,\bar\tau) = \mathrm{Tr}\,\left(q^{L_0-1}\bar q^{\tilde L_0-1}\right)\,, \tag{2.1.9}$$

[10]: : denotes normal ordering.

where the trace is performed over the Hilbert space of physical states of string theory. The parameter q is defined by $q = \exp(2i\pi\tau)$ where $\tau = \mathrm{Re}\,\tau + i\mathrm{Im}\,\tau$ is the modulus of the complex torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. This expression can be rewritten to make manifest "phases" and "modulus" as:

$$Z(\tau, \bar{\tau}) = \mathrm{Tr}\, e^{+2i\pi \mathrm{Re}\,\tau(L_0 - \bar{L}_0)} e^{-2\pi \mathrm{Im}\,\tau(L_0 + \bar{L}_0 - 2)} \,. \tag{2.1.10}$$

Thus the level-matching condition $(L_0 - \bar{L}_0) = 0$ is enforced by integration over the "phases" $\int \mathrm{dRe}\,\tau$ while the "moduli" cause a mass weighting. More precisely, the masses of the oscillator states are given by $m^2 = \frac{4}{\alpha'}\left(\frac{N+\bar{N}}{2} - 1\right)$ where N and $\bar{N}$ are the number operators for the left-moving and right-moving sector defined by $L_0 = N + \alpha' p^2/4 - 1$ and $\bar{L}_0 = \bar{N} + \alpha' p^2/4 - 1$. The lowest mass state has $N = \bar{N} = 0$; it is the tachyon, whose mass is $m^2 = -4/\alpha'$. Then come the massless states at $N = \bar{N} = 1$ which constitute the gravity sector. For $N = \bar{N} \geq 2$ the states are massive with masses $m^2 = 4(N-1)/\alpha'$.

Thus in the region of modular parameter τ where $\mathrm{Im}\,\tau \sim 1/\alpha'$, the torus looks like a long and thin wire and one has $\mathrm{Im}\,\tau(N + \bar{N} - 2) \sim m^2$. As $\alpha' \to 0$, the massive states with $N \geq 2$ give rise to exponentially suppressed contributions in the partition function; only the massless modes propagate.[11] Since all states are now massless, the level matching condition is trivial and may be integrated out; we classically recover a worldline loop of length

$$T = \alpha' \mathrm{Im}\,\tau \,, \tag{2.1.11}$$

as we explain in detail in sec. 2.2.3. In the range of complex moduli τ where $\mathrm{Im}\,\tau$ stays of order $O(1)$, the massive modes are not decoupled and dictate the UV behavior of the low energy theory. We will see later that these tori, that are well known to generate the insertion of higher order operators and counter-terms in the supergravity effective action, give rise to natural objects of tropical geometry. Although there is no trivial integration of the phase dependence in this case, one can think of these phases as phases of complex numbers of vanishingly small modulus which are integrated out as well. To summarize, the tropical nature of the $\alpha' \to 0$ of string theory is apparent if we decompose it in two steps:

Step 1 (Point-like limit) Send $\alpha' \to 0$ and distinguish between the contribution of massive states and massless states in the amplitudes,

Step 2 (Level matching) Integrate out the phases of the complex numbers that are not vanishingly small to get the contributions of the massless states, and integrate out the regions of phase space where the complex numbers are vanishingly small to get the contributions of massive states.

[11] The tachyon state $N = \bar{N} = 0$ creates an infrared divergence, that can simply be ignored here.

The technical implementation to higher genus of these well known ideas led the author in [PT3] to study the general formula

$$\lim_{\alpha'\to 0} \mathsf{A}^{(g,n)}_{\alpha'} = \int_{\mathcal{M}^{\mathrm{trop}}_{g,n}} \mathrm{d}\mu^{\mathrm{trop}} F_{g,n}\,, \tag{2.1.12}$$

which means that string theory amplitudes, written as integrals over the bosonic moduli space, are projected onto integrals over the tropical moduli space $\mathcal{M}^{\mathrm{trop}}_{g,n}$ in the $\alpha' \to 0$ limit. This was called the "tropical representation" of the field theory limit of the string theory amplitude. Later we describe in detail the tropical form of the integrand $F_{g,n}$ and the structure of tropical moduli space $\mathcal{M}^{\mathrm{trop}}_{g,n}$. Physically one can think of this space as the set of all Feynman diagrams at a particular loop-order, parametrized in terms of Schwinger proper times. Hence, the formula in eq. (2.1.12) is a compact and well-defined way to write the result of the field theory limiting procedure in string theory and quantify how strings worldsheets are degenerated to different kind of worldlines. Moreover, the amplitude is renormalized according to a particular renormalization scheme that we describe later.

Such a formula, very natural for the physicist, would by itself not be of great interest besides its curious link with a new branch of mathematics, if it did not enable us to extract new physics and do new computations. In [PT3], we managed to derive the form of the low-energy limit of the genus-two four-graviton amplitude in type II superstring written in [96]. We shall recall the essential step of the reasoning here and show at little cost that the form of the genus-three amplitude written in [72] is compatible with the explicit set of graphs found in [34, 35]. Before getting there, we would like to discuss some aspects of tropical geometry, which we will need to describe the field theory limits of the genus two and three amplitudes.

2.2 A few words about tropical geometry and its link with classical geometry

In this section, we introduce basic notions of tropical geometry of graphs, then recall the analogous classical notions for Riemann surfaces and finally come to the correspondence between classical and tropical geometry in the context of the $\alpha' \to 0$ limit of string theory amplitudes.

2.2.1 Tropical geometry

Tropical graphs From the viewpoint of particle physics, tropical graphs are Schwinger proper time parametrized graphs, where loops are allowed to degenerate to vertices with a weight indicating the number of degenerated loops. On these can be inserted operators or counterterms of the effective action of corresponding loop order, which regulate the high energy behavior of the theory. This is physically sensible since short proper times correspond to high energies.

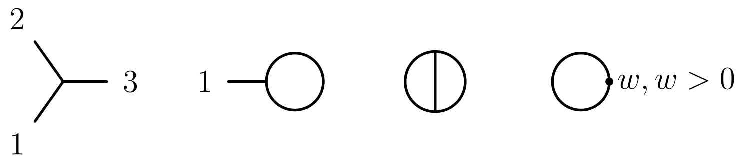


Figure 2.1: Examples of tropical graphs. From left to right: a 3-point tropical tree, a once-punctured graph of genus one, a genus-2 tropical graph, a graph of genus $1+w$.

A more formal definition is as follows. An (abstract) tropical graph is a connected graph with labeled legs, whose inner edges have a length and whose vertices are weighted. The external legs are called punctures. A *pure* tropical graph is a tropical graph that has only vertices of weight zero. Pure tropical graphs were first introduced in [97, 98], then later extended by [99, 100] to tropical graphs with weights, simply called tropical graphs here. Therefore a tropical graph Γ is a triple $\Gamma = (G, w, \ell)$ where G is a connected graph called the combinatorial type of Γ, ℓ and w are length and weight functions:

$$\begin{aligned} &\ell : E(G) \cup L(G) \to \mathbb{R}_+ \cup \{\infty\} \\ &w : V(G) \to \mathbb{Z}_+ \end{aligned} \tag{2.2.1}$$

In these two definitions, $E(G)$, $L(G)$ and $V(G)$ are respectively the sets of inner edges, legs and vertices of the graph. The total weight $|w|$ of a tropical graph is the sum of all the weights of its vertices $|w| = \sum_{V(G)} w(V)$. The genus $g(\Gamma)$ of a tropical graph $\Gamma = (G, w, \ell)$, is the number of loops $g(G)$ of G plus its total weight

$$g(\Gamma) = g(G) + |w| \, . \tag{2.2.2}$$

Hence the genus of a pure tropical graph is the number of loops of G in the usual sense. Moreover, every vertex of weight zero should have valence at least three (vertices with weight $w \geq 1$ may be of arbitrary non-zero valency). This automatically enforces a global stability condition for a given tropical graph of genus g and n punctures

$$2g - 2 + n \geq 1 \, , \tag{2.2.3}$$

which is the exact analogues of the classical stability condition.[12] Vertices weights obey natural rules under degenerations as shown in the figure 2.2. Now it should be clear that the vertices weights *keep track* of degenerated loops. It is easily checked that the genus of a graph (2.2.2) and the stability criterion (2.2.3) are stable under specialization.

[12] Strictly speaking, the local valency condition should be viewed as considering *classes* of abstract tropical graphs under the equivalence relation that contracts edges connected to 1-valent vertices of weight 0, and removes weight 0 bivalent vertices. Physically, on the worldline, this equivalence relation is perfectly sensible, since no interpretation of these 1- or 2- valent vertices of weight zero seem obvious in the absence of external classical sources.

Tropical Jacobians In this paragraph, following closely [97], we introduce tropical analogues of the classical objects, such as abelian one-forms, period matrices and Jacobians. A slight subtlety absent in the classical case comes the fact that tropical graphs of identical genus may not have the same number of inner edges. For simplicity, here, we shall only deal with pure tropical graphs, while we mention in [PT3] how this is generalized following [99].

Tropical graphs support an homology basis and corresponding one-forms. Let Γ be a pure tropical graph of genus g and $(B_1, , \dots, B_g)$ be a canonical homology basis of Γ as in figure 2.3. The linear vector space of the g independent abelian one-forms ω_I^{trop} can be canonically defined by

$$\omega_I^{\text{trop}} = \begin{cases} 1 & \text{on } B_I\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.2.4}$$

These forms are *constant* on the edges of the graph. The period matrix K_{IJ} is defined as in the classical case by integration along B cycles,

$$\oint_{B_I} \omega_J^{\text{trop}} = K_{IJ}\,. \tag{2.2.5}$$

It is a $g \times g$ positive semi-definite real valued matrix. These abelian one-forms and period matrix were already used in [96, 101] where they were observed to be the exact analogs of the classical quantities. The Jacobian variety of Γ is a real torus defined by

$$J(\Gamma) = \mathbb{R}^g / K\mathbb{Z}^g\,, \tag{2.2.6}$$

where $K\mathbb{Z}^g$ is the g-dimensional lattice defined by the g columns of the period matrix K.

The tropical version of the Abel-Jacobi map μ^{trop} of [97, 102] is defined by integration along a path γ between base point P_0 and end point P_1 of the vector of the abelian one-forms:

$$\mu_\gamma^{\text{trop}}(P_0, P_1) = \int_{P_0}^{P_1} (\omega_1^{\text{trop}}, \dots, \omega_g^{\text{trop}}) \mod K\mathbb{Z}^g\,. \tag{2.2.7}$$

Since changing γ by elements of the homology basis only results in the addition to the right hand side of elements of the lattice $K\mathbb{Z}^g$, μ^{trop} is well defined as a map in the Jacobian variety $J(\Gamma)$. Before we introduce the tropical moduli space, let us discuss two examples, taken from [97], in order to illustrate these notions.

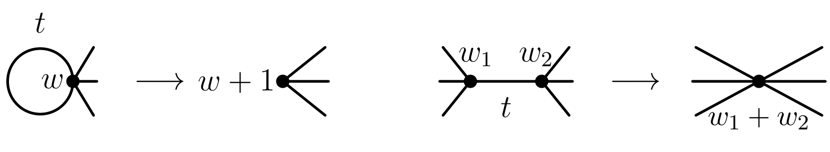


Figure 2.2: The genus of a graph is stable under degenerations $t \to 0$.

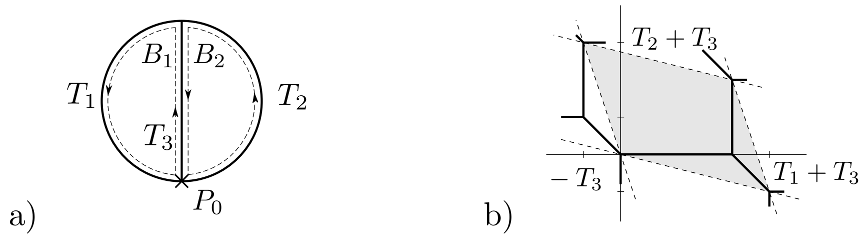


Figure 2.3: a) A $g = 2$ graph Γ with edges lengths T_1, T_2, T_3. b) The image of Γ (thick line) by the tropical Abel-Jacobi map in the Jacobian variety $J(\Gamma) = \mathbb{R}^2/K^{(2)}\mathbb{Z}^2$ (shaded area). Dashes indicate the $K^{(2)}\mathbb{Z}$ lattice.

Example 1. Let Γ be the genus two tropical graph of fig. 2.3 a) with canonical homology basis (B_1, B_2) as depicted. Using the definition (2.2.5), its period matrix is written:

$$K^{(2)} = \begin{pmatrix} T_1 + T_3 & -T_3 \\ -T_3 & T_2 + T_3 \end{pmatrix} . \tag{2.2.8}$$

Choosing P_0 as depicted, one can draw the image of Γ by the tropical Abel-Jacobi map in $J(\Gamma)$, as shown in the figure 2.3 b).

Example 2. The picture 2.4 below shows two inequivalent pure tropical graphs of genus two. The period matrix $K^{(2)}$ of the graph a) is given in (2.2.8), the period matrix of the graph b) is just $\text{Diag}(T_1, T_2)$. Thus, the Jacobian description is blind to such kind of "separating edges".

Tropical moduli space The moduli space $\mathcal{M}^{\text{trop}}(\Gamma)$ associated to a single tropical graph $\Gamma = (G, w, \ell)$ is the real cone spanned by the lengths of its inner edges modulo the discrete automorphism group of the graph [99]

$$\mathcal{M}^{\text{trop}}(\Gamma) = \mathbb{R}_+^{|E(G)|}/\text{Aut}(G) . \tag{2.2.9}$$

The moduli space of all genus-g, n-punctured tropical graphs is the space obtained from gluing all these cones together. This space is precisely the tropical moduli space introduced in [99, 100] denoted $\mathcal{M}_{g,n}^{\text{trop}}$ which enters the formula (2.1.12).

Below we describe a few examples of tropical moduli spaces. The moduli space of genus-0 tropical curves, $\mathcal{M}_{0,n}^{\text{trop}}$ is a well defined space that has the peculiar property of being itself a tropical variety of dimension $n-3$ [98, 103]. Because of the stability condition (2.2.3) one should start with $n = 3$. The space $\mathcal{M}_{0,3}^{\text{trop}}$ contains only one graph with no modulus (no inner length): the 3-pointed tropical curve. Hence $\mathcal{M}_{0,3}^{\text{trop}}$ is just a one-point set. The space $\mathcal{M}_{0,4}^{\text{trop}}$

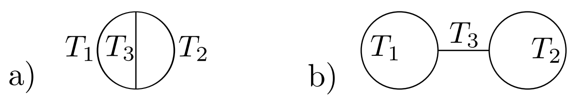


Figure 2.4: The period matrix is blind to the central length of the rightmost graph.

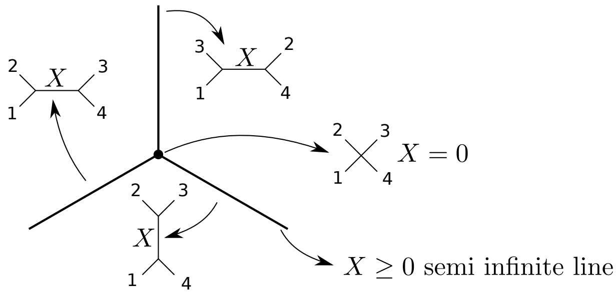


Figure 2.5: Tropical moduli space $\mathcal{M}_{0,4}^{\text{trop}}$ (thick line). Each semi infinite line corresponds to one of three inequivalent graphs. The X coordinate on these gives the length of the inner edge of the graphs. The central point with $X = 0$ is common to the three branches.

has more structure; it has the topology of the three-punctured tropical curve and contains combinatorially distinct graphs which have at most one inner length, as shown in figure 2.5.

The space $\mathcal{M}_{0,5}^{\text{trop}}$ is a two dimensional complex with an even richer structure. It is represented in figure 2.6. At genus one, $\mathcal{M}_{1,1}^{\text{trop}}$ is still easily described. A genus one tropical graph with one leg is either a loop or a vertex of weight one. Hence, $\mathcal{M}_{1,1}^{\text{trop}}$ is the half infinite line $\mathbb{R}_+$.

In general, Euler's relation gives that a given graph has at most $3g-3+n$ inner edges (and has exactly that number if and only if the graph is pure and possess only trivalent vertices). This implies that $\mathcal{M}_{g,n}^{\text{trop}}$ is of "pure (real) dimension" $3g-3+n$, which means that some of its subsets are of lower dimension

$$\dim_{\mathbb{R}} \ \mathcal{M}_{g,n}^{\text{trop}} \ \text{“} \leq \text{”} \ 3g-3+n\,. \tag{2.2.10}$$

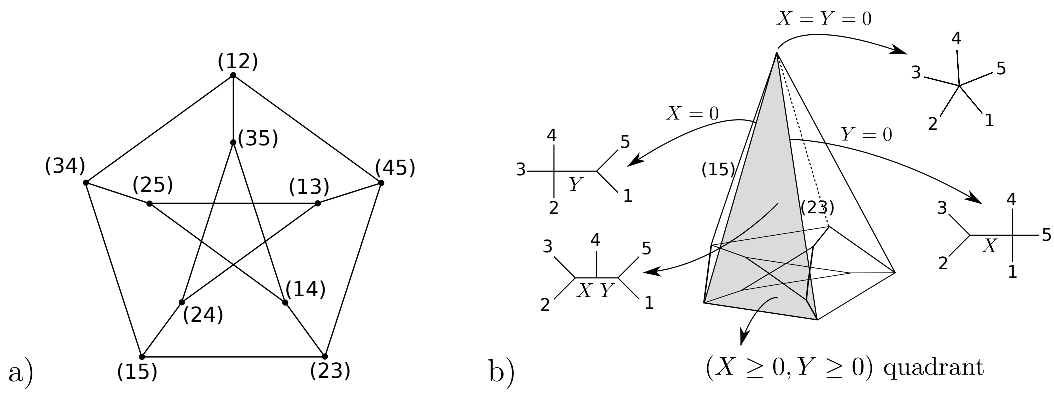


Figure 2.6: a) A slice of $\mathcal{M}_{0,5}$. The vertices (black dots) carry a two digits index, which corresponds to rays of $\mathcal{M}_{0,5}$, while edges corresponds to the 15 quadrants (one for each tree with 5 external legs and trivalent vertices). b) $\mathcal{M}_{0,5}$, with a specific quadrant in grey.

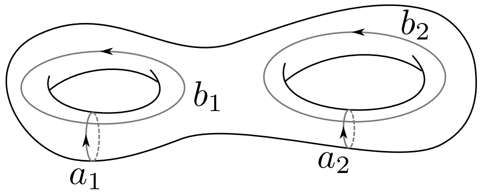


Figure 2.7: Canonical homology basis at $g = 2$.

2.2.2 Classical geometry of Riemann surfaces

We recall now classical facts about homology and Jacobian varieties of smooth Riemann surfaces. For a more elaborate introduction, we refer to the standard textbooks [104, 105]. Let Σ be a generic Riemann surface of genus g and let (a_I, b_J) $I, J = 1, \ldots, g$ be a canonical homology basis on Σ with intersection $a_I \cap b_J = \delta_{IJ}$ and $a_I \cap a_J = b_I \cap b_J = 0$ as in figure 2.7. The abelian differential ω_I, $I = 1, \ldots, g$ form a basis of holomorphic one-forms. They can be normalized along a-cycles so that their integral along the b-cycles defines the period matrix Ω_{IJ} of Σ:

$$2i\pi \oint_{a_I} \omega_J = \delta_{IJ}\,, \qquad 2i\pi \oint_{b_I} \omega_J = \Omega_{IJ}\,. \tag{2.2.11}$$

Note also Riemann's bilinear relations

$$\int_\Sigma \omega_I \wedge \bar{\omega}_J = -2i\,\mathrm{Im}\,\Omega_{IJ}\,. \tag{2.2.12}$$

The modular group $Sp(2g, \mathbb{Z})$ at genus g is spanned by the $2g \times 2g$ matrices of the form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where A, B, C and D are $g \times g$ matrices with integer coefficients satisfying $AB^t = BA^t$, $CD^t = DC^t$ and $AD^t - BC^t = 1_g$. The $g \times g$ matrix 1_g is the identity matrix. For $g = 1$, the modular group reduces to $\mathrm{SL}(2, \mathbb{Z})$. The Siegel upper half-plane $\mathcal{H}_g$ is the set of symmetric $g \times g$ complex matrices with positive definite imaginary part

$$\mathcal{H}_g = \{\Omega \in \mathrm{Mat}(g \times g, \mathbb{C}) : \Omega^t = \Omega, \mathrm{Im}\,(\Omega) > 0\}\,. \tag{2.2.13}$$

The modular group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on the Siegel upper half-plane by

$$\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1} \tag{2.2.14}$$

Period matrices of Riemann surfaces are elements of the Siegel upper half-plane and the action of the modular group on these is produced by Dehn twists of the surface along homology cycles. The Jacobian variety $J(\Sigma)$ of Σ with period matrix Ω is the complex torus

$$J(\Sigma) = \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)\,. \tag{2.2.15}$$

The classical Abel-Jacobi map μ is defined by integration along a path C between two points (divisors) p_1 and p_2 on the surface of the holomorphic one-forms

$$\mu(p_1, p_2)_C = \int_{p_1}^{p_2} (\omega_1, \ldots, \omega_g) \quad \mathrm{mod}\ \mathbb{Z}^g + \Omega \mathbb{Z}^g\,. \tag{2.2.16}$$

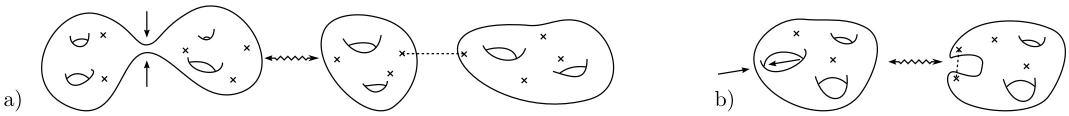


Figure 2.8: a) A separating degeneration. b) A non-separating degeneration. Dashes represent double points.

As in the tropical case, the right hand side of (2.2.16) does not depend on the integration path as it is considered only modulo the Jacobian lattice. Note that apart for the very special case of genus one where $\mu(\Sigma_1) \cong \Sigma_1$, the image of a genus $g \geq 2$ Riemann surface Σ_g by μ is strictly included in $J(\Sigma_g)$, $\mu(\Sigma_g) \subsetneq J(\Sigma_g)$.

