IPhT-t13/039

On irreducible maps and slices

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Abstract

We consider the problem of enumerating d-irreducible maps, i.e. planar maps whose all cycles have length at least d, and such that any cycle of length d is the boundary of a face of degree d. We develop two approaches in parallel: the natural approach via substitution where these maps are obtained from general maps by a replacement of all d-cycles by elementary faces, and a bijective approach via slice decomposition which consists in cutting the maps along shortest paths. Both lead to explicit expressions for the generating functions of d-irreducible maps with controlled face degrees, summarized in some elegant "pointing formula". We provide an equivalent description of d-irreducible slices in terms of so-called d-oriented trees. We finally show that irreducible maps give rise to a hierarchy of discrete integrable equations which include equations encountered previously in the context of naturally embedded trees.

1. Introduction

1.1. General introduction

The enumeration of planar maps has been of constantly renewed interest in combinatorics since Tutte's seminal papers [1-4], some of its developments arising from theoretical physics or probability theory. Among the various enumeration techniques used so far, let us mention Tutte's original recursive decomposition, the matrix integral approach [5] and the more recent use of bijections with trees [6]. While the first two approaches are fairly systematic (i.e. allow to translate almost automatically the counting problem into equations), a weakness of the third one is that it relies on some preliminary divination of the family of trees with which a bijection is to be found. In practice, one usually solves the counting problem via another approach and, guided by the knowledge of the result, discovers the bijection afterwards. For this reason, the numerous bijections found in the literature might appear as a myriad of rather *ad hoc* tools. Recently, some authors have undertaken the task of understanding the general underlying principles of the bijective approach and providing a unified framework, the "master bijection", in which all the previously known bijections appear as special cases [7-9].

On some other side, it was realized that one of the desirable feature of trees, namely that they are easy to enumerate thanks to their natural recursive structure, could be directly achieved at the level of the maps themselves via the so-called *slice decomposition* [10]. This approach, close in spirit to Tutte's approach, has the merit of relying on a simple systematic construction, which consists in cutting a map along some shortest paths. The parts obtained in this decomposition are maps with geodesic boundaries, called slices for short, and may themselves be iteratively cut into smaller slices, reproducing a recursive tree-like structure. We observe that slices were also used in [11] under the name of DMGB (discrete maps with geodesic boundaries).

In this paper, we extend the slice decomposition formalism to the case of maps with a girth constraint, i.e. with a control on the minimal length of their cycles. Such maps were already considered in the master bijection framework and we shall indeed recover some of the results of [7,8]. We actually consider the slightly more general case of *irreducible maps*, i.e. maps with a girth constraint and without separating shortest cycles (as we shall see below, it is indeed more general since the former may be recovered by setting some parameter to 0). Irreducible triangulations and quadrangulations were first enumerated respectively by Tutte [1], and by Mullin and Schellenberg [12] using a substitution approach (these authors used the denomination "simple" instead of irreducible, which is slightly misleading since a simple map is nowadays understood as a map without loops or multiple edges). These results were later promoted to bijections with trees [13,14].

We actually start by extending the Tutte-Mullin-Schellenberg substitution approach to arbitrary irreducible maps. As a key ingredient, we use an expression for the generating function of maps with a boundary which originates combinatorially from the slice decomposition. It turns out that slice decomposition somehow "commutes" with the substitution approach. We are therefore led to studying irreducible slices and their recursive decomposition.

Before presenting in more details the outline of the paper, we would like to discuss the influence of Philippe Flajolet on this work. Making an exhaustive list is an impossible task so let us focus on two particular examples. First, the slice decomposition [10] was inspired by the combinatorial theory of continued fractions [15]. This theory is far more general than the context of planar maps, but it proved to be the key to understanding the phenomenon of "discrete integrability", on which we will return below. Second, Philippe was no stranger to counting maps via substitution: in [16], he and his coauthors have shown that a universal "Airy phenomenon" occurs when, generally speaking, one decomposes a map into (multi)connected "cores". Figuring out whether the same phenomenon subsists in the present context of irreducible maps is an intriguing open question.

1.2. Definitions

A *planar map* is an embedding of a connected graph in the sphere without edge crossing, considered up to continuous deformation. It is made of *vertices*, *edges* and *faces*. A *rooted* map is a map with a distinguished oriented edge, the *root edge*. The face on the right of the root edge is called the *outer face* (whose degree is called the *outer degree* of the map), the other ones being referred to as *inner faces*. For n and d two positive integers, a *d-angular dissection of the n-gon* is a rooted map whose outer degree is n and where all inner faces have degree d.

The *girth* of a map is the minimal length (number of edges) of its cycles, a cycle being a simple closed path on the map. Note that, by this definition, trees have an infinite girth since they contain no cycle. In a map not reduced to a tree, the degree of every face is larger than or equal to the girth (which is finite).

Given a nonnegative integer d, we say that a rooted map is *d-irreducible* if its girth is at least d and any cycle of length d is the boundary of an inner face of degree d. Note that every rooted map is 0-irreducible. Furthermore, by definition, a d-irreducible map with outer degree smaller than or equal to d is either a tree (with one face of even degree $\leq d$) or is reduced to a cycle of length d (delimiting two faces of degree d). A d-irreducible d-angular dissection will be called an irreducible d-angular dissection for short. Note finally that maps of girth at least d are nothing but $(d-1)$-irreducible maps containing no $(d-1)$-valent face, and in this sense, the class of irreducible maps is more general than that of maps with controlled girth.

Let us denote by $F_n^{(d)}(z; x_{d+1}, x_{d+2}, \ldots)$ the generating function of d-irreducible maps with outer degree n, counted with a weight z per inner face of degree d and, for all $i \geq d+1$, a weight x_i per inner face of degree i. Our motivation for choosing a different notation for the weight of faces of degree d is that it plays a very different role in the forthcoming expressions. From the above remark, we have, for $n \leq d$

$$F_n^{(d)}(z; x_{d+1}, x_{d+2}, \ldots) = \begin{cases} \mathrm{Cat}(n/2) & \text{for } n < d \\ \mathrm{Cat}(d/2) + z & \text{for } n = d \end{cases} \tag{1.1}$$

where $\mathrm{Cat}(k)$ is equal to $\binom{2k}{k}/(k+1)$ (the k-th Catalan number, counting rooted trees with k edges, hence an outer degree $2k$) for integer k and 0 for noninteger k.

Let us now discuss a few interesting specializations of $F_n^{(d)}$. First, by taking all x_i, $i \geq d+1$, to 0, we obtain the generating function $f_n^{(d)}(z) = F_n^{(d)}(z; 0, 0, \ldots)$ of irreducible d-angular dissections of the n-gon, depending on the single variable z coupled to the number of inner faces. Second, by conversely taking $z = 0$, we forbid all faces of degree d hence all cycles of length d, so that $F_n^{(d)}(0; x_{d+1}, x_{d+2}, \ldots)$ coincides with the generating function of rooted maps of girth at least $d+1$ and outer degree n, as studied in [8]. Note that, in the particular case $d = 0$, $F_n(x_1, x_2, \ldots) = F_n^{(0)}(0; x_1, x_2, \ldots)$ is nothing but the generating function of arbitrary maps with outer degree n. Finally, a third specialization concerns *bipartite planar maps*, i.e. maps whose all faces have even degrees: it is obtained by taking n and d even, and setting all odd x_i to 0.

1.3. Overview of the main results

Our main result is a general expression for $F_n^{(d)}$ in terms of two auxiliary quantities, which we denote by $R^{(d)}$ and $S^{(d)}$, and which may be interpreted as d-irreducible slice generating functions. As such, these quantities are themselves determined by an explicit system of two equations, which is algebraic whenever we impose a bound on the face degrees (i.e. $x_i = 0$ for i large enough). A particularly elegant expression for $F_n^{(d)}$ is via a *pointing formula* which amounts to counting *annular maps*, i.e. rooted maps having a distinguished inner face of degree d. This pointing formula has a clear combinatorial interpretation from the slice decomposition. We now mention a number of other interesting results appearing on the way.

We find that $R^{(d)}$ and $S^{(d)}$ are particular members (corresponding essentially to the first two values $k = -1, 0$) of a larger family of generating functions $V_k^{(d)}$ $(k \geq -1)$ of so-called d-irreducible k-slices, where k controls some excess boundary length of the slice (a more precise definition will come in due time). We provide a closed system of equations for $V_k^{(d)}$ which results from an elementary recursive decomposition of k-slices. We are then able to explicitly eliminate all $V_k^{(d)}$ with $k \geq 1$, yielding the wanted system of two equations determining $R^{(d)}$ and $S^{(d)}$.

While all the enumeration is carried out in terms of slices, we also discuss, in the case of irreducible d-angular dissections, an equivalent formulation in terms of trees: more precisely, $V_k^{(d)}$ may in this case be interpreted as the generating function of so-called d-oriented k-trees, reminiscent of the $d/(d-2)$-trees considered in [7,8]. In the particular cases $d = 3$ and $d = 4$, d-oriented trees reduce respectively to ternary and binary trees and we recover the bijections of [13,14].

Finally, we consider slices subject to a control of an extra parameter, namely their maximal length. In the case of non necessarily irreducible maps, generating functions of these objects are known to be solution of a hierarchy of "discrete integrable equations" [17-19,10]. We show that this phenomenon subsists in the d-irreducible case and explicit the corresponding equations. A particular attention is paid to the cases $d = 3$ and $d = 4$ for which we recover integrable equations describing so-called naturally embedded trees, respectively in their ternary [20] and binary [21] flavour. In particular, answering a question raised by Bousquet-Mélou, we provide a combinatorial explanation of Proposition 25 in [21], in the same spirit as those previously found for well labeled trees [10] and very well labeled trees [22].

1.4. Outline of the paper

We now come to the detailed plan of the paper. We begin by describing the substitution approach to the enumeration of d-irreducible maps. For pedagogical reasons, we first address the simplest cases $d = 4$ (Sect. 2.1) and $d = 3$ (Sect. 2.2). The general case is treated in Sect. 3. We proceed by induction on d and explain in Sect. 3.1 how $F_n^{(d)}$

is related to $F_n^{(d-1)}$. We deduce in Sect. 3.2 a relation between $F_n^{(d)}$ and the generating function F_n of arbitrary maps. We then exploit known expressions for F_n to obtain $F_n^{(d)}$, first in the simpler bipartite case (Sect. 3.3), then in the general case (Sect. 3.4). This yields the first derivation of our main result, together with the pointing formula. The quantities $R^{(d)}$, $S^{(d)}$ and $V_k^{(d)}$ appear in this derivation as mere intermediate products. In Sect. 4, we take the time to discuss their combinatorial significance as slice generating functions. We recall in Sect. 4.1 the definition of slices and extend it to what we call k-slices. We then show in Sect. 4.2 that $V_k^{(d)}$ is nothing but the generating function of d-irreducible k-slices. In Sect. 5, we obtain a recursive decomposition of these slices, actually in two variants (Sects. 5.1 and 5.2). The closed system for the $V_k^{(d)}$ is deduced in Sect. 5.3 and we discuss in Sects. 5.4 (bipartite case) and 5.5 (general case) how to eliminate the $V_k^{(d)}$ with $k \geq 1$. This yields a second route to the system of equations satisfied by $R^{(d)}$ and $S^{(d)}$. Sect. 6 discusses the equivalent formulation in terms of trees (for the case of irreducible d-angular dissections): we define d-oriented k-trees in Sect. 6.1 and exhibit their one-to-one correspondence with slices. An alternative description of the correspondence as a closure algorithm is given in Sect. 6.2. Some simplifications occuring in the bipartite case are mentioned in Sect. 6.3. Sect. 7 is devoted to the bijective proof of the pointing formula: we explain in Sect. 7.1 how to build an annular map out of slices, and present in Sect. 7.2 the inverse mapping (involving the notion of "lift"). Combined with the results of Sect. 5, this provides a second derivation of our main result. Sect. 8 is devoted to discrete integrable equations: we first discuss the particular cases of irreducible quadrangular (Sect. 8.1) and triangular (Sect. 8.2) slices, related to naturally embedded trees, and we then write down the general equations in Sect. 8.3. Sect. 9 discusses other aspects of irreducibility: in Sect. 9.1 we relax the definition of d-irreducibility for maps with outer degree d and solve the corresponding enumeration problem. Sect. 9.2 deals with the enumeration of d-irreducible maps with two marked faces of degree strictly larger than d. Finally, we consider in Sect. 9.3 a generalized notion of annular maps, leading to beautiful identities extending the pointing formula.

