

An introduction to the non-perturbative renormalization group

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A comprehensive review on the subject of the course is in preparation by the lecturer.

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“Il y a un autre monde mais il est dans celui-ci.”

P. Eluard

I. INTRODUCTION

The subject of this course is the Non-Perturbative Renormalization Group (NPRG). Wilson was the pioneer in this field with his Renormalization Group (RG) ideas. Although he devised a conceptual framework that was valid at the non-perturbative level — and which he actually applied as such to the Kondo problem —, his RG calculation techniques were mainly used in a perturbative framework. The modern form that we will be studying is mainly due to Wetterich¹. Other nomenclatures to refer to the NPRG in the literature include the names ‘exact RG’ (ERG) or ‘functional RG’ (FRG).

In its full form the NPRG equation is an exact description, hence, it is applicable without conceptual limitations. In practice, however, truncations are inevitable. They have to be constructed such that the dominating effects of the physics at hand are captured well. However, by this freedom in choosing truncations the flow equations maintains its flexibility, such that a variety of fields has been covered so far. In particular in statistical systems or quantum field theories with (second order) phase transitions the flow equation is a powerful method, as comparatively simple approximations suffice to obtain quantitatively satisfying results. Moreover, the NPRG has been applied to as complex systems as quantum chromodynamics or quantum gravity, where in the latter, Weinberg’s idea of asymptotic safety, meaning the existence of a non-trivial ultraviolet fixed-point, is realised.

II. STATISTICAL MECHANICS – $O(N)$ MODELS

We present equilibrium statistical models defined on the lattice for various ferromagnetic systems with a $O(N)$ symmetry. They are related to euclidean field theories

¹ U. Ellwanger and, later, T. Morris also developed closely related ideas.

based on the same $O(N)$ symmetry², and share with them *universal* properties.

For systems at thermal equilibrium, the stationary probability distribution is entirely determined by the hamiltonian H of the system and is Gibbsian: $P_{\text{eq}}(\phi) \propto \exp(-H(\phi)/kT)$.

A. The $O(N)$ models on the lattice

Let us present the ferromagnetic $O(N)$ models defined on the lattice (d -dimensional, with lattice spacing a).

$$\begin{cases} O(3) \rightarrow \text{Heisenberg model} \\ O(2) \rightarrow \text{XY model} \\ O(1) \approx \mathcal{Z}_2 \rightarrow \text{Ising model} \end{cases}$$

We consider classical spins \mathbf{S}_i defined on lattice sites i and having N components. The simplest $O(N)$ -invariant hamiltonian is

$$H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \quad (1)$$

where $J > 0$ (ferromagnetic interaction), $\mathbf{S}_i^2 = 1$ and $\langle ij \rangle$ means summation on nearest neighbor pairs of spins. The coupling to an external source \mathbf{h} (magnetic field) contributes to the hamiltonian by a term:

$$- \sum_i \mathbf{h}_i \cdot \mathbf{S}_i. \quad (2)$$

The partition function is

$$\mathcal{Z}[\mathbf{h}] = \sum_{\{\mathbf{S}_i\}} \exp\{-H + \sum_i \mathbf{h}_i \cdot \mathbf{S}_i\}, \quad (3)$$

where the $\beta = 1/k_B T$ coefficient in front of the hamiltonian has been absorbed into the definition of H for simplicity; the symbolic sum on $\{\mathbf{S}_i\}$ adds up all spin configurations for all lattice sites (it may contain integrals).

We can define an average magnetization per lattice site,

$$\mathbf{M} = \langle \mathbf{S}_i \rangle, \quad (4)$$

which plays the role of an *order parameter* in this system, characterizing the phase transition (see below). For now, we will assume the thermodynamic limit³ in the following, because strictly-speaking there is no phase transition for a system with a finite number of degrees of freedom.

² Field theories whose potential terms are powers of φ^2 .

³ infinite number of degrees of freedom associated to an infinite volume; not the continuum limit.

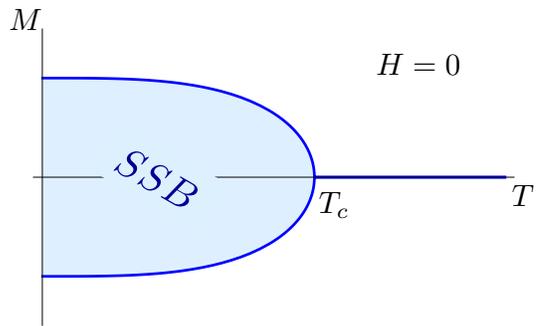


FIG. 1 Diagram magnetization-temperature. For T larger than the critical temperature T_c , $M = 0$ and the system is in its symmetric phase while for $T < T_c$, $M \neq 0$ and the system is in its spontaneously symmetry broken (SSB) phase.

There are two competing effects as regards the value of the average magnetization: the kinetic / vibrational energy associated to the temperature and the tendency of spins to align among themselves, given by the attractive spin-coupling term in the hamiltonian. At high temperature, the thermal disordering effect dominates, and we have a symmetrical phase where the average magnetization is zero. At small temperature, the spin-couplings dominate and there is a spontaneous breaking of the symmetry $O(N)$ to $O(N-1)$: spins align preferentially in one direction. These two distinct phases are separated by a second order phase transition with the critical temperature T_c of the order of $kT_c \sim J$. Figure 1 is a sketch of the behaviour of M (magnitude of the magnetization) as a function of temperature.

The *correlation function*⁴ $G_{\alpha\beta}^{(2)}(\mathbf{r}_i - \mathbf{r}_j) \equiv \langle S_i^\alpha S_j^\beta \rangle$ depends on the distance between the lattice sites i, j . For generic $T > T_c$, the correlation function shows an exponentially-decaying asymptotic behavior:

$$G_{\alpha\beta}^{(2)}(\mathbf{r}) \underset{|\mathbf{r}| \gg a}{\sim} \frac{e^{-|\mathbf{r}|/\xi(T)}}{|\mathbf{r}|^\#} \quad (5)$$

with $\xi(T)$, the *correlation length*⁵. For generic temperatures much larger than T_c , $\xi(T) \sim a$ and the system is weakly correlated. It behaves as clusters of correlated degrees of freedom of typical size ξ that are almost uncorrelated from each other. [Thus, the Central Limit

⁴ The *connected correlation function*, that we will see later on, is equal to the correlation function at zero external field and for temperatures above T_c .

⁵ At distances of the order of the lattice spacing, the system does not show rotation invariance ($O(d)$ -symmetry) and the correlation length depends in general on the direction of \mathbf{r} . When the system is close to the phase transition, $\xi \gg a$ and at large distances rotation invariance is effectively restored: The correlation length is no longer anisotropic and one can speak of a unique, direction-independent, correlation length.

Theorem (CLT) applies to the distributions of mean values: for example, the magnetization mode: $\sum_i S_i$ is gaussian distributed at large temperature (and field theory is useless)[could we refer to the average magnetization? what's the statement about field theory?, we have not introduced it yet, anyway]. On the other hand, for $T \simeq T_c$, the correlation length diverges as a power law:

$$\xi(T) \underset{T \rightarrow T_c^+}{\sim} (T - T_c)^\nu \quad (6)$$

with ν being a *critical exponent*. The system is in this case *strongly correlated* and the two-point correlation function decreases algebraically so that, up to very large distances, the spins cannot be considered independent. For $T < T_c$, and for $N > 1$, $N - 1$ massless Goldstone bosons exist and the system remains strongly correlated even far away from T_c . For $N = 1$, the correlation length in the low temperature phase is finite and it is only around T_c that the system is strongly correlated.

B. The $O(N)$ models in the continuum

Two models obtained in the continuum from the lattice $O(N)$ models will be interesting in the following: the φ^4 and the non-linear sigma models. These two models belong to the same *universality class* (see section III below) as the lattice model. They are formulated in terms of either a N -component real field $\varphi = (\varphi_1, \dots, \varphi_N)$ or a $N - 1$ -component real field π both defined in the continuum.

1. The Non-linear sigma (NL σ) model.

Starting from (3), and using delta functions, we can rewrite the sum over the configurations of the spins as integrals over unconstrained real variables:

$$\mathcal{Z}[\mathbf{h}] = \prod_i \left(\int d\varphi_i \delta(\varphi_i^2 - 1) \right) \exp \left\{ -\frac{H}{T} + \sum_j \mathbf{h}_j \cdot \varphi_j \right\}. \quad (7)$$

Here, we have now called $\mathbf{S}_i \rightarrow \varphi_i$. Furthermore, we have trivially redefined H such that the temperature is shown explicitly in order to elucidate the low-temperature expansion below. Now, we can use the relation $\varphi_i \cdot \varphi_j = -1/2(\varphi_i - \varphi_j)^2 + cte$ to take the continuum limit, which leads to kinetic terms in the action,

$$(\varphi_i - \varphi_j)^2 \underset{a \rightarrow 0}{\rightsquigarrow} (\partial_\mu \varphi)^2, \quad (8)$$

hence,

$$\mathcal{Z}[\mathbf{h}] = \int \mathcal{D}[\varphi] \delta(\varphi^2(\mathbf{r}) - 1) \exp \left\{ -\frac{1}{2T} \int_{\mathbf{r}} (\partial_\mu \varphi)^2 + \int_{\mathbf{r}} \mathbf{h} \cdot \varphi \right\}, \quad (9)$$

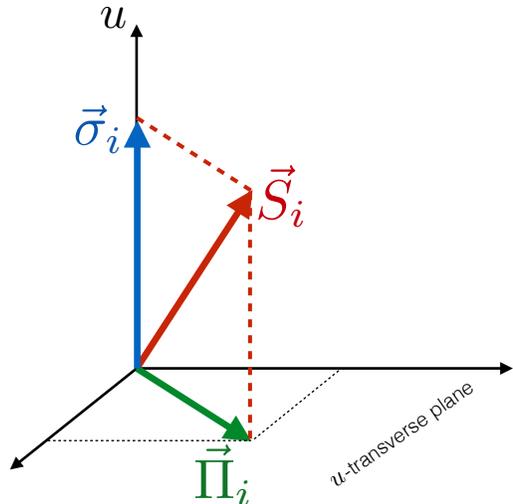


FIG. 2 Decomposition of the spin vector along the magnetization direction. The $N - 1$ -dimensional π vector corresponds to the Goldstone modes.

where

$$\int_{\mathbf{r}} = \int d^d \mathbf{r}. \quad (10)$$

In the low temperature phase of the $O(N > 1)$ model in $2d$ spontaneous symmetry breaking occurs ($O(N) \rightarrow O(N - 1)$). Here, the $\varphi(\mathbf{r})$ field can be decomposed into a longitudinal component $\sigma(\mathbf{r})$, that is, a component parallel to the direction \mathbf{u} of the spontaneous magnetization and a transverse component $\pi(\mathbf{r})$ such that $\mathbf{u} \cdot \pi(\mathbf{r}) = 0$ (which implies that $\pi(\mathbf{r})$ is a $(N - 1)$ -component field):

$$\varphi(\mathbf{r}) = \sigma(\mathbf{r})\mathbf{u} + \pi(\mathbf{r}) \quad \text{with} \quad \sigma^2(\mathbf{r}) + \pi^2(\mathbf{r}) = 1. \quad (11)$$

The projection is sketched in figure 2.

At low temperature and in the broken phase, $\langle \pi^2(\mathbf{r}) \rangle$ is small and, after the rescaling $\varphi(\mathbf{r}) \rightarrow \sqrt{T}\varphi(\mathbf{r})$, one can take $\sigma(\mathbf{r}) = \sqrt{1/T - \pi^2(\mathbf{r})}$ and neglect the constraint $|\pi(\mathbf{r})| < 1$. In the presence of a magnetic field, \mathbf{u} and \mathbf{h} are parallel and the partition function reads:⁶

$$\mathcal{Z}[\mathbf{h}] = \int \mathcal{D}[\pi] \exp \left\{ -H[\pi] + \int_{\mathbf{r}} h\sigma \right\}, \quad (12)$$

with the hamiltonian $H[\pi]$ of the π fields,

$$H[\pi] = \frac{1}{2} \int_{\mathbf{r}} \left[(\partial_\mu \pi)^2 + T \frac{(\pi \cdot \partial_\mu \pi)^2}{1 - T\pi^2} \right]. \quad (13)$$

⁶ Let us notice that the validity of the NL σ model goes beyond the low temperature phase contrary to what could be believed from the derivation given here.

2. The φ^4 model.

We start from (3), and its re-writtten (equivalent) form in the precedent section (7). We take again the continuum limit to obtain (9). Now, we change the delta into a smoothed (double) Gaussian term:

$$\delta(\varphi^2 - 1) \rightarrow \exp\{-\lambda(\varphi^2 - 1)^2\}. \quad (14)$$

The delta being a functional delta we obtain in this replacement a product at all points \mathbf{r} of a double gaussian which means an exponential of the sum (in fact, of the integral) of $-\lambda(\varphi^2 - 1)^2$. With this replacement, we obtain the φ^4 model,

$$H[\varphi] = \int d^d r \left\{ \frac{1}{2}(\partial\varphi)^2 + \frac{r_0}{2}\varphi^2 + \frac{u_0}{4!}(\varphi^2)^2 \right\}, \quad (15)$$

$$\mathcal{Z}[\mathbf{h}] = \int \mathcal{D}[\varphi] \exp\left\{-H[\varphi] + \int_{\mathbf{r}} \mathbf{h} \cdot \varphi\right\}, \quad (16)$$

where u_0, r_0 are trivial functions of λ . The gradient term is reminiscent of the scalar product $\mathbf{S}_i \cdot \mathbf{S}_j$ of the lattice model. The potential term $\frac{r_0}{2}\varphi^2 + \frac{u_0}{4!}(\varphi^2)^2$ smoothly replaces the constraint $\mathbf{S}_i^2 = 1$.