Classical Moduli $\mathcal{M}_{g,n}$ space and its Deligne-Mumford compactification $\overline{\mathcal{M}_{g,n}}$. Smooth Riemann surfaces of genus g with n punctures can be arranged in a moduli space denoted $\mathcal{M}_{g,n}$ of complex dimension is $3g-3+n$. The $3g-3+n$ complex parameters that span this space are called the moduli of the surface. This space is not compact, as surfaces can develop nodes when non-trivial homotopy cycles pinch and give rise to nodal surfaces with ordinary double points. The result of adding all such nodal curves to $\mathcal{M}_{g,n}$ is the well known Deligne-Mumford compactified moduli space of curves $\overline{\mathcal{M}_{g,n}}$ [106]. There exists two types of such degenerations. As depicted in fig. 2.8, a "separating" degeneration splits off the surface into a surface with two disconnected components that are linked by a double point, while a "non-separating" degeneration simply gives rise to a new surface with two points identified whose genus is reduced by one unit. Note that no degeneration may split off a surface that does not satisfy the stability criterion shared with tropical graphs, eq. (2.2.3). As a consequence, a maximally degenerated surface is composed of thrice punctured spheres.

These degenerations induce a stratification on $\overline{\mathcal{M}_{g,n}}$, characterized by the combinatorial structure of the nodal curves, represented by its "dual graph". It is obtained by making a line go through each pinched cycle and turning each non-degenerated component of genus $g \geq 0$ into a vertex of weight g. Finally, the legs of a dual graph are just what used to be punctures on the surface. Examples are provided in fig.2.9. The strata corresponding to maximally degenerated curves are the deepest ones. The stratum corresponding to the non-pinched curves, whose dual graphs are a vertex of weight g, is the most superficial one (it is the interior of $\overline{\mathcal{M}_{g,n}}$). We come back to this in section 2.2.3.

A surface where a node is developing locally looks like a neck whose coordinates x and y on each of its side obey the equation $xy = t$, where the complex number t of modulus $|t| < 1$ is a parameter measuring the deformation of the surface around the singularity in $\overline{\mathcal{M}_{g,n}}$. The surface is completely pinched when $t = 0$. After a conformal transformation, one sees that this surface is alternatively described by a long tube of length $-\ln|t|$ and the tropicalization procedure classically turn these tubes into edges. The exact relation in string theory involves a factor of α' such that for instance the length T of the worldloop coming from a torus is

$$T = -\alpha' \ln |q| \,, \tag{2.2.17}$$

which coincides with (2.1.11).

2.2.3 From classical to tropical geometry

Moduli Spaces In [PT3] we outlined a construction of the tropicalization of $\mathcal{M}_{g,n}$ into $\mathcal{M}_{g,n}^{\text{trop}}$, which we later applied to string theory. Here we give a shortened version of this discussion based on more physical grounds. The starting point is the following question: "How can one commute the $\alpha' \to 0$ limit and the integration symbol in (2.1.5)"? Schematically, we wonder how to give sense to

$$\lim_{\alpha'\to 0} \left(\int_{\mathcal{M}_{g,n}} \mathcal{W}_{g,n} \exp(\mathcal{Q}_{g,n}) \, \mathrm{d}\mu_{\text{bos}} \right) \stackrel{?}{=} \int_{\mathcal{M}_{g,n}} \lim_{\alpha'\to 0} \left(\mathcal{W}_{g,n} \exp(\mathcal{Q}_{g,n}) \, \mathrm{d}\mu_{\text{bos}} \right) , \tag{2.2.18}$$

Such a procedure should treat well the integration domain, i.e. should not forget regions nor double counts others. If the integration domain were compact and the integrand a well behaved funcion, standard integration theorems would allow to simply commute the symbols. Here, we cannot replace $\mathcal{M}_{g,n}$ by its compactification $\overline{\mathcal{M}_{g,n}}$ precisely because the integrand has singularities at the boundary, which correspond to the IR singularities of string theory massless thresholds.

Hence, to deal with this integral we will follow the method pioneered at one-loop by Green and Schwarz in their work [59] where they showed that maximal supergravity and maximal SYM were the massless limits of type II and I strings, respectively. At generic loop order, their approach can be formulated by splitting the integral in the left-hand side of (2.2.18) into a sum of integrals over different regions where the limit can be safely taken. These regions are open sets of $\mathcal{M}_{g,n}$, such that

$$\mathcal{M}_{g,n} = \bigsqcup_{G} \mathcal{D}_G \,, \tag{2.2.19}$$

where each $\mathcal{D}_G$ contains the nodal curve with combinatorial type G. The point is that these dual graphs correspond to particular Feynman graphs, and the

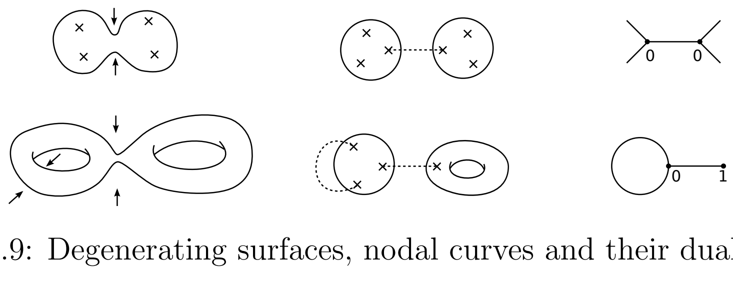


Figure 2.9: Degenerating surfaces, nodal curves and their dual graphs.

limit is obtained for each integral as

$$\int_{\mathcal{D}_G} \mathrm{d}\mu_{\mathrm{bos}}\ \mathcal{W}_{g,n}\ \exp(\mathcal{Q}_{g,n}) = \int_{\mathcal{M}^{\mathrm{trop}}(\Gamma)} \mathrm{d}\mu_{\mathrm{trop}}\ W_{g,n}\ \exp(Q_{g,n}) + O(\alpha')\,. \quad (2.2.20)$$

In [PT3] we described in great detail the limiting integration measure and integrand and show that they coincide with the contribution of the Feynman graph corresponding to Γ.

Example at genus one Before we describe these technical points, let us come back to genus one and discuss what could be a decomposition of $\mathcal{M}_{1,1}$ like the one in (2.2.19). Genus one Riemann surfaces are complex tori[13] $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ parametrized by a complex parameter, τ with positive imaginary part, $\tau \in \mathcal{H}_1 = \{\tau \in \mathbb{C},\ \mathrm{Im}\,(\tau) > 0\}$. Modding out by the action of the modular group $SL(2,\mathbb{Z})$ further restricts τ which eventually lies in an $SL(2,\mathbb{Z})$ fundamental domain. A representative one that we will use is $\mathcal{F} = \{\tau \in \mathbb{C},\ |\tau| > 1,\ -1/2 \leq \mathrm{Re}\,\tau < 1/2,\ \mathrm{Im}\,\tau > 0\}$, depicted in the figure 2.10. Therefore

$$\mathcal{M}_{1,1} \cong \mathcal{F} \quad (2.2.21)$$

There is only one singularity in $\mathcal{M}_{1,1}$, the pinched torus, at $q = 0$. Topologically, it is a sphere with three punctures, two of which are connected by a double point. The dual graph G_1 of this surface is a single loop with one external leg, and the corresponding domain should be defined such that it is possible to integrate out the real part of τ (phase of q) independently of the value of $\mathrm{Im}\,\tau$. Therefore we see that if we define $\mathcal{D}_{G_1}$ in terms of an arbitrary number $L > 1$ such that $\mathcal{D}_{G_1} = \{\tau \in \mathcal{F}, \mathrm{Im}\,\tau > L\}$, we can define families of tori with $\mathrm{Im}\,\tau = T/\alpha'$ tropicalizing to a worldloop of length T, independently of $\mathrm{Re}\,\tau$. In this way, $\mathcal{F}$ is split in two parts, $\mathcal{F}^+(L) \equiv \mathcal{D}_{G_1}$ and a lower part $\mathcal{F}^-(L)$ defined by $\mathcal{F}^-(L) = \{\tau \in \mathcal{F},\ \mathrm{Im}\,\tau \leq L\}$:

$$\mathcal{F} = \mathcal{F}^+(L) \sqcup \mathcal{F}^-(L)\,. \quad (2.2.22)$$

The dual graph corresponding to $\mathcal{F}^-(L)$ is a weight one vertex with a single leg. To the knowledge of the author, this precise decomposition was first used

[13] The complex tori is actually the Jacobian variety of the Riemann surface, but at genus one both are isomorphic. This property does not hold for higher genus curves.

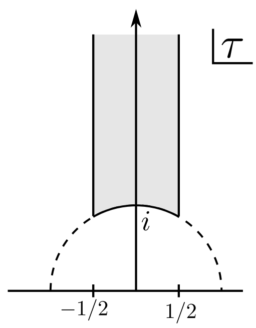


Figure 2.10: $SL(2,\mathbb{Z})$ fundamental domain of the torus.

by Green and Vanhove in [107], where $\mathcal{F}^+(L)$ and $\mathcal{F}^-(L)$ were respectively called $\mathcal{F}_L$ and $\mathcal{R}_L$. With this splitting, let us for definiteness introduce the following quantities

$$\mathsf{A}^{(1,4)}_{\alpha',+}(L) = \int_{\mathcal{F}^+(L)} \mathrm{d}\mu_{\text{bos}}\ \mathcal{W}_{g,n}\ \exp(\mathcal{Q}_{g,n}) \tag{2.2.23a}$$

$$\mathsf{A}^{(1,4)}_{\alpha',-}(L) = \int_{\mathcal{F}^-(L)} \mathrm{d}\mu_{\text{bos}}\ \mathcal{W}_{g,n}\ \exp(\mathcal{Q}_{g,n}) \tag{2.2.23b}$$

According to the previous property in (2.2.20), the $\alpha' \to 0$ limit of $\mathsf{A}^{(1,4)}_{\alpha',+}(L)$ gives the contribution of graphs with worldline loops of finite size $T = \operatorname{Im}\tau/\alpha'$. Therefore, the condition $\operatorname{Im}\tau > L$ gives the following field theory cut-off

$$T \geq T_{UV} = \alpha' L \tag{2.2.24}$$

In [107], the authors explained that since the total amplitude $\mathsf{A}^{(1,4)}_{\alpha',+}(L) + \mathsf{A}^{(1,4)}_{\alpha',-}(L)$ does not depend on L, any divergent term in $\mathsf{A}^{(1,4)}_{\alpha',+}(L)$ has to be canceled by a term coming from the field theory limit of $\mathsf{A}^{(1,4)}_{\alpha',-}(L)$. This supports the fact that these integrals produce counterterms in the effective action. In [PT3, section VI.B], we describe this cancellation in the trivial case of the quadratic R^4 one-loop 10-dimensional divergence of the four-graviton amplitudes in type II supergravity, while in this manuscript in appendix 3.A we present a one-loop computation for the R^4 logarithmic divergence in 8 dimensions for four-graviton amplitudes in heterotic string models. Several other examples are discussed in the original work [107]. As a consequence, the field theory limit of the sum of the two contributions is written as an integral over $\mathcal{M}^{\text{trop}}_{1,1}$ with a contact term inserted at $T = 0$ and gives, as claimed below eq. (2.1.12), the *renormalized* field theory amplitude. Let us now come back to the integrand in the right-hand side of eq. (2.2.20).

Back to the tropical form of the integrand. The bosonic measure $\mathrm{d}\mu_{\text{bos}}$ is a $(3g-3+n)$-dimensional measure that can be traded for an integration over the period matrices for genus $1,2,3,4$.[14] In this way, the tropical limit of the measure is given by

$$\mathrm{d}\mu_{bos} = \frac{|\prod_{1\leq I<J\leq g} \mathrm{d}\Omega_{IJ}|^2}{|\det \operatorname{Im}\Omega|^5} \prod_{i=1}^{n} \mathrm{d}^2 z_i \to \mathrm{d}\mu_{\text{trop}} = \frac{\prod_{i\in\{edges\}} \mathrm{d}\ell(i)}{|\det K|^5} \tag{2.2.25}$$

[14] We recall that the description of moduli of Riemann surfaces in terms of these of Jacobian varieties is called the classical Schottky problem (for a recent survey see [108]). Algebraically, the Jacobian varieties are characterized by $g(g+1)/2$ complex numbers that span period matrices, while Riemann surfaces have only $3g-3$ moduli. These numbers coincide for $g = 1,2,3$ (for $g = 1$ one should have $n = 1$ because of conformal invariance on the torus), for $g = 4$, $\mathcal{M}_4$ is a hypersurface in the moduli space of Jacobian variety of co-dimension one, but it is known that the zero locus of the so called "Schottky-Igusa" form furnishes a defining equation of this hypersurface. For $g \geq 5$ the problem is totally open. We describe later the Schottky-Igusa form, in chapter 3, sec. 3.2

where $\ell(i)$ is the length of the edge i. The interest of writing the measure explicitly in terms of period matrices is the appearance of the $\det \Omega^5$ factor, giving rise to $\det K$, an important element of Feynman graph as we explain below.

We also explained that the Koba-Nielsen factor descends to a tropical Koba-Nielsen factor, modulo a conjecture on a would-be "tropical" prime form which we mention below. Under this hypothesis, the bosonic propagator $\mathcal{G}$ descends to the worldline Green's function G^{trop} computed by Dai and Siegel in [101], and we have

$$\mathcal{Q}_{g,n} \underset{\alpha' \to 0}{\longrightarrow} Q_{g,n}^{\text{trop}} = -\sum_{i,j} k_i.k_j G^{\text{trop}}(Z_i, Z_j) + O(\alpha') \tag{2.2.26}$$

where details and explicit expressions can be found in [PT3, sec. V.A, eq(V.11)] and below in eq. (2.2.43).

To convince the reader that we are really describing Feynman graphs in this procedure, it is worth recalling the classical exponentiation procedure that leads to Schwinger proper time parametrized Feynman graphs. Starting from an arbitrary g-loop D-dimensional Feynman integral, we exponentiate the Feynman propagators D_i^2 and obtain

$$\begin{aligned} \int (\mathrm{d}^D p)^g \frac{n(p)}{\prod_i D_i^2} &= \int_0^\infty \prod_i \mathrm{d}a_i \int (\mathrm{d}^D p)^g \, n(p) \exp(-\sum a_i D_i^2) \\ &= \int_0^\infty \frac{\prod_i \mathrm{d}a_i}{(\det \tilde{K})^{D/2}} \langle n(a_i, \tilde{K}) \rangle \exp(-Q_{g,n}^{\text{trop}}) \end{aligned} \tag{2.2.27}$$

where we used that $\sum a_i D_i^2 = 1/2\, {}^t p \cdot \tilde{K} \cdot p + {}^t A \cdot p + c(a_i)$ is a quadratic form which produces a determinant after completing the square to

$$\sum a_i D_i^2 = 1/2\, {}^t(p + \tilde{K}^{-1}A) \cdot \tilde{K} \cdot (p + \tilde{K}^{-1}A) - 1/2 {}^t A \tilde{K}^{-1} A + c(a_i) \tag{2.2.28}$$

and D-dimensional Gaussian integration over a suitable Wick rotation of the shifted momentum $\tilde{p} = (p + \tilde{K}^{-1}A)$. The fact that the loop momentum constant part $-1/2 {}^t A \tilde{K}^{-1} A + c(a_i)$ in the exponential equals $Q_{g,n}^{\text{trop}}$ defined as in (2.2.26) is indirectly proven in [101]. The last line is the desired Schwinger proper time form of the Feynman graph, where the a_i correspond to the inner edges of the graph $\ell(i)$ in (2.2.25). In this way, it is then easy to show that $\tilde{K}$ is the period matrix K of the tropical graph as defined in (2.2.5). Finally, the bracket notation in (2.2.27) refers to the fact that $\langle n(a_i, K) \rangle$ is a Gaussian average of $n(p)$. The K-dependence comes from Gaussian integrating the terms with non-trivial loop momentum dependence in $n(p)$. The correspondence between the tropical form (2.2.20) with the measure (2.2.25) and this form is now, hopefully, clearer.[15] We use this procedure in the last chapter of this manuscript at one loop.

[15]The reader familiar with Symanzik polynomials may notice that the tropical Koba-Nielsen factor is the second Symanzik polynomial of the graph, while the $\det K$ of the proper time measure is the first Symanzik polynomial of the graph.

At this point, we have an almost complete description of the $\alpha' \to 0$ limit of string theory amplitudes between the tropical form of the integrand in (2.2.20) and the Feynman graph. The only missing point is also the most interesting one; the numerator of the Feynman graph $\langle n \rangle$, corresponding to $W_{g,n}$. Below we introduce a few technical elements necessary to tackle the tropical limit of the numerator $\mathcal{W}_{g,n}$ for $g \geq 2$.

Cohomology Thanks to the splitting of $\mathcal{M}_{g,n}$ (2.2.19), it is possible in each domain to safely define families of degenerating worldsheets, and show that their period matrices and one forms descend to their tropical analogues, as described in [PT3, sec. IV.C]. The one-forms, at a neck i_0 parametrized by a local coordinate t_0 around $t_0 = 0$, locally behave as on a very long tube:

$$\omega_I = \frac{c}{2i\pi}\frac{\mathrm{d}z}{z} + O(t_0)\,, \tag{2.2.29}$$

with $c = 1$ or 0 depending on if i belongs to the cycle b_i or not. As a consequence, the bilinear relation descends to

$$\int_\Sigma \omega_I \wedge \bar{\omega}_J = \boxed{-2i\,\mathrm{Im}\,\Omega_{IJ} \underset{\alpha'\to 0}{\sim} -2i\,\frac{K_{IJ}}{\alpha'} + O(1)}\,. \tag{2.2.30}$$

which indicates the fundamental scaling relation of the tropical limit. At one-loop, this is the relation (2.1.11), but at higher loop this gives non-trivial information on the behavior of the period matrices of the degenerating worldsheets.

Fourier-Jacobi expansions As a consequence, this provides a nice system of local coordinates in each patch $\mathcal{D}_G$ around the nodal curve G (at least for when G corresponds to deepest strata[16] of $\overline{\mathcal{M}_{g,n}}$), defined as

$$q_j = \exp(2i\pi\tau_j) \tag{2.2.31}$$

for $j \in E(G)$ such that $\mathrm{Im}\,\tau_j = \ell(j)/\alpha'$. It is then possible to perform the so-called "Fourier-Jacobi" expansion of the various quantities defined on the worldsheet in terms of these q_j's. Generically a function F of the moduli of the worldsheet admits a Fourier-Jacobi expansion of the form (neglecting the the punctures for simplicity):

$$F = \sum_{n_i,m_j} F_{hol}^{(n_1,\ldots,n_{3g-3})} q_1^{n_1}\ldots q_{3g-3}^{n_{3g-3}} F_{anti-hol}^{(m_1,\ldots,m_{3g-3})} (\bar{q}_1)^{m_1}\ldots(\bar{q}_{3g-3})^{m_{3g-3}} \tag{2.2.32}$$

where at $g = 1$, it is understood that $3g - 3$ should be replaced by 1. The general strategy to extract the tropical form of integrand $\mathcal{W}$ is to perform the Fourier-Jacobi expansion of the integrand (step 1 of sec. 2.1.2) then integrate the phase of the q_i's (step 2). The procedure tells us that it is safely possible

[16]We recall that we defined these strata before, as the ones corresponding to maximally degenerated curves made of tri-valent weight-0 vertices only.

to commute the integration and the Fourier Jacobi expansion. The outcome of this procedure is that higher order contributions in q_i vanish; only the constant terms stay and constitute the tropical form of the integrand.

At this point the reader might wonder why the phase integration is not simply redundant with the $q_j \to 0$ limit; since non-zero powers of q_j are projected out anyway, what is the point of the phase-integration of constant terms ? As a matter of fact, the integrands of string theory amplitudes do contain a partition function, whose Fourier-Jacobi expansion typically starts with *inverse* powers of q_j: q_j^{-1} for the bosonic sector of heterotic string and $q_j^{-1/2}$ for the NS sector of the superstring. Therefore, a term like $q_j(\bar{q}_j)^{-1}$ is not killed by the $q_j \to 0$ limit alone, while it is by the phase integration $\int \mathrm{d}(\mathrm{Re}\,\tau_j) q_j (\bar{q}_j)^{-1} = 0$; this is the level matching condition (step 2 of sec. 2.1.2). For maximal supergravity amplitudes, the tachyon is projected out of the spectrum by the GSO projection and the limit is easier to extract, but in general the inverse powers of the partition function do contribute via residue contributions of the form

$$\int \mathrm{dRe}\,\tau_1 \dots \mathrm{dRe}\,\tau_{3g-3} \frac{F}{q_1^{n_1} \dots q_{3g-3}^{n_{3g-3}} (\bar{q}_1)^{m_1} \dots (\bar{q}_{3g-3})^{m_{3g-3}}} = c_{n_1,\dots,m_{3g-3}} F^{\mathrm{trop}} \,. \tag{2.2.33}$$

At one-loop, there is only one q and these techniques are perfectly well under control as we review in chap. 3. They led Bern and Kosower to develop the eponymous rules which allow to compute n-gluon amplitudes [61–64]. These were first derived from the low energy limit of heterotic string fermionic models, later understood from first principles in field theory [109] then extended to gravity amplitudes [64, 110]. See also the review [111] for an exhaustive account on this worldline formalism. At higher loop, such residue formulas are still not known, and are required to extract general amplitudes, as we discuss later in the chapter dedicated to half maximal supergravity amplitudes at two loops, 3.2. The basic building block of which these generic integrands are made of is the bosonic correlator $\mathcal{G}$ and its derivatives, to which we come now.