2. First simple cases

As a preamble to Sect. 3 where we shall explain in details the substitution approach to d-irreducible maps, let us discuss in the simplest case of quadrangular and triangular dissections how this approach allows one to obtain expressions for generating functions of irreducible maps at no cost. More precisely, substitution tells us that these generating functions may be obtained from those of arbitrary maps by a simple renormalization of the weights x_i. This turns out to be sufficient to determine them fully.

2.1. Irreducible quadrangular dissections

We have at our disposal a number of expressions for the generating functions of general (non necessarily irreducible) quadrangular dissections, counted with a weight x_4 per face. Recall that $F_n(x_1, x_2, \ldots)$ denotes the generating function of arbitrary planar

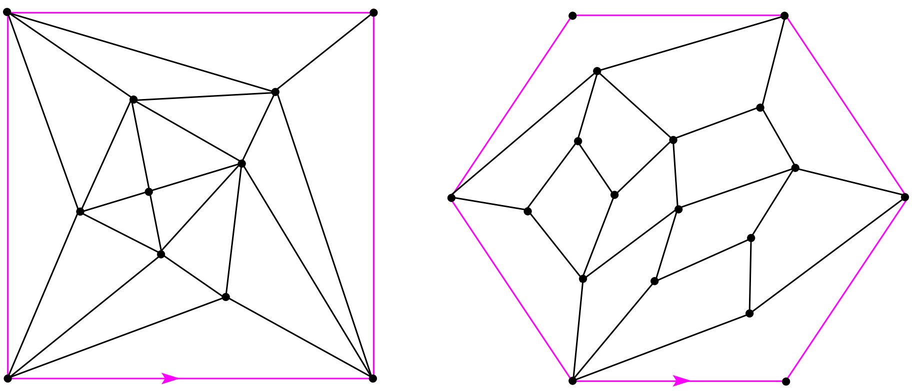

Fig. 1: Left: an irreducible triangular dissection of the square. Right: an irreducible quadrangular dissection of the hexagon.

maps with outer degree n, counted with weights x_i per inner face of degree i. In this section, we denote by $F_n = F_n(0, x_2, 0, x_4, 0, 0, \ldots)$ its specialization to $x_i = 0$ for i different from 2 or 4, that is to say we consider maps with only bivalent or tetravalent inner faces. The expression for F_n takes a particularly simple form if we introduce the series $R \equiv R(x_2, x_4)$, solution of $R = 1 + x_2 R + 3x_4 R^2$ (which itself may be understood as a generating function of some kind). We have in particular [18,10]

$$\begin{aligned} F_2 &= R - \left(x_4 R^3\right), \\ F_4 &= 2R^2 - 3R\left(x_4 R^3\right), \\ F_6 &= 5R^3 - 9R^2\left(x_4 R^3\right). \end{aligned} \tag{2.1}$$

We now claim that there exists two formal power series $X_2 \equiv X_2(z)$ and $X_4 \equiv X_4(z)$ such that the generating function $f_n^{(4)} \equiv f_n^{(4)}(z)$ of *irreducible* quadrangular dissections of the n-gon, counted with a weight z per face, is obtained from F_n by the substitution $x_2 \to X_2(z)$ and $x_4 \to X_4(z)$. Furthermore, the "renormalized face weights" X_2 and X_4 are entirely determined by the condition (1.1) for $d = 4$. We will prove these statements in greater generality in Section 3 below. At this stage, let us simply justify them heuristically by noting that a general quadrangular dissection may be obtained from an irreducible one by a renormalization procedure which, so to say, consists in replacing each (four-valent) face of the latter by a more general quadrangular dissection with outer degree 4. Actually, we also need to eliminate multiple edges which is achieved by introducing bivalent faces and renormalizing them out.

Defining $r^{(4)} = R(X_2(z), X_4(z))$, we deduce from (2.1) that

$$\begin{aligned} f_2^{(4)} &= 1 = r^{(4)} - \left(X_4(z)(r^{(4)})^3\right), \\ f_4^{(4)} &= 2 + z = 2(r^{(4)})^2 - 3\,r^{(4)}\left(X_4(z)(r^{(4)})^3\right), \\ f_6^{(4)} &= 5(r^{(4)})^3 - 9(r^{(4)})^2\left(X_4(z)(r^{(4)})^3\right). \end{aligned} \tag{2.2}$$

Now it is interesting to note that we do not need any precise expression for X_2 or X_4. Indeed, eliminating X_4 from the first two equations of (2.2) yields:

$$z + (r^{(4)})^2 - 3r^{(4)} + 2 = 0, \tag{2.3}$$

which fully determines $r^{(4)}$ as a function of z. Once $r^{(4)}$ is known, we may get $f_6^{(4)}$ from the third equation which, after elimination of X_4, reads simply:

$$f_6^{(4)} = 9(r^{(4)})^2 - 4(r^{(4)})^3 \ . \tag{2.4}$$

Differentiating both equations (2.3) and (2.4) with respect to z, we deduce in particular $(3 - 2r^{(4)})\frac{dr^{(4)}}{dz} = 1$ and

$$\frac{df_6^{(4)}}{dz} = 6r^{(4)}(3 - 2r^{(4)})\frac{dr^{(4)}}{dz} = 6r^{(4)}. \tag{2.5}$$

This result is a particular example of a more general pointing formula which will be discussed in details later.

Eq. (2.3) is more transparent upon setting

$$r^{(4)}(z) = 1 + z\,T(z), \tag{2.6}$$

as it then reads

$$T = 1 + z\,T^2. \tag{2.7}$$

This allows to identify T with the generating function of binary trees (with a weight z per inner vertex). Using $[z^n]T = \mathrm{Cat}(n)$, we arrive at

$$[z^{n+2}]f_6^{(4)} = \frac{6}{n+2}\,\mathrm{Cat}(n) \tag{2.8}$$

for the number of irreducible quadrangular dissections of the hexagon with $(n+2)$ squares, $n \geq 0$ (see Fig. 1 for an example with $n = 12$). We recover here a result of [12,14].

More general formulas are obtained along the same lines. From the expression

$$F_{2m} = \mathrm{Cat}(m)R^m - \frac{3}{m-1}\binom{2m}{m-2}x_4R^{m+2} \tag{2.9}$$

for the generating function of arbitrary quadrangular dissections of the $2m$-gon [18,10], we readily deduce that the generating function of irreducible ones reads

$$f_{2m}^{(4)}(z) = \binom{2m}{m-2}\left(\frac{3}{m-1}(r^{(4)})^{m-1} - \frac{2}{m}(r^{(4)})^m\right) \tag{2.10}$$

and satisfies the pointing formula $df_{2m}^{(4)}/dz = \binom{2m}{m-2}(r^{(4)})^{m-2}$. However the general coefficient of $f_{2m}^{(4)}(z)$ does not seem to be "nice". In contrast, Mullin and Schellenberg [12] obtained a nice general coefficient when considering irreducible dissections whose outer boundary is simple. We may recover their formula from the expression [23, Eq. (5.16)]

$$\tilde{F}_{2p} = \frac{(3p-3)!}{p!(2p-1)!}\left(px_4^{p-1}R^{3p-2} + (2-3p)x_4^p R^{3p}\right) \tag{2.11}$$

for the generating function of arbitrary quadrangular dissections of the $2p$-gon with a simple outer boundary. It is not difficult to check that this formula remains valid if we also allow for bivalent faces provided R is taken as the solution $R = 1 + x_2 R + 3x_4 R^2$ as before. Substituting $x_4 \to X_4(z)$ and $R \to r^{(4)}$ in (2.11) and noting that $X_4(z)(r^{(4)})^3 = z\,T(z)$ and $[z^k]\,T(z)^p = \frac{p}{2k+p}\binom{2k+p}{k}$, we obtain after some algebra that the number of irreducible quadrangular dissections of the $2m$-gon with k inner faces and a simple outer boundary reads

$$[z^k]\tilde{f}_{2p}^{(4)} = \frac{(3p-3)!}{(p-3)!(2p-1)!}\frac{(2k-p-1)!}{k!(k-p+1)!} \tag{2.12}$$

which, by the reparametrization $m = p-2$ and $n = k-p+1$, coincides for $m > 0$ with Mullin and Schellenberg's formula. Let us observe that, in contrast to the present approach, these authors started directly from Brown's formula [24] for the number of quadrangular dissections of the $2p$-gon which both are simple (i.e. have no multiple edges) and have a simple outer boundary.

2.2. Irreducible triangular dissections

We may now play the same game with triangular dissections. As before, we have simple expressions for the generating functions of general triangular dissections, with weight x_3 per face. In this section, $F_n = F_n(x_1, x_2, x_3, 0, 0, \ldots)$ denotes the generating function of planar maps with outer degree n and all inner faces of degree at most 3 (as obtained by specializing $x_i = 0$ for $i > 3$). Introducing the series R, S in the variables x_1, x_2, x_3 specified by the equations $R = 1 + x_2 R + 2x_3 RS$, $S = x_1 + x_2 S + x_3(S^2 + 2R)$, we have [10]

$$\begin{aligned} F_1 &= S - \left(x_3 R^2\right), \\ F_2 &= S^2 + R - 2S\left(x_3 R^2\right), \\ F_3 &= S^3 + 3RS - (3S^2 + 2R)\left(x_3 R^2\right), \\ F_4 &= S^4 + 6RS^2 + 2R^2 - (4S^3 + 8RS)\left(x_3 R^2\right). \end{aligned} \tag{2.13}$$

We now claim that there exists three formal power series $X_1 \equiv X_1(z)$, $X_2 \equiv X_2(z)$, $X_3 \equiv X_3(z)$ such that the generating function $f_n^{(3)} \equiv f_n^{(4)}(z)$ of *irreducible* triangular dissections of the n-gon, counted with a weight z per inner face, is obtained from F_n by the substitution $x_1 \to X_1(z)$, $x_2 \to X_2(z)$ and $x_3 \to X_3(z)$. Again, the renormalized face weights X_1, X_2 and X_3 are entirely determined by (1.1) for $d = 3$. Setting $r^{(3)}(z) = R(X_1(z), X_2(z), X_3(z))$ and $s^{(3)}(z) = S(X_1(z), X_2(z), X_3(z))$, we can now write

$$\begin{aligned}
f_1^{(3)} &= 0 = s^{(3)} - \left(X_3(z)(r^{(3)})^2\right), \\
f_2^{(3)} &= 1 = (s^{(3)})^2 + r^{(3)} - 2s^{(3)} \left(X_3(z)(r^{(3)})^2\right), \\
f_3^{(3)} &= z = (s^{(3)})^3 + 3r^{(3)}s^{(3)} - (3(s^{(3)})^2 + 2r^{(3)}) \left(X_3(z)(r^{(3)})^2\right), \\
f_4^{(3)} &= (s^{(3)})^4 + 6r^{(3)}(s^{(3)})^2 + 2(r^{(3)})^2 - (4(s^{(3)})^3 + 8r^{(3)}s^{(3)}) \left(X_3(z)(r^{(3)})^2\right).
\end{aligned} \tag{2.14}$$

Eliminating X_3, the first three equations lead to the following algebraic system determining $r^{(3)}$ and $s^{(3)}$ in terms of z:

$$r^{(3)} = 1 + (s^{(3)})^2 \ , \qquad z + (s^{(3)})^3 - s^{(3)} = 0 \ . \tag{2.15}$$

As for $f_4^{(3)}$, we deduce from the fourth equation

$$f_4^{(3)} = 2 + 2(s^{(3)})^2 - 3(s^{(3)})^4 \tag{2.16}$$

and, upon differentiating with respect to z

$$\frac{df_4^{(3)}}{dz} = 4s^{(3)}(1 - 3(s^{(3)})^2)\frac{ds^{(3)}}{dz} = 4\, s^{(3)} \tag{2.17}$$

since, from (2.15), $(1 - 3(s^{(3)})^2)\frac{ds^{(3)}}{dz} = 1$.