Another, exact way to the $O(N)$ model defined in terms of an unconstrained vector is given by the *Hubbard-Stratonovich transformation*. The potential thus obtained behaves as $\log(\cosh(|\varphi|))$. Once expanded at order 4 in the field this potential gives back the φ^4 model. However, the true potential involves all powers and they are a priori all important if one wants to compute non-universal quantities, e.g. T_c .

C. The free energies (generating functionals)

In the following, we are mostly interested in the continuous models, in particular in the φ^4 model. Thus, we define the quantities we used for these models. The generalization to lattice models is straightforward and we do not give them here.

We can construct two free energies from the partition function: The Helmholtz and the Gibbs free energy that are obtained from each other by a Legendre transform.

The *Helmholtz free energy*, \mathcal{W} , is a functional of the external source \mathbf{h} and reads:

$$\mathcal{W}[\mathbf{h}] = \log \mathcal{Z}[\mathbf{h}], \quad (17)$$

where we have absorbed the factor $-k_B T$ in front of the $\log \mathcal{Z}[\mathbf{h}]$ term. $\mathcal{Z}[\mathbf{h}]$ is the generating functional of the correlation functions while $\mathcal{W}[\mathbf{h}]$ generates the *connected* correlation functions. For the one-point function:

$$\phi_i(\mathbf{r}) = \langle \varphi_i(\mathbf{r}) \rangle = \frac{\delta \mathcal{W}[\mathbf{h}]}{\delta h_i(\mathbf{r})}. \quad (18)$$

For a vanishing external source ($\mathbf{h} = 0$), $\phi(\mathbf{r})$ is nothing but the order parameter. It vanishes in the high temperature phase while it is finite in the low temperature phase where spontaneous symmetry breaking occurs.

The second functional derivative of $\mathcal{W}[\mathbf{h}]$ is:

$$\frac{\delta^2 \mathcal{W}[\mathbf{h}]}{\delta h_i(\mathbf{r}) \delta h_j(\mathbf{r}')} = \langle \varphi_i(\mathbf{r}) \varphi_j(\mathbf{r}') \rangle - \langle \varphi_i(\mathbf{r}) \rangle \langle \varphi_j(\mathbf{r}') \rangle. \quad (19)$$

All connected correlation functions $G_c^{(n)}{}_{i_1 \dots i_n}[\mathbf{r}_1, \dots, \mathbf{r}_n; \mathbf{h}]$ can be obtained from $\mathcal{W}[\mathbf{h}]$ by taking n functional derivatives w.r.t. the external source $h_{i_1}(\mathbf{r}_1), \dots, h_{i_n}(\mathbf{r}_n)$:

$$G_c^{(n)}{}_{i_1 \dots i_n}[\mathbf{r}_1, \dots, \mathbf{r}_n; \mathbf{h}] = \frac{\delta^n \mathcal{W}[\mathbf{h}]}{\delta h_{i_1}(\mathbf{r}_1) \dots \delta h_{i_n}(\mathbf{r}_n)}. \quad (20)$$

Notice that, at this stage, the $G^{(n)}$ are really functional of the field \mathbf{h} . When evaluated in a uniform field configuration they become functions of the field momenta.

The *Gibbs free energy* – also called the effective action in the context of high energy physics – is obtained from \mathcal{W} by a Legendre transform⁷

$$\Gamma[\phi] + \mathcal{W}[\mathbf{h}] = \int_{\mathbf{r}} \mathbf{h} \cdot \phi. \quad (21)$$

It is important to realize that in (21), the computation of $\Gamma[\phi]$ requires to eliminate \mathbf{h} for ϕ . This is achieved by inverting the equation of state (18) yielding $\phi = \phi[\mathbf{h}]$ to get $\mathbf{h} \equiv \mathbf{h}[\phi]$.⁸

From (21) we obtain

$$\frac{\delta \Gamma[\phi]}{\delta \phi_i(\mathbf{r})} = h_i(\mathbf{r}), \quad (22)$$

which is the reciprocal of (18). This relation shows that at vanishing external source the equilibrium states ϕ are given by the minima of $\Gamma[\phi]$.

The effective action $\Gamma[\phi]$ is the generating functional of one-particle-irreducible (1PI) correlation functions (also called the vertex functions):

$$\Gamma_{i_1 \dots i_n}^{(n)}[\mathbf{r}_1, \dots, \mathbf{r}_n; \phi] = \frac{\delta^n \Gamma[\phi]}{\delta \phi_{i_1}(\mathbf{r}_1) \dots \delta \phi_{i_n}(\mathbf{r}_n)}. \quad (23)$$

The connected correlation functions can be expressed in terms of the $\Gamma^{(n)}$'s. Let us show this for $n = 2$. Consider the equation of state (18) that we differentiate with

⁷ Let us notice that the Legendre transform is often defined by: $\Gamma[\phi] = \sup_{\mathbf{h}} (\int_{\mathbf{r}} \mathbf{h} \cdot \phi - \mathcal{W}[\mathbf{h}])$. For what follows, it is useless to compute the supremum with respect to \mathbf{h} and we explain in the following how to proceed without this requirement.

⁸ Notice that in the broken phase $\mathbf{h} = 0$ corresponds to infinitely many ϕ fields having all the same modulus and the equation $\phi = \phi[\mathbf{h}]$ is thus ambiguous in the limit $\mathbf{h} \rightarrow 0$. However, even in this case, (22) remains well-defined.

respect to the field \mathbf{h} . We obtain

$$\begin{aligned} \delta(1-2) &= \frac{\delta}{\delta h(2)} \frac{\delta \Gamma[\phi]}{\delta \phi(1)} \\ &= \int d3 \frac{\delta^2 \Gamma[\phi]}{\delta \phi(1) \delta \phi(3)} \frac{\delta \phi(3)}{\delta h(2)} \\ &= \int d3 \frac{\delta^2 \Gamma[\phi]}{\delta \phi(1) \delta \phi(3)} \frac{\delta^{(2)} \mathcal{W}[\mathbf{h}]}{\delta h(3) \delta h(2)} \\ &= \int d3 \Gamma_{i_1 i_3}^{(2)}[\mathbf{r}_1, \mathbf{r}_3; \phi] G_{i_3 i_2}^{(2)}[\mathbf{r}_3, \mathbf{r}_2; \mathbf{h}] \quad (24) \end{aligned}$$

(we use the notations $1 \equiv (\mathbf{r}_1, i_1)$, $\int d1 = \int d^d r_1 \sum_{i_1}$, etc.). In matrix form this relation is expressed as

$$\Gamma^{(2)} = G_c^{-1}. \quad (25)$$

The inverse of the two-point function is the full (field-dependent) propagator, G_c . Note that here we drop the superscript (2) in G_c .

In the following, we will need this function evaluated in a uniform, that is, constant, field configuration $\phi(\mathbf{r}) = \phi$. In general, a uniform field configuration ensures translational invariance in position space. In momentum space this leads to momentum conservation in any n -point functions. Hence, an n -point function depends only on $(n-1)$ momenta. When ϕ is a constant, we call ρ the $O(N)$ invariant

$$\rho = \phi^2/2. \quad (26)$$

The functions $\Gamma^{(n)}$ are called one-particle-irreducible (1PI) because their perturbative expansion involves only graphs that are 1PI. This means that they remain connected when any of their internal line is cut.

III. UNIVERSALITY

A natural question in strongly correlated systems is to know whether their statistical properties are insensitive to their small distance features⁹.

Strikingly, some properties are indeed independent of the microscopic details. As a consequence, vastly different physical systems can exhibit similar behaviour close to criticality. According to this feature of *universality* these systems are said to be in the same *universality class*. Determining the specific universality class of the physics at hand is non-trivial. It has to be studied for

⁹ The situation looks somewhat involved for strongly correlated systems especially when they show *damage spreading* (the reader is encouraged to look this term up). When it is the case, the system is chaotic and one could naively think that all microscopic details matter. Although true for the time evolution of the system, universality can, nevertheless, emerge for averaged quantities

each particular case. However, symmetries and dimensionality can provide a practical guideline.

A salient feature of the behavior of the thermodynamic quantities close to a second order phase transition is the fact that they are all power laws either as functions of the temperature, the size of the system or the length scale at which it is studied. The *critical exponents* of these power laws are among the best known universal quantities: the correlation length, specific heat and the correlator of the Ising model at T_c behave with the same power law as those of water around its critical point or that of the demixing (i.e. phase separation) transition: They lie in the same universality class.

For this set of systems, for $T \rightarrow T_c^+$, we have:

$$\xi(T) \sim (T - T_c)^{-\nu}, \quad (27)$$

$$G^{(2)}(\mathbf{r}) \underset{\xi \gg |\mathbf{r}| \gg a}{\sim} \frac{e^{-|\mathbf{r}|/\xi(T)}}{|\mathbf{r}|^{d-2+\eta}} \underset{F.T.}{\sim} \frac{1}{|\mathbf{r}|^{d-2+\eta}} \frac{1}{q^{2-\eta}}, \quad (28)$$

$$\chi(T) \equiv \left. \frac{\partial M}{\partial h} \right|_{h=0} \sim (T - T_c)^{-\gamma}, \quad (29)$$

where $\chi(T)$ is the (magnetic, for Ising) susceptibility and η is the anomalous dimension. Other universal quantities at a second order phase transition are amplitude ratios.

On the other hand, properties such as the existence of a finite T_c (in the negative case, there should still exist a “transition” at $T = 0$) and its value, are non-universal but dependent on the microscopic details.

IV. MEAN FIELD THEORY

Landau’s idea of mean field (MF) theory is to approximate the functional integral itself by proposing an ansatz for the effective action $\Gamma[\phi]$. MF theory constitutes a classical approximation of a quantum field theory in the sense that fluctuations around the expectation value of the field are not taken into account. Instead, only the classical field configuration is considered. Indeed, referring to the integrand (argument) of the functional integral, MF theory is also known as the method of steepest descent, the saddle-point approximation or simply the classical approximation (see next section).

The ansatz is constructed such that

- $\Gamma[\phi]$ retains the $O(N)$ symmetry,
- $T \approx T_c$ is assumed and, hence, $\phi = \langle \varphi \rangle$ is small,
- $\Gamma[\phi]$ derives from a density of Gibbs free energy, and

- $\Gamma[\phi]$ is analytic and we can expand and retain only the smallest terms, which corresponds to assuming only mild fluctuations around the mean field (MF) value.

The easiest non-trivial ansatz is given by an expansion up to fourth order in the field¹⁰,

$$\begin{aligned} \Gamma[\phi] &= \int d^d x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + U(\phi^2) \right\} \\ &\approx \int d^d x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{r_0}{2} \phi^2 + \frac{u_0}{4!} (\phi^2)^2 \right\}. \end{aligned} \quad (30)$$

Note that the $O(N)$ symmetry is satisfied by $U(\phi^2)$ and the original space-time dimensional symmetry is preserved by the kinetic term.

MF theory often serves as a starting point in the investigation of new models, because of its advantageous features: it does certainly exhibit phase transitions, shows universality and yields power laws as criticality is approached. However, as it stands, this approximation is often too crude to describe the physics at hand accurately. With regard to $O(N)$ -models it incorrectly predicts that neither the existence of a phase transition, nor the values of the critical exponents, depend on N or d . Clearly, this is in contradiction with the $1d$ Ising model, which shows no phase transition, or the Mermin–Wagner theorem, which dictates that continuous symmetries cannot be broken spontaneously in 2 dimensions.

Contrary to our last hypothesis, the long wavelength excitations present in the vicinity of the 2nd order phase transition produce non-analiticities (in the thermodynamic limit), .

V. THE PERTURBATIVE RG

One way to improve the MF approach is to approximate the integrand of the functional integral $Z[h]$, instead of the effective action itself. We will consider a perturbative expansion around the dominating classical trajectory¹¹. Hence, this procedure is usually referred to as perturbation theory (PT). The zeroth order of this approximation corresponds to the classical approximation or MF theory.

A convenient way to introduce PT follows: for the generating function defined in (16), we decompose the

Hamiltonian into its Gaussian components,

$$H_0(\varphi) = \frac{1}{2} \int_x \left((\partial_\mu \varphi)^2 + r_0 \varphi^2 \right), \quad (31)$$

and Taylor expand¹² the interaction term H_I ,

$$H = H_0 + H_I \approx H_0 + \int_x \frac{u_0}{4!} \varphi^4(x). \quad (32)$$

The generating functional is now given as a series in the bare coupling, u_0 . This allows for a computation of correlation functions of arbitrary order. The resulting expressions can be illustrated with Feynman diagrams. After the perturbative expansion, we still have to perform the functional integral itself, so that an actual sum over different field configurations is carried out, unlike in MF theory.

On its part, the idea of *perturbative renormalisation* is to include the effect of fluctuations which perturb the system mildly via a change of parameters defining the theory. Hence, we reparametrise $u_0 \rightarrow u_R$, $r_0 \rightarrow r_R$ (this coupling corresponds to the square of the mass, m_R , of the field ϕ) and $\varphi_0 \rightarrow \varphi_R$, as we know that the bare expressions are ill-defined (they lead to divergent terms in the UV cut-off). The reparametrisation leads to 'renormalised' correlation functions, $\Gamma_R^{(n)}(\{p_i\}, m_R, u_R)$, which are expressed in terms of the renormalised quantities and the momenta p_i carried by the external legs of the vertex.

Naturally, in a series expansion, it is necessary to check for its convergence. Actually, the perturbative expansion has bad properties, as it leads to asymptotic series at best¹³. As an example where standard PT predicts the wrong behaviour, we may consider the beta-function of the coupling of φ^4 theory in $d = 3$ (the beta function describes the change of the coupling under a change of renormalization scale). In fact, PT yields a divergent series. Nevertheless, in this case we can still apply the idea of a small perturbation around the free theory as we know the perturbative expansion up to high orders. Via a reordering of the series, a Borel resummation can be successfully employed: The resulting series of approximants is well-behaved and approaches the correct result with increasing expansion order.