A tropical prime form ? In this paragraph, we describe the *first term* of the Fourier-Jacobi expansion of bosonic Green's function on Riemann surfaces. Its complete expression is [85, 94][17]

$$\mathcal{G}(z_1, z_2) = -\frac{\alpha'}{2} \ln\left(|E(z_1, z_2)|\right) - \frac{\alpha'}{2} \left(\int_{z_2}^{z_1} \omega_I\right) (\mathrm{Im}\,\Omega^{-1})^{IJ} \left(\int_{z_2}^{z_1} \omega_J\right) \tag{2.2.34}$$

It is defined in terms of the prime form E, whose definition requires to introduce first the classical Riemann theta function:

$$\theta(\zeta|\Omega) = \sum_{n \in \mathbb{Z}^g} e^{i\pi n^t \Omega n} e^{2i\pi m^t \zeta} \tag{2.2.35}$$

[17] The normalization differs from the one used in [PT3] by the factor α' that we keep inside $\mathcal{G}$ here.

where $\zeta \in J(\Sigma)$ and $\Omega \in \mathcal{H}_g$. Theta functions with characteristics are defined by

$$\theta \begin{bmatrix} \beta \\ \alpha \end{bmatrix} (\zeta|\Omega) = e^{i\pi\beta^t\Omega\beta+2i\pi\beta^t(\zeta+\alpha)}\theta(\zeta + \Omega\beta + \alpha|\Omega) \qquad (2.2.36)$$

where α and β are g dimensional vectors of $\frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^{2g}$ called the theta-characteristics. The prime form E is then defined by [85, 112, 113]

$$E : (x, y) \in \Sigma \times \Sigma \longrightarrow E(x, y|\Omega) = \frac{\theta \left[\begin{smallmatrix} \beta \\ \alpha \end{smallmatrix}\right] (\mu(x, y)|\Omega)}{h_\kappa(x)h_\kappa(y)} \in \mathbb{C}\,, \qquad (2.2.37)$$

with the requirement that $\kappa = \left[\begin{smallmatrix} \beta \\ \alpha \end{smallmatrix}\right] \in \frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^{2g}$ should be a non-singular odd[18] theta-characteristics and h_κ the half-differentials defined on Σ by $h_\kappa(z) = \sqrt{\sum_{i=1}^{g} \omega_I(z)\partial_I\theta \left[\begin{smallmatrix} \beta \\ \alpha \end{smallmatrix}\right] (0|\Omega)}$. Also, μ is the classical Abel-Jacobi map defined in (2.2.16). Defined in this way, the prime form is a differential form of weight $(-1/2, -1/2)$ which do not depend on the spin structure κ chosen. In a sense, it is the generalization of the map $x, y \in \mathbb{C}^2 \mapsto x - y$ to arbitrary Riemann surfaces. In particular, $E(x, y)$ vanishes only along the diagonal $x = y$ and locally behaves as

$$E(x, y) \underset{x\to y}{\sim} \frac{x - y}{\sqrt{dx}\sqrt{dy}}(1 + O(x - y)^2) \qquad (2.2.38)$$

It is multi-valued on $\Sigma \times \Sigma$ since it depends on the path of integration in the argument of the theta function. More precisely, it is invariant up to a sign if the path of integration is changed by a cycle a_I, but it picks up a multiplicative factor when changing the path of integration by a cycle b_J

$$E(x, y) \to \exp(-\Omega_{JJ}/2 - \int_x^y \omega_J)E(x, y)\,. \qquad (2.2.39)$$

In $\mathcal{G}$, it is easily checked that the additional terms with holomorphic forms precisely cure this ambiguity.

In [PT3] was proposed a definition of a the tropical prime form as the result of the following limit:

$$E^{\text{trop}}(X, Y) := - \lim_{\alpha'\to 0} \left(\alpha' \ln \left|E(x_{\alpha'}, y_{\alpha'}|\Omega_{\alpha'})\right|\right) \qquad (2.2.40)$$

where $\Omega_{\alpha'}$ are the period matrices of a family of curves tropicalizing as in (2.2.30), $x_{\alpha'}, y_{\alpha'}$ are two families of points on the surface whose image by the Abel-Jacobi map tropicalizes as in (2.2.29) and X and Y are the two limit points on the tropical graph. Inspired by [101], we made the conjecture that the tropical prime form defined in this way corresponds to the graph distance $d_\gamma(X, Y)$ between X and Y along a path γ:

$$E^{\text{trop}}(X, Y) = d_\gamma(X, Y) \qquad (2.2.41)$$

[18]Odd means that $2^{2n}\alpha \cdot \beta \equiv 1[2]$ and "non-singular" that $\theta[\kappa](\zeta|\Omega)$ vanishes exactly to first order at $\zeta = 0$. Even characteristics are these for which $2^{2n}\alpha \cdot \beta \equiv 0[2]$

This object is also ill-defined on the graph since it depends on γ. To prove this conjecture, the first ingredient to use would be tropical theta functions with characteristics. Tropical theta functions without characteristics were introduced in [97] and it is easy to show directly that they arise in the limit of the classical ones;

$$\Theta^{\text{trop}}(Z|K) = \lim_{\alpha' \to 0} -\alpha' \ln |\theta(\zeta_{\alpha'}|\Omega_{\alpha'})| \tag{2.2.42}$$

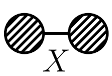
where $(\zeta_{\alpha'}) = \mu(x_{\alpha'}, y_{\alpha'})$ is sent to $Z = \mu^{\text{trop}}(X, Y)$ as defined previously. So far, the author has not managed to prove this property in the case of tropical theta functions with characteristics, as defined in [97]; this is a crucial missing step. As this limit is not fully under control, it does not make sense to try to describe higher order corrections in the Fourier-Jacobi expansion of the prime form, which would enter residue formulas as (2.2.33).

The other class of terms in the right-hand side of (2.2.34) are easily dealt with by replacing the one-forms and period matrix by their tropical analogues, and, using (2.2.41) we obtain the $\alpha' \to 0$ limit of the bosonic correlator of (2.2.34) is the following quantity

$$G(Z_1, Z_2) = \lim_{\alpha' \to 0} \mathcal{G}(z_1, z_2) = -\frac{1}{2} E^{\text{trop}}(Z_1, Z_2) - \frac{1}{2} \left(\int_{Z_2}^{Z_1} \omega^{\text{trop}} \right) K^{-1} \left(\int_{Z_2}^{Z_1} \omega^{\text{trop}} \right) \tag{2.2.43}$$

which is precisely the expression computed by Dai and Siegel in [101]. Note that it is now well defined on the graph.

A Remark On Contact Terms Before closing the section, let us clarify a point concerning contact-terms. In the usual perturbative expansion of quantum field theory, the Feynman rules include vertices of valency four in non-abelian gauge theories and arbitrarily high in gravity theories, to guarantee gauge invariance. What is referred to as "contact-term" in string theory is different. It is the vertex that results from integrating out the length of a separating edge in a one-particle-reducible graphs:

$\int \Big($  $\Big) \mathrm{d}X = c_0 \times$  (2.2.44)

In the tropicalization procedure, we do not perform these integrations. Therefore, higher valency vertices (of weight zero) are present in our considerations, but only as boundaries between domains in $\mathcal{M}_{g,n}^{\text{trop}}$ of maximal codimension and should not carry any localized contribution in the integrands, unlike in Feynman rules where they carry a distinct structure compared to the lower valency vertices.

Furthermore, in string theory, these type of contributions only arise from configurations where two neighboring vertex operators V_i and V_j collide towards one another, $z_i - z_j \ll 1$. It can be shown in full generality that a contact term can arise only if the string integrand $\mathcal{W}_{g,n}$ contains a factor of $|\partial \mathcal{G}(z_i - z_j)|^2$. The basic argument is that in the region where the position of the punctures collide, the local behavior of E (2.2.38) grants that ∂G has a first order pole $\partial \mathcal{G} \sim 1/(z_i - z_j)$ and the change of variable[19] $z_i - z_j = e^{-X/\alpha'} e^{i\theta}$ can be used to take the limit of $\int \mathrm{d}^2 z_i |\partial \mathcal{G}(z_i - z_j)|^2 \exp\big(- k_i.k_j \alpha' \ln(z_i - z_j)\big)$ in the string amplitude to obtain the following integral over $\mathrm{d}X$, the length of the separating edge:

$$\int \mathrm{d}^2 z_i |\partial \mathcal{G}(z_i - z_j)|^2 e^{-2k_i.k_j \alpha' \ln|z_i - z_j|} = -\frac{2\pi}{\alpha'} \int \mathrm{d}X e^{-2Xk_i \cdot k_j} \tag{2.2.45}$$

after integrating the phase $\mathrm{d}\theta$. The crucial point here is that if $|\partial \mathcal{G}(z_i - z_j)|^2$ had not been in the integrand, either the local behavior would have failed or the phase integration would have killed the contribution.

The "Analytic" and the "Non-Analytic" domains.

For simplicity let us exclude the punctures of that discussion. The authors of [107] introduced the splitting (2.2.22) because it actually decomposes the string amplitude into its analytic and non-analytic parts, respectively obtained from the lower- and upper-domain integration. In [PT3] we proposed an extension of these "lower" and "upper" domains for higher genus. We defined the analytic and non-analytic domains in $\mathcal{M}_{g,n}$ by the requirement that the first should correspond to the more superficial stratum of $\overline{\mathcal{M}}_g$ and the second should correspond to the deepest strata of $\overline{\mathcal{M}}_g$ in the decomposition (2.2.19). These strata where defined in paragraph concerning the structure of $\overline{\mathcal{M}_{g,n}}$.

Therefore, the analytic domain is defined by removing all neighborhoods around the singularities of $\mathcal{M}_g$; it is a compact space. In this region, the string integrand has no singularity and the limit may be safely commuted with the integration, where the factor α' present in the definition of $\mathcal{Q}_{g,n}$ via $\mathcal{G}$ simply sends $\exp(\mathcal{Q}_{g,n})$ to 1. This reasoning justifies why in an important part of the literature, "taking the low energy limit" is often translated as getting rid of the Koba-Nielsen factor. This may be done only modulo these non-trivial geometric assumptions.

This also suggests that to compute the primary divergence of an amplitude, it is possible to compute the string integral over the analytic domain, as illustrated in the one-loop example of secs.2.2.3 and 3.A. Understanding the role of the precise form of the boundary of this domain is an open interesting question. Regarding the non-analytic domains, they provide the contribution of the pure tropical graphs, made of trivalent vertices only. Summed over, they give rise to the *unrenormalized* field theory amplitude, with all of its sub-divergences.

2.3 Extraction of supergravity amplitudes

The $\alpha' \to 0$ limit of tree-level amplitudes is sketched later in this text when we discuss a formula related to the BCJ duality at tree-level in sec. 4.2. One-loop amplitudes are also discussed in some detail in the following chapter 3.

[19]In [PT3] we discussed this tree-level behavior in detail

Here we discuss the somewhat non-trivial and interesting cases of $g = 2, 3$ four-graviton amplitudes in type II string theory.

2.3.1 Two-loops field theory limit in maximal supergravity

The two-loop four-graviton type II amplitude in 10 dimensions has been computed explicitly in the RNS formalism by D'Hoker and Phong in [43–49] and later obtained in the pure spinor formalism in [114, 115]. The normalizations between the two results was carefully observed to match in [116]. We reproduce the RNS form here:

$$\mathsf{A}^{(2,4)}_{\alpha'} = \frac{t_8 t_8 R^4}{2^{12}\pi^4}\int_{\mathcal{F}_2} \frac{|\prod_{I\leq J} \mathrm{d}\Omega_{IJ}|^2}{(\det \operatorname{Im}\Omega)^5}\int_{\Sigma^4} |\mathcal{Y}_S|^2 \exp\big(\mathcal{Q}_{2,4}\big) \tag{2.3.1}$$

where $\int_{\Sigma^4}$ denotes integration over the surface Σ of the position of the four punctures and $t_8t_8R^4$ is the only supersymmetric invariant in maximal supergravity made of four powers of the Riemann tensor (see [77, Appendix 9.A]). The domain $\mathcal{F}_2$ is an $Sp(4,\mathbb{Z})$ fundamental domain, isomorphic to $\mathcal{M}_2$. The quantity $\mathcal{Y}_S$ arises from several contributions in the RNS computation and from fermionic zero-mode saturation in the pure spinor formalism. Its expression is given in terms of bilinears in the holomorphic one-forms $\Delta(z,w) = \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z)$ as follows

$$3\mathcal{Y}_S = (k_1 - k_2)\cdot(k_3 - k_4)\,\Delta(z_1,z_2)\Delta(z_3,z_4) + (13)(24) + (14)(23)\,. \tag{2.3.2}$$

Thus, $|\mathcal{Y}_S|^2$ is a top-form on Σ^4. In [PT3], we checked the conjecture of [96] on the low energy limit of the string theory amplitude, starting from the field theory amplitude derived in [13] rewritten in a worldline language. This concerns only the non-analytic domain of the amplitude. The essence of the demonstration is to find the tropical form of $\mathcal{Y}_S$. As noted in the previous section, in amplitudes where maximal supersymmetry is not broken, the NS tachyons are projected out of the spectrum by the GSO projection, and there is no non-trivial residue to extract. The tropical form of $\mathcal{Y}_S$ is then immediately obtained:

$$3\mathcal{Y}_S \to 3Y_S = (k_1 - k_2)\cdot(k_3 - k_4)\,\Delta^{\mathrm{trop}}(12)\Delta^{\mathrm{trop}}(34) + (13)(24) + (14)(23)\,. \tag{2.3.3}$$

where Δ^{trop} descends from Δ by replacing ω by ω^{trop}. As explained in [PT3, section VI.C], it is not difficult to see that Y_S has the simple behavior summarized in table 2.1.

In total, the non-analytic part of the amplitude is written as

$$\mathsf{A}^{(2,4)}_{\mathrm{non-ana}}(L) = \mathcal{N} t_8 t_8 R^4 \int_{K_{22}>K_{11}\geq\alpha' L}^{\infty} \frac{\prod_{I\leq J}\mathrm{d}K_{IJ}}{(\det K)^5}\int_{\Gamma^4} Y_S^2 \exp\big(Q^{\mathrm{trop}}_{2,4}\big), \tag{2.3.4}$$

where $\mathcal{N}$ is a global normalization factor, $\int_{\Gamma^4}$ represents the integration of the positions of the four punctures on the graph and $\int_{K_{22}>K_{11}\geq\alpha' L}$ represents a pos-

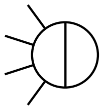
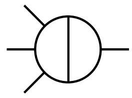
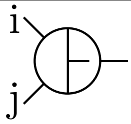
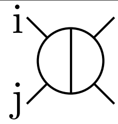
Graph				
Y_S	0	0	$-s_{ij}$	$-s_{ij}$

Table 2.1: Numerators for the two-loop four-graviton worldline graphs.

sible choice for the boundaries of the non-analytic domain described before.[20] This object coincides with the one derived in [96, eq. 2.12] from the two-loop field theory computation of [13], thus it is the two-loop unrenormalized four-graviton amplitude.

The other domains of $\mathcal{M}_2$ have been studied as well, but for the moment the author is missing some technology for genus-2 modular integrals, which hopefully would be resolved once the questions raised in [117] are answered. To be complete, we should also mention that the absence of $|\partial \mathcal{G}|^2$ terms forbids the appearance of contact-terms.

2.3.2 New results at three loops

Recently a four-graviton amplitude three-loop amplitude in type II superstring was proposed in [72] in the pure spinor formalism. This amplitude passes a very important consistency check by matching the S-duality prediction of [118] confirmed in [68] for the coefficient of the $\nabla^6 R^4$ in the effective action in ten dimensions after carefully matching normalizations.[21] Here we propose new results concerning the set of graphs that appear (or rather, the ones that do not) in the field theory limit of this amplitude in the non-analytic domains. There are two different vacuum topologies of genus 3 graphs, depicted in the figure 2.11. Let us reproduce the structure of this genus three amplitude. In our notations, up to a global normalization factor $\mathcal{N}_3$, it writes

$$\mathsf{A}^{(3,4)}_{\alpha'}(\epsilon_i, k_i) = \mathcal{N}_3 \int_{\mathcal{M}_3} \frac{|\prod_{I\leq J} \mathrm{d}\Omega_{IJ}|^2}{(\det \operatorname{Im} \Omega)^5} \int_{\Sigma^4} \left[\langle |\mathcal{F}|^2\rangle + \langle |\mathcal{T}|^2\rangle\right] \exp\left(\mathcal{Q}_{3,4}\right), \quad (2.3.5)$$

where $\int_{\Sigma^4}$ is again the integration of the positions of the four punctures. The integrand is a top form and $\mathcal{F}$ and $\mathcal{T}$ are correlation functions of the bosonic pure spinor ghosts $\lambda, \bar\lambda$, including kinematic invariants, polarization tensors, derivatives of the genus three Green's function and holomorphic one-forms $\omega_I(z_i), \bar\omega_J(z_j)$, where $I, J = 1, 2, 3$ and $i, j = 1, \dots, 4$. The one-forms appear in objects generalizing the genus-two bilinears Δ defined by:

$$\Delta(z_i; z_j; z_k) = \epsilon^{IJK} \omega_I(z_i)\omega_J(z_j)\omega_K(z_k)\,, \quad (2.3.6a)$$

$$\Delta^\mu(z_i, z_j; z_k; z_l) = \epsilon^{IJK} (\Pi\omega)^\mu_I(z_i, z_j)\omega_J(z_k)\omega_K(z_l)\,, \quad (2.3.6b)$$

[20]Looking back at the explicit parametrization of K in (2.2.8), this contribution sets the length of both B_1 and B_2 loops to be greater than the cutoff scale.

[21]A subtle reasoning on the symmetries of genus-three surfaces led the authors of [72] to include a global factor of 1/3 *a posteriori*. A first-principle computation or a cross-check appears necessary to ensure the validity of this result.

Figure 2.11: The two vacuum topologies at three loops: the Mercedes one and the ladder (hyperelliptic) one, endowed with the choice of a particular homology.

where $(\Pi\omega)^\mu_I := \Pi^\mu_I \omega_I(z_i)\omega_I(z_j)$ (no sum on I) and the index $\mu = 0, \dots, 9$ is the target spacetime index. The quantity Π^μ_I is the zero mode part of the momentum Π^μ that flows through the cycle B_I. One-forms are also present in the derivatives of the Green's function, since $\partial_{z_i} G(z_i, z_j) = \sum_{I=1}^3 \omega_I \partial_{\zeta_I} G(z_i, z_j)$ where ζ_I is the I-th component of $\mu(z_i, z_j)$. Finally, $\mathcal{F}$ is solely defined in terms of Δ and derivatives of the Green's function (not mentioning the tensorial structure involving polarization vectors, momenta and pure spinor ghosts) while $\mathcal{T}$ is only defined in terms of Δ^μ and does not contain derivatives of the Green's function.

This being said, what we want to show here is that in the tropical limit, $\mathcal{F}$ and $\mathcal{T}$ vanish *before integration* for both topologies of graphs in fig.2.11 where three or more particles are on the same edge of the graph, possibly via a tree-like contact-term. For the quantity, $\mathcal{F}$, this property follows from the antisymmetry of the tropical version of $\Delta(z_i; z_j; z_k)$, Δ^{trop}, defined by replacing the ω's by they tropical counterparts

$$\Delta^{\mathrm{trop}}(i, j, k) = \epsilon^{IJK} \omega_I^{\mathrm{trop}}(i) \omega_J^{\mathrm{trop}}(j) \omega_K^{\mathrm{trop}}(k) \tag{2.3.7}$$

Whenever two particles, for instance 1 and 2 are on the same edge, one has $\omega_I^{\mathrm{trop}}(1) = \omega_I^{\mathrm{trop}}(2)$ and $\Delta^{\mathrm{trop}}(1, 2, i)$ vanishes by antisymmetry. Therefore, when three particles (or more) are on the same edge, any triplet of particles (i, j, k) necessarily involves two particles inserted on the same edge and Δ always vanishes. As regards $\mathcal{T}$, the vanishing follows from symmetry properties of its defining building blocks rather than on these of the Δ^μ's. We reproduce the definition of $\mathcal{T}$ given in eq. (3.26) in [72]:

$$\begin{aligned} \mathcal{T} = \quad & T^\mu_{1234} \Delta^\mu(z_1, z_2; z_3; z_4) + T^\mu_{1324} \Delta^\mu(z_1, z_3; z_2; z_4) + T^\mu_{1423} \Delta^\mu(z_1, z_4; z_2; z_3) \\ & + T^\mu_{2314} \Delta^\mu(z_2, z_3; z_1; z_4) + T^\mu_{2413} \Delta^\mu(z_2, z_4; z_1; z_3) + T^\mu_{3412} \Delta^\mu(z_3, z_4; z_1; z_2) \end{aligned} \tag{2.3.8}$$

where

$$T^\mu_{1234} = L^\mu_{1342} + L^\mu_{2341} + \frac{5}{2} S^\mu_{1234} \,. \tag{2.3.9}$$

We do not need any detail about L and S but their symmetry properties. The quantity L_{ijkl} is antisymmetric in $[ijk]$, which is enough to ensure vanishing of the L part in $\mathcal{T}$ when three particles are on the same edge of the graph. However S^μ_{ijkl} is only symmetric in (ij) and antisymmetric in $[kl]$ and we need additional identities. It is indeed possible to show the following relation

$$S^\mu_{1234} + S^\mu_{1324} + S^\mu_{2314} = 0 \tag{2.3.10}$$

from the properties of the defining constituents[22] of S^μ, and this identity eventually brings the desired results after a few manipulations of the indices i, j, k, l. The integrand does vanish in the more general case where *at least one B cycle is free of particles*, while it is not trivially zero in the other cases, we arrive at the aforementioned property. Before concluding, we shall also mention that the regions of the moduli space where vertex operators collide to one-another here provide non-vanishing contributions. The required $\partial\mathcal{G}$ terms are present in $\mathcal{F}$, which leaves room for contact-terms to arise in the field theory limit of (2.3.5).

The conclusion is the following; the tropical limit of the amplitude (2.3.5) describes the same set of 12 graphs as the one used in the computation of the four graviton three-loop amplitude in maximal supergravity of Bern *et. al.* in [34]. The complete extraction of the tropical form of the integrand would be a very interesting thing to do.

[22]The author is grateful to Carlos Mafra for a discussion and sharing results on that point.

Chapter 3

Half-Maximal Supergravity

In this chapter, we turn to the theory of half-maximal supergravity and its one- and two-loop amplitudes. We recall that this theory is interesting because of its UV behavior and because of its richer structure than $\mathcal{N} = 8$ since it can be coupled to $\mathcal{N} = 4$ SYM matter fields.

We review in sec. 3.1 the one-loop computation of [PT2], and focus in particular on the amplitudes computed in CHL orbifolds of the heterotic string. Then in sec. 3.2 we describe the two-loop analysis of [PT1] concerning the UV behavior of half-maximal supergravity. We also provide unpublished material on the genus-two partition function in CHL models and propose a genuine worldline description of the field theory limit of these two-loop amplitudes. Finally we present in appendix 3.A an example of computation of one-loop logarithmic divergence in the case of half-maximal supergravity four-graviton amplitudes $D = 8$. This illustrates the discussion of the previous chapter on the cancellation of divergences between the analytic and non-analytic contributions at one-loop.