Eq. (2.15) for $s^{(3)}$ is more transparent upon setting

$$s^{(3)}(z) = z\, T(z), \tag{2.18}$$

as it then reads

$$T = 1 + z^2\, T^3. \tag{2.19}$$

This allows to identify T with the generating function of ternary trees (with a weight z^2 per inner vertex). Using $[z^{2n}]T = \binom{3n}{n}/(2n+1)$, we arrive at

$$[z^{2n+2}]f_4^{(3)} = \frac{2}{n+1} \times \frac{\binom{3n}{n}}{2n+1} \tag{2.20}$$

for the number of irreducible triangular dissections of the square with $2n + 2$ triangles, $n \geq 0$ (the number of triangles in a dissection of the square must be even - see Fig. 1 for an example with $n = 7$). We recover here a result of [1].

3. Approach by substitution

A natural approach to irreducible maps is via substitution. Intuitively speaking, a d-irreducible map is obtained by erasing the contents of all cycles of length d in a general map. This naive viewpoint can be made more precise by combining the following two observations:

(i) rooted maps of girth at least d and outer degree n are obtained from $(d-1)$-irreducible maps with outer degree n by forbidding all inner faces of degree $d-1$;

(ii) rooted maps of girth at least d and outer degree n are alternatively obtained from d-irreducible maps with outer degree n by *substituting* each inner face of degree d with an arbitrary rooted map of girth d and outer degree d.

Observation (i), which was already made in Sect. 1.2, implies that the generating function of rooted maps of girth at least d and outer degree n is equal to $F_n^{(d-1)}(0; x_d, x_{d+1}, \ldots)$. Observation (ii), which we will justify in the forthcoming subsection, implies that the same generating function is equal to $F_n^{(d)}(G_d(x_d, x_{d+1}, \ldots); x_{d+1}, \ldots)$ where

$$G_d(x_d, x_{d+1}, \ldots) = F_d^{(d-1)}(0; x_d, x_{d+1}, \ldots) - \mathrm{Cat}(d/2) \tag{3.1}$$

is the generating function of rooted map of girth d and outer degree d (indeed a map of girth at least d and outer degree d has girth exactly d unless it is reduced to a tree). Since we are expressing the same quantity in two manners, we get the basic identity

$$F_n^{(d-1)}(0; x_d, x_{d+1}, \ldots) = F_n^{(d)}(G_d(x_d, x_{d+1}, \ldots); x_{d+1}, \ldots). \tag{3.2}$$

Let us now complete the proof of this identity before explaining how it allows to compute practically $F_n^{(d)}$.

3.1. The basic substitution relation

We now justify the observation (ii) made above. More precisely, we shall prove that we have a bijection between, on the one hand, the set of rooted maps of girth at least d and outer degree n and, on the other hand, the set of pairs of the form $(\mathcal{M}, (m_f)_{f\in\mathcal{F}_d(\mathcal{M})})$ where $\mathcal{M}$ is a d-irreducible map of outer degree n, $\mathcal{F}_d(\mathcal{M})$ is the set of its d-valent inner faces and, for $f \in \mathcal{F}_d(\mathcal{M})$, m_f is a rooted map of girth d and outer degree d.

Starting from such a pair $(\mathcal{M}, (m_f)_{f\in\mathcal{F}_d(\mathcal{M})})$, we define a rooted map $\mathcal{M}'$ by "gluing" inside each face $f \in \mathcal{F}_d(\mathcal{M})$ the map m_f. More precisely, we identify clockwise each edge of m_f incident to the outer face with an edge of $\mathcal{M}$ incident to f, starting from the root edge of m_f which is identified with an edge of f selected in some canonical manner (for instance by breadth-first search from the root of $\mathcal{M}$). Note that the boundary of f and that of the outer face of m_f are both simple, thus by identification they yield a cycle of $\mathcal{M}'$. $\mathcal{M}'$ is a rooted map (with the same root as $\mathcal{M}$) of outer degree n and we claim that its girth is at least d. The proof relies on two lemma, the first of which will also be useful later on.

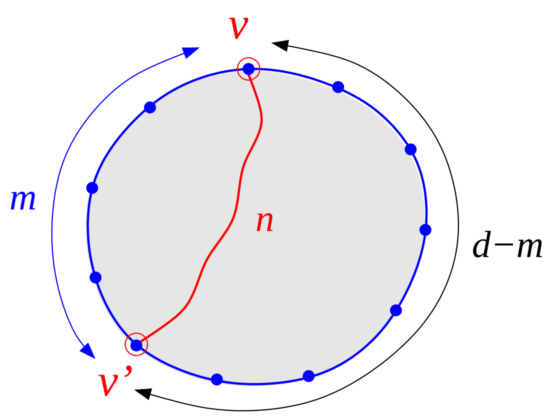

Fig. 2: Illustration of the no-shortcut lemma: v and v' are two vertices m edges away in one direction (hence $d-m$ in the other direction) along the outer boundary (blue) of a map of girth d and outer degree d. A simple path from v to v' (red) containing at least one inner edge necessarily has length $n \geq \max(m, d-m)$.

No-shortcut lemma: in a map of girth d and outer degree d, if v and v' are two outer vertices m edges away in one direction or the other along the boundary, then any simple path from v to v' containing at least one inner edge has length larger than or equal to $\max(m, d-m)$ (which itself is larger than or equal to $d/2$).

Proof: consider a simple path from v to v' containing at least one inner edge (see Fig. 2), and denote its length by n. Adding the shorter boundary between v' and v, we obtain a closed path of length $n + \min(m, d-m)$, which is not necessarily simple but encircles at least one inner face. Thus, by possibly removing some edges we obtain a cycle of length at most $n + \min(m, d-m)$. Since the map has girth d, we deduce $n \geq d - \min(m, d-m) = \max(m, d-m)$, Q.E.D.

Encircling lemma: given a cycle C' of $\mathcal{M}'$, there exists a cycle C of $\mathcal{M}$ such that C' lies in the closed region bounded by C, and such that the length of C is at most that of C'.

Proof: let C' be a cycle of $\mathcal{M}'$ of length ℓ. If C' remains within a single d-valent face f of $\mathcal{M}$, then the statement is clearly true: C' is a cycle of m_f, so $\ell \geq d$ and we may take C to be the boundary of f, which has length d. Otherwise, C' necessarily visits at least two vertices of $\mathcal{M}$. We orient C' counterclockwise, and denote by $v_1, v_2, \ldots, v_p$ $(2 \leq p \leq \ell)$ the successive vertices of $\mathcal{M}$ visited along C'. We then let γ'_i $(1 \leq i \leq p)$ be the part of C' comprised between v_i and v_{i+1} (with $v_{p+1} = v_1$). γ'_i is a simple open oriented path which is either reduced to a single edge of $\mathcal{M}$, or made of edges not in $\mathcal{M}$ that are all inside a same d-valent face f_i of $\mathcal{M}$. In the former case we let $\gamma_i = \gamma'_i$, while in the latter case we let γ_i be the part of the boundary of f_i going from v_i to v_{i+1} in the counterclockwise direction: the no-shortcut lemma ensures that the length of γ_i is at most that of γ'_i. Upon concatenating $\gamma_1, \cdots, \gamma_p$ together we obtain a closed path $\tilde{C}$ on $\mathcal{M}$ of length at most ℓ. However, we are not assured that it is a cycle encircling C' so we cannot yet conclude. Instead, we let R' be the closed region bounded by C'

and, for $1 \le i \le p$, r_i be the closed region bounded by $\gamma_i \cup \gamma'_i$ if $\gamma_i \ne \gamma'_i$, or the empty set otherwise. Then, $R = R' \cup r_1 \cup \cdots \cup r_p$ is a simply connected closed region whose boundary is the cycle C we are looking for. Indeed, note first that it obviously contains C'. Secondly, if r_i is nonempty, then γ'_i has r_i on its right and R' on its left, thus the interior of R is simply connected and, furthermore, any edge of C' not in $\mathcal{M}$ cannot be on the boundary of R. Therefore, the boundary of R is a cycle C whose edges form a subset of those of $\tilde{C}$, thus its length is at most ℓ, Q.E.D.

The encircling lemma immediately implies that the girth of $\mathcal{M}'$ is at least d (more precisely it is equal to the girth of $\mathcal{M}$ which is at least d, with equality iff $\mathcal{F}_d(\mathcal{M})$ is nonempty). Furthermore, since $\mathcal{M}$ is irreducible, this lemma also implies that any cycle of length d in $\mathcal{M}'$ necessarily remains within a single d-valent face of $\mathcal{M}$: the boundaries of d-valent faces of $\mathcal{M}$ are thus precisely the *outermost* cycles of length d in $\mathcal{M}'$. This shows that, starting from $\mathcal{M}'$, we may recover $\mathcal{M}$ by erasing all the edges and vertices that are interior to the outermost d-cycles. Then, for a given $f \in \mathcal{F}_d(\mathcal{M})$, the edges and vertices of $\mathcal{M}'$ that lie within f form the map m_f. In conclusion, the mapping that maps $(\mathcal{M}, (m_f)_{f \in \mathcal{F}_d(\mathcal{M})})$ to $\mathcal{M}'$ is injective. It remains to check that it is surjective.

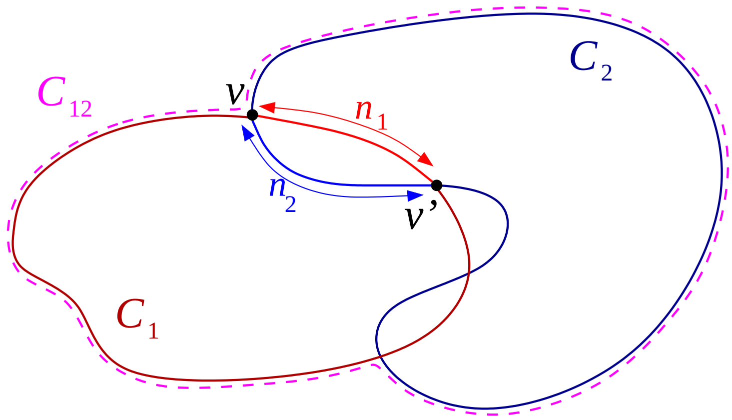

Fig. 3: Illustration of the proof that two outermost cycles of length d in a map of girth d do not overlap. If two cycles C_1 and C_2 of length d overlap (hence we may find two vertices v and v' such that the part of C_1 between v and v', of length n_1 lies in the interior of C_2 and vice versa), we can build a cycle C_{12} (dashed line) encircling them both, and having length at most $2d - (n_1 + n_2) \le d$, hence equal to d.