¹⁰ As the mean of φ vanishes at the phase transition, going to higher order in powers of ϕ does not improve the description of the critical phenomena.

¹¹ Note that the assumption of a small coupling is the major difference with the NPRG, which can be applied for strong couplings too.

¹² Note that the interchange of the functional integral with the series may yield (insurmountable) problems.

¹³ The reason why perturbative expansions are applied is indeed a practical one, not a mathematical one: In many physical systems, PT has been successfully applied even to high orders in the coupling. Most prominently in QED the perturbative expansion yields astonishingly good agreement with experiments.

More generally, we can summarise the cases in which PT is expected to work or fail, respectively. The latter situations motivate a non-perturbative description in terms of the NPRG.

The perturbative approach is expected to work well, if either

- the coupling is small; in this case the results are supposedly qualitatively and potentially even quantitatively good even at low order: This is the case of QED; or
- the coupling is not small, many terms of the perturbative expansion have been computed and the series is Borel-summable. In this case, the Borel transformed expansion can be resummed (e.g. by Padé approximants) and one can build a sequence of approximants that become more and more accurate. This is the case of the φ^4 theory in three dimensions where the series expansion of the β function is known at six loops.

In contrast, we can encounter the pitfalls of perturbation theory in the following cases:

- The system is strongly coupled and the series are either non Borel summable or not known at high enough orders to be resummed. This is the case of the non-linear σ -model in $d = 3$, where the perturbative renormalisation in $d = 2 + \epsilon$ with $\epsilon \rightarrow 1$ yields series expansion that cannot be resummed and that are almost useless quantitatively for the $O(N)$ models (for other non-linear σ -models, they can even be qualitatively wrong).
- The system is strongly coupled but the series are Borel summable and known at high enough orders to be resummed but the resummed series are quantitatively very far from the exact result. This is the case of the φ^4 theory in $d = 2$ ($N = 1$) where the critical exponents have been computed at five loops order but do not match the exact Onsager's results: The anomalous dimension found perturbatively is $\eta_{\text{PT}} = 0.145(15)$, and disagrees with the exact result, $\eta_{\text{exact}} = .25$. The mismatch can even be of a qualitative nature, e.g. the $O(4)$ -model in $d = 2$ predicts an $\eta_{\text{PT}} \simeq 0.1$, but due to the Mermin-Wagner theorem there is no phase transition, and $\eta_{\text{exact}} = 0$.
- Another possible failure of perturbation theory may occur when genuinely non-perturbative phenomena exist, that is, when one computes physical quantities that are not Taylor expandable.¹⁴ Such a situa-

tion is encountered in the RG context when a fixed point is not connected to the gaussian fixed point by any RG trajectory: Perturbation theory cannot find it since it amounts to an expansion of the RG flow around the gaussian fixed point.

In all of the latter cases the NPRG improves the description of or even solves the physics at hand.

VI. NPRG

The main difference between the non-perturbative and the perturbative formulations of the RG is that it does not rely on an expansion around the Gaussian model¹⁵. Again, the idea is to improve the (MF) approach of Landau, however, in this occasion, extending the RG idea of Wilson.

We will be interested in calculating the effective action $\Gamma[\phi]$, (21). Doing so requires summing all the *fluctuations i.e.*, all possible values that the *fluctuating field* φ can assume.¹⁶

Before we embark on the derivation of the underlying NPRG equation, we outline the distinct scales that occur in the computation of the generating functional when starting from the microscopic interactions encoded in H . We will pick the example of the φ^4 -model.

A. Scales

We introduce an artificial ultraviolet cutoff in momentum space, Λ . With this choice we only allow for modes with momenta $p < \Lambda$ to contribute. The choice of the cutoff is motivated by lattice field theory, where the lattice spacing, a , constitutes a smallest resolvable distance in position space. This is equivalent to the choice of a largest resolvable momentum, $\Lambda \sim 1/a$, due to the relation of the two representations by a Fourier-transformation. Dropping modes with momenta $p > \Lambda$ is exact in the discretised spacetime. In contrast to this, in the continuum, we can think of the effects from large momenta being absorbed into the definition of H within a renormalisation procedure.

By construction, when looking at the theory at the scale Λ we are not sensitive to the effects that arise from smaller momenta. Therefore, at this scale we can identify the free energy with the Hamiltonian of the

¹⁴ As a toy model, consider the function $f(x) = \exp(-1/x)$. This function is not expandable around $x = 0$ since $f^{(n)}(0) = 0$ for all n and is therefore found to be 0 at all orders in the expansion in x .

¹⁵ Hence, the NPRG does not suffer from the potential problems that arise in the interchange of the series and functional integral as performed in the perturbative RG.

¹⁶ According to the physics under study, the fluctuations can be either quantum or statistical. We need not specify since the field theory formalism treats the two in the same way.

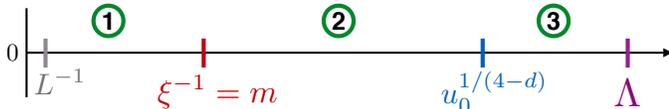


FIG. 3 Relevant scales between the microscopic dynamics given by the hamiltonian and the free energy.

system, i.e. $\Gamma|_{k=\Lambda} = H$. As already stressed, we want to compute the macroscopic theory, given by full effective action, Γ , from the microscopic dynamics. In the course of this computation we encounter particular scales that will dominate and which will make the properties of the system change qualitatively. We will now sketch the distinct scales and outline their effects.

The relevant scales are set by Λ , the couplings in the action (mass and coupling constant) and, for a finite-size system, the inverse of the size of the system, L^{-1} . For an illustration of the hierarchy of scales in momentum space see figure 3. Note that the identity $\Gamma = H$ holds only at the MF approximation and the different scales remain being given by the bare parameters, r_0 and u_0 only at this approximation. Beyond MF they have to be read off from the effective action. Notice also that bringing the system close to criticality requires to fine tune the bare parameters since, otherwise, the renormalized mass of the order of the large ultra violet scales.

With the finite scale Λ we have defined the resolution of our system in the ultraviolet¹⁷. For many systems, all bare length scales are given in terms of Λ which is the fundamental UV scale and, in these cases, all microscopic scales are of order Λ , e.g. $u_0^{1/(4-d)}$. (this is the case of the $O(N)$ model derived from the lattice ferromagnetic model with the Hubbard-Stratonovich transformation). It can happen that it is not so and that $u_0^{1/(4-d)}$ is independent of Λ and can therefore be much smaller (by orders of magnitude).

In order to avoid considering finite-size effects we take in the following $L \rightarrow \infty$.

Below Λ , the first smaller scale in momentum space is the Ginzburg scale. For the theory at hand it is defined by the coupling, $u_0^{1/(4-d)}$. As we aim at studying critical physics, the mass, which is the next distinct scale, has to be small compared to the other scales. Therefore, it must be well separated from the Ginzburg scale. Note

¹⁷ If necessary, we assume that an UV renormalisation has been performed and the renormalised parameters at scale Λ define the Hamiltonian.

that only in the limit $u_0 \rightarrow \infty$ and $\Lambda \rightarrow \infty$ we can have a scale invariant theory for $T \rightarrow T_c$ for all momenta.

The behaviour of the system is qualitatively different in the regions given in figure 3. As an example we study the two-point function, $\Gamma^{(2)}(p)$, defined in (25). We have implicitly assumed a uniform field configuration and hence, because of translation invariance, the correlator only depends on (the absolute value of) one momentum, p , see section II.C.

In region 3, $u_0^{1/(4-d)} < p (\ll \Lambda)$, the mass-scale is negligible and, thus, the system is dominated by the scale $u_0^{1/(4-d)}$. In this domain no universal scaling behaviour related to critical physics emerges and PT works well.

In region 2, $m \ll p \ll u_0^{1/(4-d)}$, we find (close-to-)critical behaviour: compared to the value of the momentum, the mass is negligible, and, at the same time, the Ginzburg scale is large. As a result, the propagator exhibits a scaling behaviour according to its anomalous dimension, cf. (28). In the critical domain MF fails because the contributions of fluctuations on all scales belonging to this region add up coherently and their effects become strong. PT resummed by means of the RG finds the right scaling behavior for the 2-point function with an accurate determination of η if computed at large orders (at least three loops).

In region 1, $p \ll m$ and, even though close to criticality, the system looks weakly correlated: the comparatively large mass suppresses deviations from MF and non-analyticities do not show up. As a consequence, MF theory (with possibly perturbative corrections), $\Gamma^{(2)}(p) = c_1(p^2 + m^2) + c_2 p^4 + \dots$, works well. In a certain sense, MF theory is tailored for this regime. Obviously, if $m \rightarrow 0$, this region vanishes and Landau's idea of MF theory is not applicable.

B. NPRG flow: General idea

In order to study the critical phenomena we have to remove the infrared regulator, i.e. we have to take the limit $m \rightarrow 0$, or equivalently $\xi \rightarrow \infty$ or $T \rightarrow T_c$. There are different ways to regularise the non-analyticities that show up in this procedure.

The first possibility is to consider a non-vanishing mass: We bring the system out of criticality, and then study the behaviour approaching the critical limit. This amounts to analysing the change of the system with respect to a change in the mass. In a differential formulation, i.e. taking the derivative $\frac{\partial}{\partial m}$, this leads to the Callan-Symanzik RG equation.

Another possibility is to put the system in a finite box of spatial extent $L < \infty$. By means of a scaling analysis with respect to L we can identify the behaviour as

the system approaches the critical limit, i.e. $L \rightarrow \infty$. By the nature of the simulational setup, this procedure is most conveniently applied in lattice Monte-Carlo computations.

The third way to regularise the infrared non-analyticities is realised in the NPRG formalism. The idea is to sum over the fluctuations existing on all scales between Λ and 0 in a better way than perturbatively. To this aim, we construct a family of models that interpolate smoothly and in the most convenient way between Λ and 0¹⁸. As we will see later, the NPRG has the form of an evolution equation with respect to a momentum scale k , which we introduce as an artificial scale. The initial condition is set in the ultraviolet, $k = \Lambda$, where $\Gamma|_{\Lambda} = H$. By slightly lowering the scale k we sum over fluctuations between Λ and $\Lambda - dk$. These fluctuations modify the effective action and, by iterating this step we finally reach the limit $k \rightarrow 0$, where we all fluctuations on all scales have been taken into account. We are left with the (full) effective action, $\Gamma_{k=0} = \Gamma$. In summary, at a finite scale k , the running effective action Γ_k is a precursor of the effective action satisfying

$$\begin{cases} \Gamma_{k=\Lambda} = H \\ \Gamma_{k=0} = \Gamma \end{cases} . \quad (33)$$

Due to this interpolation the NPRG equation is also known as flow equation: The effective action flows through the momentum interval. The trajectory in this interpolation in between $k = \Lambda$ and $k = 0$ depends on the details of the way fluctuations are summed over (choice of regulator function) as we will see below. However, the limits (33) are unique. This method of considering only part of the interval in figure 3 in each step relates to Wilson's idea of integrating fluctuations momentum shells by momentum shells.

In order to specify the idea of the NPRG, we construct a deformation of the original model, Z_k , by adding a term to the Hamiltonian,

$$Z_k = \int \mathcal{D}\varphi e^{-H - \Delta H_k + \int_x h \cdot \varphi} . \quad (34)$$

The deformation, ΔH_k , can be chosen to be quadratic in the fields. In principle, also terms of higher order in the fields would be possible. However, the quadratic term is the easiest one: in this case the NPRG equation has

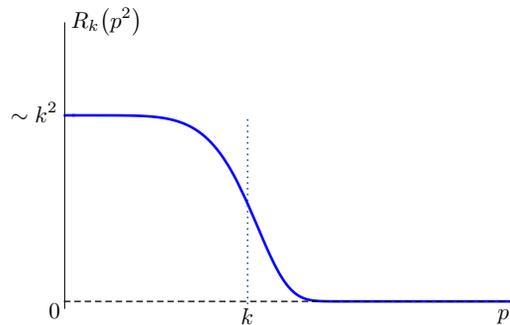


FIG. 4 Typical form of the regulator function $R_k(q)$: R_k vanishes for $q \gg k$, i.e. it does not modify the action at scales larger than k , and acts like an additional mass term in the infrared.

one-loop structure, as we will see below. Thus, we define

$$\begin{aligned} \Delta H_k &= \frac{1}{2} \int_x \varphi(x) R_k(x-y) \varphi(y) \\ &\xrightarrow{F.T.} \frac{1}{2} \int_q \varphi(q) R_k(q) \varphi(-q) , \end{aligned} \quad (35)$$

with the so-called regulator, $R_k(q)$, which can be chosen freely as long as it does not contradict the conditions imposed on Γ_k , (33). We choose the regulator as a momentum-dependent mass-like term that drives the system away from criticality when $k > 0$. In the limit, $k \rightarrow 0$, the regulator must not modify the theory at any scale. As a consequence, it must strictly vanish in this limit $\Delta H_{k=0} = 0$. These properties are met by taking the regulator of a shape similar to figure 4.¹⁹ In fact, it is usually expressed as q^2 times a dimensionless function: $R_k(q) = q^2 r(q^2/k^2)$. The small value of $R_k(q)$ for $q^2/k^2 \gg 1$ reflects the fact that it does not change the theory at large momentum scales.

The modes with large momenta compared to k are referred to as 'rapid' modes, in analogy with the Brownian motion. Instead, the 'slow' modes in the infrared, $q \ll k$, are frozen by the regulator. For $k \sim \Lambda$ the regulator term is of the order of Λ^2 for all q and all fluctuations are frozen.