The one-loop analysis of this chapter does not require the technical material exposed in the previous section since the techniques involved were fully described already in the Bern-Kosower works [61–64]. In contrast, the analysis of the two-loop amplitude is what originally led the author to look for more advanced mathematical tools.[23]

3.1 String theory models and their one-loop amplitudes.

To start this chapter on gravity amplitudes on a concrete basis, we begin by recalling some details of the computation of one-loop amplitudes in string theory. At four-point, in heterotic or type II string, they write as a correlation

[23]The author would like to thank here the mathematician Samuel Grushevsky for suggesting him to look at tropical geometry.

function of a product of vertex operators

$$\mathcal{A}^{\text{1-loop}}_{\text{string}} = N \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\operatorname{Im}\tau^2} \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{\mathrm{d}^2 z_i}{\operatorname{Im}\tau} \langle V_1(z_1)V_2(z_2)V_3(z_3)V_4(z_4)\rangle \,, \tag{3.1.1}$$

where the normalization constant N depends on the details of the string theory model. The domain of integration $\mathcal{F}$ has been defined in the previous chapter and the z_i belong to $\mathcal{T} = \{z \in \mathbb{C}, -1/2 < \operatorname{Re} z \leq 1/2, 0 < \operatorname{Im} z < \operatorname{Im}\tau\}$. One of the vertex operators is fixed to $z_4 = i \operatorname{Im}\tau$ by conformal invariance. The unintegrated vertex operators have a holomorphic part and an anti-holomorphic part:

$$V(z) =: V^{(L)}(z)V^{(R)}(\bar{z})e^{ikX(z,\bar{z})} : , \tag{3.1.2}$$

where $V^{(L)}$ and $V^{(R)}$ are the chiral vertex operators for the left- and right-moving sectors.[24] In heterotic string, the anti-holomorphic chiral vertex operators for gravitons are the bosonic vertex operators, normalized as in [60]:

$$V^{(L)}_{(0)}(\bar{z}) = i\sqrt{\frac{2}{\alpha'}}\, \varepsilon_\mu \bar{\partial} X^\mu(\bar{z}) \,, \tag{3.1.3}$$

while the right-moving are supersymmetric chiral vertex operators:

$$V^{(R)}_{(0)}(z) = \sqrt{\frac{2}{\alpha'}}\varepsilon_\mu(k) \left(i\partial X^\mu + \frac{\alpha'}{2}(k\cdot\psi)\psi^\mu \right) . \tag{3.1.4}$$

Type II graviton vertex operators are obtained by choosing both chiral vertex operators to be supersymmetric vertex operators.

The periodicity conditions for the fermionic fields $\psi^\mu, \bar{\psi}^\nu$ upon transport along the a- and b-cycles, corresponding to the shifts $z \to z+1$ and $z \to z+\tau$, respectively, define *spin structures*, denoted by two integers $a, b \in \{0,1\}$ such that

$$\psi^\mu(z+1) = e^{i\pi a}\psi^\mu(z) \,, \qquad \psi^\mu(z+\tau) = e^{i\pi b}\psi^\mu(z) \,. \tag{3.1.5}$$

All of these sectors should be included for modular invariance of the string integrand. The GSO projection indicates relative signs between the corresponding partition functions. The partition function of a supersymmetric sector in the spin structure a, b writes

$$\mathcal{Z}^{ab}(\tau) \equiv \frac{\theta \left[{}^a_b \right] (0|\tau)^4}{\eta^{12}(\tau)} \,, \tag{3.1.6}$$

where the Dedekind η function is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \,, \tag{3.1.7}$$

and the theta functions with characteristics have been defined in (2.2.36). The GSO projection gives rise to supersymmetric cancellation identities on

[24] The vertex operators V_i can all be chosen in the (0) superghost picture since the superghost background charge is zero on the torus.

the worldsheet, which generically go under the name of "Riemann identities" [112, 113] of which we reproduce two below;

$$\sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}Z_{a,b}(\tau)=0\,, \tag{3.1.8a}$$

$$\sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}Z_{a,b}(\tau)\prod_{i=1}^{4}S_{a,b}(z_i-z_{i+1}|\tau)\quad=-(2\pi)^4\qquad(\text{with } z_5\equiv z_1)\,, \tag{3.1.8b}$$

The first identity ensures the vanishing of the string self-energy, as expected in supersymmetric theories. The second identity involves fermionic correlators $S_{a,b}=\langle\psi^\mu(z)\psi^\nu(w)\rangle_{a,b}$ in the spin structure a,b and is the consequence of supersymmetric simplifications on the worldsheet in the RNS formalism.[25] In amplitudes of maximally supersymmetric theories, these identities kill the terms in the correlator (3.1.1) with less than four bilinears of fermions $:\psi\psi:$. They produce the t_8F^4 tensor when there are exactly four of them. Details on these identities can be found in appendix A of [PT2].

In orbifold compactifications, the GSO boundary conditions can be mixed with target-space shifts and the fields X^μ and ψ^μ acquire non-trivial boundary conditions, mixing the standard spin structures with more general (g,h)-orbifold sectors [119, 120];

$$\begin{aligned}X^\mu(z+1)&=(-1)^{2h}\,X^\mu(z)\,, &\psi^\mu(z+1)&=-(-1)^{2a+2h}\psi^\mu(z)\,,\\ X^\mu(z+\tau)&=(-1)^{2g}\,X^\mu(z)\,, &\psi^\mu(z+\tau)&=-(-1)^{2b+2g}\,\psi^\mu(z)\,.\end{aligned} \tag{3.1.9}$$

The string theory four-graviton scattering amplitude is then computed using Wick's theorem as a sum in the various GSO/orbifold sectors in terms of the two points correlators $\langle X(z)X(w)\rangle$ and $\langle\psi(z)\psi(w)\rangle_{a,b}$. It assumes the following general form

$$\mathcal{A}^{\text{string}}_{\text{1-loop}}=N\int_{\mathcal{F}}\frac{d^2\tau}{(\operatorname{Im}\tau)^{D/2-3}}\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{d^2z_i}{\operatorname{Im}\tau}\times\left(\sum_{s,\tilde{s}}\mathcal{Z}^{s\tilde{s}}\left(\mathcal{W}^{(L)}_s(z)\,\overline{\mathcal{W}^{(R)}_{\tilde{s}}}(\bar{z})+\mathcal{W}^{L-R}_{s,\tilde{s}}(z,\bar{z})\right)e^{\mathcal{Q}}\right)\,, \tag{3.1.10}$$

where we dropped the $(1,4)$ index in the Koba-Nielsen factor $\mathcal{Q}_{1,4}$, $s=(a,b,g,h)$ and $\tilde{s}=(\tilde{a},\tilde{b},\tilde{g},\tilde{h})$ label the GSO and orbifold sectors of the theory with their corresponding partition function $\mathcal{Z}_{s,\tilde{s}}$ and conformal blocks $\mathcal{W}^{(L/R)}_{s,\tilde{s}}$, $\mathcal{W}^{L-R}_{s,\tilde{s}}(z,\bar{z})$. Explicit expressions for these terms can be found in [PT2] for the heterotic and type II orbifold models. The term $\mathcal{W}^{L-R}_{s,\tilde{s}}(z,\bar{z})$ contains contractions between left- and right- moving fields like

$$\langle\partial X(z_1,\bar{z}_1)\bar{\partial}X(z_2,\bar{z}_2)\rangle=-\alpha'\pi\delta^{(2)}(z_1-z_2)+\frac{\alpha'}{2\pi\operatorname{Im}\tau}\,. \tag{3.1.11}$$

[25] In the space-time supersymmetric formalisms, there are no sums over spin structures since there are no worldsheet fermions and these simplifications occur from zero mode saturation.

In a generic compactification, the partition function contains a lattice sum corresponding to the Kaluza-Klein states and an oscillator part. For illustrative purposes, let us write the basic compactification lattice sum for a toroidal compactification to $D = 10 - d$ dimensions

$$\Gamma_{d,d}(G,B) = \sum_{P_L,P_R} \bar{q}^{\frac{P_L^2}{2}} q^{\frac{P_R^2}{2}} \,, \tag{3.1.12}$$

where the momenta P_L and P_R span the Narain lattice of the compactification (see chapter 4.18.5 and appendix D of the textbook [79] for more details). In our toroidal compactifications, we will always be in a regime where the Kaluza-Klein states are decoupled. For this it is sufficient to choose the radii of compactification R_i to be of the order of the string-length $R_i \sim \sqrt{\alpha'}$. Therefore we will always set $\Gamma_{d,d}$ to 1 in the following.

The field theory limit is extracted by following the two-step procedure described in the previous section. Here we are interested only in the non-analytic part of the amplitude where the condition $\mathrm{Im}\,\tau \geq L$ gives the field theory cutoff $T \geq \alpha' L$. The heterotic string and type II orbifolds partition function respectively exhibit $1/q$ and $1/\sqrt{q}$ poles. When they hit the integrand, the amplitude picks up non-zero residues upon the phase integration as in (2.2.33). More precisely, in the heterotic string case, the following identities have been used in [PT2, eqs.(III.33), (III.38)]:[26]

$$\int_{-1/2}^{1/2} \mathrm{d}(\mathrm{Re}\,\tau)\mathrm{d}(\mathrm{Re}\,z)\frac{1}{\bar{q}}(\partial\mathcal{G}(z))^2) = (\alpha' i\pi)^2 \,, \tag{3.1.13a}$$

$$\int_{-1/2}^{1/2} \mathrm{d}(\mathrm{Re}\,\tau)\prod_i \mathrm{d}(\mathrm{Re}\,z_i)\,\frac{1}{\bar{q}}\prod_j \partial\mathcal{G}(z_j - z_{j+1}) = (\alpha' i\pi)^4 \,, \tag{3.1.13b}$$

They describe how derivatives of the propagator are eaten up by an inverse power of q. Note that other type of identities can be shown to produce vanishing contributions. Later we connect this to supersymmetric simplifications. Once all phases (real parts of τ and z_i's) are integrated out, the tropical variables corresponding to $\mathrm{Im}\,\tau$ and $\mathrm{Im}\,z$ are obtained by

$$\begin{aligned} \mathrm{Im}\,\tau &= T/\alpha' \,, & T &\in [\alpha' L; +\infty[\,, \\ \mathrm{Im}\,z_i &= T u_i/\alpha' \,, & u_i &\in [0; 1[\,. \end{aligned} \tag{3.1.14}$$

After repeated use of (3.1.13), we obtain the tropical form of $\mathcal{W}$ obtained by turning all $\partial\mathcal{G}$'s which have not been eaten-up in the process to derivatives of the worldline propagator (2.2.43) which writes explicitly at one-loop as

$$G(u_i, u_j) = T(|u_i - u_j| - (u_i - u_j)^2) \,. \tag{3.1.15}$$

Its derivatives with respect to the unscaled variables $t_i = T u_i$ indicated by dots write

$$\begin{aligned} \dot{G}(u_i, u_j) &= \mathrm{sign}(u_i - u_j) - 2(u_i - u_j) \,, \\ \ddot{G}(u_i, u_j) &= \frac{2}{T}\left(\delta(u_i - u_j) - 1\right) \,, \end{aligned} \tag{3.1.16}$$

[26] These identities were obtained in [PT2] in a normalization where α' was set to 1/2.

In supersymmetric sectors, the fermionic propagators left-over from Riemann identities are also subject to residues identities involving $1/\sqrt{q}$ poles. Normally the propagators escaping these two simplifications descend to their worldline analogues $G_F(u_i, u_j) = \text{sign}(u_i - u_j)$. In the computations [PT2], these remaining terms eventually appeared in squares and disappeared of the final expressions. In conclusion, the field theory limit of our expressions can be recast as a worldline integrand W_X which schematically writes solely in terms of $\dot{G}$ and $\ddot{G}$ as $\sum_{n,m,i,j,k,l} C_{n,m}(\ddot{G}_{ij})^m(\dot{G}_{kl})^n$. The monomials satisfy the power counting

$$(\dot{G})^n(\ddot{G})^m \sim \frac{u_i^n}{T^m} \longleftrightarrow \ell^{n+2m}\,, \tag{3.1.17}$$

which can be proven by Gaussian integration of ℓ as explained in (2.2.27) and also in [121, 122]. Eventually, one obtains the following type of worldline integrals for the contribution of the multiplet X to the low energy limit of the string amplitudes

$$\mathcal{M}_X^{1-\text{loop}} = \frac{\pi^4 t_8 t_8 R^4}{4}\,\frac{\mu^{2\epsilon}}{\pi^{D/2}}\int_0^\infty \frac{\mathrm{d}T}{T^{D/2-3}}\int_0^1\prod_{i=1}^3 \mathrm{d}u_i\, W_X\, e^{-\pi T Q^{\text{trop}}} \tag{3.1.18}$$

where μ is an infrared mass scale and the factor $t_8 t_8 R^4$ encompasses the polarization dependence of these supersymmetric amplitudes. Moreover, we traded the hard cut-off $T \geq \alpha' L$ for dimensional regularization to non-integer dimension D. Now that this general discussion of the low energy limit of string theory one-loop amplitudes is complete, let us come to particular models.

3.1.1 CHL models in heterotic string

Chaudhuri-Hockney-Lykken models [50–52] are asymmetric $\mathbb{Z}_N$ orbifolds of the heterotic string compactified on a $T^5 \times S^1$ manifold that preserve all of the half-maximal supersymmetry.[27] They act geometrically by rotating N groups of ℓ bosonic fields $\bar{X}^a$ belonging to the internal T^{16} of the heterotic string or to the T^5 and produce an order-N shift on the S^1. More precisely, if we take a boson $\bar{X}^a$ of the $(p+1)$-th group $(p = 0, \dots, N-1)$ of ℓ bosons we have $a \in \{p\ell, p\ell+1, \dots, p\ell+(\ell-1)\}$ and for twists $g/2, h/2 \in \{0, 1/N, \dots, (N-1)/N\}$ we get

$$\bar{X}^a(\bar{z}+\bar{\tau}) = e^{i\pi gp/N}\bar{X}^a(\bar{z})\,, \qquad \bar{X}^a(\bar{z}+1) = e^{i\pi hp/N}\bar{X}^a(\bar{z})\,. \tag{3.1.19}$$

The massless spectrum is then composed of the half-maximal supergravity multiplet coupled to n_v maximal SYM matter multiplets. The number of matter vector multiplets is found to be

$$n_v = \frac{48}{N+1} - 2\,. \tag{3.1.20}$$

In [PT1, PT2], we restricted to prime N and considered the models with $N = 1, 2, 3, 5, 7$ displayed in the upper part of tab. 3.1. Here we also observe

[27] There also exists type IIA duals [52, 123, 124].

N	ℓ	k	n_v	Gauge group
1	12	10	22	$U(1)^{22}$
2	8	6	14	$U(1)^{14}$
3	6	4	10	$U(1)^{10}$
5	4	2	6	$U(1)^{6}$
7	3	1	4	$U(1)^{4}$
11	2	0	2	$U(1)^{2}$
23	1	-1	0	$\emptyset$

} Unphysical ? (rows $N = 11$ and $N = 23$)

Table 3.1: Adapted from [128]; CHL orbifolds geometry and massless spectrum.

that it is in principle possible to formally define models with $N = 11$ as noted by [125, footnote 2], but also $N = 23$. This model would have $n_v = 0$, meaning that it would describe pure half-maximal supergravity. To achieve this in full rigor, one should actually compactify the theory further on a $T^6 \times S^1$ and $T^7 \times S^1$ to 3 and 2 dimensions, respectively. [28]

Finally, these models have the following moduli space:

$$\Gamma\backslash SU(1,1)/U(1) \times SO(6, n_v; \mathbb{Z})\backslash SO(6, n_v)/SO(6) \times SO(n_v)\,, \tag{3.1.21}$$

where the Γ's are discrete subgroups of $SL(2, \mathbb{Z})$ defined in appendix A.3 of [PT2]. The scalar manifold $SU(1,1)/U(1)$ is parametrized by the axion-dilaton in the $\mathcal{N} = 4$ gravity supermultiplet. The $U(1)$ duality symmetry is known to be an anomalous symmetry [54], whose intriguing implications in the UV behavior of the theory [129, 130] have not been clarified yet .

In loop amplitudes, supergravity is realized by the following combination of the bosonic and supersymmetric sectors;

$$(1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4,\text{vect.}} \times (1_{\mathbf{1}}, 1/2_{\mathbf{0}}, 0_{\mathbf{0}})_{\mathcal{N}=0} = (2_{\mathbf{1}}, 3/2_{\mathbf{4}}, 1_{\mathbf{6}}, 1/2_{\mathbf{4}}, 0_{\mathbf{1}+\bar{\mathbf{1}}})_{\mathcal{N}=4,\text{grav.}} \tag{3.1.22}$$

From the worldsheet point of view, the supersymmetric sector of the amplitude is left untouched by the orbifold and is computed as usual with Riemann identities which reduce the holomorphic integrand to the t_8F^4 tensor.

Hence, the non-trivial part of the computation concerns the bosonic sector. The orbifold partition function writes as a sum of the twisted partition functions in the orbifold twisted and untwisted sectors:

$$\mathcal{Z}^{(n_v)}_{(4,0)\text{het}}(\tau) = \frac{1}{N}\sum_{(g,h)} \mathcal{Z}^{h,g}_{(4,0)\text{het}}(\tau)\,. \tag{3.1.23}$$

[28]We did not make any additional comment on that point, as we already had a type II superstring compactification with $(4, 0)$ supersymmetry that had $n_v = 0$. Here we note that CHL models also appear to be related to the Mathieu Moonshine program (see [126] and references therein), where in particular the order N of the orbifold should relate to the conjugacy class of the Mathieu group $\mathbb{M}_{24}$ via the duality with type II orbifolds of $K3$. To the understanding of the author, despite that an $N = 23$ model might exist, it has not been constructed yet. This putative model, similarly to the observation of [125] for the $N = 11$ one, should act non-geometrically, thus it would not be described by the previous geometric analysis (see also the review of [127] on the classical symmetries of the Mathieu group).

At a generic point in the moduli space, Wilson lines give masses to the adjoint bosons of the $E_8 \times E_8$ or $SO(32)$ gauge group, and decouple the $(6,24)$ Kaluza-Klein lattice sum. The low energy limit also decouples the massive states of the twisted sector ($h \neq 0$) of the orbifold. As a result, only the ℓ gauge bosons of the $U(1)^\ell$ group left invariant by the orbifold action stay in the massless spectrum. The untwisted ($g = h = 0$) partition function reduces to the bosonic string partition function

$$\mathcal{Z}^{0,0}_{(4,0)\text{het}}(\tau) = \mathcal{Z}_{\text{bos}} = \frac{1}{\bar\eta^{24}(\bar\tau)}\,, \qquad (3.1.24)$$

and the partition functions describing the quantum fluctuations of the massless sectors of the theory with $g \neq 0$ are independent of g and write

$$\mathcal{Z}^{g,0}_{(4,0)\text{het}}(\tau) = \frac{1}{f_k(\bar\tau)}\,. \qquad (3.1.25)$$

The modular form $f_k(\tau)$ has weight[29] $\ell = k+2 = 24/(N+1)$ and is defined by:

$$f_k(\tau) = (\eta(\tau)\eta(N\tau))^{k+2}\,. \qquad (3.1.26)$$

In total, the low energy limit of the CHL partition function writes

$$\mathcal{Z}^{n_v}_{CHL} = \frac{1}{N}\left(\frac{1}{(\bar\eta(\bar\tau))^{24}} + \frac{N-1}{f_k(\bar\tau)}\right) = \frac{1}{\bar q} + (n_v+2) + O(\bar q)\,, \qquad (3.1.27)$$

where for the first time we encounter explicitly this $1/\bar q$ pole which was advertised.

At the next step of the computation, we need to write the conformal block $\overline{\mathcal{W}}^B$ coming from Wick contractions[30] of the bosonic chiral vertex operators. It is defined by (3.1.3)

$$\overline{\mathcal{W}}^B = \frac{\langle \prod_{j=1}^4 \epsilon^j \cdot \bar\partial X(z_j) e^{ik_j\cdot X(z_j)}\rangle}{\langle \prod_{j=1}^4 e^{ik_j\cdot X(z_j)}\rangle}\,, \qquad (3.1.28)$$

which can be schematically rewritten as

$$\overline{\mathcal{W}}^B \sim \sum (\bar\partial \mathcal{G})^4\,. \qquad (3.1.29)$$

The $\bar\partial\mathcal{G}$'s come from OPE's between the $\partial\bar X$ and the plane-waves, but also from integrating by parts the double derivatives created by $\bar\partial X \bar\partial X$ OPE's. The coefficients of the monomials are not indicated but carry the polarization dependence of the amplitude.

[29] We recall that a modular form of weight w transforms as $f(\frac{az+b}{cz+d}) = (cz+d)^w f(z)$ for $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$

[30] At four-point, supersymmetry in the right-moving sector does not allow for left-right contractions.

Putting everything together and using residue identities of the form given in (3.1.13), we find that the part of the integrand contributing to the low energy limit of the CHL amplitudes is given by

$$\mathcal{Z}_{(4,0)\text{het}}\overline{\mathcal{W}}^B e^{\mathcal{Q}} \to \underbrace{\left(\overline{\mathcal{W}}^B e^{\mathcal{Q}}\right)|_{\bar q}}_{\dot G^0,\dot G^2} + (n_v+2) \underbrace{\left(\overline{\mathcal{W}}^B e^{\mathcal{Q}}\right)|_{\bar q^0}}_{\dot G^4} + O(\bar q)\,. \qquad (3.1.30)$$

This formula already exhibits the organization of the amplitude by the field theory limit. As indicated by the braces, the first terms give rise to worldline polynomials of degree $\dot G^0$ and $\dot G^2$, due to the $1/\bar q$ pole, while the second term is not reduced of full degree $\dot G^4$. Using the dictionary of (3.1.17), these respectively correspond to numerators with ℓ^0, ℓ^2 and ℓ^4 homogeneous polynomials in loop momentum.