Let us now start conversely with an arbitrary rooted map $\mathcal{M}'$ of outer degree n and girth at least d and consider the set $\mathcal{C}_d(\mathcal{M}')$ of its *outermost* cycles of length d , i.e. those cycles of length d whose interior is not strictly included into the interior of another cycle of length d. For $C \in \mathcal{C}_d(\mathcal{M}')$, the edges and vertices of $\mathcal{M}'$ that lie within C form a map m_C which clearly has girth d and outer degree d (we discuss its rooting below). Now we have the crucial property that two distinct cycles in $\mathcal{C}_d(\mathcal{M}')$, say C_1 and C_2, cannot overlap, i.e. the intersection of their interiors is necessarily empty. Indeed since the interior of one cycle cannot be included in the interior of the other, if we assume that these interiors have a nonempty intersection, there exist two vertices v and v' at the intersection of C_1 and C_2 such that one of the parts of C_1 between v and v' lies in the interior of C_2 and one of the parts of C_2 between v and v' lies in the interior of C_1 (see Fig. 3). Calling n_1 and n_2 the lengths of these parts, we have $n_1 \geq d/2$ and $n_2 \geq d/2$ by the no-shortcut lemma applied to m_{C_2} and m_{C_1} respectively. We can then build a cycle C_{12} by following the outer boundary of the union of the interiors of C_1 and C_2 (note that this union is connected but not necessarily simply connected – see Fig. 3). This cycle has length at most $(d-n_1)+(d-n_2) \leq d$, hence it has length d from the girth condition on $\mathcal{M}'$. The interiors of C_1 and C_2 are then strictly included in the interior of the cycle C_{12} of length d, a contradiction. Since outermost cycles of length d do not overlap, we may unambiguously replace the content of each such cycle by a simple face of degree d, resulting in a rooted map $\mathcal{M}$ of outer degree n and girth at least d, such that all cycles of length d are the boundary of an inner face of degree d, hence a d-irreducible map (note that the outer face of $\mathcal{M}'$ is unaffected by the substitution since it cannot belong to the interior of a cycle). In particular, we may identify $\mathcal{F}_d(\mathcal{M})$ with $\mathcal{C}_d(\mathcal{M}')$. For each $f \in \mathcal{F}_d(\mathcal{M})$, associated with $C \in \mathcal{C}_d(\mathcal{M}')$, we select an edge of $\mathcal{M}$ incident to f by the same canonical procedure as before: this provides a canonical rooting of $m_f \equiv m_C$. Obviously, applying the previous construction to $(\mathcal{M}, (m_f)_{f\in\mathcal{F}_d(\mathcal{M})})$ restores $\mathcal{M}'$, hence the mapping from $(\mathcal{M}, (m_f)_{f\in\mathcal{F}_d(\mathcal{M})})$ to $\mathcal{M}'$ is surjective. It is therefore a bijection.

As a final remark, note that this bijection preserves the following parameters:

- the number of d-valent inner faces of $\mathcal{M}'$ is equal to the total number of d-valent inner faces in all m_f, $f \in \mathcal{F}_d(\mathcal{M})$,
- for each $k > d$, the number of k-valent inner faces of $\mathcal{M}'$ is equal to the total number of k-valent inner faces in $\mathcal{M}$ and all m_f, $f \in \mathcal{F}_d(\mathcal{M})$.

It follows that the generating function of rooted map of girth at least d and outer degree n is indeed equal to $F_n^{(d)}(G_d(x_d, x_{d+1}, \ldots); x_{d+1}, \ldots)$, which concludes the proof of the basic identity (3.2).

3.2. General strategy

We turn to discussing the practical use of the basic identity (3.2) in computing $F_n^{(d)}$. We first claim that there exists a series $X_d(z; x_{d+1}, \ldots)$ such that

$$G_d(X_d(z; x_{d+1}, \ldots), x_{d+1}, \ldots) = z, \tag{3.3}$$

in other words G_d admits a *compositional inverse* with respect to its first variable. Observe indeed that G_d is a series in $x_d, x_{d+1}, \ldots$ whose constant term is zero (since any map contributing to G_d contains at least one inner face) and whose coefficient of x_d is 1 (corresponding to the map reduced to a single d-gon), i.e.

$$G_d(x_d, x_{d+1}, \ldots) = A(x_{d+1}, \ldots) + x_d(1 + B(x_{d+1}, \ldots)) + x_d^2 C(x_d, x_{d+1}, \ldots) \tag{3.4}$$

where A, B are formal power series in $x_{d+1}, x_{d+2}, \ldots$ without constant term. Since $1+B$ is invertible in the ring of formal power series in $x_{d+1}, x_{d+2}, \ldots$, the series $x_d(1 + B) +$

$x_d^2 C$, viewed as a series in x_d whose coefficients are series in $x_{d+1}, x_{d+2}, \ldots$, admits a compositional inverse D with respect to the variable x_d, satisfying $D(0, x_{d+1}, x_{d+2}, \ldots) = 0$. The wanted series is then $X_d(z; x_{d+1}, \ldots) = D(z - A(x_{d+1}, \ldots), x_{d+1}, \ldots)$, the substitution being well defined as A has no constant term. Note that, when specializing $x_i = 0$ for all odd i, we have $G_d(0, x_{d+1}, \ldots) = A(x_{d+1}, \ldots) = 0$ for d odd (since there are no maps with odd outer degree and all inner faces of even degree), hence $X_d(0; x_{d+1}, \ldots) = D(0, x_{d+1}, \ldots) = 0$.

Replacing x_d by $X_d(z; x_{d+1}, \ldots)$ in (3.2), we obtain the reciprocal identity

$$F_n^{(d)}(z; x_{d+1}, \ldots) = F_n^{(d-1)}(0; X_d(z; x_{d+1}, \ldots), x_{d+1}, \ldots). \tag{3.5}$$

By iterating this relation d times, we relate $F_n^{(d)}$ to the generating function $F_n(x_1, x_2, \ldots) = F_n^{(0)}(0; x_1, x_2, \ldots)$ of arbitrary maps with outer degree n. Namely, we have the general substitution relation

$$F_n^{(d)} = F_n(X_1^{(d)}, X_2^{(d)}, \ldots, X_d^{(d)}, x_{d+1}, \ldots) \tag{3.6}$$

where the series $X_j^{(d)}(z; x_{d+1}, \ldots)$, $1 \le j \le d$, are defined inductively by

$$X_j^{(d)}(z; x_{d+1}, \ldots) = \begin{cases} X_d(z; x_{d+1}, \ldots) & \text{for } j = d, \\ X_j(0; X_{j+1}^{(d)}, \ldots, X_d^{(d)}, x_{d+1}, \ldots) & \text{for } j < d. \end{cases} \tag{3.7}$$

Note that, in the bipartite case (n, d even, $x_i = 0$ for i odd), we showed above that $X_j(0; x_{j+1}, \ldots) = 0$ for j odd so that, by (3.7), $X_j^{(d)} = 0$. In other words, (3.6) relates the generating function of bipartite d-irreducible maps to the generating function of bipartite maps without irreducibility constraints (alternatively, this can be shown by writing the bipartite analogue of (3.2), relating directly $F_n^{(d-2)}$ and $F_n^{(d)}$).

While F_n is a well studied quantity for which convenient expressions are known (see below), we have *a priori* no such expressions for $X_1^{(d)}, \ldots, X_d^{(d)}$ which appear in (3.6). However, these d unknown quantities may in principle be *determined* by the conditions (1.1), and then *eliminated* from the expression of $F_n^{(d)}$. We have seen in Sect. 2 two cases where this elimination can been carried out smoothly, and this can be done in general, as we shall see in the following subsections. We first concentrate on the bipartite case where the expressions are somewhat simpler.

3.3. Elimination in the bipartite case

In the bipartite case, both the outer degree n and the girth d are even integers, hence we write $n = 2m$, $d = 2b$. The generating function F_{2m} of bipartite, non necessarily irreducible, maps is given by [18,10]

$$F_{2m} = \mathrm{Cat}(m) R^m - \sum_{j \ge 1} \sum_{k=1}^{\min(m, j-1)} \frac{2k+1}{2m+1} \binom{2m+1}{m-k} \binom{2j-1}{j+k} x_{2j} R^{m+j} \tag{3.8}$$

where R is the formal power series determined by the equation

$$R = 1 + \sum_{j \geq 1} \binom{2j-1}{j} x_{2j} R^j. \tag{3.9}$$

For concreteness, let us mention that the general coefficient of R reads explicitly

$$\Big[\prod_{j\geq 1} (x_{2j})^{n_j}\Big] R = \frac{\left(\sum_{j\geq 1} j n_j\right)!}{\left(1+\sum_{j\geq 1}(j-1)n_j\right)!} \prod_{j\geq 1} \frac{1}{n_j!} \binom{2j-1}{j}^{n_j} \tag{3.10}$$

as seen by applying the Lagrange inversion formula [25-27]. We may rewrite (3.8) in a more compact form by introducing the shorthand notations

$$A_{m,k} = \frac{2k+1}{2m+1} \binom{2m+1}{m-k} \tag{3.11}$$

(note that $\mathrm{Cat}(m) = A_{m,0}$) and

$$U_k = \sum_{j\geq k+1} \binom{2j-1}{j+k} x_{2j} R^{j+k} \tag{3.12}$$

so that

$$F_{2m} = R^m \sum_{k=0}^{m} A_{m,k} \left(\delta_{k,0} - U_k R^{-k}(1-\delta_{k,0})\right). \tag{3.13}$$

We now apply the general substitution relation (3.6), to get

$$F_{2m}^{(d)} = (R^{(d)})^m \sum_{k=0}^{m} A_{m,k} \left(\delta_{k,0} - U_k^{(d)} (R^{(d)})^{-k} (1-\delta_{k,0})\right) \tag{3.14}$$

where $R^{(d)}$ and $U_k^{(d)}$ are the series obtained by substituting, for all j between 1 and b, the formal variable x_{2j} by the series $X_{2j}^{(d)}$ in R and U_k respectively (recall that $X_j^{(d)} = 0$ for j odd in the bipartite case). In particular, since the variables x_{2j} with $j > b$ are unaffected by the substitution, we have

$$U_k^{(d)} = \sum_{j\geq k+1} \binom{2j-1}{j+k} x_{2j} (R^{(d)})^{j+k} \qquad \text{for } k \geq b. \tag{3.15}$$

We are therefore left with the b unknown quantities $R^{(d)}$ and $U_k^{(d)}$, $k = 1, \ldots, b-1$, which replace the original unknowns $X_{2j}^{(d)}$, $j = 1, \ldots, b$.

We then observe that the condition (1.1) implies that $F_{2m}^{(d)} = \mathrm{Cat}(m)$ for $1 \le m \le b-1$, which may be viewed as a system of $b-1$ *linear* equations for $U_1^{(d)}, \dots, U_{b-1}^{(d)}$ (note that the condition $F_0^{(d)} = 1$ is readily satisfied since $F_0 = 1$). It may easily be solved by introducing the inverse $B = (B_{n,m})_{n,m\ge 0}$ of the semi-infinite unitriangular matrix $A = (A_{m,k})_{m,k\ge 0}$, whose coefficients read explicitly

$$B_{n,m} = (-1)^{n+m}\binom{n+m}{2m}. \tag{3.16}$$

Multiplying (3.14) by $B_{n,m}(R^{(d)})^{-m}$ and summing over m from 0 to n, we obtain that

$$U_n^{(d)} = -\sum_{m=0}^{n} B_{n,m}\mathrm{Cat}(m)(R^{(d)})^{n-m} \qquad \text{for } 1 \le n \le b-1. \tag{3.17}$$

Plugging (3.15) and (3.17) into (3.14) yields

$$F_{2m}^{(d)} = \sum_{k=0}^{b-1}\sum_{\ell=0}^{k} A_{m,k}B_{k,\ell}\mathrm{Cat}(\ell)(R^{(d)})^{m-\ell} - \sum_{k=b}^{m}\sum_{j\ge k+1} A_{m,k}\binom{2j-1}{j+k}x_{2j}(R^{(d)})^{m+j}. \tag{3.18}$$

This expression may be further simplified using the two hypergeometric identities

$$\sum_{k=\ell}^{b-1} A_{m,k}B_{k,\ell} = (-1)^{b-\ell-1}\frac{b-\ell}{m-\ell}\binom{2m}{m-b}\binom{b+\ell}{2\ell}, \tag{3.19}$$

$$\sum_{k=b}^{m} A_{m,k}\binom{2j-1}{j+k} = \frac{b+j}{m+j}\binom{2m}{m-b}\binom{2j-1}{j+b}, \tag{3.20}$$

which are routinely obtained via Gosper's algorithm and may easily be checked by induction on b, resulting in the expression

$$\begin{aligned} F_{2m}^{(d)} = \binom{2m}{m-b}\Bigg(&\sum_{\ell=0}^{b-1}(-1)^{b-\ell-1}\frac{b-\ell}{m-\ell}\binom{b+\ell}{2\ell}\mathrm{Cat}(\ell)(R^{(d)})^{m-\ell} \\ &- \sum_{j\ge b+1}\frac{b+j}{m+j}\binom{2j-1}{j+b}x_{2j}(R^{(d)})^{m+j}\Bigg). \end{aligned} \tag{3.21}$$