Remarks:

- In principle, one could take $R_k(p)$ to be a step function that vanishes for $p > k$ and takes a positive infinite value for $p < k$. This is known to be an ultra sharp cut-off and corresponds to the "block-spin" version of RG. In practice, it is a potential source

¹⁸ Note that the NPRG provides an exact equation for the effective action, however, in practical applications approximations are inevitable. Hence, the 'most convenient' way of regularisation has to be chosen with respect to a given truncation scheme.

¹⁹ Note that in the standard nomenclature the scale is written as an index, however, the regulator is a function of both k and momentum, $R_k(q) = R(q, k)$.

of difficulties when approximations are to be made and could lead to bad results when calculating, for example, the anomalous dimension.

- In the (original version of the) Callan–Symanzik RG approach one modifies the Hamiltonian by adding the a mass, $\int_x \Delta m^2 \varphi^2(x)$, which would correspond to $R_k(q) \equiv \Delta m^2$ in NPRG language. Therefore, it suppresses the fluctuations of all scales at once. In this sense one may see Wetterich NPRG as a refined version of Callan–Symanzik RG.

C. Wetterich’s equation

As a next step we derive the equation that describes the evolution of Γ_k . Before doing this in detail, we shall precise the definition of Γ_k . Starting from the partition function of modified model \mathcal{Z}_k , see above (34, 35), we define as usual

$$\mathcal{W}_k = \log \mathcal{Z}_k \quad (36)$$

and its Legendre transform

$$\Gamma_k^{Leg}[\phi] + \mathcal{W}_k[\mathbf{h}] = \int_{\mathbf{r}} \mathbf{h} \cdot \phi. \quad (37)$$

However, we shall define Γ_k using the following modified Legendre transform

$$\Gamma_k[\phi] + \mathcal{W}_k[\mathbf{h}] = \int_{\mathbf{r}} \mathbf{h} \cdot \phi - \Delta H_k[\phi], \quad (38)$$

or, equivalently,

$$\Gamma_k[\phi] = \Gamma_k^{Leg}[\phi] - \Delta H_k[\phi]. \quad (39)$$

This modification is necessary in order that Γ_k satisfies the limiting conditions (33). Indeed, at $k = 0$ no difference has been made: $\Gamma_0 = \Gamma_0^{Leg} = \Gamma$ since ΔH_0 vanishes; While at $k = \Lambda$, it is easy to see that $\Gamma_{\Lambda}^{Leg}[\phi] \sim H[\phi] + \Delta H_{\Lambda}[\phi]$ (because at this scale, the MF approximation is almost exact, thanks to the regulator ΔH_k which presents a mass $m \sim \Lambda$ large enough to suppress all fluctuations); Therefore, only after subtracting $\Delta H_k[\phi]$ would we have $\Gamma_{\Lambda} \sim H$ as desired²⁰. Now, we state the cornerstone of the NPRG approach, the equation of Wetterich. This is a central piece of this course.

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \text{circle with } \partial_k R_k \text{ and } (\Gamma_k^{(2)} + R_k)^{-1}$$

FIG. 5 Diagrammatic representation of Wetterich’s equation (40). The line denotes the (full) propagator, $(\Gamma_k^{(2)} + R_k)^{-1}$, the crossed circle illustrates the derivative of the regulator function, $\partial_k R_k$.

Theorem (Wetterich) $\Gamma_k[\phi]$ satisfies the following equation

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x,y) \left(\Gamma_k^{(2)}[\phi] + R_k \right)_{(x,y)}^{-1} \quad (40)$$

where

$$\Gamma_k^{(2)}[\phi](x,y) = \left. \frac{\delta^2 \Gamma_k}{\delta \phi(x) \delta \phi(y)} \right|_{\phi}$$

is the 1PI propagator of the modified effective action. $R_k(x,y) = R_k(x-y)$ is the regulator, and $(\Gamma_k^{(2)} + R_k)^{-1}$ is the inverse of $\Gamma_k^{(2)}(x,y) + R_k(x,y)$ in the sense of integral kernel linear operator, i.e. $B(x,y) = A_{(x,y)}^{-1} \Leftrightarrow \int_z B(x,z)A(z,y) = \delta^d(x-y)$.

Its derivation is conceptually straightforward and we leave it to VI.C.2. Another, convenient form of the equation is:

$$\partial_k \Gamma_k = \frac{1}{2} \tilde{\partial}_k \text{Tr} \log \left(\Gamma_k^{(2)} + R_k \right) \quad (41)$$

where $\tilde{\partial}_k$ acts only on the k -dependence of R_k and not on $\Gamma_k^{(2)}$:

$$\tilde{\partial}_k = \frac{\partial R_k}{\partial k} \frac{\partial}{\partial R_k}, \quad (42)$$

and the trace means integral over spatial or (momentum) indices (and for more complex theories, summation over any internal index).

Wetterich’s equation (40) can be expressed in a diagrammatic form²¹: Under the integral the full propagator is multiplied by the derivative of the regulator function and in the language of diagrams this can be expressed as a closed loop representing the propagator with the insertion of a cross representing the derivative of the regulator. This is given in figure 5.

Below, we make several remarks on Wetterich’s equation.

²⁰ In fact, it may be necessary for the equality $\Gamma_{\Lambda} = H$ to hold strictly, e.g. in the computation for non-universal quantities. Then, the regulator must diverge at Λ . As a result, Γ_{Λ}^{Leg} diverges also while Γ_{Λ} stays finite.

²¹ Albeit their graphical similarities, the diagrammatic representations of flow equations do not involve Feynman diagrams, because the latter ones refer to perturbative quantities in a strict sense.

- Wetterich’s approach can be generalised to $O(N)$ models with $N > 1$ and fermionic field theories without conceptual problems.
- Wetterich’s equation (40), like any other RG equation, is to be interpreted as a dynamical system: The role of time is played by k which runs from Λ to 0²². In this sense, it must be completed by the initial condition (33).
- Non-polynomial forms of the initial condition H do not lead to technical difficulties contrary to the perturbative approach where it is difficult (and even often impossible) to deal with infinitely many interaction terms. For example, as already mentioned in section II.B.2, the potential of the Ising model has the complicated form $V(\varphi) \sim \log \cosh \varphi$. Obviously, a low-order polynomial expansion in PT fails to approximate this form and thus non universal quantities cannot be computed perturbatively at least in a reliable way.²³
- It is a partial differential equation (PDE) in the sense that $\Gamma_k[\phi] = \Gamma[k; \phi]$ depends on two variables. As a result, one has to deal with difficulties intrinsic to PDE study (*e.g.* stability, reliability of the numerical solutions). Even worse, the right hand side of (40) is *non-linear* and is *functional* in the field variable. Nevertheless, it is better posed in mathematical terms than a path-integral in the continuum.
- The quantity $G_{c,k} = \left(\Gamma_k^{(2)}[q, \phi] + R_k(q) \right)^{-1}$ is the full (functional) propagator at scale k (in particular, it is the full functional propagator of the theory for $k = 0$). This is made clear in the derivation VI.C.2.
- When the background field ϕ is uniform, the (running) effective action $\Gamma_k[\phi(x) = \phi]$ becomes the (running) effective potential (up to volume factor) and the right hand side of (40) Fourier transforms into an integral over a single momentum. In the $O(N)$ case where $\Gamma_k^{(2)}$ is a scalar, the inverse of $\Gamma_k^{(2)}(q, \phi) + R_k(q)$ boils down to $1/(\Gamma_k^{(2)}(q, \phi) + R_k(q))$ and

$$\partial_k \Gamma_k[\phi(x) = \phi] = \frac{1}{2} \int_q \frac{\partial_k R_k(q)}{\Gamma_k^{(2)}(q, \phi) + R_k(q)}.$$

The integrand is under control because of the presence of $\partial_k R_k$ in the numerator and of $R_k(q)$ in the denominator (see figure 4): Only the region where $q \sim k$ is really contributing²⁴. This implements Wilson’s RG idea of momentum shell integration of fluctuations.

- The (linearly realized) symmetries of the bare action are preserved by the RG flow as long as the regulator term ΔH_k is invariant. This is the case of the $O(N)$ symmetry for instance. In gauge theories, however, the regulator breaks gauge invariance, because it acts like a mass term in the infrared. In this case, the breaking of gauge invariance along the RG flow can be controlled by modified Slavnov–Taylor identities which take into account the regulator term²⁵.
- To make contact with PT, it is convenient to reintroduce the \hbar factors and to expand ((41)) at the lowest, non trivial order in \hbar . Taking into account that the right-hand-side of the flow equation is proportional to \hbar , it is sufficient to replace $\Gamma_k^{(2)}$ by its mean-field approximation $H^{(2)}$. In this case, $\tilde{\partial}_k$ becomes ∂_k and thus:

$$\partial_k \Gamma_k = \frac{1}{2} \hbar \partial_k \text{Tr} \log \left(H^{(2)} + R_k \right). \quad (43)$$

As a consequence, the flow becomes a total derivative and we can (trivially) integrate between 0 and Λ :

$$\Gamma - H = \frac{1}{2} \hbar \text{Tr} \log \left(H^{(2)} \right) + \text{const} + \mathcal{O}(\hbar^2). \quad (44)$$

This is the 1-loop result for the effective action. Equation (44) implies that any sensible approximation of Γ_k will lead to a flow equation which is at least 1-loop exact. Equation (43) can also be read the other way around: it is remarkable that substituting in this equation the bare inverse propagator $H^{(2)} + R_k$ by the full inverse functional propagator $\Gamma_k^{(2)} + R_k$ turns this one-loop equation into the exact flow equation.

- We shall see in the following that for $N \geq 2$, any sensible approximation of Γ_k leads to flows that are also one-loop exact in $d = 2 + \epsilon$, that is, the low-temperature expansion obtained from the non-linear σ model is also retrieved automatically *with*

²² This is the direction to follow in statistical physics, *i.e.* where one tries to deduce the macroscopic behavior from the microscopic description of the system. To deal with other problems of physics, *e.g.* quantum gravity, it may be the other way around.

²³ The non polynomial character of the bare potential is actually not the only difficulty encountered perturbatively in the computation of non universal quantities.

²⁴ Indeed, for $q \gg k$, $R_k(q)$ almost vanishes and, hence, so does $\partial_k R_k(q)$. Moreover, the $R_k(q)$ term in the denominator regularizes the infrared divergences, if any.

²⁵ It is however non trivial to preserve the modified Slavnov–Taylor identities as well as unitarity when approximations are performed.

the same set of flow equations that are one-loop exact in $d = 4 - \epsilon$. This means that the critical exponent ν , for instance, computed with the simplest sensible ansatz for Γ_k (see the following) will automatically be one-loop exact around four *and* around two dimensions. We can therefore expect that the flow equations for the $O(N)$ models obtained from very simple approximations of Γ_k will be at least clever interpolations between a one-loop result in $d = 4 - \epsilon$ and another one-loop result in $d = 2 + \epsilon$. We show in the following that very accurate results can indeed be obtained with NPRG. Let us also emphasize that this property is very non-trivial (but perhaps in the limit $N \rightarrow \infty$) and is not recovered at any finite order of PT: even computed at large orders, the perturbative RG equations of the φ^4 model for $N > 1$ are not able to reproduce Mermin-Wagner's theorem for instance and those obtained from the non-linear σ model cannot predict that four is the upper critical dimension of the $O(N)$ systems.

- In the same vein, we shall show that any sensible approximation of Γ_k leads to flows of the coupling constant that are exact in the limit $N \rightarrow \infty$. This implies in particular that the critical exponents obtained from any sensible approximation of the NPRG equations are exact in the limit $N \rightarrow \infty$.

In summary, the NPRG approach transforms a (quantum/statistical) field theory problem, which is usually formulated as a (functional) integral problem, into a (non-linear functional integro partial) differential equation. In the general form this equation is exact, *i.e.*, the integration of (40) with the initial condition $\Gamma_\Lambda = H$ gives the full solution $\Gamma = \Gamma_{k=0}$ of the theory. However, in practice approximations have to be employed. It is certainly possible to expand (40) around its Gaussian solution: this leads to PT. The key-point is to construct approximation schemes that go beyond PT. We shall discuss in the next section two such schemes: the derivative expansion (DE) and the Blaizot–Mendez–Wschebor (BMW) scheme. The rest of this section is devoted to the derivation of Wetterich's equation (40) and a historical remark on the Wilson–Polchinski approach to RG; neither of them is necessary to understand the rest of the lecture.

1. Derivation of (33): $\Gamma_{k=\Lambda} \sim H$

The meaning of the equation $\Gamma_{k=\Lambda} \sim H$ is that at scale Λ all fluctuations are (almost) frozen and thus the mean-field approximation is valid. Let us show this in more details.

