

The formalisms of quantum mechanics: an introduction

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PRELIMINARY VERSION

Updated and slightly enlarged – but still preliminary – version of the lecture notes “A short introduction to the quantum formalism[s]” that I gave in 2012 at IPhT , and available at arXiv:1211.5627.

These notes present an introductory, but hopefully coherent, view of the main formalizations of quantum mechanics, of their interrelations and of their common physical underpinnings: causality, reversibility and locality/separability. The approaches covered are mainly: (i) the canonical formalism; (ii) the algebraic formalism; (iii) the quantum logic formulation. Other subjects: quantum information approaches, quantum correlations, contextuality and non-locality issues (Bell’s inequalities, hidden-variable models), quantum measurements, interpretations of quantum mechanics and alternate quantum theories, quantum gravity, are only very briefly and superficially discussed.

Most of the material is not new, but is presented in an original, homogeneous and hopefully not technical or abstract way. I try to define simply all the mathematical concepts used and to justify them physically. Some emphasis is put on the concept of reversibility.

These notes should be accessible to young physicists (graduate level) with a good knowledge of the standard formalism of quantum mechanics, and hopefully be of some interest for theoretical physicists and mathematicians.

These notes do not try to cover the historical and philosophical aspects of quantum physics.

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