Interestingly, this expression involves only the (so far unknown) series $R^{(d)}$. This quantity is in turn determined by the last case of condition (1.1), namely that $F_d^{(d)} = \mathrm{Cat}(b) + z$, which may be rewritten in the form

$$z + \sum_{\ell=0}^{b}(-1)^{b-\ell}\binom{b+\ell}{2\ell}\mathrm{Cat}(\ell)(R^{(d)})^{b-\ell} + \sum_{j\ge b+1}\binom{2j-1}{j+b}x_{2j}(R^{(d)})^{b+j} = 0. \tag{3.22}$$

In particular, if we impose a bound on the face degrees (i.e. we take $x_{2j} = 0$ for j larger than some fixed M), then $R^{(d)}$ hence $F_m^{(d)}$ are algebraic. This is notably the case for irreducible d-angular dissections, where we specialize $x_{2j} = 0$ for all j, keeping z as the only formal variable, to get

$$f_{2m}^{(d)}(z) = \binom{2m}{m-b} \sum_{\ell=0}^{b-1} (-1)^{b-\ell-1} \frac{b-\ell}{m-\ell} \binom{b+\ell}{2\ell} \mathrm{Cat}(\ell) (r^{(d)})^{m-\ell} \tag{3.23}$$

where $r^{(d)}(z)$, the corresponding specialization of $R^{(d)}$, satisfies the algebraic equation

$$z + \sum_{\ell=0}^{b} (-1)^{b-\ell} \binom{b+\ell}{2\ell} \mathrm{Cat}(\ell) (r^{(d)})^{b-\ell} = 0. \tag{3.24}$$

A remarkable formula follows from differentiating (3.21) wrt $R^{(d)}$ (keeping the x_{2j}, $j > b$, fixed): observe that

$$\frac{\partial F_{2m}^{(d)}}{\partial R^{(d)}} = \binom{2m}{m-b} (R^{(d)})^{m-b} \frac{\partial F_d^{(d)}}{\partial R^{(d)}}. \tag{3.25}$$

Since $F_d^{(d)} = \mathrm{Cat}(b) + z$, it follows, by multiplying both sides by $\frac{\partial R^{(d)}}{\partial z}$, that we have the *pointing formula*

$$\frac{\partial F_{2m}^{(d)}}{\partial z} = \binom{2m}{m-b} (R^{(d)})^{m-b} \tag{3.26}$$

and, in particular, for $m = b + 1$,

$$\frac{\partial F_{d+2}^{(d)}}{\partial z} = (d+2) R^{(d)}. \tag{3.27}$$

These formulas generalize, in some sense, the expression [10]

$$F_{2m}^{\bullet} = \binom{2m}{m} (R^{(d)})^m \tag{3.28}$$

for the generating function of *pointed* rooted bipartite (non necessarily irreducible) planar maps, which we recover in the case $d = 0$ (upon understanding z as a weight per vertex).

Let us now give some examples. Taking $b = 2$ and $x_i = 0$ for all i, we recover the case of irreducible quadrangular dissections discussed in Sect. 2.1: (3.23) yields the expression (2.10) for the generating function of irreducible quadrangular dissections of the $2m$-gon and (3.24) yields the algebraic equation (2.3) for $r^{(4)}$. Taking now $b = 3$ and

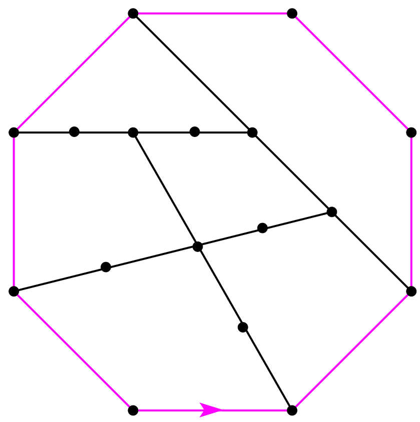

Fig. 4: An example of irreducible hexangular dissection of the octagon.

still $x_i = 0$ for all i, we obtain generating functions of irreducible hexangular dissections (see Fig. 4), namely

$$f_{2m}^{(6)}(z) = \binom{2m}{m-3}\left(\frac{10}{m-2}(r^{(6)})^{m-2} - \frac{12}{m-1}(r^{(6)})^{m-1} + \frac{3}{m}(r^{(6)})^{m}\right) \tag{3.29}$$

where $r^{(6)}$ satisfies

$$z - (r^{(6)})^3 + 6\,(r^{(6)})^2 - 10\,r^{(6)} + 5 = 0. \tag{3.30}$$

The first few terms read

$$\begin{aligned} r^{(6)} &= 1 + z + 3\,z^2 + 17\,z^3 + 120\,z^4 + 948\,z^5 + 8022\,z^6 + \cdots \\ f_8^{(6)} &= 14 + 8\,z + 4\,z^2 + 8\,z^3 + 34\,z^4 + 192\,z^5 + 1264\,z^6 + 9168\,z^7 + \cdots \\ f_{10}^{(6)} &= 42 + 45\,z + 45\,z^2 + 105\,z^3 + 450\,z^4 + 2547\,z^5 + 16785\,z^6 + 121815\,z^7 + \cdots \end{aligned} \tag{3.31}$$

consistently with the pointing formula.

3.4. Elimination in the general case

We now repeat the same strategy in the general case, i.e. when the maps are non necessarily bipartite. Our starting point is the expression found in [10] for the generating function F_n of general maps with outer degree n. It involves generating functions of *three-step paths*, i.e. lattice paths in Z^2 made of three types of steps: up-steps $(1,1)$, down-steps $(1,-1)$ and level-steps $(1,0)$. We denote by

$$\begin{aligned} P_k(n;r,s) &= \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{n!}{j!(j+k)!(n-2j-k)!} r^j s^{n-2j-k} \\ P_k^+(n;r,s) &= \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(k+1)\,n!}{j!(j+k+1)!(n-2j-k)!} r^j s^{n-2j-k} \end{aligned} \tag{3.32}$$

the generating polynomials of respectively arbitrary and nonnegative three-step paths from $(0,0)$ to (n,k), counted with a weight r per down-step and s per level step (a path is said nonnegative if it only visits vertices with nonnegative ordinates). Note that, for $k \leq 0$, $P_k(n;r,s)$ is well defined (the sum over j then starts in practice at $j=-k$) and equals $r^{-k}P_{-k}(n;r,s)$. Then, we have [10]

$$F_n = P_0^+(n;R,S) - \sum_{k\geq 1} P_k^+(n;R,S)V_k \tag{3.33}$$

where V_k is defined by

$$V_k = \sum_{j\geq k+2} x_j P_{-k-1}(j-1;R,S) \tag{3.34}$$

and R, S are formal power series determined by the equations

$$R = 1 + V_0 \qquad S = V_{-1}. \tag{3.35}$$

Clearly, (3.33) yields (3.13) in the bipartite case where $S = 0$, $P_{2k}^+(2m;R,0) = A_{m,k}R^{m-k}$, $V_{2k} = U_k$, $V_{2k+1} = 0$.

Applying the general substitution relation (3.6), we get

$$F_n^{(d)} = P_0^+(n;R^{(d)},S^{(d)}) - \sum_{k\geq 1} P_k^+(n;R^{(d)},S^{(d)})V_k^{(d)} \tag{3.36}$$

where $R^{(d)}$, $S^{(d)}$ and $V_k^{(d)}$ are the series obtained by substituting, for all j between 1 and d, the formal variable x_j by the series $X_j^{(d)}$ in R, S and V_k respectively, namely

$$\begin{aligned} R^{(d)} &= R(X_1^{(d)}, X_2^{(d)}, \ldots, X_d^{(d)}, x_{d+1}, \ldots) \\ S^{(d)} &= S(X_1^{(d)}, X_2^{(d)}, \ldots, X_d^{(d)}, x_{d+1}, \ldots) \\ V_k^{(d)} &= V_k(X_1^{(d)}, X_2^{(d)}, \ldots, X_d^{(d)}, x_{d+1}, \ldots) \ . \end{aligned} \tag{3.37}$$

Note that, in particular,

$$V_k^{(d)} = \sum_{j\geq k+2} x_j P_{-k-1}(j-1;R^{(d)},S^{(d)}) \qquad \text{for } k \geq d-1. \tag{3.38}$$

We are left with d unknowns $R^{(d)}$, $S^{(d)}$ and $V_1^{(d)},\ldots,V_{d-2}^{(d)}$ which replace the original unknowns $X_1^{(d)},\ldots,X_d^{(d)}$. Similarly to the previous section, we may determine these quantities using the conditions (1.1). Here, we need the inverse of the semi-infinite unitriangular matrix $(P_k^+(n;r,s))_{n,k\geq 0}$, which is denoted by $(Q_{n,k}(r,s))_{n,k\geq 0}$ with the explicit form

$$Q_{n,k}(r,s) = \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} \frac{(n-j)!}{k!j!(n-2j-k)!}(-r)^j(-s)^{n-2j-k}. \tag{3.39}$$

Multiplying (3.36) by $Q_{k,n}(R^{(d)}, S^{(d)})$ and summing over n, the conditions (1.1) amount to

$$V_k^{(d)} = -\sum_{j=0}^{\lfloor k/2 \rfloor} Q_{k,2j}(R^{(d)}, S^{(d)}) \, \mathrm{Cat}(j) - \delta_{k,d} z \qquad \text{for } 1 \leq k \leq d \tag{3.40}$$

(recall that $\mathrm{Cat}(j) = 0$ for noninteger j). In particular, comparing this expression with (3.38) for $k = d - 1$ or d yields the equations

$$\begin{aligned} \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} Q_{d-1,2j}(R^{(d)}, S^{(d)}) \, \mathrm{Cat}(j) + \sum_{j \geq d+1} x_j P_{-d}(j-1; R^{(d)}, S^{(d)}) = 0 \\ z + \sum_{j=0}^{\lfloor d/2 \rfloor} Q_{d,2j}(R^{(d)}, S^{(d)}) \, \mathrm{Cat}(j) + \sum_{j \geq d+2} x_j P_{-d-1}(j-1; R^{(d)}, S^{(d)}) = 0 \end{aligned} \tag{3.41}$$

which determine the power series $R^{(d)}$ and $S^{(d)}$. As previously, if we impose a bound on the face degrees ($x_j = 0$ for $j > M$) then $R^{(d)}$ and $S^{(d)}$ are algebraic. It follows that $F_n^{(d)}$ is also algebraic, since by (3.40) (for $k \leq d-2$) and (3.38) (for $k \geq d-1$), (3.36) is a polynomial in $R^{(d)}$ and $S^{(d)}$. This polynomial admits an expression similar to (3.18) (*mutatis mutandis*) which we do not find particularly illuminating: no simplification as nice as (3.21) has been found. Remarkably however, a pointing formula stills holds in the form

$$\frac{\partial F_n^{(d)}}{\partial z} = P_d(n; R^{(d)}, S^{(d)}) \tag{3.42}$$

and may be viewed as a generalization of the formula [10]

$$F_n^{\bullet} = P_0(n; R, S) \tag{3.43}$$

for the generating function of pointed rooted maps. We will prove (3.42) in Sect. 7 below using a combinatorial argument.