We start from the definition of Z_k , (34), and of Γ_k ,

(38). We find:

$$h_x = \frac{\delta \Gamma_k}{\delta \phi_x} + \int_y R_k(x-y) \phi_y. \quad (45)$$

Thus, by substituting (38) and (45) into the definition of W_k we obtain:

$$e^{-\Gamma_k[\phi]} = \int \mathcal{D}\varphi \exp \left\{ -H[\varphi] + \int_x \frac{\delta \Gamma_k}{\delta \phi_x} (\varphi - \phi)_x + \right. \\ \left. - \frac{1}{2} \int_{x,y} (\varphi - \phi)_x R_k(x-y) (\varphi - \phi)_y \right\}. \quad (46)$$

If we choose a function $R_k(q)$ that diverges for all q as $k \rightarrow \Lambda$ we find :

$$\exp \left(-\frac{1}{2} \int (\varphi_x - \phi_x) R_{k=\Lambda}(x-y) (\varphi_y - \phi_y) \right) \sim \delta(\varphi - \phi) \quad (47)$$

that is, it behaves as a functional Dirac delta. Therefore,

$$\Gamma_k[\phi] \rightarrow H[\varphi = \phi] \quad \text{as } k \rightarrow \Lambda, \quad (48)$$

if the cut-off R_k is such that it diverges in this limit. If R_k does not diverge and is only very large,

$$\Gamma_{k=\Lambda} \approx H. \quad (49)$$

2. Derivation of (40)

We recall from (34, 35) that

$$\mathcal{Z}_k[\mathbf{h}] = \int \mathcal{D}\varphi \exp \left(-H - \Delta H_k + \int_x \mathbf{h} \cdot \varphi \right).$$

Taking ∂_k of both sides, we have

$$\partial_k \mathcal{Z}_k[\mathbf{h}] = - \int \mathcal{D}\varphi \partial_k \Delta H_k[\varphi] e^{-H - \Delta H_k + \mathbf{h} \cdot \varphi},$$

where

$$\partial_k \Delta H_k[\varphi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x,y) \varphi(x) \varphi(y)$$

since $\Delta H_k = \frac{1}{2} \int_x \varphi(x) R_k(x,y) \varphi(y)$, see (35). In terms of $\mathcal{W}_k = \log \mathcal{Z}_k$, this implies

$$\begin{aligned} & \partial_k \mathcal{W}_k[\mathbf{h}] \\ &= - \frac{1}{\mathcal{Z}_k[\mathbf{h}]} \int \mathcal{D}\varphi \int_{x,y} \frac{1}{2} \partial_k R_k(x,y) \varphi(x) \varphi(y) e^{-H - \Delta H_k + \mathbf{h} \cdot \varphi} \\ &= - \frac{1}{2} \int_{x,y} \partial_k R_k(x,y) \langle \varphi(x) \varphi(y) \rangle_{k,\mathbf{h}}, \end{aligned} \quad (50)$$

where $\langle \varphi(x) \varphi(y) \rangle_{k,\mathbf{h}}$ is the (full) 2-point correlator of the modified theory ($H \rightarrow H + \Delta H_k$) with external source. In terms of connected correlators,

$$\langle \varphi(x) \varphi(y) \rangle_{k,\mathbf{h}} = \langle \varphi(x) \rangle_{k,\mathbf{h}} \langle \varphi(y) \rangle_{k,\mathbf{h}} + G_{c,k}[x,y;\mathbf{h}]$$

where $G_{c,k}[x, y; \mathbf{h}] := \frac{\delta^2 \mathcal{W}_k}{\delta \mathbf{h}(x) \delta \mathbf{h}(y)} \Big|_{\mathbf{h}}$ is the connected 2-point function of the modified theory. Plugging back to (50) and noting that $\langle \varphi(x) \rangle_{k, \mathbf{h}} = \phi(x)$, we have

$$\partial_k \mathcal{W}_k[\mathbf{h}] = -\frac{1}{2} \int_{x,y} \partial_k R_k(x, y) (\phi(x) \phi(y) + G_{c,k}[x, y; \mathbf{h}]) \quad (51)$$

To obtain the evolution of Γ_k we need to Legendre transform this equation. Notice that the left hand side $\partial_k \mathcal{W}_k[\mathbf{h}]$ needs more than a simple Legendre transform. Indeed, the partial derivative ∂_k is taken while keeping \mathbf{h} fixed: $\partial_k \mathcal{W}_k[\mathbf{h}] = \partial_{k|\mathbf{h}} \mathcal{W}_k[\mathbf{h}]$ while in (40), $\partial_k \Gamma_k := \partial_{k|\phi} \Gamma_k$ is taken keeping ϕ fixed. They are not the same partial derivative but are related by $\partial_{k|\mathbf{h}} = \partial_{k|\phi} + \int_x \partial_k \phi(x) \frac{\delta}{\delta \phi(x)}$. Therefore,

$$\begin{aligned} \partial_k \mathcal{W}_k[\mathbf{h}] &= -\partial_{k|\mathbf{h}} \Gamma_k^{Leg}[\phi] + \partial_{k|\mathbf{h}} \left(\int_x \mathbf{h}(x) \phi(x) \right) \\ &= -\partial_{k|\phi} \Gamma_k^{Leg}[\phi] - \int_x \partial_k \phi(x) \frac{\delta \Gamma_k^{Leg}[\phi]}{\delta \phi(x)} \\ &\quad + \int_x \mathbf{h}(x) \partial_k \phi(x). \end{aligned}$$

But the two integrals cancel each other since $\delta \Gamma_k^{Leg} / \delta \phi(x) = \mathbf{h}(x)$, so

$$\begin{aligned} \partial_{k|\mathbf{h}} \mathcal{W}_k[\mathbf{h}] &= -\partial_{k|\phi} \Gamma_k^{Leg}[\phi] \\ &= -\partial_k \Gamma_k[\phi] - \partial_k \Delta H_k[\phi] \\ &= -\partial_k \Gamma_k[\phi] - \frac{1}{2} \int_{x,y} \partial_k R_k(x, y) \phi(x) \phi(y). \end{aligned}$$

Comparing this with (51) we get

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_{x,y} \partial_k R_k(x, y) G_{c,k}[x, y; \mathbf{h}]. \quad (52)$$

Now, since \mathcal{W}_k and Γ_k^{Leg} are related by a Legendre transformation, we have

$$G_{c,k}[\mathbf{h}](x, y) = \left(\frac{\delta^2 \Gamma_k^{Leg}}{\delta \phi \delta \phi} \Big|_{\phi} \right)_{(x,y)}^{-1}.$$

Here, the inverse is in the integral kernel sense²⁶. Finally, differentiating $\Delta H_k = \frac{1}{2} \int_x \varphi(x) R_k(x, y) \varphi(y)$, see (35), gives

$$\begin{aligned} &\frac{\delta^2 \Gamma_k^{Leg}}{\delta \phi(x) \delta \phi(y)} \\ &= \frac{\delta^2 \Gamma_k}{\delta \phi(x) \delta \phi(y)} + \frac{\delta^2 \Delta H_k}{\delta \phi(x) \delta \phi(y)} \\ &= \Gamma_k^{(2)}[\phi](x, y) + R_k(x, y) \end{aligned}$$

where $\Gamma_k^{(2)}[\phi](x, y) = \frac{\delta^2 \Gamma_k}{\delta \phi(x) \delta \phi(y)} \Big|_{\phi}$. So

$$G_{c,k}[\mathbf{h}](x, y) = \left(\Gamma_k^{(2)}[\phi] + R_k \right)_{(x,y)}^{-1} \quad (53)$$

Plugging this back to (52) gives (40).

VII. APPROXIMATION SCHEME I: THE DERIVATIVE EXPANSION

We will study two approximation schemes for the NPRG equation (40). The Blaizot–Mendez–Wschebor (BMW) method (Section VIII) at order n aims at computing the full momentum dependence of the functions $\Gamma_k^{(m)}(\mathbf{p}_1, \mathbf{p}_2, \dots)$ with $m \leq n$. The derivative expansion (DE) is less ambitious and aims at describing the physics at zero external momentum only. This is enough to capture the thermodynamics of the system, e.g., critical exponents, phase diagrams and so on.

The scale k in the NPRG formalism acts as an infrared regulator (for $k \neq 0$) somewhat similar to a box of finite size $\sim k^{-1}$. Thus, for $k > 0$, there is no phase transition and thus no singularity in the free energy Γ_k , which can therefore be power-expanded safely²⁷. We can therefore conclude that

- (i) the singularities of Γ build up as k is lowered and are thus smoothed by k in Γ_k ,
- (ii) the precursor of the critical behavior should already show up at finite k for $|p| \gg k$. Here and below, p is momentum of the probe (say the argument in the two-point function $\Gamma_k^{(2)}(p)$).

An important consequence of the regularity of Γ_k at $k > 0$ is that it can be expanded in a power series of $\partial \phi(x)$. For slowly varying fields $\phi(x)$ this expansion is expected to be well-behaved (see the following for a discussion of the validity of the DE). This is the basis of the *derivative expansion* that consists in proposing an *ansatz* for Γ_k involving only a finite number of derivatives of the field.

The underlying idea is that we are mostly interested (for the study of critical phenomena) [This seems inconsistent with something said in a previous paragraph: the precursor of critical behaviour shows up for $|p| \gg k$, and here we are saying that we only care about $|p| \rightarrow 0$ for the study of critical phenomena.] [Stricto sensu we should first take $k \rightarrow 0$ and then $p \rightarrow 0$ but it is too hard by (ii) above. So let us assume something about the double limit

²⁶ Reminder: This is a general property of Legendre transform. If $f(x)$ and $g(p)$ are Legendre transform of each other, i.e., $p = f'(x)$ and $x = g'(p)$ are inverse to each other, $f''(x) = dp/dx$ and $g''(p) = dx/dp$ so $f''(x) = (g''(p))^{-1}$.

²⁷ There can actually exist nonanalyticities showing up in the flow at finite k that do not result from the existence of a second order phase transition. This is what occurs for instance in the Random Field Ising Model at sufficiently low dimension or in the Pair Contact Process with Diffusion.

$(p, k) \rightarrow 0$ (scaling hypothesis enters here) and approach it by first taking $p \rightarrow 0$ and then $k \rightarrow 0$. Yes if we are not careful or nature is not kind (say letting scaling hypothesis break down) we may miss completely the point by [reversing limit order] in the long distance physics, that is the $|p| \rightarrow 0$ region of the correlation functions. Thus, we keep only the lowest orders of the expansion of Γ_k in $\partial\phi$ while we keep, for the moment being, all orders in the field ϕ . For the Ising case:

$$\Gamma_k = \int d^d x \left(U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x)) (\partial\phi)^2 \right) + \mathcal{O}(\partial^4) \quad (54)$$

The term U_k is the potential. The function $Z_k(\phi)$ is called the field renormalisation or wave-function renormalisation function²⁸. When $k \rightarrow 0$, U_k will become the *effective potential*.

For several fields in the $O(N)$ models, because of the $O(N)$ symmetry, any function of local fields can only depend on its ($O(N)$ -invariant) norm

$$\rho(x) := \frac{1}{2} \phi(x)^2 = \frac{1}{2} \sum_{i=1}^N \phi_i(x) \phi_i(x). \quad (55)$$

The approximation of Γ_k at order ∂^2 of the DE is

$$\Gamma_k = \int d^d x \left(U_k(\rho) + \frac{1}{2} Z_k(\rho) (\partial\phi)^2 + \frac{1}{4} Y_k(\rho) (\phi \cdot \partial\phi)^2 + \mathcal{O}(\partial^4) \right). \quad (56)$$

Notice that the term with coefficient $Y_k(\rho)$ is specific to $O(N)$ -models with $N \neq 1$: when $N = 1$ it can be absorbed into the term $\sim Z_k(\rho) \dots$.

To derive flow equations for the coefficients (U_k, Z_k, \dots) , we need to read them off from Γ_k . As it turns out, they can be derived from Γ_k or its functional derivatives evaluated at a uniform field configuration (with given norm ρ), as we can see below. Moreover, due to $O(N)$ -symmetry, the 'direction' of the constant vector in field-space is ad libitum. A particularly simple choice is to take only the zero-component in field-space being non-vanishing, i.e.,

$$\phi_{unif.}(\rho) := \begin{pmatrix} (2\rho)^{1/2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ with } \rho = \text{constant}. \quad (57)$$

- Evaluating Γ_k directly in a uniform field configuration, (57), kills all ∂ 's in (56) and yields

$$\Omega_d U_k(\rho) = \Gamma_k[\phi_{unif.}(\rho)] \quad (58)$$

²⁸ The function $Z_k(\phi)$ has, of course, nothing to do with the partition function $Z_k[h]$, although it is customary to use the same symbol for both functions.

with Ω_d the volume of space-time.

- The DE ansatz (56) implies that the 1PI 2-point function in the uniform configuration (57) becomes²⁹

$$\Gamma_{k,ij}^{(2)}(q) = U'_k(\rho) \delta_{ij} + \phi_i \phi_j U''_k(\rho) + Z_k(\rho) q^2 \delta_{ij} + \frac{1}{2} Y_k(\rho) \phi_i \phi_j q^2. \quad (59)$$

For components $i = j > 2$ in the above equation, only the first and the third terms contribute. Finally, deriving with respect to q^2 and evaluating at $q = 0$, we extract Z_k as the (running) coefficient of the quadratic term in q :

$$Z_k(\rho) = \partial_{q^2} \Gamma_{k,ij}^{(2)}[\phi_{unif.}(\rho)](q) \Big|_{q=0}. \quad (60)$$

Effectively, the derivative with respect to q^2 removes the terms of lower order in momentum, whereas by evaluation at vanishing momentum the terms of higher powers are neglected. Note that the above is valid up to any order in the DE.

- Generalisation to further coefficients follows the same recipe.

At this point, the derivation of a flow equation for any of the coefficients $U_k(\rho)$, $Z_k(\rho)$, etc. is straightforward (although, perhaps, tedious). For example, for U_k : take ∂_k of (58), apply Wetterich's equation (40) to $\partial_k \Gamma_k$ at the right hand side, and apply the ansatz (56) to the right hand side of Wetterich's equation. This yields

$$\begin{aligned} \partial_k U_k = & \frac{1}{2} \int_q \partial_k R_k(q) \left((N-1) (U'_k + Z_k q^2 + R_k(q))^{-1} \right. \\ & \left. + (U'_k + 2\rho U''_k + Z_k q^2 + Y_k \rho q^2 + R_k(q))^{-1} \right). \end{aligned} \quad (61)$$

We show the derivation in subsection VII.A.1. As can be imagined, the calculation of $\partial_k Z_k$ is more involved and we will again relegate it to subsection VII.A.2. The set of flow equations for U_k , Z_k (and Y_k) is a closed system constituting the DE truncation of Wetterich's equations up to order $O(\partial^4)$. We emphasise that the procedure involves no conceptual difficulty and can be automatized and generalized to higher derivative orders.