Asymmetric orbifolds of type II superstrings In [PT2], we presented an analysis of the low energy limit of four-graviton amplitudes in the asymmetric orbifold of type II superstrings models with $(4,0)$ supersymmetry of [131–133]. One of these models has the property that matter is totally decoupled [132, 133]. The physical and technical content of this analysis being highly redundant with the heterotic and symmetric orbifold cases, we shall skip it here.

3.1.2 Symmetric orbifolds of type II superstrings

Here we briefly discuss $(2,2)$ models of four-dimensional $\mathcal{N}=4$ supergravity. These models can be obtained from the compactification of type II string theory on symmetric orbifolds of $K_3\times T^2$. The difference with the heterotic CHL models is that the scalar parametrizing the coset space $SU(1,1)/U(1)$ that used to be the axio-dilaton S is now the Kähler modulus of the two-torus T^2 for the type IIA case or complex structure modulus for the type IIB case. The non-perturbative duality relation between these two classes of models is discussed in detail in [124, 131].

The way in which these models are constructed structurally forbids the possibility to decouple completely the matter states. Indeed, supersymmetry is realized by the tensor product between two $\mathcal{N}=2$ vector multiplet theories, which yield the $\mathcal{N}=4$ gravity multiplet plus two $\mathcal{N}=4$ matter vector states

$$\begin{aligned}(1_{\mathbf{1}},1/2_{\mathbf{2}},0_{\mathbf{2}})_{\mathcal{N}=2,\text{vect.}}\times(1_{\mathbf{1}},1/2_{\mathbf{2}},0_{\mathbf{2}})_{\mathcal{N}=2,\text{vect.}} &= (2_{\mathbf{1}},3/2_{\mathbf{4}},1_{\mathbf{6}},1/2_{\mathbf{4}},0_{\mathbf{1}+\bar{\mathbf{1}}})_{\mathcal{N}=4,\text{grav.}}\\ &+\mathbf{2}\,(2_{\mathbf{0}},3/2_{\mathbf{0}},1_{\mathbf{1}},1/2_{\mathbf{4}},0_{\mathbf{6}})_{\mathcal{N}=4,\text{matt.}}\end{aligned} \qquad (3.1.31)$$

The same phenomenon arises when trying to construct pure gravity from pure Yang-Mills:

$$(1_{\mathbf{1}},1/2_{\mathbf{0}},0_{\mathbf{0}})_{\mathcal{N}=0,\text{YM}}\times(1_{\mathbf{1}},1/2_{\mathbf{0}},0_{\mathbf{0}})_{\mathcal{N}=0,\text{YM}} = (2_{\mathbf{1}},3/2_{\mathbf{0}},1_{\mathbf{2}},1/2_{\mathbf{0}},0_{\mathbf{1}})\,, \qquad (3.1.32)$$

Therefore, if an $N=23$ CHL model was constructed, it would be interesting to understand the mechanism that decouples the matter fields and translate it in a type II symmetric duals. This might shed light on how to build pure gravity amplitude directly from Yang-Mills amplitudes [134].

Regarding the structure of the partition function, no novelties arise in this construction compared to the previous analysis. However, a new element enters the computation of the integrand where reduced supersymmetry on both sectors now leave enough room for Wick contractions between the holomorphic and anti-holomorphic sectors of the theory.

3.1.3 Worldline limit

The outcome of these three computations is first that the amplitudes computed in each model do match for identical n_v's. Second, the $\mathcal{N}=4$ supergravity coupled to n_v $\mathcal{N}=4$ vector supermultiplets field theory amplitude is always decomposed as follows;

$$\mathcal{M}^{\text{1-loop}}_{(\mathcal{N}=4,\text{grav})+n_v\,(\mathcal{N}=4\text{matt.})} = \mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8,\text{grav}} - 4\,\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{matt}} + (n_v+2)\,\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{matt}}\,. \tag{3.1.33}$$

Explicit integrated expressions for the integrals can be found in [PT2, eqs. (IV.11), (IV.23),(IV.25)]. These match the known results of [135–137]. For ease, the computation was performed in a helicity configuration $(1^-,2^-,3^+,4^+)$, called the MHV configuration.[31] We set as well the reference momenta q_i's of graviton $i=1,\dots,4$ as follows, $q_1=q_2=k_3$ and $q_3=q_4=k_1$. In that fashion, the covariant quantities t_8F^4 and $t_8t_8R^4$ are written in the spinor helicity formalism[32] $2t_8F^4=\langle 12\rangle^2[34]^2$, and $4t_8t_8R^4=\langle 12\rangle^4[34]^4$, respectively. In this helicity configuration, no triangles or bubbles can be generated from neighboring vertex operators as in (2.2.45) in the symmetric construction.[33] We display below the integrands that were found:

$$W_{\mathcal{N}=8,\text{grav}} = 1\,, \tag{3.1.34a}$$
$$W_{\mathcal{N}=6,\text{matt}} = W_3\,, \tag{3.1.34b}$$

in both models. The matter contributions assume structurally different forms in the two models:

$$W^{\text{asym}}_{\mathcal{N}=4,\text{matt}} (= W^B) = W_1 + W_2\,, \tag{3.1.35a}$$
$$W^{\text{sym}}_{\mathcal{N}=4,\text{matt}} = W_3^2 + W_2/2\,, \tag{3.1.35b}$$

as a consequence of the different ways supersymmetry is realized in string theory, as apparent in eqs.(3.1.22), (3.1.31). Moreover, the factor W_2 in (3.1.35b) comes from the left-right mixing contractions allowed by half-maximal supersymmetry on both sectors of the symmetric orbifold. In the asymmetric models W_2 is simply present in double derivatives in W^B. The explicit worldline

[31] At four points in supersymmetric theories, amplitudes with more + or − helicity states vanish.

[32] See [138] for an introduction to the Spinor-Helicity formalism.

[33] Supersymmetry discards them in the asymmetric models from the start.

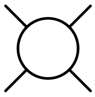
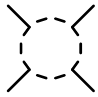
$$\mathcal{M}^{\text{1-loop}}_{(\mathcal{N}=4,\text{grav})+n_v\,(\mathcal{N}=4\text{matt.})} =$$  $+\,n_v$ 

Figure 3.1: One loop worldline description of $\mathcal{N} = 4$ gravity amplitudes coupled to matter fields. Straight lines indicate $\mathcal{N} = 4$ gravity states, dashes indicate $\mathcal{N} = 4$ matter states.

numerators write

$$\begin{aligned}
W_1 &= \frac{1}{16}(\dot{G}_{12} - \dot{G}_{14})(\dot{G}_{21} - \dot{G}_{24})(\dot{G}_{32} - \dot{G}_{34})(\dot{G}_{42} - \dot{G}_{43})\,, \\
W_2 &= -\frac{1}{u}(\dot{G}_{12} - \dot{G}_{14})(\dot{G}_{32} - \dot{G}_{34})\ddot{G}_{24}\,, \\
W_3 &= -\frac{1}{8}\Big((\dot{G}_{12} - \dot{G}_{14})(\dot{G}_{21} - \dot{G}_{24}) + (\dot{G}_{32} - \dot{G}_{34})(\dot{G}_{42} - \dot{G}_{43})\Big)\;.
\end{aligned} \tag{3.1.36}$$

Notice that the $1/u$ factor in the definition of W_2 is dimensionally present since the double derivative $\ddot{G}_{24}$ in W_2 contains a $1/T$. Alternatively, integration by parts of the double derivative would bring down powers of $k_i \cdot k_j$ from the exponential of the tropical Koba-Nielsen factor (2.2.26) and trade $1/u\ddot{G}$ for terms like $s/u(\dot{G})^2$.

Now that all the quantities entering the decomposition (3.1.33) are defined, we can look back at eq. (3.1.30). We confirm a posteriori the link between decreasing powers of $\dot{G}^n$ due to residue identities and the degree of supersymmetry of the multiplets running in the loop. This obeys the qualitative empirical power-counting in gravity amplitudes, which states that the maximal degree of loop momentum in a $(n = 4)$-point one-loop amplitude should be related to the number of supersymmetries $\mathcal{N}$ by[34]

$$\ell^{2n-\mathcal{N}}\,. \tag{3.1.37}$$

Finally, we found interesting to associate to the expansion in eq. (3.1.30) a worldline description in terms of the $(\mathcal{N} = 4)$ gravity and $(\mathcal{N} = 4)$ matter multiplets, as depicted in fig. 3.1. This description extends to the two-loop analysis that we propose now.

3.2 Two loops

The techniques and results described in the previous sections are well under control and widely used since the 80's. In this section, we describe an attempt to push them at the second loop order, where almost nothing similar has been constructed so far. Our starting point is the two-loop heterotic string four-graviton amplitude of [43–49], adapted in CHL models. It assumes the general

[34]Another possibility for power-counting seems to be compatible: $\ell^{4s-\mathcal{N}}$.

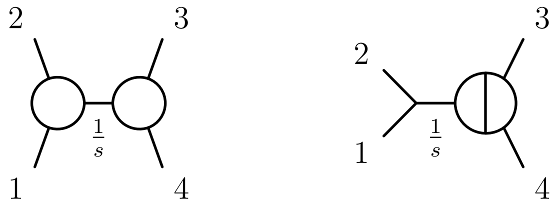


Figure 3.2: Potential pole singularities that would eat up the factorised ∂^2.

form in $D = 10$:

$$\mathcal{M}^{(n_v)}_{4,2-loop} = \mathcal{N}_2 \frac{t_8 F^4}{64\pi^{14}} \int_{\mathcal{F}_2} \frac{|d^3\Omega|^2}{(\det \operatorname{Im}\Omega)^5} \mathcal{Z}_2^{(n_v)} \int \prod_{i=1}^{4} d^2\nu_i \, \overline{\mathcal{W}}^{(2)} \, \mathcal{Y}_s \, e^{\mathcal{Q}} , \quad (3.2.1)$$

where $\mathcal{Z}_2^{(n_v)}$ is the full genus-two partition function of the model under consideration which contains an oscillator and a lattice part[35], $\mathcal{N}_2$ is a normalization constant and $\overline{\mathcal{W}}^{(2)}$ is defined as $\overline{\mathcal{W}}^{B}$ in (3.1.28) in terms of the genus two propagators and we dropped the index $(2,4)$ in $\mathcal{Q}$. The only difference between this amplitude and the heterotic one of [43–49] is that the chiral bosonic string partition function has been replaced with $\mathcal{Z}_2^{n_v}$ and the integration domain is now an $Sp(4,\mathbb{Z})$ fundamental domain (as in sec. 2.2.2).

In [PT1], we used this set-up to argue that there existed a non-renormalization theorem for the R^4 counterterm at two loops in *pure* half-maximal supergravity. The argument goes as follows. First, the $\mathcal{Y}_S$ term factors two derivatives out of the integral. Second, no $1/s_{ij}$ poles as in fig. 3.2 can appear to cancel this factorization in regions where $|z_i - z_j| \ll 1$. The reason for this is the absence of terms like $|\partial\mathcal{G}_{ij}|^2$ in the integrand of (3.2.1). Finally, the matter multiplet contributions, described solely by the partition function $\mathcal{Z}_2^{n_v}$ similarly to the one-loop case, do not prevent this factorization, therefore we may do as if there were none.

The bottom line of this non-renormalization theorem is a string theory explanation, based on worldsheet supersymmetry, for the cancellation of the 3-loop divergence of $\mathcal{N} = 4$ pure supergravity in four dimensions [53]. Since R^4 is ruled out, the results of [25] on $\nabla^2 R^4$ being a full-superspace integral make this term a valid counter-term in $\mathcal{N} = 4$ supergravity, which signals that a four-loop divergence should happen. This divergence has now been explicitly observed in [130], we shall come back on this result in the last chapter of this manuscript, chap. 5, where we discuss future directions of research.

In the rest of this chapter, we provide a novel analysis on the worldline structure of the low energy limit of the amplitude (3.2.1).

Worldline in the tropical limit

The amplitude (3.2.1) has a rather simple structure, in spite of the complexity of the RNS computation performed to derive it. In the supersymmetric sector,

[35] See explicit expressions in [67, 139] for the case of toroidal compactifications. More details on the twisted sectors of genus two string orbifolds and corresponding partition functions and propagators have been worked out in [140, 141] based on the classical references [142, 143].

cancellations due to genus-two Riemann identities produced $\mathcal{Y}_S\, t_8 F^4$. Therefore, in analogy with the one-loop case, the essential step of the computation consists in understanding the partition function of the bosonic sector and its influence on the $\mathcal{W}^{(2)}$ when applying residue formulas.

The partition function The chiral genus-two partition function of the $G = E_8 \times E_8$ or $SO(32)$ bosonic sector of the heterotic string in ten dimensions writes as the product of the G lattice theta function Θ_G by the bosonic string partition function (quantum oscillator part)

$$\mathcal{Z}^G_{g=2} = \frac{\overline{\Theta}_G}{\overline{\Phi}_{10}}\,, \tag{3.2.2}$$

where Φ_{10} is a cusp modular form of weight 12, known as the Igusa cusp form [144]. It is the analogue of the genus-one cusp form η^{24} and its explicit expression is given by the product of theta functions with even characteristics

$$\Phi_{10} = 2^{-12} \prod_{\delta\,\text{even}} (\theta[\delta](0,\Omega))^2\,. \tag{3.2.3}$$

The genus-two lattice theta functions for $E_8\times E_8$ and $SO(32)$ have the following explicit expressions generalizing the one-loop ones (3.A.7)

$$\Theta_{E_8\times E_8}(\Omega) = \Big(\frac{1}{2}\sum_{\delta\ \text{even}} \left(\theta\left[\delta\right](0|\Omega)\right)^8\Big)^2, \qquad \Theta_{SO(32)}(\Omega) = \frac{1}{2}\sum_{\delta\ \text{even}} \left(\theta\left[\delta\right](0|\Omega)\right)^{16}\,, \tag{3.2.4}$$

Similarly to the one-loop case (3.A.8), it is possible to show the equality between these two objects

$$\Theta_{E_8\times E_8}(\Omega) = \Theta_{SO(32)}(\Omega)\,, \tag{3.2.5}$$

which ensures that the partition functions of the two heterotic strings are identical.[36]

Now that we have written down all explicit expressions, a `Mathematica` computation provides the first few terms of the Fourier-Jacobi expansion of these partition functions:

$$\Theta_{E_8\times E_8} = 1 + 480 \sum_{1\le i<j\le 3} q_i q_j + 26880\, q_1 q_2 q_3 + O(q_i^4)\,, \tag{3.2.6}$$

$$\frac{1}{\Phi_{10}} = \frac{1}{q_1q_2q_3} + 2\sum_{1\le i<j\le 3} \frac{1}{q_iq_j} + 24\sum_{i=1}^{3}\frac{1}{q_i} + 0 + O(q_i)\,, \tag{3.2.7}$$

[36] As a side comment, this identity is still valid at $g=3$. At $g=4$, the identity does not hold for all period matrices Ω, but only for the subset of these which precisely correspond to actual Riemann surfaces. We recall that at $g=4$, the space of symmetric $g\times g$ matrices with positive definite imaginary parts, called $\mathcal{A}_4$, is 10-dimensional, while $\mathcal{M}_4$ is 9-dimensional. The Schottky problem consists in identifying the locus of $\mathcal{M}_g$ inside $\mathcal{A}_g$, which is solved in $g=4$ since this locus is precisely the zero locus of the modular form defined by $\Theta_{E_8\times E_8} - \Theta_{SO(32)}$. For $g\ge 5$ no solution is known. The question of a connection between this five and the one of $\mathfrak{M}_5$ is, to the understanding of the author, an open question.

combine into:

$$\mathcal{Z}_2^{E_8\times E_8} = \frac{1}{\bar q_1\bar q_2\bar q_3} + 2\sum_{1\leq i<j\leq 3}\frac{1}{\bar q_i\bar q_j} + 504\sum_{i=1}^{3}\frac{1}{\bar q_i} + 29760 + O(q_i)\,. \qquad (3.2.8)$$

Before making any further comments, let us observe that when compactifying the theory on a d-dimensional torus we can introduce Wilson lines and break the heterotic gauge group to its Cartan subgroup $U(1)^{16}$. The partition function of this model is then simply equal to the quantum oscillator part

$$\mathcal{Z}_2^{U(1)^{16}} = \mathcal{Z}_2^{\text{bos}} \sim \frac{1}{\overline{\Phi}_{10}}\,. \qquad (3.2.9)$$

where it is understood that the previous identity is an equality when considering that a lattice partition function $\Gamma_{d,d}$ for the genus two amplitude as in eq. (3.1.12) reduces to unity due to a choice of vanishing radii of compactification $R_i \sim \sqrt{\alpha'}$ which causes both the Kaluza-Klein states and the $E_8\times E_8$ or $SO(32)$ gauge bosons and higher mass modes to decouple. Therefore (3.2.9), is not the partition function of the full CFT but simply the quantum oscillator part, while the numerator, which should ensure a correct modular weight, has been decoupled.

We shall come back later on the form of the corresponding partition functions for the CHL models. For now, these two partition functions are sufficient to observe interesting consequences on the worldline limit.

Worldline limit The analysis of chap. 2 indicates the kind of residue formula analogous to (3.1.13) we should look for at two loops:

$$\int\prod_{i=1}^{3}\mathrm{d}(\mathrm{Re}\,\tau_i)\left(\frac{1}{q_1^{n_1}q_2^{n_2}q_3^{n_3}}\partial\mathcal{G}(z_{ij})^2\right) = c_{n_1,n_2,n_3} \qquad (3.2.10)$$

with n_1,n_2,n_3 being either 0 or 1, and similar kind of relations where $\partial\mathcal{G}(z_{ij})^2$ is replaced by a term of the form $\partial\mathcal{G}^4$. The author confesses his failure so far in deriving the values of the coefficients c_{n_1,n_2,n_3} from a direct computation, although he hopes that the tropical geometry program will help in this quest. One of the main issues in this computation was to obtain expressions independent on the odd spin-structure δ chosen to define $\mathcal{G}$ (see eq. (2.1.8)).

In the following, we simply assume that we are given such a set of identities. They render the extraction of the field theory limit of the amplitude (3.2.1) expressible in the following schematic worldline form, similarly to the one-loop case in eq. (3.1.30):

$$\lim_{q_1,q_2,q_3\to 0}\int\prod_{i=1}^{3}\mathrm{d}(\mathrm{Re}\,\tau_i)\left(\mathcal{Z}_2^X\overline{\mathcal{W}}^{(2)}e^{\mathcal{Q}}\right) = \begin{cases} c_2 + c_1\dot G^2 + 29760\,\dot G^4 & \text{if } X = E_8\times E_8\\ \tilde c_2 + \tilde c_1\dot G^2 + 0 & \text{if } X = U(1)^{16}\end{cases} \qquad (3.2.11)$$

where in particular the constant term present in (3.2.8) gives that the $E_8\times E_8$ worldline integrand possesses a term of full degree $\dot G^4$, while the $U(1)^{16}$ integrand has only a $\dot G^2$.

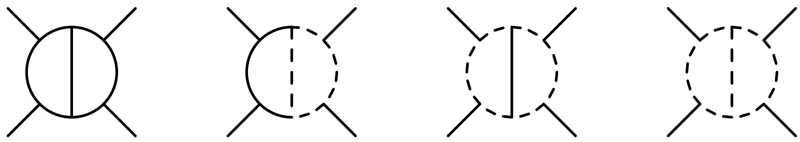


Figure 3.3: Two loop worldline diagrams $\mathcal{N}=4$ matter-coupled supergravity amplitudes. Plain lines are $\mathcal{N}=4$ gravity states, dashes are $\mathcal{N}=4$ YM matter states.

Let us try to understand the implications of this remark. Following the dictionary (3.1.17), these integrands respectively correspond to loop momentum polynomials of maximum degree 4 and 2. However, the presence of a factorized $\nabla^2 R^4$ operator outside of the integrals does not allow for loop-momentum numerators of degree higher than two, as shown in the introduction in eq. (1.1.10). The situation is all the more puzzling that we already argued that no pole would transmute the ∂^2 to an ℓ^2, creating a total ℓ^4 in the numerators. Moreover, this implies that the $E_8 \times E_8$ integrand has a worse ultraviolet behavior than the $U(1)^{16}$ model, so the issue is definitely not innocent.

The solution to this apparent paradox is linked to the spectrum content and interactions of the $E_8 \times E_8$ (or $SO(32)$) model. This model indeed describes 16 abelian gauge bosons but also 480 *non-abelian* gauge bosons, which can create diagrams such as the rightmost one in fig.3.3. This diagram is not dressed with a κ_D^6 but with a $\kappa_D^4 g_{YM}^2$. Since the coupling constants are related via

$$2\kappa_D = \sqrt{\alpha'}\, g_{\rm YM} \tag{3.2.12}$$

we now realize that an additional power of α' counterbalances the apparent superabundant ℓ^4 in the model with non-abelian gauge interactions. In addition, numerology indicates us that the numerical factor $29760 = 480 \times 496/8$ is related to the interactions of the non-abelian gauge bosons in one way or another. Therefore this divergence only arises in the mixed gravitational-Yang-Mills sector. This does not affect the discussion of the divergences in purely gravitational sector of $\mathcal{N}=4$ supergravity with or without vector-multiplets. This reasoning brushes aside the potential UV issue with the ℓ^4 term in the pure half-maximal supergravity amplitudes. In addition, it gives a heuristic argument on the form of the partition function for CHL models.

CHL models

In [125] were used the so called Siegel genus-two modular forms of weight k generalizing Φ_{10} as $\mathcal{N}=4$ CHL Dyon partition functions. We give below their Fourier-Jacobi expansion, as obtained from [125]:

$$\frac{1}{\Phi_k} = \frac{1}{q_1 q_2 q_3} + 2 \sum_{1\leq i<j\leq 3} \frac{1}{q_i q_j} + \frac{24}{N+1} \sum_{i=1}^{3} \frac{1}{q_i} + \frac{48N}{(N-1)(N+1)} + O(q_i)\,. \tag{3.2.13}$$

for $N = 2,3,5,7$ with conjectural extension to $N = 11, 23$. See again (3.2.7) for $N = 1$. These forms are the analogues of the $f_k(\tau)$ defined in (3.1.26) at

$g = 1$ and enter the computation of the genus two partition function, as we will see in an explicit example for $N = 2$ below.