We now conclude this section by some examples. Taking $d = 3$ and $x_i = 0$ for all i, we recover the case of irreducible triangular dissections discussed in Sect. 2.1: (3.41) yields the algebraic equations $1 - r^{(3)} + (s^{(3)})^2 = 0$ and $z + 2r^{(3)} s^{(3)} - (s^{(3)})^3 - 3s^{(3)} = 0$, which amount to (2.15). The pointing formula (3.42) yields (2.17) for $n = 4$, which in turn implies the expression (2.20) for the number of irreducible triangular dissections of the square. Taking now $d = 5$ and still $x_i = 0$ for all i, we obtain the case of irreducible pentagular dissections. From (3.41) we find that $r^{(5)}$ and $s^{(5)}$ are determined by the algebraic equations

$$\begin{aligned} (r^{(5)})^2 - 3r^{(5)}(s^{(5)})^2 + (s^{(5)})^4 - 3r^{(5)} + 6(s^{(5)})^2 + 2 = 0 \\ z - s^{(5)} \left(3(r^{(5)})^2 - 4r^{(5)}(s^{(5)})^2 + (s^{(5)})^4 - 12r^{(5)} + 10(s^{(5)})^2 + 10 \right) = 0 \end{aligned} \tag{3.44}$$

and their first few terms read

$$r^{(5)} = 1 + 3z^2 + 73z^4 + 3015z^6 + 151842z^8 + 8493934z^{10} + 507165545z^{12} + \dots$$
$$s^{(5)} = z + 12z^3 + 422z^5 + 19780z^7 + 1062275z^9 + 61781482z^{11} + 3786534059z^{13} + \dots . \tag{3.45}$$

By the pointing formula $df_6^{(5)}/dz = 6s^{(5)}$, the numbers of irreducible pentagular dissections of the hexagon with up to 14 inner faces are read off

$$f_6^{(5)} = 5+3z^2+18z^4+422z^6+14835z^8+637365z^{10}+30890741z^{12}+1622800311z^{14}+\dots . \tag{3.46}$$

4. Combinatorial interpretation via slices

In this section, we provide a combinatorial interpretation for some of the quantities that appear in the approach by substitution. We are led to define some particular classes of maps called *slices*, generalizing the notion introduced in [10, Appendix A].

4.1. General slices

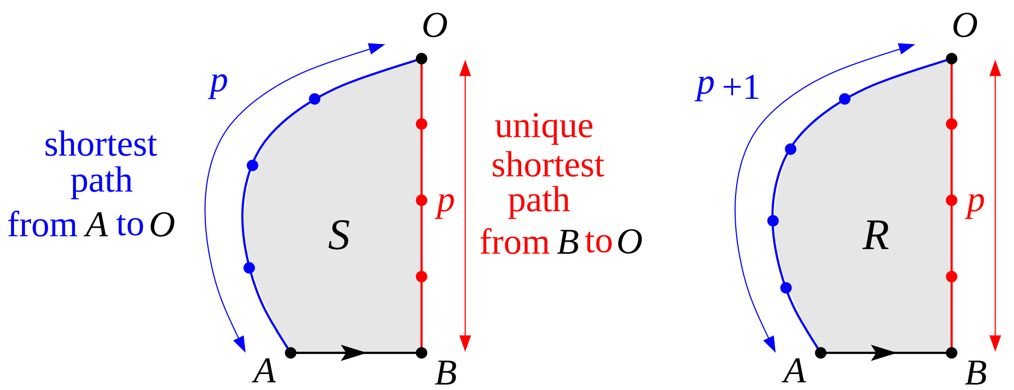

Fig. 5: Schematic picture of slices of type p/p (left) and type $p/p+1$ (right).

As explained in [10], eqs. (3.33) and (3.43) have a direct combinatorial interpretation resulting from a decomposition of the maps enumerated respectively by F_n and $F_n^\bullet$ into more primitive components called *slices*, of which R and S are generating functions. More precisely, a slice is defined as a rooted map with a marked vertex O (later called the apex) incident to its outer face, and which is of the type displayed in Fig. 5, namely satisfies:

- the right boundary of the map, defined as the path joining the endpoint of the root edge to O by following the outer face counterclockwise around the rest of the map, is the unique shortest path in the map between these two points;

- the left boundary of the map, defined as the path joining the origin of the root edge to O by following the outer face clockwise around the rest of the map, is a shortest path in the map between these two points;
- the vertex O is the only vertex common to both the right and left boundaries.

Clearly, if we denote by $p \geq 0$ the length of the right boundary, that of the left boundary is either p or $p+1$ (it cannot be $p-1$ as otherwise, the right boundary would not be the unique shortest path). We shall refer to these slices as being of type p/p or $p/p+1$ accordingly. It was shown in [10] that $R(x_1, x_2, \ldots)$ is precisely the generating function of slices of type $p/p+1$ for some (unfixed) p, counted with weights x_i per inner face of degree i, while $S(x_1, x_2, \ldots)$ is the generating function of slices of type p/p for some p. Note that R incorporates a term 1 accounting for the slice of type $0/1$ reduced to the root edge (with no inner face and with outer face of degree 2), O being the endpoint of this edge.

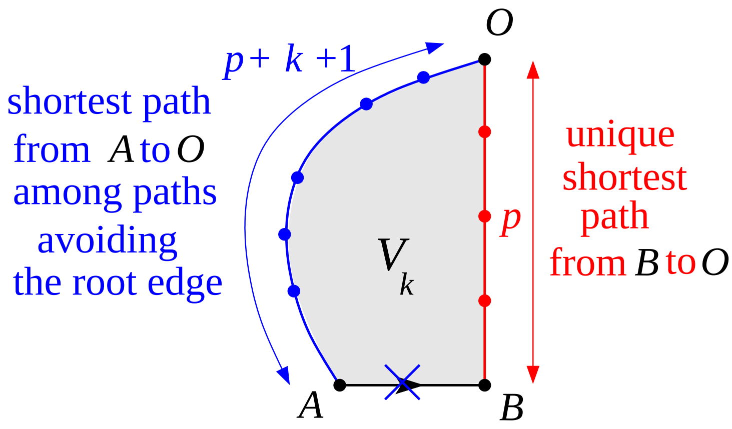

Fig. 6: Schematic picture of a k-slice of type $p/p+k+1$.

In this paper, we extend this notion of slices to what we shall call *k-slices*, defined as follows: as displayed in Fig. 6, a k-slice is, for a given integer k, a rooted map with a marked vertex O (the apex) incident to its outer face, satisfying the following requirements:
- the right boundary of the map is the unique shortest path in the map between the endpoint of the root edge and O;
- the left boundary of the map is a shortest path in the map among all paths which join the origin of the root edge to O and do not pass via the root edge;
- the difference of the lengths of the left and right boundaries is $k+1$. More precisely, if the right boundary has length p for some $p \geq 0$, the left boundary then has length $p+k+1$ and we say that the k-slice is of type $p/p+k+1$;
- the vertex O is the only vertex common to both the right and left boundaries;
- the slice has at least one inner face.

The interest of this definition lies in the following:

Lemma: For all $k \geq -1$, the generating function of k-slices, counted with weights x_i per inner face of degree i, is equal to the quantity V_k defined by (3.34).

Remark: the statement does not hold for $k < -1$, for which there are no k-slices (if the left boundary has length strictly less than p, the right boundary, of length p, cannot be the unique shortest path between the endpoint of the root edge and O), while V_k is nonzero.

Proof: It is easily seen that the lemma holds in the cases $k = -1$ and $k = 0$: the second requirement for a k-slice is then equivalent to demanding that the left boundary itself be a shortest path among all paths in the map between the origin of the root edge and O, since any path between these two points passing via the root edge has length larger than $p + 1$. We immediately deduce that the notion of slices of type p/p and $p/p + 1$ introduced above matches precisely that of -1-slices of type p/p and 0-slices of type $p/p+1$ respectively (with the slight discrepancy that, due to the last requirement that a k-slice has at least one inner face, the slice of type $0/1$ reduced to the root edge is not considered as a 0-slice). We conclude that the generating function of -1-slices is equal to S and that of 0-slices is equal to $R - 1$. By (3.35), these are equal to V_{-1} and V_0 respectively. Actually, that R and S satisfy (3.35) may itself be proved via via slices [10, Appendix A], and we now adapt the argument to the case of an arbitrary $k \geq -1$.

Starting from a k-slice ($k \geq -1$), we consider the face to the left of the root edge: this face has degree $j \geq k + 2$ as otherwise, a path of length $j - 1 + p$ (obtained by going around the face at hand), hence strictly smaller than $p + k + 1$ and avoiding the root edge, would join the origin of the root edge to the apex. Considering the successive vertices clockwise around this face and recording their distance to the apex in the k-slice with the root edge removed creates a three-step path of length $j - 1$ starting at height $p+k+1$ (= distance from the origin of the root edge to the apex) and ending at height p (= distance from the endpoint of the root edge to the apex), see Fig. 7. We shift all heights down by $-p - k - 1$ so as to obtain a three-step path starting at height 0 and ending at height $-k - 1$, as counted by $P_{-k-1}(j - 1; r, s)$. Now, for each vertex around the face, we draw the leftmost shortest path from it to the apex. Cutting along all these shortest paths results into a decomposition of the map into connected components, each component being either a 0-slice attached to a down-step, since the lengths of the boundaries differ by 1 in this case, or a -1-slice attached to a level-step, since the length of the boundaries are then identical. Note that some steps do not give rise to a slice: up-steps never do as the leftmost shortest path begins by following the boundary of the face counterclockwise (see Fig. 7), while for some down-steps it might occur that a leftmost shortest path follows the boundary of the face clockwise (in contrast, every level-step gives rise to a nonempty -1-slice). It is not difficult to check that, conversely, gluing a sequence of 0- and -1-slices attached respectively to some down- and all level-steps of a path of length $j - 1$ with total height decrease of $k + 1$, and closing the path by adding an extra root edge, thus creating a new face of degree j, rebuilds a k-slice. Translating this construction in the language of generating function, it follows that the generating function of k-slices ($k \geq -1$) is equal to $\sum_{j\geq k+2} x_j P_{-k-1}(j - 1; R, S) = V_k$, Q.E.D.

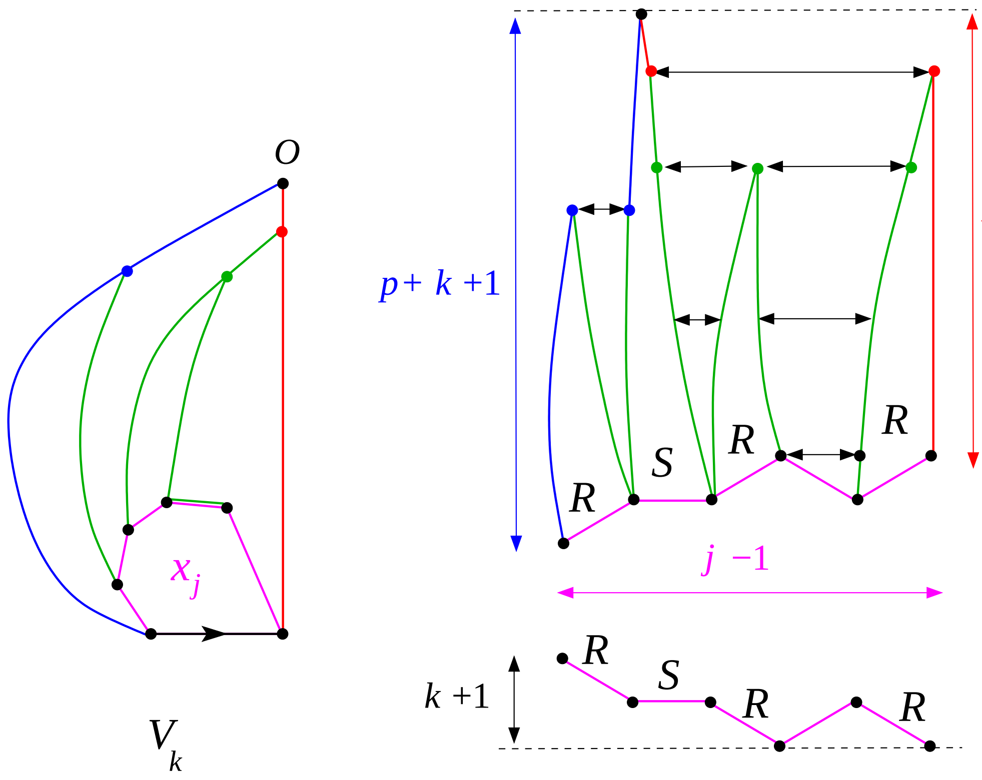

Fig. 7: Schematic picture of the slice decomposition of a k-slice by cutting along the leftmost shortest paths from the vertices incident to the face on the left of the root edge (of degree $j = 6$ here) to the apex O. Recording the lengths of these paths creates a three-step path of length $j - 1$ with height decrease $k+1$ (bottom right) with -1-slices (resp. 0-slices) associated with level-steps (resp. down-steps). Since we represent the slice with O on top, the three-step path appears vertically reflected in the slice.