The remaining of this section is organised as follows. VII.A explains technical details in the derivation of flow equations in DE at order $O(\partial^4)$ and can be skipped without affecting the understanding of the rest of the lecture. VII.B introduces the Local Potential Approximation (LPA) and discusses its application to the Ising

²⁹ For the derivation of it see VII.A.1, (64).

model. VII.C discusses an important refinement of LPA that is related to the anomalous dimension. We conclude with a brief discussion of field expansion approximations on top of the DE, in VII.F.

A. Derivation of flow equations

1. Flow equation for U_k , (61)

Recall the DE ansatz (56) in terms of the $O(N)$ -invariant (55),

$$\Gamma_k = \int d^d u \left(U_k(\rho) + \frac{1}{2} Z_k(\rho) (\partial\phi)^2 + \frac{1}{4} Y_k(\rho) (\phi \cdot \partial\phi)^2 + O(\partial^4) \right), \quad (62)$$

where $\rho = \rho(u) = \frac{1}{2} \sum_l \phi_l(u) \phi_l(u)$. We now calculate its order-2 functional derivatives and evaluate in a uniform field configuration: Therefore, the only way for a term containing two $\partial\phi$ factors to survive is that each of them is “hit” by one functional derivative. So we can proceed as if $Z_k(\rho)$ and $Y_k(\rho)$ were field independent, which simplifies the calculation considerably. With this in mind, we formally apply the functional derivative $\delta\phi_l(u)/\delta\phi_i(x) = \delta_{il}\delta^d(x-u)$ and usual rules of calculus (in particular linearity and the chain rule) to each term of the integrand in (56)

$$\begin{aligned} \left. \frac{\delta^2 U_k(\rho(u))}{\delta\phi_i(x)\delta\phi_j(y)} \right|_{\phi_{unif.}} &= (U'_k \delta_{ij} + U''_k \phi_i \phi_j) \delta_{ux,uy} \\ \left. \frac{\delta^2 Z_k(\rho) \partial\phi(u) \cdot \partial\phi(u)}{2\delta\phi_i(x)\delta\phi_j(y)} \right|_{\phi_{unif.}} &= Z_k(\rho) \delta_{ij} \partial_u \delta_{ux} \partial_u \delta_{uy} \\ \left. \frac{\delta^2 Y_k(\rho) (\phi(u) \cdot \partial\phi(u))^2}{4\delta\phi_i(x)\delta\phi_j(y)} \right|_{\phi_{unif.}} &= \frac{1}{2} Y_k(\rho) \phi_i \phi_j \partial_u \delta_{ux} \partial_u \delta_{uy}, \end{aligned} \quad (63)$$

with $\delta_{xy} = \delta^d(x-y)$ and $\delta_{ux,yu} = \delta_{ux}\delta_{yu}$. Then we integrate each of them with respect to u ³⁰ and sum the individual terms to obtain

$$\begin{aligned} \left. \frac{\delta^2 \Gamma_k}{\delta\phi_i(x)\delta\phi_j(y)} \right|_{\phi_{unif.}} &= (U'_k(\rho) \delta_{ij} + U''_k(\rho) \phi_i \phi_j) \delta_{xy} \\ &\quad - \left(Z_k(\rho) \delta_{ij} + \frac{1}{2} Y_k(\rho) \phi_i \phi_j \right) \partial_x^2 \delta_{xy} \end{aligned}$$

Taking Fourier transform gives

$$\begin{aligned} \Gamma_{k,ij}^{(2)}(q) &= U'_k(\rho) \delta_{ij} + \phi_i \phi_j U''_k(\rho) \\ &\quad + Z_k(\rho) q^2 \delta_{ij} + \frac{1}{2} Y_k(\rho) \phi_i \phi_j q^2. \end{aligned} \quad (64)$$

³⁰ In the last two cases we need to integrate by parts: Since ρ, ϕ_i, ϕ_j are norm and components of $\phi_{unif.}$ so are u independent they reduce to the same calculation $\int_u \partial_u \delta^d(u-x) \partial_u \delta_{uy} = -\int_u \partial_u^2 \delta_{ux} \delta_{uy} = -\partial_x^2 \delta_{xy}$.

Thanks to $O(N)$ symmetry, we can take $\phi_{unif.}$ to be $\phi_{unif.}(\rho) = (\sqrt{2\rho}, 0, \dots, 0)^T$. We assume also that the regulator is $O(N)$ invariant, *i.e.*, a scalar matrix $R_{k,ij}(q) = R_k(q) \delta_{ij}$. Then $(\Gamma_k^{(2)}(q) + R_k(q))_{ij}$ becomes a diagonal matrix whose only non-zero entries are

$$\begin{aligned} (\Gamma_k^{(2)}(q) + R_k(q))_{11} &= U'_k(\rho) + 2\rho U''_k(\rho) + \\ &\quad + q^2 (Z_k(\rho) + \rho Y_k(\rho)) + R_k(q), \\ (\Gamma_k^{(2)}(q) + R_k(q))_{jj, j>1} &= U'_k(\rho) + q^2 Z_k(\rho) + R_k(q). \end{aligned} \quad (65)$$

Plugging its inverse into the right hand side of Wetterich’s equation (40) gives

$$\begin{aligned} \partial_k \Gamma_k[\phi_{unif.}(\rho)] &= \\ &= \frac{\Omega_d}{2} \int_q \partial_k R_k(q) \left\{ (N-1) (U'_k + Z_k q^2 + R_k)^{-1} \right. \\ &\quad \left. + (U'_k + 2\rho U''_k + Z_k q^2 + Y_k \rho q^2 + R_k)^{-1} \right\}. \end{aligned} \quad (66)$$

According to (58) the left hand side is just $\Omega_d \partial_k U_k(\rho)$, so we have obtained (61).

2. Flow equation for Z_k

Since Z_k is read off from the 1PI 2-point function, see (60), we need to calculate $\partial_k \Gamma_k^{(2)}$ to obtain its flow equation. We shall do this from Wetterich’s equation in its compact form (41)

$$\Gamma_k[\phi] = \frac{1}{2} \text{Tr} (\partial_k R_k G_k[\phi]) \quad (67)$$

where

$$G_k[\phi] := (\Gamma_k^{(2)}[\phi] + R_k)^{-1},$$

where its diagrammatic representation, see figure (5), will prove particularly useful.

We shall regard G_k, R_k and $\Gamma_k^{(2)}$ as “super” matrices whose rows and columns are indexed by $\{(p, i) : p \in \mathbf{R}^d, i = 1, \dots, N\}$ (and for which multiplication or contraction means integrating over p and summing over i) and apply formally calculus for finite matrices. For example, we can apply $\partial(A^{-1}) = -A^{-1}(\partial A)A^{-1}$ to derive $G_k[\phi]$ respect to $\phi_i(p)$:

$$\frac{\delta G_k}{\delta\phi_i(p)} = -G_k \frac{\delta \Gamma_k^{(2)}}{\delta_i(x)} G_k = -G_k \Gamma_{i,k}^{(3)} G_k. \quad (68)$$

Similarly, we can take functional derivative of (67) to obtain the flow of one-point function:

$$\partial_k \frac{\delta \Gamma_k}{\delta\phi_i(p)} = -\frac{1}{2} \text{Tr} (\partial_k R_k G_k \Gamma_{i,k}^{(3)} G_k), \quad (69)$$

$$\partial_k \frac{\delta \Gamma_k}{\delta \phi_i(p)} = -\frac{1}{2} \text{Diagram}$$

FIG. 6 Diagrammatic representation of the flow equation for the one-point function, (69). The shaded blob denotes the three-point vertex, $\Gamma^{(3)}$.

$$\partial_k \frac{\delta^2 \Gamma_k}{\delta \phi_j(-p) \delta \phi_i(p)} = -\frac{1}{2} \text{Diagram 1} + \text{Diagram 2}$$

FIG. 7 Diagrammatic representation of the flow of the two-point function, cf. (70). Shaded blobs denote vertex functions, $\Gamma^{(3)}$ and $\Gamma^{(4)}$.

which is given diagrammatically in figure 6.

Now taking another functional derivative yields the flow equation for the two-point functions, $\Gamma_{k,ij}^{(2)}$. This time, the functional derivative can hit one of the two propagators, hence, relation (68) can be applied again. This results in diagram involving two three-point functions, where the overall-sign is positive and the combinatorial factor of 2 emerges from symmetry. In addition, the derivative can act on the $\Gamma_k^{(3)}$ to give a four-point function, $\Gamma_k^{(4)}$. As a consequence, there are two distinct diagrams contributing to the flow of the two-point function. Omitting the $O(N)$ -indices and momenta for brevity, the equation is schematically given by

$$\Gamma^{(2)} = -\frac{1}{2} \text{Tr} \left(\Gamma_k^{(4)} G_k R_k G_k \right) + \text{Tr} \left(\Gamma_k^{(3)} G_k \Gamma_k^{(3)} G_k R_k G_k \right), \quad (70)$$

which is diagrammatically represented in figure 7. We can read off from each of the diagrams the corresponding contribution to $\Gamma_k^{(2)}(p)$. For example, the second diagram gives the term

$$\sum_q \int \Gamma_k^{(3)}(\{p, i\}, \{-q, i'\}, \{q-p, i''\}) \Gamma_k^{(3)}(\{-p, j\}, \{q, j'\}, \{p-q, j''\}) G_{k,i'j'}(q) G_{k,i''l}(q-p) G_{k,lj''}(q-p) \partial_k R_{k,ll}(q-p).$$

Here \sum denotes a summation over all indices other than i and j . (Note that in order to preserve $O(N)$ symmetry, R_k is always taken to be a diagonal (indeed scalar) with respect to inner indices. Hence we sum only over $R_{k,ll}$.)

So far we have been doing *exact* calculation for $\Gamma^{(2)}(q)$. Now we shall replace Γ by its DE ansatz (56) and then invoke (60) to extract the flow of Z_k .

B. Local potential approximation: Application to the Ising model

The LPA further simplifies the DE ansatz (56) by setting $Z_k \equiv 1$ and $Y_k \equiv 0$:

$$\Gamma_k = \int d^d x \left(U_k(\phi(x)) + \frac{1}{2} (\partial \phi)^2 \right) \quad (71)$$

Then (61) is the only RG flow equation that remains in the hierarchy. In the case of the Ising model ($N = 1$), it simplifies further to

$$\partial_k U_k = \frac{1}{2} \int_q \partial_k R_k (q^2 + R_k + U'_k + 2\rho U''_k)^{-1} \quad (72)$$

Its initial condition $U_{k=\Lambda}$ is the potential term in the Hamiltonian of the Ising model. As discussed in a previous section (see (14) in II.B.2), a sensible choice can be

$$U_\Lambda(\rho) = \frac{\lambda_\Lambda}{2} (\rho - \rho_{min})^2. \quad (73)$$

It is now a good time to discuss the choice of regulator $R_k(q)$. In the *exact* NPRG equation (40), the choice of $R_k(q)$ only affects the path of the flow $\Gamma_\Lambda \rightsquigarrow \Gamma_0$ but not its endpoints. This will no longer be the case when approximations are made, and the dependence of Γ_0 on R_k becomes in turn a good indicator of the error provoked by the approximation being made. We draw this idea in figure 8. More generally speaking, for any physical

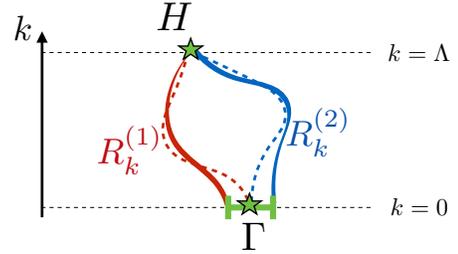


FIG. 8 Dependence of the RG flow on the regulator choice, with or without (dotted lines) approximation.

quantity Q , we should have

$$\frac{\delta Q}{\delta R_k} = 0$$

in the exact case. So as a thumb rule, we should *minimise* such dependence when making approximations. This is called the *principle of minimal sensitivity* (PMS). Another guiding rule is the *principle of fastest apparent convergence* (PFAC), which states that approximation schemes should be optimised so that higher order corrections are the smallest possible (compared to previous

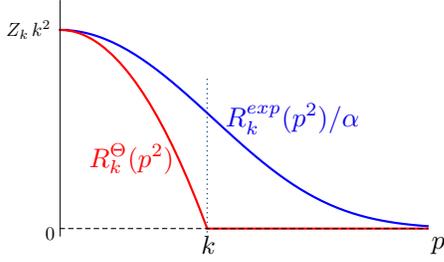


FIG. 9 Optimised and exponential regulators, (74) and (75).

terms). In general, the two rules are expected to be consistent with each other: The contrary indicates usually that the approximation is problematic.

Back to (72), we propose two choices of R_k (see figure 9 for plots), which are both known to yield quantitatively accurate critical exponents and are, furthermore, convenient for analytic and numerical applications:

$$R_k^\Theta(q) := Z_k(k^2 - q^2)\theta(k^2 - q^2), \quad (74)$$

$$R_k^{exp}(q) := \alpha Z_k \frac{q^2}{e^{q^2/k^2} - 1}. \quad (75)$$

Here θ denotes the Heaviside step function, α in (75) is a free parameter to optimise the regulator (according to the thumb rules discussed above). Note that, although $Z_k = 1$ in the LPA, we have kept Z_k explicit on the above expressions to use them in the following section, where the LPA' will be discussed.