The reasoning of the previous section indicates that the constant term of the partition function should vanish in the absence of non-abelian interactions in the massless spectrum, and that the dependence on the n_v should be linear in n_v+2. This requirement and the knowledge of the Fourier-Jacobi expansion of the partition functions at $N = 1$ and $N = 2$ will be enough to prove that they should generally have the following Fourier-Jacobi expansion;

$$\mathcal{Z}_2^{CHL_N} = \frac{1}{\bar{q}_1\bar{q}_2\bar{q}_3} + 2\sum_{1\leq i<j\leq 3}\frac{1}{\bar{q}_i\bar{q}_j} + (n_v+2)\sum_{i=1}^{3}\frac{1}{\bar{q}_i} + 0 + O(q_i)\,. \tag{3.2.14}$$

up to the lattice factor that reduce to one in the limit we are considering.

This relationship holds true for $N = 1$. Below we provide a short computation based on the derivation in [132] of the $N = 2$ CHL partition function in the context of dyon counting, after the classic reference [142]. The evaluation of the twisted quantum oscillator determinants is performed through the use of a double covering of the genus two surface by a Prym variety, and the dependence on the Prym period ultimately cancels and yield the following result for the partition function with a twist in a particular A cycle;

$$Z_{twisted} = \frac{1}{\Phi_6(\Omega)} + \frac{1}{16}\frac{1}{\Phi_6'(\Omega)} - \frac{1}{16}\frac{1}{\Phi_6''(\Omega)}. \tag{3.2.15}$$

where the theta function lattice (explicitly computed in [128]) have been replaced by 1's, since the gauge group is broken by Wilson lines, and the corresponding lattice partition function which also reduces to one have not been written. The Siegel modular forms Φ_6, Φ_6' and Φ_6'' are images of Φ_6 under modular transformations and their explicit expressions in terms of theta functions are given in [128], eqs. (4.32)-(4.34). The Fourier-Jacobi expansion of Φ_6 is given in (3.2.13), and we also computed explicitly the other two;

$$\begin{aligned}\Phi_6' &= \frac{16}{\sqrt{q_1}q_2\sqrt{q_3}} + \frac{128}{q_2} - 256\\ \Phi_6'' &= \frac{16}{\sqrt{q_1}q_2\sqrt{q_3}} - \frac{128}{q_2} + 256\end{aligned} \tag{3.2.16}$$

In total we obtain;

$$Z_{twisted} = \frac{1}{q_1q_2q_3} + \frac{2}{q_1q_3} + \frac{2}{q_2q_3} + \frac{2}{q_1q_2} + \frac{8}{q_1} + \frac{8}{q_3} + \frac{24}{q_2} + 0 + O(q_i) \tag{3.2.17}$$

As is, it is not symmetric under the exchange the q_i's together, which is required to ultimately yield the correct symmetry of the edges of the worldline graphs. Indeed, this partition function has been obtained for a particular twisted sector of the orbifold, along the A_2 cycle. Summing over all sectors, and including the untwisted one, with appropriate weight yields;

$$\frac{1}{4}\left(\frac{1}{\Phi_{10}} + (Z_{twisted} + (q_2 \leftrightarrow q_1) + (q_2 \leftrightarrow q_3))\right) \tag{3.2.18}$$

which has the Fourier-Jacobi expansion given in (3.2.14).

Finally, the worldline arguments developed before imply that we expect the dependence on n_v to be linear in these models, therefore having two points ($N = 1, 2$) is enough to show that the genus two partition function of the other models ($N \geq 3$) should have a Fourier-Jacobi expansion given by (3.2.14).

3.A Appendix on the one-loop divergence in $D = 8$ in CHL models

The section 3.1 was dedicated to the extraction of field theory amplitudes from the $\alpha' \to 0$ limit of the non-analytic part of string theory amplitudes, meaning that we focused on the part the moduli space restricted to the upper domain $\mathcal{F}^+(L)$ defined in eq. (2.2.22) and fig. 2.10.

In this section, we compute the 8-dimensional R^4 logarithmic divergence of these half-maximal supergravity amplitudes from both the non-analytic and analytic parts of the string theory amplitudes. As global normalizations between the two computation remain unfixed, relative normalizations between the contribution of the vectors multiplets and gravity multiplet agree. This section is intended to be a simple example of the reasoning of [107] described previously, supplementing the trivial computation given in [PT3] and the explicit examples given in the seminal paper. We expect that the $\ln(L)$ divergence coming from the integral over $\mathcal{F}^+(L)$ will be canceled by a term coming from $\mathcal{F}^-(L)$. The starting point is the four-graviton CHL amplitude

$$\mathcal{M}^{(n_v)}_{(4,0)het} = \mathcal{N} \left(\frac{\pi}{2}\right)^4 t_8 F^4 \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^{D/2-3}} \int_{\mathcal{T}} \prod_{1\leq i<j\leq 4} \frac{\mathrm{d}^2 z_i}{\operatorname{Im}\tau}\, e^{\mathcal{Q}}\, \mathcal{Z}^{(n_v)}_{(4,0)het} \bar{\mathcal{W}}^B\,, \tag{3.A.1}$$

which we split into the sum of two integrals as in (2.2.23) that we denote $\mathcal{M}^{(n_v)}_{(4,0)het}(L, \pm)$.

3.A.1 Divergence in the non-analytic terms

The procedure described in the previous section produced explicit expressions for the D-dimensional worldline integrands of half-maximal supergravity scattering amplitudes descending from $\mathcal{M}^{(n_v)}_{(4,0)het}(L, +)$, given in (3.1.34), (3.1.35). All we have to do here is to extract the divergence piece of the corresponding integrals in eight dimensions.

The integration is most easily performed in dimensional regularization to $D = 8 - 2\epsilon$ dimensions, using the standard techniques described in [111, 145–

147] The leading $1/\epsilon$ divergence of these integrals is found to be:

$$\begin{aligned}
\mathcal{M}^{spin\ 2}_{\mathcal{N}=8}\Big|_{D=8+2\epsilon,\,div} &= \frac{i}{(4\pi)^4}\langle 12\rangle^4[34]^4\left(\frac{1}{2\epsilon}\right)\\
\mathcal{M}^{spin\ 3/2}_{\mathcal{N}=6}\Big|_{D=8+2\epsilon,\,div} &= \frac{i}{(4\pi)^4}\langle 12\rangle^4[34]^4\left(\frac{1}{24\epsilon}\right)\\
\mathcal{M}^{spin\ 1}_{\mathcal{N}=4}\Big|_{D=8+2\epsilon,\,div} &= \frac{i}{(4\pi)^4}\langle 12\rangle^4[34]^4\left(\frac{1}{180\epsilon}\right)
\end{aligned} \tag{3.A.2}$$

where we expect the $1/\epsilon$ term to match the $\ln(\alpha' L)$ divergence. These divergences match the expressions of [148]. These of [149] are recovered as well after flipping a sign for the $\mathcal{N}=6$ spin-3/2 divergence. The divergence of the half-maximal supergravity multiplet is obtained from the decomposition (3.1.33) in $D=8+2\epsilon$ with n_v vector multiplets:

$$\mathcal{M}^{n_v}_{\mathcal{N}=4}\Big|_{div} = \frac{i}{(4\pi)^4}\langle 12\rangle^4[34]^4\left(\frac{62+n_v}{180\,\epsilon}\right) \tag{3.A.3}$$

which matches eq (3.8) of [148] with the identitifaction $n_v = D_s - 4$. The normalizations are the ones of [PT4, eq. 5.16].

3.A.2 Divergence in the analytic terms

Let us now consider $\mathcal{M}^{(n_v)}_{(4,0)het}(L,-)$, defined by the integral (3.A.1) restricted to the region $\mathcal{F}^-(L)$. We already argued that since τ is of order $O(1)$, it is possible to safely take the $\alpha' \to 0$ limit of the string theory integrand, which results in dropping the Koba-Nielsen factor.[37] Following the classical reference [150, appendix A,B], the resulting integrals involve terms of the form[38]

$$\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{\mathrm{d}^2 z_i}{\mathrm{Im}\,\tau}\left(\partial\mathcal{G}(z_{12})\right)^2\left(\partial\mathcal{G}(z_{34})\right)^2 = \left(\frac{2\pi}{12}\hat{E}_2(\tau)\right)^4, \tag{3.A.4a}$$

$$\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{\mathrm{d}^2 z_i}{\mathrm{Im}\,\tau}\left(\partial\mathcal{G}(z_{12})\right)\left(\partial\mathcal{G}(z_{23})\right)\left(\partial\mathcal{G}(z_{34})\right)\left(\partial\mathcal{G}(z_{41})\right) = \frac{(2\pi)^4}{720}E_4(\tau)\,, \tag{3.A.4b}$$

where a global factor of α'^4 has not been displayed. Up to permutations of the indices, any other combination of propagators integrates to zero. The non-holomorphic Eisenstein series $\hat{E}_2$ writes

$$\hat{E}_2 = E_2 - \frac{3}{\pi\mathrm{Im}\,\tau}\,. \tag{3.A.5}$$

[37] Actually one should here also make sure that no triangle like contribution may arise from colliding vertex operators, which is the case.

[38] The integrals involving double derivatives in $\mathcal{W}_2$ can always be turned into such kind of integrals after integration by parts.

in term of the holomorphic Eisenstein series E_2, which together with E_4 write

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = 1 - 24q + O(q^2)\,, \tag{3.A.6a}$$

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n} = 1 + 240q + O(q^2)\,. \tag{3.A.6b}$$

Eisenstein series are also related to partition functions of toroidal lattice sums or lattice theta functions:

$$\begin{aligned}\Theta_{E_8\times E_8}(\tau) = E_4(\tau)^2 = \frac{1}{2}(\theta_2(0,\tau)^8 + \theta_3(0,\tau)^8 + \theta_3(0,\tau)^8)\,,\\ \Theta_{SO(32)}(\tau) = E_8(\tau) = \frac{1}{2}(\theta_2(0,\tau)^{16} + \theta_3(0,\tau)^{16} + \theta_3(0,\tau)^{16})\,.\end{aligned} \tag{3.A.7}$$

The identity $E_4(\tau)^2 = E_8(\tau)$ ensures that the one-loop partition functions of the $E_8 \times E_8$ and $SO(32)$ heterotic string are identical:

$$\Theta_{E_8\times E_8} = \Theta_{SO(32)}\,. \tag{3.A.8}$$

Coming back to the amplitude and collecting the previous results, we obtain

$$\mathcal{M}^{(n_v)}_{(4,0)het}(L,-) = \mathcal{N}\left(\frac{\pi}{2}\right)^4 \int_{\mathcal{F}^-(L)} \frac{d^2\tau}{\mathrm{Im}\,\tau}\,\mathcal{A}(\mathcal{R},\tau) \tag{3.A.9}$$

where the reader should pay attention to the fact that we replaced D with $D = 8$, which explains the factor of $1/\mathrm{Im}\,\tau$ in the integrand. The quantity $\hat{\mathcal{A}}(\mathcal{R},\tau)$ is obtained from the heterotic string elliptic index of [150] by changing the bosonic string partition function $1/\eta(\tau)^{24}$ to the CHL partition function of eq. (3.1.27):

$$\hat{\mathcal{A}}(\mathcal{R},\tau) = \mathcal{Z}^{n_v}_{CHL}\left(\frac{1}{2^7\cdot 3^2\cdot 5}\,E_4\,t_8\,\mathrm{tr}\,\mathcal{R}^4 \;+\; \frac{1}{2^9\cdot 3^2}\,\hat{E}_2^2\,t_8\,(\mathrm{tr}\,\mathcal{R}^2)^2\right)\,. \tag{3.A.10}$$

where the normalization is adjusted so that the $t_8\mathrm{tr}\,\mathcal{F}^4$ term has coefficient 1. The t_8, $\mathrm{tr}\,^4$ and $(\mathrm{tr}\,^2)^2$ tensors are related by the following identity [151]:

$$t_8t_8R^4 = 24t_8\mathrm{tr}\,R^4 - 6t_8(\mathrm{tr}\,R^2)^2\,. \tag{3.A.11}$$

The logic now is to compute the integral of eq. (3.A.9) and extract the $\ln L$ term. This could be done in full rigor by following the argument of [27] relating the coefficient of counterterms in the Einstein frame to the coefficient of the logarithm of the D-dimensional string coupling constant in the string frame. This coefficient has be exactly computed for integrals of the form of eq. (3.A.10) with a $\Gamma_{2,2}$ included and can be found in [152, Appendix E], [153], or by using the new methods developed in [139, 154–156] . The result of this procedure can be obtained by a shortcut where one attributes exclusively the coefficient of $\ln L$ in (3.A.9) to the logarithmic divergence created by the $1/\mathrm{Im}\,\tau$ term in the expansion of $\hat{\mathcal{A}}$. This term writes precisely

$$\hat{\mathcal{A}}(\mathcal{R},\tau)\Big|_{1/\mathrm{Im}\,\tau} = \frac{1}{\mathrm{Im}\,\tau}\Bigg(\frac{1}{2^7\cdot 3^2\cdot 5}((n_v+2)+240)t_8\mathrm{tr}\,\mathcal{R}^4+ \frac{1}{2^9\cdot 3^2}\left((n_v+2)-48\right)(\mathrm{tr}\,R^2)^2\Bigg)\,. \tag{3.A.12}$$

Going to the MHV configuration thanks to the following identities

$$24t_8\mathrm{tr}\,R^4 = \frac{3}{8}\times[12]^4\langle 34\rangle^4\,,\qquad -6t_8(\mathrm{tr}\,R^2)^2 = -\frac{1}{8}\times[12]^4\langle 34\rangle^4\,. \tag{3.A.13}$$

gives the coefficient of $\ln L$

$$\mathcal{M}^{n_v}_{\mathcal{N}=4}\Big|_{div} = c_0'\langle 12\rangle^4[34]^4(62+n_v)\ln L \tag{3.A.14}$$

This result matches the one in (3.A.3) up to a global normalization constant which has not been fixed rigorously. An important consistency check that this example passes is that the relative coefficients between the vector multiplets and gravity contribution are identical in both approaches.

A similar computation is given in [PT3] for the case of the quadratic divergence of maximal supergravity in 10 dimensions, where exact matching is precisely observed. Moreover, several other examples are discussed in the original paper [107].

Chapter 4

BCJ double-copy in string theory

The domain of scattering amplitudes in quantum field theories is at the heart of high energy physics and bridges the gap between theory and collider experiments led nowadays at the Large Hadron Collider. It has been developing fast for the last twenty years, mostly pioneered by the work of Bern, Dixon and Kosower. For moderns reviews on scattering amplitudes, we refer to [138, 157]. In this context, gravitational scattering amplitudes are not directly related to precision physics[39] but rather to more conceptual aspects of the perturbative structure of quantum gravity. These can also serve as consistency checks for certain string theory computations.

The basic difficulty with gravity amplitudes is their complicated kinematical structure, partly due to the presence of arbitrarily high-valency vertices which make the number of diagrams grow very fast. The main idea to simplify these computations is to implement that some of the gravity (spin-2) degrees of freedom are described by the tensorial product of two Yang-Mills spin-1 fields. In string theory, this can be done very efficiently at tree-level, where the Kawai-Lewellen-Tye (KLT) relations [159] relate closed string amplitudes to a product of open strings amplitudes. The paradigm can be loosely formulated as

$$\text{“open} \times \text{open} = \text{closed”}\,. \tag{4.0.1}$$

The modern version of the KLT relations, known as the monodromy relations [160–163] led to the so-called “Momentum Kernel” construction of [164]. The latter relates closed-string amplitudes to open-string amplitudes at any multiplicity *via* the Momentum Kernel $\mathcal{S}_{\alpha'}$ as

$$\mathcal{M}^{\text{closed}}_{n,\text{tree}} = \mathcal{A}^{\text{open}}_{n,\text{tree}} \cdot \mathcal{S}_{\alpha'} \cdot \mathcal{A}^{\text{open}}_{n,\text{tree}}\,. \tag{4.0.2}$$

In the $\alpha' \to 0$ limit, this relation provides a similar relation between gravity and Yang-Mills amplitudes:

$$\mathcal{M}^{\text{gravity}}_{n,\text{tree}} = \mathcal{A}^{\text{YM}}_{n,\text{tree}} \cdot \mathcal{S} \cdot \mathcal{A}^{\text{YM}}_{n,\text{tree}} \tag{4.0.3}$$

[39] Except the work [158] in which scattering amplitude methods are used to re-derive the first $\hbar$ correction to the Newtonian potential.

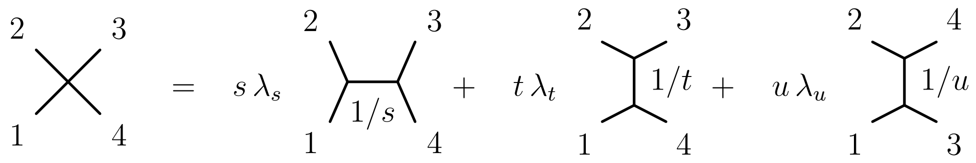


Figure 4.1: Generic ambiguities with blowing-up the contact-terms, with 2-parameters freedom determined by $\lambda_s + \lambda_t + \lambda_u = 1$. The diagrams are dressed with $1/p^2$ propagators.

where $\mathcal{S}$ is the field theory limit of $\mathcal{S}_{\alpha'}$. In this way, the computation of gravity amplitudes is considerably simplified, as it is reduced to that of gauge theory amplitudes, which is done with 3- and 4-valent vertices only.

At loop-level, the analytic structure of the S-matrix is not compatible with squaring. A $\ln(s)$ in a one-loop Yang-Mills amplitude does not indicate the presence of a $\ln(s)^2$ in any one-loop gravity amplitude – this would trivially violate unitarity of the theory. However, a squaring behavior similar to KLT was early observed at the level of the unitarity cuts of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ amplitudes [13].

The Bern-Carrasco-Johansson duality and double-copy construction [33, 34] provide all-at-once a working algorithm to reduce gravity amplitudes to a cubic-graph expansion,[40] built from gauge theory amplitudes and working at loop level. These gauge theory amplitudes have to be written in a particular representation, satisfying the so-called BCJ duality. The analysis and discussion of current understanding of this construction in string theory is the subject this chapter.

4.1 Review of the BCJ duality and double-copy.

The BCJ duality between color and kinematics in gauge theory amplitudes is defined in tree and loop amplitudes written in terms of cubic-graphs only. This reduction induces a first level of ambiguity when the quartic contact-terms are blown-up to cubic vertices by multiplying and dividing by momentum invariants, as shown in fig. 4.1. In this way, gauge theory amplitudes write

$$\mathcal{A}_n^L = i^L g^{n+2L-2} \sum_{\text{cubic graphs } \Gamma_i} \int \prod_{j=1}^{L} \frac{\mathrm{d}^d \ell_j}{(2\pi)^d} \frac{1}{S_i} \frac{c_i\, n_i(\ell)}{D_i(\ell)}\,, \tag{4.1.1}$$

where the sum runs over distinct non-isomorphic cubic graphs. The denominator $D_i(\ell)$ is the product of the Feynman propagators of the graph and the integral is performed over L independent D-dimensional loop momenta. Finally, the symmetry factors $1/S_i$ remove over counts from summing over the different configurations of the external legs. The c_i are the color factors of

[40] Cubic graphs are graphs made of trivalent vertices only.

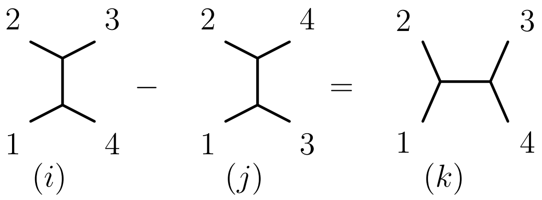


Figure 4.2: Jacobi identity for the color or numerator factors.

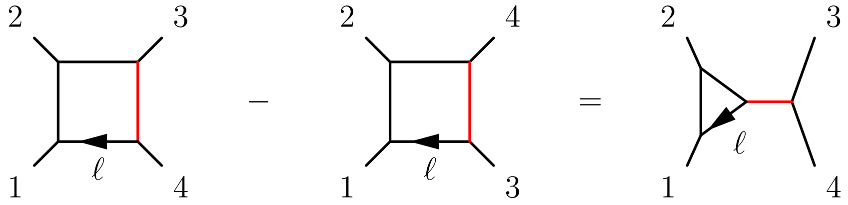


Figure 4.3: Sample Jacobi identity for one-loop numerators

the graph obtained by dressing each vertex with the structure constants of the gauge group $\tilde{f}^{abc}$ defined by

$$\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \mathrm{tr}\left([T^a, T^b] T^c\right). \tag{4.1.2}$$

The n_i's are the kinematic numerators of the graph. This representation of the amplitude satisfies the BCJ duality if the Jacobi relations of the color factors are also satisfied by the corresponding kinematic numerators;

$$c_i - c_j = c_k \quad \Rightarrow \quad n_i - n_j = n_k \,, \tag{4.1.3}$$

as depicted in fig. 4.2. Let us emphasize that this property is really not restricted to tree-level four-point diagrams, but should hold for any situation where the graphs of fig. 4.2 are embedded in a bigger graph. An example is show in fig. 4.3 at one-loop. Note that the loop momentum dependence should be traced with care and the "external legs" of the central edge on which the Jacobi relation is being applied should keep their momentum constant.

Such representations do not trivially follow from blowing-up the contact-terms randomly, but rather necessitate an important reshuffling of the amplitude. This is possible thanks to an additional freedom that possess BCJ representations, called "generalized gauge invariance". This freedom corresponds to the fact that a set of BCJ numerators $\{n_i\}$ can be deformed by any set of quantities that leave the Jacobi relations (4.1.3) invariant. If one defines $n'_m = n_m + \Delta_m$ for $m = i, j, k$, the numerators n'_m obey (4.1.3) as long as $\Delta_i - \Delta_j = \Delta_k$. This freedom can be used to reduce the non-locality of the BCJ numerators.

Once a BCJ duality satisfying representation is found, the double-copy procedure prescripts to replace the color factors c_i in (4.1.1) by another set of kinematic numerators $\tilde{n}_i$ to obtain the gravity amplitude:

$$\mathcal{M}_n^{L-\text{loop}} = i^{L+1} \left(\frac{\kappa}{2}\right)^{n+2L-2} \sum_{\text{cubic graphs } \Gamma_i} \int \prod_{j=1}^{L} \frac{\mathrm{d}^d \ell_j}{(2\pi)^d} \frac{1}{S_i} \frac{n_i(\ell)\, \tilde{n}_i(\ell)}{D_i(\ell)} \,. \tag{4.1.4}$$

Due to generalized gauge invariance of the first set of numerators $\{n_i\}$, the set $\{\tilde{n}_i\}$ does not need to be in a BCJ representation [165]. The duality has been demonstrated to hold classically by construction of a non-local Lagrangian [165]. In [166], it was observed to be more restrictive than the strict KLT relations, and later understood in open string theory by Mafra, Schlotterer and Stieberger in [167] by means of worldsheet integration by part (IBP) techniques. Part of our work [PT4] heavily relies on this "Mafra-Schlotterer-Stieberger" (MSS) construction to which we come back in sec. 4.2.