4.2. d-irreducible slices

Lemma: for all $k \geq 1$, the generating function of d-irreducible k-slices, counted with a weight z per inner face of degree d and, for all $i \geq d+1$, a weight x_i per inner face of degree i, is equal to the quantity $V_k^{(d)}$ defined by (3.37).

Proof: let $\tilde{V}_k^{(d)}(z; x_{d+1}, x_{d+2}, \ldots)$ be the generating function of d-irreducible k-slices. Let us show that the basic substitution relation (3.2) may be adapted to k-slices, namely that we have

$$\tilde{V}_k^{(d-1)}(0; x_d, x_{d+1}, \ldots) = \tilde{V}_k^{(d)}(G_d(x_d, x_{d+1}, \ldots); x_{d+1}, \ldots) \tag{4.1}$$

with G_d as in (3.1). Indeed, a k-slice of type $p/p+k+1$ is a nothing but a particular instance of rooted map with outer face of degree $n = 2p + k + 2$ to which we may apply our substitution approach, keeping the apex unchanged under substitution. By following the same line of arguments as in Sect. 3.1, we relate k-slices of type $p/p+k+1$ and girth at least d (l.h.s) to d-irreducible k-slices of the same type (r.h.s). That the type $p/p+k+1$ of slice remains the same is clear since the substitution does not alter the lengths of the right and left boundaries of the map. The only nontrivial property which must be verified is that these boundaries remain shortest paths within the desired path sets. Here we use again the no-shortcut lemma: starting from a d-irreducible k-slice of type $p/p+k+1$, we easily deduce from the lemma that substituting faces of degree d by rooted maps with girth d and outer degree d does not alter the distances between the originally existing vertices (we apply the same idea as in the proof of the encircling lemma: given a simple oriented path between two such vertices, we may construct a path in the original map lying to its left, having the same endpoints, and having a smaller or equal length). It follows that the right boundary remains the unique shortest path (of the same length) between its endpoints after substitution (indeed, having another path of smaller or equal length after substitution would imply having to the left of this path another path of smaller or equal length which existed before substitution, a contradiction). As for the left boundary, it remains a shortest path (of the same length) between its endpoints among all paths which avoid the root edge (indeed the existence of a strictly shorter path avoiding the root edge after substitution would imply the existence to the left of this path – hence also avoiding the root edge – of a path with even shorter length already present in the original map, a contradiction). Conversely, erasing the interior of the outermost cycles of length d in a k-slice of type $p/p+k+1$ with girth at least d does not modify the distances between the remaining vertices so the conditions on the two boundaries remain satisfied and these boundaries keep the same lengths.

Taking $x_d = X_d(z; x_{d+1}, \ldots)$ as in (4.1) and iterating, we obtain that

$$\tilde{V}_k^{(d)}(z; x_{d+1}, \ldots) = \tilde{V}_k^{(d-1)}(0; X_d, x_{d+1}, \ldots) = \cdots = \tilde{V}_k^{(0)}(0; X_1^{(d)}, \ldots, X_d^{(d)}, x_{d+1}, \ldots) \tag{4.2}$$

where $X_k^{(d)}(z; x_{d+1}, \ldots)$, $1 \leq k \leq d$, is defined as in (3.7). But $\tilde{V}_k^{(0)}(0; x_1, x_2, \ldots)$ is the generating function of 0-irreducible, i.e. arbitrary k-slices which, from Sect. 4.1, is nothing but $V_k(x_1, x_2, \ldots)$. Thus, by (3.37), $\tilde{V}_k^{(d)}$ and $V_k^{(d)}$ coincide, Q.E.D.

To conclude this section, let us discuss the simplifications arising in the case of bipartite maps, i.e. when d is even and $x_i = 0$ for odd i. Then the degree $2p+k+2$ of the outer face of a k-slice of type $p/p+k+1$ is necessarily even, which implies that $V_k^{(d)} = 0$ for odd k, and in particular $S^{(d)} = V_{-1}^{(d)} = 0$. We then have

$$V_{2k}^{(d)} = U_k^{(d)} \tag{4.3}$$

where $U_k^{(d)}$ is the quantity introduced in Sect. 3.3.

5. Recursive decomposition of d-irreducible slices

In this section, we shall show that, as generating functions of d-irreducible k-slices, the quantities $V_k^{(d)}$ for $k < d-1$ satisfy a system of nonlinear equations which express a recursive decomposition of the k-slices into smaller components, themselves m-slices of some kind. This provides an alternative route for computing them without recourse to the substitution procedure. Here, we shall assume that

$$d \geq 1 \qquad \text{and} \qquad -1 \leq k \leq d-2. \tag{5.1}$$

Then, since the outer face of a k-slice of type $p/p+k+1$ has degree $2(p+1)+k$, the requirement of girth at least d implies that $p \geq (d-2-k)/2$. In particular the value $p=0$ may be realized in the range (5.1) that we consider only when $k=d-2$. In this case the condition of d-irreducibility implies that the only d-irreducible $(d-2)$-slice of type $0/d-1$ has a single inner face of degree d. We now describe how to decompose slices of type $p/p+k+1$ with $p \geq 1$.

5.1. The binary decomposition procedure

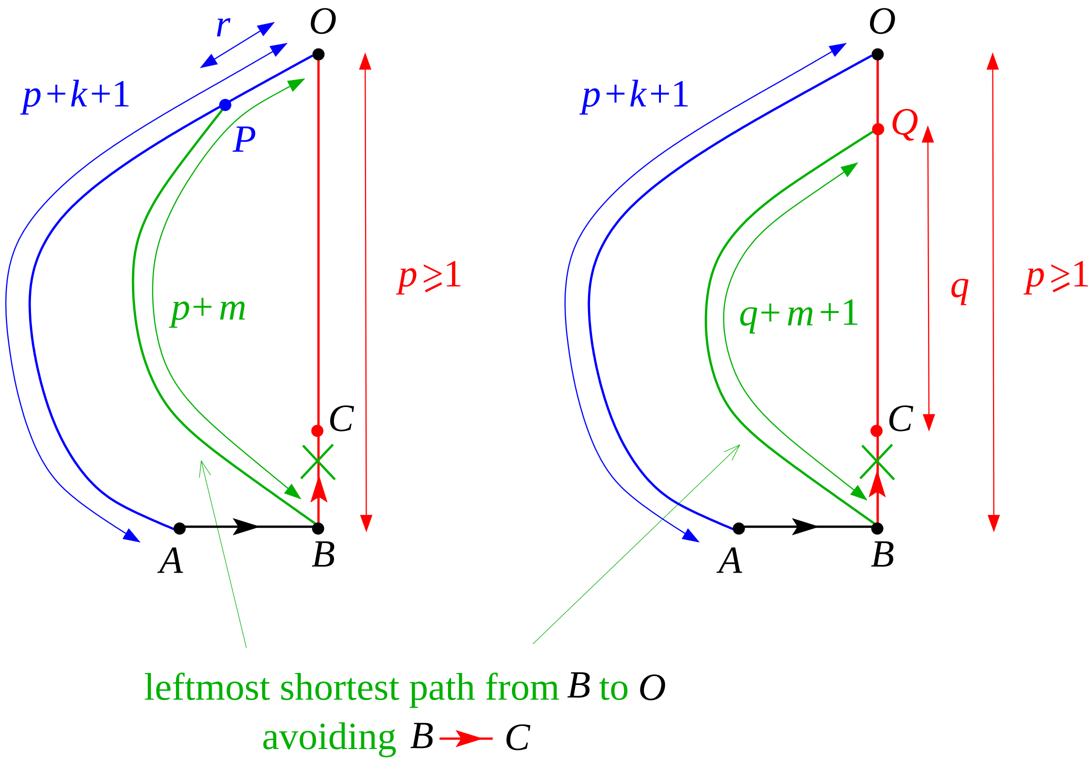

Fig. 8: Binary decomposition of a d-irreducible k-slice of type $p/p+k+1$ with root edge AB, for $p \geq 1$: we cut along the leftmost shortest path from B to the apex O which avoids the first edge BC of the right boundary. This path may either merge first with the left boundary (left) or with the right boundary (right).

Let us consider a d-irreducible slice of type $p/p+k+1$ with $p \geq 1$, and let AB be its root edge, see Fig. 8. We then singularize the first edge BC of the right boundary, which serves as root for a new map obtained by cutting the k-slice along the leftmost shortest path from B to the apex O which avoids the edge BC. This *marked path* has a length $p+m$, with $m \geq 1$, since the right boundary is the unique shortest path from B to O, and $m \leq k+2$, since the left boundary gives rise to a path of length $p+k+2$ between B and O and avoiding BC. Clearly the part of the map lying in-between the marked path and the right boundary is a d-irreducible m-slice (with root edge BC). This m-slice may be of type $q/q+m+1$ for any $q \leq p-1$ as the marked path may hit the right boundary at a point Q between C and O (note that the marked path "sticks" to the boundary after hitting it). If $m = k+2$, the marked path starts with BA and necessarily coincides with the left boundary of the k-slice, thus the decomposition amounts to rerooting the slice on BC, changing its type to $p-1/p+k+2$. If $m \leq k+1$, the part lying in-between the marked path and the left boundary is a d-irreducible $(k-m)$-slice. It is of type $q'/q'+k-m+1$, where q' is the length of the portion of marked path from B to the point P where it hits the left boundary (thus $q' = p+m-r$ where $r \geq 0$ is the distance PO, see again Fig. 8). Since the marked path either hits the right or the left boundary, we have either $q = p-1$ or $q' = p+m$, i.e. $p = \max(q+1, q'-m)$.

To summarize, a d-irreducible slice of type $p/p+k+1$ with $p \geq 1$ is decomposed into either a d-irreducible slice of type $p-1/p+k+2$ or into a pair of d-irreducible slices, one of type $q/q+m+1$ and one of type $q'/q'+k-m+1$, where m is an integer between 1 and $k+1$ and q, q' are two nonnegative integers such that $p = \max(q+1, q'-m)$.

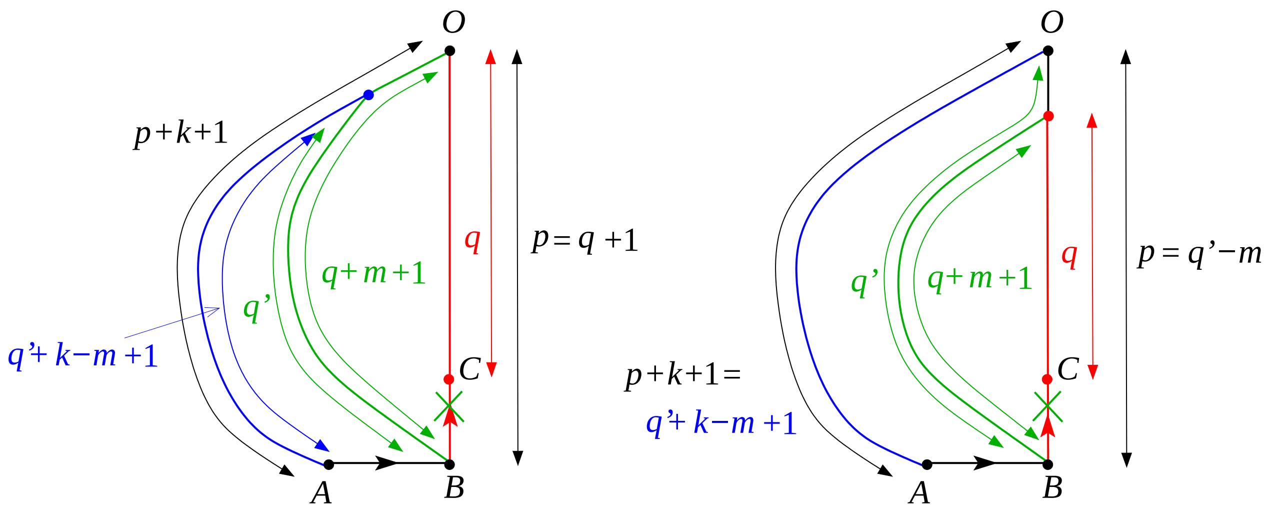

Fig. 9: The gluing of a d-irreducible m-slice of type $q/q+m+1$ with a d-irreducible $(k-m)$-slice of type $q'/q'+k-m+1$ yields a d-irreducible k slice of type $p/p+k+1$ with $p = \max(q'-m, q+1)$.