The regulator R_k^Θ is particularly convenient because it allows evaluating the momentum integral in (72) easily:

$$k\partial_k U_k = \frac{4v_d}{d} \frac{k^d}{1 + (U' + 2\rho U'')k^{-2}}, \quad (76)$$

with a dimension-dependent constant³¹ Let us now show and comment snapshots of the evolution flow of U_k that can be obtained by integrating the PDE (76), and possible scenarios that can be obtained by varying parameters in the initial condition (73). Note that the temperature enters via the value of the bare parameters.

We have already mentioned the assumption that we start with a "Mexican hat"-potential as initial condition, see figure 10. Note that we will typically refer to one of its minima, as they are all equivalent. At the beginning of the flow, $k = \Lambda - \epsilon$, the integration of fluctuations puts

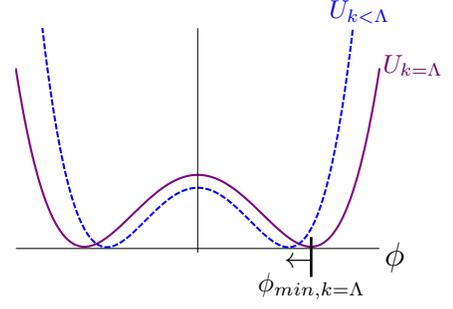


FIG. 10 Initial condition for the flow of the potential $U_k(\phi)$. Fluctuations put more disorder in the system and, hence, the minima tend to smaller (absolute) values in the field.

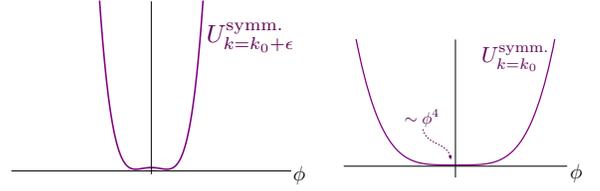


FIG. 11 Degeneration of the minima of the effective potential at a non-vanishing scale $k_0 > 0$ in the case of symmetry restoration.

more disorder to the system and thus makes the minima of U_k decrease towards $\phi = 0$ (we are considering a zero average magnetization as the measure of 'total disorder').

We now sketch the three distinct, possible scenarios corresponding to the temperature being above, below or precisely at its critical value:

- Figure 11 illustrates the case, where the minima of U_k become degenerate at $\phi = 0$ for a non-vanishing scale, $k_0 > 0$. This relates to the transition from an ordered to a disordered regime. Fluctuations from lower scales $k < k_0$ increase the disorder in the system further. As a result, at the end of the flow the minimum of the potential remains at zero field expectation value, see figure 12. This means that we are in the high temperature phase $T > T_c$. The scale k_0 is related to the correlation scale $k_0 \sim \xi^{-1}$.
- Another possibility is that the minima of U_k are again driven towards smaller field expectation values, however, they never reach $\phi = 0$. That means, that below a scale k_0 the location of the minima remain fixed. From the scale k_0 and below, it is the value of the potential for $|\phi| < \phi_{min}$ that decreases, so that U_0 becomes convex in the limit $k \rightarrow 0$, i.e. $U_{k=0}$ is completely flat for $|\phi| < \phi_{min}$. This shape of the potential signals the low temperature phase $T < T_c$. Here, ϕ_{min} is the spontaneous magnetisation. This situation is depicted in figure 13.

³¹ The volume factor v_d stems from the variable transformation $x = q^2$ in loop-integrals,

$$\int d^d q f(q) = 2v_d \int_0^\infty dx x^{\frac{d}{2}-1} f(x). \quad (77)$$

Obviously, this relation simplifies integrations where rotational symmetry only allows for quadratic dependence on momentum. v_d .

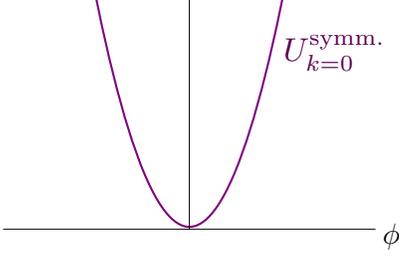


FIG. 12 Shape of the effective potential $U_{k=0}$ in the symmetric phase.

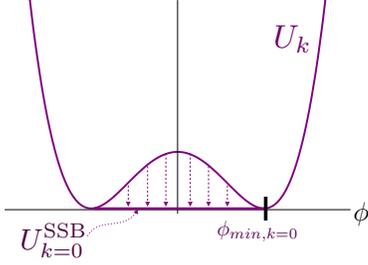


FIG. 13 Flow of the effective potential with $\phi_{min} > 0$ in the case of spontaneous symmetry breaking.

- The third distinct case is the point for which the temperature is precisely at the critical value, $T = T_c$. In this case the minima of the potential vanish for $k = 0$, $\phi_{min,k=0} = 0$. However, for every non-vanishing scale the potential has a non-vanishing expectation value, i.e. $\forall \epsilon > 0 : \phi_{min,k=\epsilon} > 0$. The fact that criticality can be seen only at $k = 0$ makes practical applications intricate. We discuss these issues in subsection VII.D.

C. LPA' and anomalous dimension

As was already mentioned, LPA' is a refinement of LPA. It allows the field normalisation Z_k to flow, but independently of the local field value

$$\Gamma_k = \int d^d x \left(U_k(\phi(x)) + \frac{1}{2} Z_k (\partial\phi)^2 \right) \quad (78)$$

It is also called the $O(\partial^2)$ approximation or the leading approximation. Recall that in the DE ansatz (56), the expression (60) is used to extract $Z_k(\rho)$ from Γ_k . Now in LPA', $Z_k(\rho)$ is taken to be independent of ρ and there is a choice to make. We shall adopt the following choice:

$$\begin{aligned} Z_k(\rho) &\rightsquigarrow Z_k(\rho_{min}) \\ \left. \frac{\partial U_k(\rho)}{\partial \rho} \right|_{\rho_{min}} &= 0. \end{aligned} \quad (79)$$

where the second equation determines implicitly ρ_{min} . *Remark.* As a general rule, when we Taylor expand a

function of ρ , we do it around ρ_{min} where $U_k(\rho)$ attains its minimum. The reason is that we are interested in the critical regime, where the external source necessarily vanishes: $h = 0$. Now with the DE Ansatz for Γ_k it is not hard to see that $h = \delta\Gamma_k/\delta\phi$ becomes proportional to $U'_k(\rho)$ when evaluated in a uniform field configuration³². Hence $h = 0 \Leftrightarrow \rho = \rho_{min}$. For LPA' in particular, we Taylor expand to order 0, i.e., replace Z_k by its value at the minimum of the effective potential.

The advantage of LPA' (over LPA) is the possibility to access the anomalous dimension η . Recall that η can be read from the p dependence of the two-point function $\Gamma_{k=0}^{(2)}(p) \sim p^{2-\eta}$ (see (25) and (28)). Based on the scaling hypothesis, we show below VII.C.1 that

$$\Gamma_k^{(2)}(p, u_0) - \Gamma_k^{(2)}(0, u_0) \sim p^2 k^{-\eta}. \quad (80)$$

(The order of limits being taken here is: first send $p \rightarrow 0$ keeping k finite, and then look at the k -dependence). Now LPA' (78) implies $\Gamma_k^{(2)}(p, u_0) - \Gamma_k^{(2)}(0, u_0) \sim p^2 Z_k$. Compared to (80), we see that $Z_k \sim k^{-\eta}$. Therefore, LPA ($Z_k = 1$) will give the mean field value $\eta = 0$ while with LPA', one can get non-trivial η values.

1. Scaling of 1PI propagator (Derivation of 80)

At criticality ($\phi = 0$ and $h = 0$), we consider the following ratio

$$\frac{\Gamma_k^{(2)}(p, u_0)}{\Gamma_k^{(2)}(0, u_0)} = \tilde{f} \left(\frac{p}{k}, \frac{k}{u_0^{1/(4-d)}} \right) \sim f \left(\frac{p}{k} \right) \quad (81)$$

where we have used that the ratio needs to be a dimensionless quantity and finite in the limit $0 < k \ll u_0^{1/(4-d)}$.

By dimensional analysis, we find

$$\Gamma_k^{(2)}(0, u_0) = k^2 g \left(\frac{k}{u_0^{1/(4-d)}} \right) \sim k^2 \left(\frac{k}{u_0^{1/(4-d)}} \right)^{-x} \quad (82)$$

where, in the last step, we have introduced the *scaling hypothesis*, that is, the fact that at criticality we expect scale invariance which implies that quantities depending on scales are power laws.

Then, using (81), we find

$$\Gamma_k^{(2)}(p, u_0) \sim k^2 \left(\frac{k}{u_0^{1/(4-d)}} \right)^{-x} f \left(\frac{p}{k} \right) \quad (83)$$

which, in the limit $k \rightarrow 0$ with $p > 0$, must be finite. We conclude that for $p/k \gg 1$

$$f \left(\frac{p}{k} \right) \sim \left(\frac{p}{k} \right)^{2-x} \quad (84)$$

³² A reader finding it hard to see may want to learn taking functional derivatives in the beginning of subsection VII.A.1

which implies for $k \ll p \ll u_0^{1/(4-d)}$

$$\Gamma_k^{(2)}(p, u_0) \sim p^{2-x} \quad (85)$$

and, thus, x is the anomalous dimension: $x = \eta$.

In the other limit where $p \ll k$ (with $k \ll u_0^{1/(4-d)}$) the theory is regularized and the function $f(p/k)$ is the function f can therefore be expanded

$$f\left(\frac{p}{k}\right) = 1 + c\left(\frac{p}{k}\right)^2 + \dots \quad (86)$$

Using (82), we can now compute the behavior of $\Gamma_k^{(2)}(p, u_0)$, in the opposite limit of (85) where $p \ll k$ (and $k \ll u_0^{1/(4-d)}$) which is the domain of validity of the DE:

$$\begin{aligned} \Gamma_k^{(2)}(p, u_0) - \Gamma_k^{(2)}(0, u_0) &\sim \Gamma_k^{(2)}(0, u_0) \left(f\left(\frac{p}{k}\right) - 1 \right) \\ &\underset{p \ll k}{\sim} p^2 k^{-\eta}. \end{aligned} \quad (87)$$

Since, within the DE,

$$\Gamma_k^{(2)}(p, u_0, \phi = 0) = U''(\phi = 0) + Z_k p^2 + O(p^4), \quad (88)$$

we find that at criticality and in the limit $k \ll u_0^{1/(4-d)}$:

$$Z_k \sim k^{-\eta} \quad (89)$$

which shows that η can be computed either from the p -dependence of $\Gamma_k^{(2)}(p, u_0)$ when $u_0^{1/(4-d)} \gg p \gg k$ (which is unreachable within the DE but that can be obtained in the BMW scheme) or from the k -dependence of Z_k which is computable from the DE.

D. The co-moving frame, self-similarity and fixed-points

In this section we address and circumvent problems which arise in practical studies of the flow of the potential in LPA close to criticality. At first, we review why the naive approach necessarily fails. In a second step, we introduce the co-moving frame and the (closely related) notions of self-similarity and fixed-points. We'll be again using the $O(N)$ -models in LPA and LPA' as templates.

The straightforward way to study the flow of the potential is to discretise the space and compute the values of the potential at fixed grid points, hence, $\phi(x) \rightarrow \phi_i := \phi(x_i)$, where the index i refers to the lattice sites. Those need to be chosen such that the form of the potential, cf. section VII.B, is approximated accurately. For the system far away from criticality and a sufficiently narrow grid this can be done easily: up

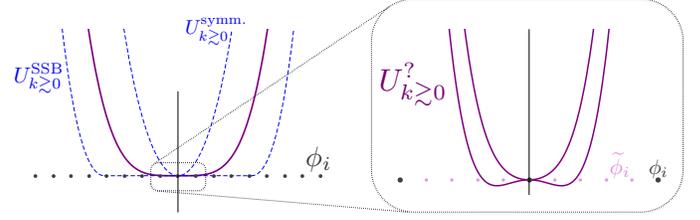


FIG. 14 Rescaling of the grid-resolution $\phi_i \rightarrow \tilde{\phi}_i$ to study the effective potential of a system close to criticality.

to the accuracy of the grid-spacing we can determine the position of the minima. For the system right at criticality the minima must reach $\phi = 0$ only in the strict limit $k = 0$. That means, that for arbitrarily small k the grid must be fine enough to resolve if the minima are degenerate or distinct. At some point, however, the distance of the minima from the origin inevitably becomes smaller than the grid-spacing and, hence, we can not tell if we are right at the phase transition. At this point we must redefine the grid to smaller spacing. This is sketched in figure 14. In order to change the grid we have two possibilities, which correspond to active or passive transformations of the reference frame: we can either zoom the fields or shrink the lattice spacing.

This paragraph may not read well for some readers.

The co-moving frame is a clever idea to zoom the fields ϕ as the minima approach the origin with decreasing scale k , simultaneously. Remember that in the flow at given k we are sensitive to fluctuations $p \approx k$. The idea of the co-moving frame is to rescale this region to its original size, so instead of looking on the flow with k from the outside we move together with the scale to smaller regions. In other words, instead of measuring the momenta p and k as dimensionful quantities, we introduce *dimensionless variables*, $\tilde{p} = p/k$. In that sense, we want to measure all quantities in units of k . From the scaling hypothesis we know that for a system close to criticality all quantities scale with a power-law. By dimensional analysis we can find and divide by the corresponding power in order to express all quantities in dimensionless variables.

As a consequence of the fact that there is only one scale k (with which we move), we have lost the explicit dependence on k in the flow equation. In other words, the flow does not know at which scale it is. Therefore, we can re-interpret the flow of the potential as its evolution in a dimensionless "RG-time"³³, $t := \log k/\Lambda$. When the shape of the potential does not change anymore in the evolution it has reached its *fixed-point*: $U^* : \partial_t U_k^* = 0$.

³³ Note that the system evolves from $t = 0 \rightarrow -\infty$.