The BCJ duality was successfully applied in the hunt for UV divergences of supergravity theories, at three and four loops in $\mathcal{N} = 8$ [34, 35]. In half-maximal supergravity, the vanishing of the three-loop R^4 divergence in $D = 4$ was observed in a direct computation [53] and the four-loop logarithmic divergence created by the $\nabla^2 R^4$ in $D = 4$ explicitly determined [130]. More broadly, it was also applied to compute $\mathcal{N} \geq 4$ supergravity amplitudes at various loop orders [137, 148, 168, 169] and even for pure Yang-Mills and pure gravity theories at one and two loops [170].

The existence of BCJ satisfying representations at any loop order is an open question. In particular, at five loops in $\mathcal{N} = 4$, no BCJ representation has yet been found, despite tenacious efforts [36].[41] At one loop there exist constructive methods to build some class of BCJ numerators in $\mathcal{N} = 4$ SYM [171] and orbifolds thereof [172, 173]. Nevertheless, the generic method to find BCJ representations for numerators is to use an ansatz for the numerators, which is solved by matching the cuts of the amplitude [172, 174]. The free-parameters that remain (if any) after satisfying all the constraints are a subset of the full generalized gauge invariance. In [PT4], we also studied some aspects of the string theory viewpoint on the ansatz approach.

4.2 Tree-level string theory understanding of BCJ

We already described that the KLT relations in string theory relate open to closed strings amplitudes. However, this does not directly relate color to kinematics at the integrand level. In [PT4] we argued that this can be done by slightly modifying the paradigm of (4.0.1) to the following purely closed-string one:

$$\text{“left-moving sector} \times \text{right-moving sector = closed”} \,. \tag{4.2.1}$$

which means that instead of focusing on an definite string theory, we consider as a freedom the possibility to plug different CFT's in both sectors of the closed string. These are tied together by the low energy limit and realize various theories, as illustrated in Table 4.1, where "Color CFT" and "Spacetime CFT" refer to the respective target-space chiral polarizations and momenta of the scattered states.

[41] We recall that 5-loop in $\mathcal{N} = 8$ is crucial to understand the UV behavior of the theory, see again fig. 1.3.

Left-moving	Right-moving	Low-energy limit	Closed string theory
Spacetime	Color	Gauge theory	Heterotic
Spacetime	Spacetime	Gravity theory	Type II, (Heterotic)
Color	Color	Cubic color scalar	Bosonic

Table 4.1: Different string theories generate various field theories in the low-energy limit. "Spacetime" and "color" refer to the CFT's plugged in the left- and right- moving sectors, respectively.

A gauge theory is realized by the closed string when one of the chiral sectors of the external states is polarized in an internal color space, this is the basic mechanism of the heterosis [175]. The use of heterotic string in this context was first described in [162] where it was realized that it sets color and kinematics on the same footing. In this sense our work descends from these ideas. A gravity theory is realized when both the left- and right-moving polarizations of the gravitons have their target space in Minkowski spacetime, as it can be done both in heterotic and type II string.[42]

The last line of the table, mostly shown as a curiosity, deserves a comment. As we mention later, this cubic scalar theory is the result of compactifying the bosonic string on a $(T^{16}\times\mathbb{R}^{1,9})\times(T^{16}\times\mathbb{R}^{1,9})$ background where the T^{16} is the internal torus of the heterotic string. At tree-level, the bosonic string tachyon can be decoupled by hand, and the remaining massless states bi-polarized in the T^{16} give rise to these cubic color scalar interactions. At loop-level the tachyon cannot be easily decoupled and the construction probably cannot be given much sense.[43]

Our starting point for the following analysis is the open string construction of MSS [176]. We recall that MSS have shown how a worldsheet IBP procedure in the open string leads to a particular representation of the open string integrand in terms of $(n-2)!$ conformal blocks. From this representation, it is explained how to extract the BCJ numerators for gauge theory amplitudes at any multiplicity. In [PT4], we argued that this construction can be recast in the closed string sector and gives rise to a somewhat stronger result, where we get all-at-once the Jacobi identities of MSS but also the double-copy form of the gravity amplitudes. Our reasoning was mostly supported by an explicit five-point example that we worked out explicitly. We outlined a n-point proof of the systematics of the result.

Below we give a more detailed account on this systematics. Regarding the *material* available in the literature[44], we shall base our reasoning on the fact

[42]Neither in the paper nor in this text have we described the gravity sector of the heterotic string, as it is always non-symmetric. Instead, we focused on the symmetric orbifolds of the type II string described in chapter 3 to obtain symmetric realizations of half-maximal supergravity.

[43]Anyway the cubic scalar theory is possible to deal with by standard techniques.

[44]*Addendum*: After the first version of this manuscript was written, Schlotterer and Mafra proposed in [177] a formalism for describing the systematics of the tree combinatorics based on "multi-particle vertex operators", which can be used for the present problem.

that type I and II string amplitudes are known at n-point in the pure spinor formalism [176, 178–180] and their field theory limits have been extensively studied in [176, 181] as well as their α' expansion in [181–184]. Hence we start with an n-point closed string theory amplitude, written as:

$$\begin{aligned}\mathcal{A}_n^{\text{string}} =& |z_{1,n-1}z_{n-1,n}z_{n,1}|^2\times \\ &\times \left\langle V_1(z_1)V_{n-1}(z_{n-1})V_n(z_n)\int\prod_{i=2}^{n-2}\mathrm{d}^2z_iV_2(z_2)\dots V_{n-2}(z_{n-2})\right\rangle .\end{aligned} \tag{4.2.2}$$

A global normalization $g_c^{n-2}8\pi/\alpha'$, where g_c is the closed string coupling constant, has been omitted. The factor $|z_{1,n-1}z_{n-1,n}z_{n,1}|^2$ comes from gauging the $SL(2,\mathbb{C})$ conformal invariance of the sphere by fixing the positions of 3 vertex operators, here z_1, z_{n-1} and z_n. The unintegrated vertex operators have a holomorphic part and an anti-holomorphic part:

$$V(z) = :V^{(L)}(z)V^{(R)}(\bar z)e^{ikX(z,\bar z)}:, \tag{4.2.3}$$

as already described in the beginning of chap 3. The anti-holomorphic part of the heterotic gauge-boson vertex operators are made of a current algebra

$$V^{(R)}(\bar z) = T^a\, J_a(\bar z)\,. \tag{4.2.4}$$

The currents satisfy the following operator product expansion (OPE):

$$J^a(\bar z)J^b(0) = \frac{\delta^{ab}}{\bar z^2} + if^{abc}\frac{J^c(\bar z)}{\bar z} + \dots, \tag{4.2.5}$$

where the f^{abc}'s are defined in (4.1.2).

On the sphere, there is a $(+2)/(+2,+2)$ superghost background charge in heretoric/type II string that needs to be canceled, therefore we need to provide expressions for the kinematic parts of the vertex operators in the (-1) picture:

$$V^{(L)}_{(-1)}(z) = \varepsilon_\mu(k)\,e^{-\phi}\psi^\mu\,. \tag{4.2.6}$$

The complete vertex operators for gluons or gravitons (4.2.3) are obtained by plugging together the pieces that we described, following tab. 4.1.

The essential point of the discussion is that the correlation function (4.2.2) can be split off as a product of a holomorphic and of an anti-holomorphic correlator thanks to the "canceled propagator argument". As explained in the classical reference [60, sec. 6.6], the argument is an analytic continuation which makes sure that Wick contractions between holomorphic and anti-holomorphic operators

$$\langle\partial X(z,\bar z)\bar\partial X(w,\bar w)\rangle = -\alpha'\pi\delta^{(2)}(z-w)\,, \tag{4.2.7}$$

provide only vanishing contributions at tree-level. At loop-level, the left- and right-moving sectors couple via the zero-mode of the $X(z,\bar z)$ field and, as we saw (3.1.35b), the left-right contractions are necessary to produce correct amplitudes. These terms are the subject of the one-loop analysis of [PT4], which we review in sec. 4.3.

Kinematic sector We start with the open-string kinematic correlator of MSS, who proved that it can decomposed in terms of $(n-2)!$ basis elements

$$\langle V_1(z_1)\dots V_n(z_n)\rangle = \sum_{\sigma\in S_{n-2}} \frac{\tilde{\mathcal{K}}_\sigma}{z_{1,\sigma(2)}z_{\sigma(2),\sigma(3)}\cdots z_{\sigma(n-2),\sigma(n)}z_{\sigma(n),n-1}z_{n-1,1}}, \tag{4.2.8}$$

where S_{n-2} is the set of permutations of $n-2$ elements and $\tilde{\mathcal{K}}_\sigma$ are kinematical objects whose explicit expression do not concern us here.[45] The procedure used to reach this representation solely relies on worldsheet IBP's and fraction-by-part identities.

As emphasized above, the use of the canceled propagator argument grants us that performing the same IBP's on the chiral heterotic string correlator does not yield contact terms due to ∂ derivatives hitting $\bar{\partial}\mathcal{G}$ for instance.[46] Therefore, one can legitimately consider that the formula written above in eq. (4.2.8) is also the expression of the chiral closed string kinematical correlator in full generality.

Planar color sector Following [185], we write the *planar sector*[47] of the n-point correlator for the color currents from the basic OPE (4.2.5) as follows:

$$\langle J^{a_1}(z_1)\dots J^{a_N}(z_N)\rangle_{\text{plan.}} = \\ -2^{n-3}\sum_{\sigma\in S_{n-2}} \frac{f^{a_1 a_{\sigma(2)} c_1} f^{c_1 a_{\sigma(3)} c_2}\dots f^{c_{n-3} a_{\sigma(n)} a_{n-1}}}{z_{1,\sigma(2)}z_{\sigma(2),\sigma(3)}\cdots z_{\sigma(n-2),\sigma(n)}z_{\sigma(n),n-1}z_{n-1,1}} \tag{4.2.9}$$

Pay attention to the special ordering of the last terms of the denominator; it is designed so that one obtains directly the $(n-2)!$ element of the MSS basis. The low-energy limit of these correlators was thoroughly described in [185], and proven to reproduce the color ordering usually produced by color ordering along the boundary of the open string disk.

Low energy limit Now we need to describe how the two sectors of the closed string are tied together by the field theory limit. In [PT4] we carried the explicit procedure at five points and gave details on how the 5-punctured sphere degenerates into thrice punctured spheres connected by long tubes. Here, we rather focus on the similarities between the field theory limit in open string and the one in closed string, at tree-level.

This procedure is by now well understood and the limit can be described by the following rule [181]. In the open string, a given gauge theory cubic diagram Γ receives contributions from the color-ordered amplitude $\mathcal{A}_n^{\text{open}}(1,\sigma(2),\dots,\sigma(n-$

[45] It was determined in terms of the $(n-2)$ elements $\mathcal{K}^l_\sigma$ of [167, eq. (3.5)] for $l=1,\dots,n-2$ and $\sigma\in S_{n-3}$. Here we implicitly relabeled these in terms of the elements of S_{n-2}.

[46] In the open string the IBP's on the boundary of the disk yield contact-terms which are also discarded by use of the canceled propagator argument.

[47] We decouple by hand the gravitational sector which creates non planar-corrections in heterotic string vacua.

$2), n, \sigma(n-1))$ only from the integrals

$$\mathcal{I}^O_{\rho,\sigma} = \int_{z_1=0<z_{\sigma(2)}<\cdots<z_{\sigma(n-1)}=1} \prod_{i=2}^{n-1} \mathrm{d}z_i \frac{\prod_{i<j}(z_{ij})^{-\alpha' k_i.k_j}}{z_{1,\rho(2)}z_{\rho(2),\rho(3)}\cdots z_{\rho(n-2),\rho(n)}z_{\rho(n),n-1}z_{n-1,1}} \tag{4.2.10}$$

where the ordering of ρ and σ are compatible with the cubic graph G under consideration, see the section 4 of [181] for details and precise meaning of the compatibility condition. We can then write:

$$\mathcal{I}^O_{\rho,\sigma} = \sum_{\Gamma|(\rho\wedge\sigma)} \frac{1}{s_\Gamma} + O(\alpha') \tag{4.2.11}$$

where the summation is performed over the set of cubic graphs Γ compatible with both σ and ρ and s_Γ is the product of kinematic invariants associated to the pole channels of Γ.

In closed string, we first consider a heterotic gauge-boson amplitude. The latter has to match the result obtained from the field theory limit of the open string amplitude. Therefore, if we select a particular color-ordering σ for the open-string, we can identify the corresponding terms in the heterotic string color correlator (4.2.9). Actually, only one of them does, precisely the one with the permutation σ. This is actually sufficient to see that the mechanism that describes the low energy limit of closed string amplitudes has to be the following one: the field theory limit of the integrals

$$\mathcal{I}^C_{\sigma,\rho} = \int \prod_{i=2}^{n-1} \mathrm{d}^2 z_i \prod_{i<j} |z_{ij}|^{-2\alpha' k_i.k_j} \Big(\frac{1}{z_{1,\rho(2)}z_{\rho(2),\rho(3)}\cdots z_{\rho(n-2),\rho(n)}z_{\rho(n),n-1}z_{n-1,1}} \times \frac{1}{\bar{z}_{1,\sigma(2)}\bar{z}_{\sigma(2),\sigma(3)}\cdots \bar{z}_{\sigma(n-2),\sigma(n)}\bar{z}_{\sigma(n),n-1}\bar{z}_{n-1,1}} \Big) \tag{4.2.12}$$

contribute to the set of cubic diagrams which are compatible (in the sense mentioned above) with ρ and σ. This gives the same formula as for the open string

$$\mathcal{I}^C_{\rho,\sigma} = \sum_{\Gamma|(\rho\wedge\sigma)} \frac{1}{s_\Gamma} + O(\alpha') \tag{4.2.13}$$

up to factors of 2π created by phase integration of the z_{ij}'s. A direct proof in the sense of [181] would require to work out the complete combinatorics. This could be done, though it does not appear necessary as we are simply describing generic features of these amplitudes.

The formula (4.2.13) can now be applied to more general amplitudes, as long as their chiral correlators are recast in the MSS representation:

$$\langle V_1^{(L)}(z_1)\dots V_n^{(L)}(z_n)\rangle = \sum_{\sigma\in S_{n-2}} \frac{a^L_\sigma}{z_{1,\sigma(2)}z_{\sigma(2),\sigma(3)}\cdots z_{\sigma(n-2),\sigma(n)}z_{\sigma(n),n-1}z_{n-1,1}} \tag{4.2.14a}$$

$$\langle V_1^{(R)}(z_1)\dots V_n^{(R)}(z_n)\rangle = \sum_{\sigma\in S_{n-2}} \frac{a^R_\sigma}{\bar z_{1,\sigma(2)}\bar z_{\sigma(2),\sigma(3)}\cdots \bar z_{\sigma(n-2),\sigma(n)}\bar z_{\sigma(n),n-1}\bar z_{n-1,1}} \tag{4.2.14b}$$

In these formulas, the $a^{(L/R)}$ variables are independent of the z_i and carry color or kinematical information, they write as tensorial products between the group structure constants f^{abc} or polarization ε_i and momenta k_j of the external states. The total contribution to a given graph Γ of the low energy limit of closed string amplitude made of these chiral correlators is found to be given by the following sum

$$\begin{aligned}\frac{N_\Gamma}{s_\Gamma} &= \lim_{\alpha'\to 0} \sum_{\rho,\sigma\in\{\sigma_1,\dots\sigma_p\}} \mathcal{I}^C_{\rho,\sigma} a^{(L)}_\rho a^{(R)}_\sigma \\ &= \frac{1}{s_\Gamma}\underbrace{\left(\sum_{\rho\in\{\sigma_1,\dots\sigma_p\}} a^{(L)}_\rho\right)}\times\underbrace{\left(\sum_{\sigma\in\{\sigma_1,\dots\sigma_p\}} a^{(R)}_\sigma\right)} \\ &= \frac{1}{s_\Gamma}\left(\quad n^{(L)}_\Gamma \quad\times\quad n^{(R)}_\Gamma\quad\right),\end{aligned} \tag{4.2.15}$$

where $\{\sigma_1,\dots\sigma_p\}$ is the set of permutations compatible with Γ. We see that the numerator of the graph N_Γ splits as a product of two numerators corresponding to each sector of the theory. Summing over all cubic graphs produces the total n-point field theory amplitude as:

$$\mathcal{A}^{\text{tree}}_n(L,R) = \sum_{\substack{\text{cubic graphs}\\ \Gamma_i}} \frac{n^{(L)}_{\Gamma_i} n^{(R)}_{\Gamma_i}}{s_{\Gamma_i}}\,, \tag{4.2.16}$$

where a global normalization factor of $(g_{YM})^{n-2}$ or $(\kappa_D/2)^{n-2}$ should be included according to what L and R vertex operators were chosen.[48]

This formula have been written without referring to the actual theories plugged in the left-moving and in the right-moving sector of the closed string, hence we have the possibility to choose the string theory we want, following the table 4.1. Therefore, $\mathcal{A}^{\text{tree}}_n(L,R)$ could be either a gauge theory amplitude, if for instance, we had been doing the computation in heterotic string where the ($L=$ col) and $n^{(L)}_{\Gamma_i} = c_{\Gamma_i}$ are color factors while ($R=$ kin) so that $n^{(R)}_{\Gamma_i} = n_{\Gamma_i}$ are kinematic factors. It could as well be a gravity amplitude if we had been

[48]We recall that $g_c = \kappa_D/2\pi = (\sqrt{\alpha'}/4\pi)g_{YM}$. The appearance of $(2\pi)^{n-2}$ factors is compensated in the final result in the field theory limit by phase integrations for the tropicalized z_i.

doing the computation in type II where ($L = R = \text{kin}$) so that both $n_{\Gamma_i}^{(L)}$ and $n_{\Gamma_i}^{(R)}$ are kinematic numerators. Another possibility would be to choose both $n_{\Gamma_i}^{(L)}$ and $n_{\Gamma_i}^{(R)}$ to be color factors, in which case $\mathcal{A}_n^{\text{tree}}(\text{col}, \text{col})$ corresponds to the scattering amplitude between n color cubic scalars. Note also that these relations do not depend on supersymmetry nor on spacetime dimension.

Finally, let us note that recently, Cachazo, He and Yuan proposed a new prescription to compute scalar, vector and gravity amplitudes at tree-level [186–188]. This prescription was elucidated from first principles by Mason and Skinner in [189], where a holomorphic worldsheet sigma model for the so-called "Ambitwistor strings" was demonstrated to produce the CHY prescription at tree-level.[49] The CHY prescription at tree-level is naturally a closed string type of construction, although there are no right movers, and the way by which color and kinematics are generated is very similar to the one that we reviewed in this section; the authors of [189] built "type II" and "heterotic" Ambitwistor string sigma models. In [190], formulas for type II Ambitwistor n-graviton and heterotic Ambistwistor string n-gluon amplitudes have been proposed. The properties of the gravity amplitude form a very interesting problem on its own. It is also important to understand to what extent the heterotic Ambitwistor string have to or can be engineered in order to produce *pure* $\mathcal{N} = 4$ Yang-Mills amplitudes at one-loop, where the couplings to $\mathcal{N} = 4$ gravity are suppressed. These are a traditional issue encountered in Witten's twistor string [191].

4.3 Towards a string theoretic understanding in loop amplitudes

At loop-level, the form of the amplitudes integrand depends on the spectrum of the theory. We already emphasized the simplicity of maximally supersymmetric Yang-Mills and gravity theories. This simplicity here turns out to be a problem in the sense that the one- and two-loop four-gluon and four-graviton amplitudes are *too* simple to obtain non-trivial insight on a stringy origin of the BCJ duality. The box numerators reduce to 1 at one-loop in SYM and maximal supergravity [59] and they are given by s, t, u and s^2, t^2, u^2 at two loops (result of [13, 96] which we discussed in eq. (2.3.1)). In addition, there are no triangles and the Jacobi identities 4.3 are satisfied without requiring any special loop momentum identities besides the trivial $1 - 1 = 0$ and $s - s = 0$.

To increase the complexity of the amplitudes, it is necessary to introduce a non-trivial dependence in the loop momentum. Considering the empirical power counting of eq. (3.1.37), this could be achieved in two ways; either by increasing the number of external particles, or by decreasing the level of supersymmetry. Five-point amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity were recently discussed in [192] in open and closed string. The appearance of

[49] Although the preprint [PT5] deals with this issues, as already emphasized it is not the intention of the author to discuss it in this text.

left-right mixing terms was observed to be crucial in the squaring behavior of the open string integrand to the closed string one. These terms are central in our one-loop analysis as well.

In [PT4], we investigated the reduction of supersymmetry and studied four-graviton amplitudes obtained from the symmetric product of two $\mathcal{N}=2$ SYM copies. We already discussed in eq. (3.1.31) that these constructions structurally produce matter couplings in the gravity theory. Both in our string theory construction and in our direct BCJ construction, the contribution of the ($\mathcal{N}=2$) vector-multiplet running in the loop is realized as

$$\mathcal{A}^{1-\text{loop}}_{(\mathcal{N}=2 \text{ vect.})} = \mathcal{A}^{1-\text{loop}}_{(\mathcal{N}=4 \text{ vect.})} - 2\,\mathcal{A}^{1-\text{loop}}_{(\mathcal{N}=2 \text{ hyper})} \tag{4.3.1}$$

in analogy with the similar relation in $\mathcal{N}=4$ gravity in eq. 3.1.33. It can be seen in tab. 1.1, that these identities are coherent with respect to the spectrum of the multiplets. This implies that the non-trivial loop momentum part of the integrands is described by the following product

$$(\mathcal{N}=2 \text{ hyper}) \times (\mathcal{N}=2 \text{ hyper}) = (\mathcal{N}=4 \text{ matter}) \tag{4.3.2}$$

which is therefore the important sector of the four-graviton amplitude on which we will focus from now on. Each of the hyper-multiplet copies will carry an ℓ^2 dependence in the loop momentum, respectively an $\dot{G}^2$ in the worldline integrand.