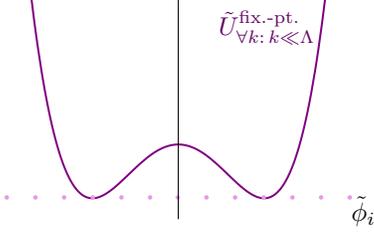


FIG. 15 Self-similar fixed-point potential \tilde{U}_k at criticality in the co-moving frame. The zoom in the fields is done such that \tilde{U}_k does not change its shape for smaller k . In particular the minima are fixed at non-zero positions, even though the system is critical.

It is now *self-similar*, i.e. it is the same for all scales if we move with the scale. Figure 15 shows the self-similar fixed-point potential in the co-moving frame, where the grid is defined in the new variables $\tilde{\phi}$. To emphasise the connection to figure 14 we stress that the fixed-point potential is unchanged for arbitrarily small k .

We now turn to the co-moving frame for the $O(N)$ -models in the LPA and advance to the LPA' below.

For the LPA it turns out to be sufficient to go to dimensionless variables to get rid of the explicit k -dependence in the flow equation. Here, on the left hand side of the flow of the potential³⁴ for an optimised regulator, (74),

$$k\partial_k U_k(\rho) = \frac{4v_d k^d}{d} \left\{ \frac{1}{1 + (U'_k + \rho U''_k) k^{-2}} + \frac{N-1}{1 + U'_k k^{-2}} \right\}. \quad (90)$$

Note that we have also turned the derivative ∂_k into a dimensionless derivative, $k\partial_k = \partial \log k$, as we transform the equation to dimensionless quantities. By dimensional analysis in units of k , we find

$$[x] = k^{-1}, [p] = k, [\phi] = k^{\frac{d-2}{2}}, [U_k] = k^d. \quad (91)$$

The transformation to dimensionless quantities (tilded symbols) is given by

$$\begin{aligned} p &= \tilde{p} k, \quad x = \tilde{x}/k, \\ \phi(x) &= k^{\frac{d-2}{2}} \tilde{\phi}(\tilde{x}), \\ U_k(\phi(x)) &= k^d \tilde{U}_k(\tilde{\phi}(\tilde{x})). \end{aligned} \quad (92)$$

As a result, the flow equation of the potential is given by

$$\begin{aligned} \partial_t \tilde{U}_k &= -d \tilde{U}'_k + (d-2) \tilde{p} \tilde{U}''_k + \\ &+ \frac{4v_d}{d} \left\{ \frac{1}{1 + \tilde{U}'_k(\tilde{\rho}) + 2\tilde{p} \tilde{U}''_k(\tilde{\rho})} + \frac{N-1}{1 + \tilde{U}'_k(\tilde{\rho})} \right\}. \end{aligned} \quad (93)$$

³⁴ We refrain from deriving (90) explicitly here as it can be easily inferred from (61) and (76).

The first two terms encode the dimensions of the potential and the fields, whereas the last term contains the dynamics from the radial and Goldstone modes.

By transforming to dimensionless variables we have succeeded in dropping the k -dependence in LPA. In the LPA' (and any higher order), however, we have to also consider the scaling of the wave-function renormalisation Z_k . The flow equation for the potential in LPA' is simply given by (61) with the condition $Y_k = 0$, hence,

$$\begin{aligned} \partial_t U_k(\rho) &= \frac{1}{2} \int_q \partial_t R_k(q) \left(\frac{N-1}{Z_k q^2 + R_k(q) + U'_k(\rho)} \right. \\ &\quad \left. + \frac{1}{Z_k q^2 + R_k(q) + U'_k(\rho) + 2\rho U''_k(\rho)} \right) \end{aligned} \quad (94)$$

Indeed, for a system near criticality, we have the scaling $Z_k \stackrel{k \rightarrow 0}{\sim} k^{-\eta}$ (where η is the anomalous dimension) so we will take this into account in our dimensional counting in k . It is convenient to define the running anomalous dimension η_k by

$$\eta_k = -\frac{k\partial_k Z_k}{Z_k}. \quad (95)$$

Now, the scaling behavior of Z_k implies $\eta_k \stackrel{k \rightarrow 0}{\rightarrow} \eta$, i.e., the fixed point of η_k is the anomalous dimension³⁵.

Expressed in the new variable, η_k , we find for the optimised regulator, (74),

$$k\partial_k R_k^\Theta(q) = -\eta_k R_k^\Theta(q) + 2Z_k k^2 \Theta(k^2 - p^2). \quad (96)$$

Nevertheless, k will appear explicitly in the flow equation due again to the non-trivial running of Z_k , even in dimensionless variables. We cannot factor Z_k in both the numerator and the denominator of the flow equation, because the terms $\sim U'_k, \rho U''_k$ scale differently than the term quadratic in momentum. However, by a change of variables, we can eliminate the dependence on Z_k : we modify the field renormalisation in a k -dependent way to express the flow in renormalised (and dimensionless) quantities. The transformation of the field is given by

$$\phi(x) = Z_k^{-1/2} k^{\frac{d-2}{2}} \tilde{\phi}(\tilde{x}). \quad (97)$$

This ensures that for (92) with the modified definition of $\tilde{\phi}$, (97), the terms $\sim \tilde{U}'_k, \tilde{p} \tilde{U}''_k$ scale in the same way³⁶ as $Z_k q^2$. Therefore, the factor of Z_k cancels between the numerator and the denominator in the flow equation, leaving only the dimensionless η_k . As a result, for

³⁵ The initial condition for η_k at $k = \Lambda$ is given uniquely by the running. Indeed, as the anomalous dimension only changes in the critical region of small k , usually the initial condition $\eta_\Lambda = 0$ is applicable.

³⁶ The derivative is now taken with respect to the renormalised

a general regulator $R_k(q) = Z_k q^2 r(q^2/k^2)$, we obtain the flow of the potential

$$\begin{aligned}
k\partial_k \tilde{U}_k &= -d\tilde{U}_k + (d-2 + \eta_k) \tilde{\rho} \tilde{U}'_k \\
&\quad -v_d \int_0^\infty dy y^{d/2} (\eta_k r(y) + 2yr'(y)) \\
&\quad \times \left\{ \frac{1}{y(1+r(y)) + \tilde{U}'_k + 2\tilde{\rho}\tilde{U}''_k} + \frac{N-1}{y(1+r(y)) + \tilde{U}'_k} \right\}, \tag{99}
\end{aligned}$$

with $y := q^2/k^2$ and $\tilde{U}'_k = \tilde{U}'_k(\tilde{\rho})$ and similarly for $U''(k)$. The structure of the equation is similar to the one in LPA, (93): On the right hand side, the first term encodes the dimension of the potential while the second term considers the dimension of the field, which is now taking into account the non-trivial scaling of the propagator, η_k . The dynamical part of the flow is given by the loop-integral.

The question is left, whether such a fixed-point potential exists. Indeed, for $d = 3$ (and arbitrary N) we find exactly one solution for \tilde{U}^* (aside from the trivial solution $\tilde{U}_k^* = \text{constant}$), which is the famous *Wilson-Fisher* fixed-point. The existence of only one solution is highly-nontrivial, because, in principle, an infinite number of fixed-point potentials mathematically could exist. In the present case, however, only a finite number is globally defined: except for a discrete number³⁷ all other solutions have singularities for a finite ρ .

Two comments address the question of convexity and dimensional dependence of the fixed-point potential:

- The fixed-point potential does not need to be convex (unlike the normal, dimensionful potential), because for a divergent correlation length the comoving frame is an *infinite* zoom in the fields,

$$\phi(x) = Z_k^{-1/2} k^{\frac{d-2}{2}} \tilde{\phi}(\tilde{x}) \stackrel{k \rightarrow 0}{\sim} \underbrace{k^{\frac{\eta+1}{2}}}_{\xrightarrow{k \rightarrow 0} 0} \tilde{\phi}(\tilde{x}). \tag{100}$$

for $d = 3$ and small η . Therefore, a finite range in $\tilde{\phi}$ corresponds to zero range in ϕ .

- The number of solutions depends on the dimension, even though the equation is the same. For

field, hence, in a slight abuse of notation, we find

$$\begin{aligned}
\rho(x) &= Z_k^{-1} k^{d-2} \tilde{\rho}(\tilde{x}), \\
k^d U'_k(\rho) &= \tilde{U}'_k(\tilde{\rho}) \frac{\delta \tilde{\rho}}{\delta \rho} = Z_k k^2 \tilde{U}'_k(\tilde{\rho}), \\
k^d \rho U''_k(\rho) &= k^d Z_k^{-1} k^{d-2} \tilde{\rho} \tilde{U}''_k(\tilde{\rho}) \left(\frac{\delta \tilde{\rho}}{\delta \rho} \right)^2 = Z_k k^2 \tilde{\rho} \tilde{U}''_k(\tilde{\rho}), \tag{98}
\end{aligned}$$

i.e. it scales like the quadratic part in the fields.

³⁷ Note that the discrete number of possible solutions is not necessarily finite.

	MC	7-loop PT	$O(6)$	$O(4)$	$O(2)$	LPA
ν	0.63001(10)	0.6304(13)	0.6303(1)	0.632	0.628	0.6506
η	0.0362(2)	0.0335(25)	0.0357(2)	0.033	0.044	0
ω	0.832		0.827			

TABLE I Comparison of critical exponents for $N = 1$, $d = 3$ from Monte-Carlo lattice simulations, 7-loop PT and different orders of the DE in NPRG.

$O(N)$ -models in LPA', in $d = 2.9$ there are two non-trivial fixed-point potentials. For $d = 2$ there is an infinite number of possible fixed-point potentials, which constitutes a so-called *multi-critical fixed-point*. This result has been known from conformal field theories.

E. Quantitative results of the NPRG

In this subsection we compare quantitative results obtained in the NPRG with other methods. For $N = 1$, $d = 3$ we focus on the critical exponents ν , (6), and η , (80), and the *correction-to-scaling* exponent, ω , defined as to correction to the dominant scaling of the correlation length, viz.

$$\xi \sim |T - T_c|^{-\nu} \left(1 + A(T - T_c)^{-\omega} \right), \tag{101}$$

where A is a constant.

In principle, the results should depend on three approximations that have been made: the initial condition, the order of the DE and the choice of the regulator function. However, as we consider the self-similar fixed-point potential, the dependence on the initial condition is lost. The dependence on the choice of the regulator is mild, even for very different classes of regulators as given in (74), (75), thus, we shall not comment on it further. As a consequence, the only significant errors stem from truncational artifacts.

Monte-Carlo lattice computations serve as the benchmark, hence, we measure the accuracy with respect to results obtained there. In addition, we state the 7-loop PT predictions, in comparison with which we see the strength of the NPRG, as even comparatively simple truncations³⁸ yield good accuracy. The results are given in table I. [[@ Bertrand: I took the numbers from your review, Lect.Notes Phys. 852 \(2012\) 49-132. Are those up-to-date? You said in the lecture that the PT results](#)

³⁸ Note that in $d = 2$ the results are much worse. [[Is there a \(simple\) reason for this? I assume that going to slightly higher order in the DE is not sufficient to improve?](#)] For example, the exact result of Onsager $\eta^{\text{Onsager}} = 1/4$ is approximated by the $O(6)$ value by $\eta^{O(6)} = 0.237$ only.

were 6-loop, in your review it says 7-loop? Are they the same, or was that the comment you made that 'parts of the 7-loop contributions are taken into account there'?

F. Field expansions

On top of the DE, we can make a field expansion of U_k, Z_k, \dots . The simplest such truncation consists in using the LPA and in keeping only the first two terms of the expansion of U_k in powers of ϕ :

$$U_k(\rho) = g_{2,k}\rho + g_{4,k}\rho^2 + \dots \quad (102)$$

But as is explained above (see remark after (79)), it is better to expand U_k around its minimum instead

$$U_k(\rho) = u_{2,k}(\rho - \rho_{min})^2 + u_{3,k}(\rho - \rho_{min})^3 + \dots$$

With these kinds of ansatz, the RG equation on Γ_k becomes a set of ordinary differential equations for the couplings retained in the *ansatz*:

$$\partial_t g_{n,k} = \beta_n(\{g_{p,k}\}) . \quad (103)$$

If U_k, Z_k, \dots are not truncated in a field expansion, the RG equation on Γ_k becomes a set of coupled partial differential equations for these functions (as we have seen in the Ising model example VII.B). The initial condition at scale Λ is, as always, given by the Hamiltonian of the model.

An important remark follows. Let us notice that if the k -dependence of the couplings was neglected, the *ansatz* of (102) would exactly coincide with the ansatz chosen by Landau to study second order phase transitions. We

know that it would lead to the mean field approximation which overall fails to predict the critical exponents. This is still the case considering a more 'refined' ansatz of the kind of the LPA', provided that we have removed the k -dependence. Hence, let us stress the remarkable fact that much of the physics is actually contained in the k -dependence.

VIII. THE BMW EXPANSION

So far, only $\Gamma_k^{(2)}(\mathbf{p})$ has been computed for the $O(N)$ models.

Appendix A: Gaussian free theory, perturbation theory, Mean Field and the classical approximation

We want to shed some light on these concepts, which are interrelated. Let's consider a theory with just one coupling constant like $g\varphi^4$. *Perturbation theory* stands for a perturbative expansion in the coupling constant or, equivalently, in \hbar :

$$\Gamma^{(2)} = p^2 + m^2 + \sum_i ((g\hbar)^i \dots) \quad (A1)$$

$$\Gamma^{(4)} = g + \sum_i ((g\hbar)^i \dots) \quad (A2)$$

The *gaussian free theory* is the limit $g \rightarrow 0$, so no interactions and only $\Gamma^{(2)}$ is non zero. On its part, *Mean Field = the classical approximation* stand for the limit $\hbar \rightarrow 0$, which is also the *tree-level* of perturbation theory.