4.3.1 BCJ ansatz for ($\mathcal{N}=2$) hyper multiplets.

The ansatz that we used to find a BCJ satisfying representation of $\mathcal{N}=2$ gauge theory amplitudes is described in great detail in [PT4, sec.4]. The first constraint that we apply is our choice to start with two master boxes, corresponding to the (s,t) and (t,u) channels, the (s,u) channel being obtained from the (s,t) box by the exchange of the legs $3 \leftrightarrow 4$.

The second physical requirement was to stick as much as possible to our string theoretic construction which in particular has no triangle nor bubble integrals in the field theory limit. Since the Jacobi identities between boxes force triangles to be present, the best we could do was to require all bubbles to vanish. To our surprise, this turned out to be sufficient to force the triangles to vanish at the integrated level, despite a non-trivial loop-momentum numerator structure.

In total, after solving all the D-dimensional unitarity cuts constraints on the ansatz, only two free coefficients remain from the original definition of the ansatz, called α and β in the paper. They parametrize residual generalized gauge invariance in our representation. The total number of diagrams is therefore 9; three boxes and six triangles. Their explicit expressions may be found in [PT4, eqs. (4.20)-(4.21)]. As expected from power counting, the box numerators of these $\mathcal{N}=2$ gauge theory amplitudes have degree $(4-\mathcal{N})=2$ in the loop momentum. In addition, we provide in [PT4, appendix C] a short explicit computation for the vanishing of a particular gauge theory triangle after integration. An important additional feature of our ansatz, generally present in other ansatzes as well [172], is that it requires the inclusion of parity-odd terms $i\epsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho \ell^\sigma$ for consistency. In gauge theory amplitudes, they vanish due to Lorentz invariance since the vector $i\epsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho$ is orthogonal to any of the momenta of the scattered states k_i^σ. Combined with the triangles, these terms are invisible to the string theory amplitude because they vanish when the loop momentum is integrated. An essential feature of the BCJ double-copy is that these terms do contribute to the gravity amplitudes after squaring.

4.3.2 String theoretic intuition

We proposed in [PT4] a possible origin for this mechanism in string theory. Our physical intuition is based on the fact that in string theory gravity amplitudes possess additional terms coming from Wick contractions between the left- and right-moving sectors. Furthermore, these left-right moving contractions are absent in gauge theory amplitudes in heterotic string because the two CFT's (color and kinematical) have different target spaces and do not communicate. Therefore we naturally expect that these additional terms in BCJ and worldline gravity amplitudes have to be related, this is indeed what was shown in [PT4].

For illustrative purposes, we display below the form of the one-loop amplitudes in gauge theory and gravity as obtained from the generic four-point string theory amplitude in eq. (3.1.10) with the vertex operators described along the text:

$$\mathcal{A}^{\text{1-loop}}_{\text{gauge}} = \int_0^\infty \frac{\mathrm{d}T}{T^{d/2-3}} \int_0^1 \mathrm{d}^3u \cdot \left(W^{(L,\,\text{kin})}\, W^{(R,\,\text{col})}\right) \cdot e^{-TQ}\,, \tag{4.3.3a}$$

$$\mathcal{M}^{\text{1-loop}}_{\text{gravity}} = \int_0^\infty \frac{\mathrm{d}T}{T^{d/2-3}} \int_0^1 \mathrm{d}^3u \cdot \left(W^{(L,\,\text{kin})}\, W^{(R,\,\text{kin})} + W^{(L-R,\,\text{kin})}\right) \cdot e^{-TQ}\,. \tag{4.3.3b}$$

where the transparent abbreviations col and kin follow from the terminology used in the previous section. The form of the gravity amplitude has been discussed before, where the kinematic numerators $W^{(\cdot,\text{kin})}$ were described as polynomials in $\dot{G}$ and $\ddot{G}$. On the other hand, the form of the gauge theory worldline amplitude deserves a comment. The presence of a current algebra in the left-moving sector of the gauge boson heterotic-string CFT not only prevents mixed contractions, but also produces color ordered amplitudes, so that $W^{(R,\,\text{col})}$ writes

$$W^{(R,\,\text{col})} = \sum_{\sigma\in S_{n-1}} \mathrm{Tr}\,(T^{a_{\sigma(1)}}\dots T^{a_{\sigma(n-1)}}T^{a_n})H(u_{\sigma(1)} < \cdots < u_{\sigma(n-1)} < u_n)\,, \tag{4.3.4}$$

where H is a boolean Heaviside step function. This was demonstrated by Bern and Kosower in [61–64], where they proved that $1/\bar{q}$ residue identities tie particular combinations of the color factors to a given ordering of the external legs along the loop.[50]

[50] In open string gauge theory amplitudes, color-ordering naturally follows from ordering

It should be recalled now that the left-right mixing contractions present in the worldline integrand $W^{(L-R,\,\mathrm{kin})}$ descend from string theory contractions such as $\langle \partial X(z,\bar z)\bar\partial X(w,\bar w)\rangle$ as in eq. (3.1.11). In the field theory limit, they solely provide a $1/T$ factor, since the $\delta^{(2)}$-function drops out of the amplitude by the canceled propagator argument just like at tree-level:

$$\langle \partial X(z,\bar w)\bar\partial X(z,\bar w)\rangle \underset{\alpha'\to 0}{\longrightarrow} -\frac{2}{T} \tag{4.3.5}$$

up to a global factor of α'^2 required for dimensionality. We give below the explicit worldline numerators for the ($\mathcal{N}=2$) hyper multiplet[51] and also recall the form of the symmetric worldline integrand for the ($\mathcal{N}=4$) matter multiplets

$$\begin{aligned} W_{(\mathcal{N}=2),\ \mathrm{hyper}} &= W_3\,,\\ W_{(\mathcal{N}=4),\ \mathrm{matt.}} &= W_3^2 + 1/2 W_2\,, \end{aligned} \tag{4.3.6}$$

where the worldline integrands W_2 and W_3 were defined in (3.1.36).

4.3.3 Comparing the integrands

In [PT4], we carried the comparison of the integrands coming from the BCJ construction to the worldline one by turning the loop momentum representation to a Schwinger proper time representation.[52] This procedure was already sketched in chap. 2, eq. (2.2.27), when we needed to illustrate the generic form of a worldline integrand in terms of more common Feynman graphs quantities. We defined $\langle n\rangle$ to be the result of loop-momentum Gaussian-integration of a given numerator $n(\ell)$ after exponentiating the propagators. For a detailed account at one-loop, the reader is referred to the section 6.1 of [PT4]. For definiteness, let us simply reproduce here the defining equation for the bracket notation $\langle n\rangle$:

$$\int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{n(\ell)}{\ell^2(\ell-k_1)^2\ldots(\ell-\sum_{i=1}^{n-1}k_i)^2} = \frac{(-1)^n i}{(4\pi)^{D/2}}\int_0^\infty \frac{\mathrm{d}T}{T^{\frac{D}{2}-(n-1)}}\int\prod_{i=1}^{n-1}\mathrm{d}u_i\langle\mathbf{n}\rangle e^{-TQ} \tag{4.3.7}$$

which appears in boldface for readability in this equation. The left-hand side of this formula is a n-leg ($n=3,4$ here) one-loop Feynman integral in momentum space while the right-hand side is its Schwinger proper time representation. We recall that the u_i parameters are rescaled (see eq. (3.1.14)) so that they belong to $[0,1]$. Their ordering along the worldloop corresponds to the ordering indicated by the Feynman propagators in the left-hand side.

of the external states along the boundary of the annulus.

[51] Similar computations as these performed in the gravity amplitudes can be performed to derive this polynomial in $\mathcal{N}=2$ orbifolds of the heterotic string, in which case one should make sure to decouple the gravitational multiplets by hand. Another possibility is to use $W_{\mathcal{N}=4,\mathrm{vect}}=1$ in the identity ($\mathcal{N}=2$, hyper) $\times$ ($\mathcal{N}=4$, vector) $=$ ($\mathcal{N}=6$, spin-3/2) to obtain $W_{\mathcal{N}=2,\mathrm{hyper}} = W_{\mathcal{N}=6,\mathrm{spin}\text{-}3/2}$.

[52] In principle, it would have been desirable to perform the inverse procedure. However we faced technical obstacles in doing so, because of the quadratic nature of the gauge theory loop-momentum polynomials. Furthermore, the absence of triangles in string theory was also a severe issue to match the BCJ loop momentum triangles.

Gauge theory The first step of the analysis is to compare the gauge theory box integrand $\langle n_{\text{box}} \rangle$ obtained from the BCJ procedure to the string based numerator W_3.[53] We observe matching of the two quantities only up to a new total derivative that we call δW_3:

$$\langle n_{\text{box}} \rangle = W_3 + \delta W_3 \,. \tag{4.3.8}$$

This δW_3 integrates separately to zero in each color ordered sector of the amplitude. Moreover, it is sensitive to the subset of generalized gauge invariance left-over from solving the unitarity cut-constraints for the ansatz as it depends on α and β. A natural interpretation for this term is that it carries some information from the BCJ representation to the string integrand and indicates that the correct BCJ representation in string theory is not W_3 but $W_3 + \delta W_3$.

From our experience of the MSS procedure at tree-level, we would expect the addition of this total derivative term to be the result of worldsheet integration by part. However, in [PT4] we argued that this is not the case; $W_3 + \delta W_3$ cannot be the result of any chain of IBP's. The argument is based on a rewriting δW_3 as a worldline polynomial in the derivatives of the Green's function,[54] followed by the observation that this polynomial cannot be integrated away because of the presence of squares $\dot{G}_{ij}^2$ not paired with the required $\ddot{G}_{ij}$ which would make them originating from $\partial_i(\dot{G}_{ij} e^{-TQ})$.[55] The reason why there are no room for such terms as $\ddot{G}$ in δW_3 is related to the form of our box numerators, whose quadratic part in the loop-momentum turns out to be traceless. Ultimately, this is again a consequence of our restriction to discard bubble integrals in our gauge theory ansatz.

The first conclusion of this gauge theory analysis is that the BCJ representation is visible at the integrand level in string theory, as shows the necessity to select a particular representation. The second conclusion is that, contrary to the intuition from the MSS procedure, there seem to exist particular BCJ representations which cannot be reached directly from string theory, or at least not with solely "naive" IBP's.

Gravity At the gravity level, we compare the BCJ double-copy and string-based integrated results. They give schematically:

$$\int \sum \langle n_{\text{box}}^2 \rangle + \sum \langle n_{\text{tri}}^2 \rangle = \int W_3^2 + 1/2 W_2 \,. \tag{4.3.9}$$

The physical intuition that we have been following so far tells us that loop momentum total derivatives in the BCJ representation in gauge theory, which contribute after squaring, should match the new left-right mixing term W_2 arising in the string-based gravity amplitude. Therefore, we expect the triangles $\langle n_{\text{tri}}^2 \rangle$ and the parity-odd terms present in $\langle n_{\text{box}}^2 \rangle$ and $\langle n_{\text{tri}}^2 \rangle$ to be related to W_2.

[53] We recall that the gauge theory triangle integrand vanish once the loop momentum is integrated, in other words we have $\langle n_{\text{tri}} \rangle = 0$ for all BCJ triangles.

[54] The complete expression may be found in appendix D of [PT4].

[55] See the discussion above and below (6.24) in [PT4].

To understand this relation, it is necessary to use our knowledge gained in the analysis of the gauge theory integrands to first relate $\langle n_{\text{box}}^2\rangle$ to W_3^2. Since we already argued that no IBP procedure may transform W_3 to $\langle n_{\text{box}}\rangle$, the best we can do is to introduce and remove by hand δW_3 in (4.3.9), which transforms W_3 to $W_3 + \delta W_3 = \langle n_{\text{box}}\rangle$ while the W_2 is modified to turn $W_2 \to W_2 + \delta W_2$ with

$$\delta W_2 = -2(2\delta W_3 W_3 + W_3^2)\,. \tag{4.3.10}$$

Contrary to δW_3, this new term *is not* a total derivative. This is expected, since its integral does not vanish. In total we obtain

$$\boxed{\int W_2 + \delta W_2 = \int \sum \langle n_{\text{tri}}^2\rangle + \left(\langle n_{\text{box}}^2\rangle - \langle n_{\text{box}}\rangle^2\right)} \tag{4.3.11}$$

An interesting combination, $(\langle n_{\text{box}}^2\rangle - \langle n_{\text{box}}\rangle^2))$, appears in the right-hand side of the previous equation. This term is computed in detail in subsection 6.3.2 of [PT4], by Gaussian integration of the loop momentum. In particular it contains contribution coming from the parity-odd terms and other total derivatives. However, its appearance is more generic than this and actually signals the non-commutativity of the squaring operation in loop momentum space and in Schwinger proper time space. Therefore, any string theory procedure supposed to explain the origin of the BCJ double-copy should elucidate the nature of these “square-correcting terms”.

The difficulties caused by the non-IBP nature of δW_3 and δW_2 prevented us from pushing the quantitative analysis much further. However, in our conclusive remarks below we provide a qualitative statement based on the fact that the square-correcting terms are always of order $1/T$ at least (this can be proven by direct Gaussian integration).

Before, let us make one more comment. So far we did not describe the worldline properties of δW_2 and δW_3, besides explaining that we could rewrite δW_3 as a polynomial of in the derivatives of the worldline propagator. This implies that the same can be done for δW_2. By doing so, we mean that these polynomials, δW_2 and δW_3, are well defined worldline quantities and we are implicitly pretending that they descend from certain string theoretic ancestors, obtained by turning the G’s for $\mathcal{G}$’s. However, nothing grants us from the start that the corresponding $\delta\mathcal{W}_2$ and $\delta\mathcal{W}_3$ would not produce triangles or bubbles in the field theory limit due to vertex operator colliding as in eq. (2.2.45). This would spoil a correct worldline interpretation for these corrections. Hence we had to carefully check this criterion for both polynomials, which they turn out to pass; in [PT4], this property was referred to as the *string-ancestor-gives-no-triangles* criterion. The conclusion of this paragraph gives strength to interpreting the δW’s as “stringy” reactions to the BCJ change of parametrization in gauge an gravity amplitudes.

Conclusive remarks We can now conclude. The formula eq. (4.3.11) illustrates that the modified left-right moving contractions, $W_2+\delta W_2$, are related to two terms in field theory; the BCJ triangles squared and the square-correcting terms.

Noting that the square correcting terms do contain in particular the squares of the parity-odd terms, we are lead to our first conclusion, which confirms our physical intuition; the left-right mixing contractions in string theory, modified by the BCJ representation, account for the need to include total derivatives in the BCJ representation of gauge theory amplitudes.

The second important conclusion is linked to the change of representation that we found, which we argued to be a non-IBP type of modification. At tree-level, the MSS paradigm consists in performing integrations by parts on the gauge theory integrands to put them in a particular representation (see eq. (4.2.8)). At one-loop, integrations-by-part produce additional left-right mixing contractions when ∂ derivatives hit $\bar{\partial}\mathcal{G}$ terms, which eventually give rise to worldline terms with $1/T$ factors (see eq. (4.3.5)). In view of our previous comment on the $1/T$ order of the square-correcting terms, it is natural to expect that these terms actually indicate missing worldsheet IBP's in the term $W_2 + \delta W_2$. Therefore, we face a paradox; on the one hand, no IBP can be done to produce the δW's, on the other hand the final result seem to lack IBP's.

A possible way out might lie in the definition of the ansatz itself. More precisely, the issues might be caused by a mutual incompatibility of the gauge choice in string theory producing the worldline integrand W_3 and forbidding triangle/bubble-like contributions with the choice of an ansatz constrained by discarding all bubbles, thereby producing BCJ triangles as total derivatives only. Put differently, the absence of triangle contributions in the string-based computation that lead us to consequently restrict the full generalized gauge invariance is possibly not the most natural thing to do from string theory viewpoint on the BCJ double-copy. Then what have we learned ? It seems that string theory is not compatible with certain too stringent restrictions of generalized gauge invariance. A more general quantitative analysis of this issue will certainly give interesting results on which of BCJ-ansatzes are natural from string theory and which are not, hopefully helping to find new ansatzes.

Chapter 5

Outlook

One of the aims of this manuscript was to draw a coherent scheme in the work of the author during his years of PhD. Their remain open questions after these works, in addition to the new ones that were raised. I would like to describe a few of them now. As they are related to several chapters at the same time, there is not point anymore in sectioning the text according to the previous chapters.

Role of the $U(1)$ anomaly We emphasized in the text that half-maximal supergravity has a $U(1)$ anomalous symmetry of the axio-dilaton parametrizing the coset $SU(1,1)/U(1)$ [54]. The direct computation of the four-loop $\nabla^2 R^4$ divergence in $D = 4 - 2\epsilon$ dimensions of [130] using the double-copy $(\mathcal{N} = 0) \times (\mathcal{N} = 4)$ also shows traces of this anomalous behavior, according to the authors of this work. Let us reproduce the amplitude here in order to recapitulate their reasoning:

$$\mathcal{M}_{n_v}^{\text{four-loop}}\Big|_{\text{div.}} = \frac{(\kappa_D/2)^{10}}{(4\pi)^8} \frac{(n_v + 2)}{2304} \left[\frac{6(n_v + 2)n_v}{\epsilon^2} + \frac{(n_v + 2)(3n_v + 4) - 96(22 - n_v)\zeta_3}{\epsilon} \right] \mathcal{T} \,,$$

where $\mathcal{T}$ encodes the polarization dependence of the amplitude in a covariant manner. The $(n_v+2)\zeta_3$ contribution is the important term here. On one hand, $(n_v + 2)$ was argued to be typical of anomalous one-loop amplitudes [129], on the other hand ζ_3 is a 3-loop object, therefore the four-loop divergence carried by $(n_v + 2)\zeta_3$ does seem to be caused by the anomaly. It would be really interesting to investigate this issue further, below we describe possible topics of research related to it.

Extract exactly the coupling of R^4 and $\nabla^2 R^4$ in the CHL heterotic string action ? A computation that would shed light in this direction is to determine the exact value of the R^4 and $\nabla^2 R^4$ couplings in the effective action of CHL models. The program in $\mathcal{N} = 8$ led to major advances both in physics and in mathematics, and it is very reasonable to expect that the similar program in $\mathcal{N} = 4$ would imply the discovery of new kind of automorphic forms for orthogonal groups.

What is the significance of the $N = 23$ CHL orbifold ? This point is more speculative. We mentioned that an $N = 23$ CHL model would decouple totally the matter fields, hence producing pure half-maximal supergravity from the start. The Mathieu moonshine program seems to indicate that there may exist such a model, as a consequence of a putative $\mathbb{M}_{24}$ fundamental symmetry of ... something. At the moment, it is not clear what theory the Mathieu group $\mathbb{M}_{24}$ could be a symmetry group of. It is known however that it cannot be the symmetry group of K_3 sigma models, preventing naive interpretations of this sort [127]. Maybe uncovering deeper aspects of these connexions may lead to powerful group theoretical arguments on the low energy effective action of pure half-maximal supergravity ?

Build some $4 \leq \mathcal{N} < 8$ orbifolds models in pure spinor superstring and extract non-renormalization theorems via zero-mode counting ? Another option to understand the role of the $U(1)$ anomaly, suggested by the authors of [130], would be to perform similar type of analysis in $\mathcal{N} \geq 5$ supergravities, where the anomalous symmetry is not present. From the superstring point of a view, such an analysis would most easily be performed by constructing asymmetric orbifolds models in the pure spinor superstring and perform systematically the zero-mode counting in the lines of [17].

Extract exactly the three-loop four-graviton amplitude in type II ? Going to the tropical limit program now, a very important computation to do is to extract explicitly the worldline numerators for the three-loop computation in type II of [72]. In addition to the intrinsic interest of such a computation, it may help to understand the apparent paradox between the supermoduli space non-projectedness issues in the RNS formalism and the bosonic moduli space integration of pure spinor formalism.

Extract exactly the two-loop four-graviton amplitude in CHL models of heterotic string ? The genus two case is really the turning point in terms of the technical machinery involved in extracting the tropical limit of string amplitudes formulated as integrals over $\mathcal{M}_{g,n}$. Therefore, developing the tropical limit technology enough to be able to extract the complete form of the worldline integrand of the two-loop heterotic string amplitude would settle the last subtleties with this aspect of the $\alpha' \to 0$ limit (at least in the non-analytic domains).

Towards a super-tropical geometry ? The analysis of [86–91] has shown that the non-projectedness of $\mathfrak{M}_{g,n}$ implies that the IR behavior of RNS superstring theory is naturally described by means of super-dual-graphs which characterize the holomorphic degenerations of the super-Riemann surfaces.

They basically account for what kind of states, NSNS, RNS, NSR and RR are exchanged through the pinching necks. The corresponding super-dual-graphs in type II for instance are then built out of the following vertices and their

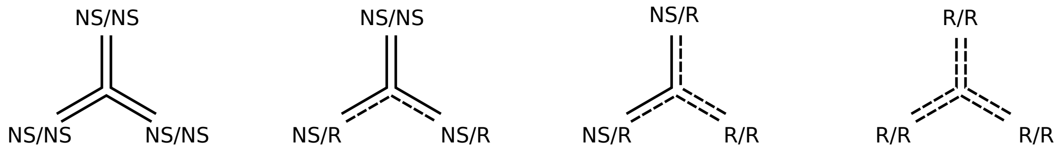


Figure 5.1: "Superworldline Feynman rules" in type II.

weighted n-point generalization. It would be interesting and certainly helpful to formulate in more details this super worldline picture for arbitrary RNS string theory amplitudes. Comparison with the pure spinor *worldline* formalism of [26] may then help to understand the connexions between the various perturbative formalisms in string theory.

Double-copy; find a constructive way at one-loop ? The question of understanding the nature of the generalized gauge invariance in string theory is conceptually important, as it may be used as a guideline for the direct ansatz approaches. Another result that hopefully may follow from a string theory analysis would be a procedure to derive BCJ numerators at loop level from first principles, in the lines of the tree-level MSS construction.

Is there any string theoretic understanding of the difficulties at five loop ? In the paradigm where we consider string theory as a natural framework where to understand the BCJ duality, it would be natural to assume that the supermoduli space discussion of [86–91] may have an impact on the BCJ duality, for instance by involving variations of the Jacobi identities ? A way to probe this statement would be to identify an amplitude in the RNS formalism that has to involve the super-graph picture in the low energy limit, and investigate if there are signs of a breakdown or alteration of the duality or of the double-copy.

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