Conversely, gluing as in Fig. 9 an arbitrary d-irreducible m-slice of type $q/q+m+1$, $1 \leq m \leq k+1$, with root edge BC, and an arbitrary d-irreducible $(k-m)$-slice of type $q'/q'+k-m+1$, with root-edge AB, creates a rooted map of type $p/p+k+1$ where $p = \max(q'-m, q+1) > 0$. Here the apex O of the concatenated map is chosen as being the apex of the m-slice if $q' \leq q+m+1$ and that of the $(k-m)$-slice otherwise. This map is clearly d-irreducible: indeed, in a d-irreducible slice, any simple path joining two vertices of the left (resp. right) boundary and not entirely included in the boundary has length at least (resp. strictly larger than) $d/2$ (as otherwise closing this path with the, necessarily shorter, portion of the boundary between the two vertices would create a cycle of length strictly less than d), which ensures that the concatenation of d-irreducible slices along their boundaries is still d-irreducible. Moreover, since $m \geq 1$, the right boundary of the resulting map is clearly the unique shortest path in the map from the point B to the apex. As for the left boundary of the concatenated map, it is clearly a shortest path from A to O among all paths which avoid both the AB and BC edges. To make sure we have a k-slice, we must guarantee that this is also true among paths which avoid AB but pass via BC, which requires that the length of any path from A to B avoiding AB has length at least $k+1$. This is again guaranteed by the condition of girth at least d: any path from A to B avoiding AB has length at least $d-1$ which is larger than or equal to $k+1$ in the range (5.1). This shows that our decomposition is a bijection (in the case of a slice of type $p-1/p+k+2$, its unique pre-image of type $p/p+k+1$ is recovered by a simple rerooting).

5.2. The iterated decomposition procedure

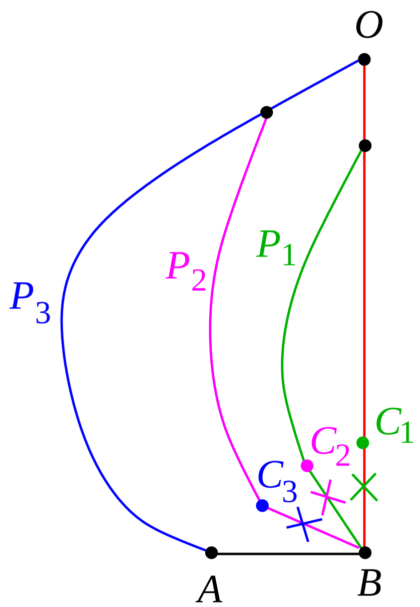

Fig. 10: Iterated decomposition of a d-irreducible k-slice with root AB and apex O. We start with the binary decomposition of Fig. 8, i.e. pick the leftmost shortest path P_1 from B to O avoiding the first edge BC_1 of the right boundary. We then iterate the procedure by picking the leftmost shortest path P_2 from B to O staying to the left of P_1 and avoiding the first edge BC_2 of P_1. The process ends after a finite number of iterations whenever the selected leftmost shortest path (here P_3) matches the left boundary of the slice.

It is useful to consider a variant of the above binary decomposition procedure (we dub it "binary" since it splits a slice into at most two parts). Simply put, it consists in iterating the decomposition on all subslices with the same root edge as the original slice. More precisely, starting again from a slice of type $p/p+k+1$ ($p \geq 1$) with root edge AB, we decompose it as follows:

- as in the binary decomposition, we pick the leftmost shortest path P_1 among all paths from B to the apex O which do not pass via the first edge BC_1 of the right boundary. Since the right boundary is the only shortest path between B and O, the length of P_1 is $p+m_1$ for some $m_1 \geq 1$. We call C_2 the extremity of the first edge of P_1 (see Fig. 10).
- we then iterate by picking the leftmost shortest path P_2 among all paths from B to the apex O which stay to the left of P_1 and do not pass via the edge BC_2. Since P_1 is a leftmost shortest path between B and O, the length of P_2 is $p+m_1+m_2$ for some $m_2 \geq 1$. We call C_3 the extremity of the first edge of P_2.
- we continue the process until the sum $\sum_i m_i$, which increases strictly at each step, reaches $k+2$. The corresponding leftmost path P_q, of length $p+k+2$, then necessarily follows the left boundary of the slice. In other words, the extremity C_{q+1} of the first edge of P_q coincides with the origin A of the root edge.

Cutting along all the paths P_j, $j=1,\cdots,q-1$, results in a decomposition of the slice into q pieces. Upon rooting the i-th piece on the edge BC_i, this piece is clearly, by construction, a d-irreducible m_i-slice (the lengths of its right and left boundaries differ by m_i+1). Note again that, if $q=1$, i.e. $m_1=k+2$, the slice is kept uncut but rerooted at the edge BC_1 so that we obtain a $(k+2)$-slice whose right boundary length is reduced by 1 and whose left boundary length is increased by 1. To summarize, the iterated decomposition transforms a slice of type $p/p+k+1$ for any $p \geq 1$ into a sequence of q slices, the i-th slice being a m_i-slice, with $m_i \geq 1$ and $m_1+\cdots+m_q=k+2$.

If we now define the *size* of a slice as being the number of its *non-left edges*, i.e. those edges which do not belong to the left boundary, then it is clear that, in the iterated decomposition, the sum of the sizes of all the subslices is exactly 1 less than the size of the original slice. This is because all non-left edges of the original slice are non-left edges of exactly one subslice, except the root edge.

5.3. Recursive equations for the generating functions

We now translate the above decompositions into equations. At this stage, we are interested in generating functions of d-irreducible k-slices, irrespectively of their precise type (i.e. we disregard the value of p for now). Then, the binary decomposition states that a k-slice (distinct from the trivial $(d-2)$-slice reduced to a single d-valent face) is in one-to-one correspondence with either a $(k+2)$-slice or an ordered pair formed by a m-slice and $(k-m)$-slice, with $1 \leq m \leq k+1$. This yields immediately

$$V_k^{(d)} = z\delta_{k,d-2} + \sum_{m=1}^{k+1} V_m^{(d)} V_{k-m}^{(d)} + V_{k+2}^{(d)} \ , \qquad -1 \leq k \leq d-2 \tag{5.2}$$

where the term $z\delta_{k,d-2}$ is the contribution from the only slice of type $0/k+1$ which, as noted above, is reduced to a single d-valent face, with $k = d-2$. As for the iterated decomposition, it yields the equivalent form

$$V_k^{(d)} = z\delta_{k,d-2} + \sum_{q\geq 1} \sum_{\substack{m_i\geq 1,\ i=1,\cdots,q \\ m_1+\cdots+m_q=k+2}} \prod_{i=1}^{q} V_{m_i}^{(d)}, \qquad -1 \leq k \leq d-2. \tag{5.3}$$

Both systems of equations determine $V_k^{(d)}$ for all $-1 \leq k \leq d-2$ from the data of $V_{d-1}^{(d)}$ and $V_d^{(d)}$. These quantities are themselves obtained from (3.38), namely

$$\begin{aligned} V_{d-1}^{(d)} &= \sum_{j\geq d+1} x_j P_{-d}(j-1; R^{(d)}, S^{(d)}) \\ V_d^{(d)} &= \sum_{j\geq d+2} x_j P_{-d-1}(j-1; R^{(d)}, S^{(d)}). \end{aligned} \tag{5.4}$$

This allows to determine $V_k^{(d)}$ for all $-1 \leq k \leq d$ in terms of z, the x_i $(i \geq d+1)$, $R^{(d)}$ and $S^{(d)}$. Equating the obtained expressions for $V_{-1}^{(d)}$ and $V_0^{(d)}$ to the values

$$V_{-1}^{(d)} = S^{(d)}, \qquad V_0^{(d)} = R^{(d)} - 1, \tag{5.5}$$

obtained by specializing (3.35), we get algebraic equations which determine $R^{(d)}$ and $S^{(d)}$ themselves.

It is interesting to note that Eq. (5.2) can be extended to the value $k = d-1$ in the case $z = 0$, i.e. when we consider maps of girth at least $(d+1)$ but not necessarily $(d+1)$-irreducible. Examining the decomposition of $(d-1)$-slices, we arrive at

$$\begin{aligned} V_{d-1}^{(d)}(0; x_{d+1}, \ldots) = G_{d+1}(x_{d+1}, \ldots) + \sum_{m=1}^{d} V_m^{(d)}(0; x_{d+1}, \ldots) V_{d-1-m}^{(d)}(0; x_{d+1}, \ldots) \\ + V_{d+1}^{(d)}(0; x_{d+1}, \ldots) \end{aligned} \tag{5.6}$$

with $V_{d+1}^{(d)}(0; x_{d+1}, \ldots) = \sum_{j\geq d+3} x_j P_{-k-1}(j-1; R^{(d)}(0; x_{d+1}, \ldots), S^{(d)}(0; x_{d+1}, \ldots))$. Here we simply used the fact that the $p = 0$ contribution to $V_{d-1}^{(d)}(0; x_{d+1}, \ldots)$ is precisely the generating function $G_{d+1}(x_{d+1}, \ldots)$ of maps of outer degree $d+1$ and of girth $d+1$ (hence the first term on the r.h.s.). Eq. (5.6) allows eventually to obtain $G_{d+1}(x_{d+1}, \ldots)$ and recover the result of [8].

As an illustration, let us consider the case of 5-angular irreducible dissections ($d = 5$ and $x_i = 0$ for all $i \geq 6$). Using, as before, lower case letters for generating functions specialized at $x_i = 0$, Eq. (5.2) gives

$$\begin{aligned} v_{-1}^{(5)} &= v_1^{(5)} \\ v_0^{(5)} &= v_1^{(5)} v_{-1}^{(5)} + v_2^{(5)} \\ v_1^{(5)} &= v_1^{(5)} v_0^{(5)} + v_2^{(5)} v_{-1}^{(5)} + v_3^{(5)} \\ v_2^{(5)} &= v_1^{(5)} v_1^{(5)} + v_2^{(5)} v_0^{(5)} + v_3^{(5)} v_{-1}^{(5)} + v_4^{(5)} \\ v_3^{(5)} &= z + v_1^{(5)} v_2^{(5)} + v_2^{(5)} v_1^{(5)} + v_3^{(5)} v_0^{(5)} + v_4^{(5)} v_{-1}^{(5)} + v_5^{(5)} \end{aligned} \tag{5.7}$$

with $v_4^{(5)} = v_5^{(5)} = 0$ from (5.4) and, from (5.5), $v_{-1}^{(5)} = s^{(5)}$ and $v_0^{(5)} = r^{(5)} - 1$. This leads, by elimination, to the system of equations

$$\begin{aligned} 0 &= 2 + (r^{(5)})^2 + 6(s^{(5)})^2 + (s^{(5)})^4 - 3r^{(5)}(1 + (s^{(5)})^2) \\ z &= s^{(5)} \left(2(r^{(5)})^2 + 4(2 + (s^{(5)})^2) - r^{(5)}(9 + (s^{(5)})^2) \right) \end{aligned} \tag{5.8}$$

which is equivalent to (3.44).

5.4. Solving the recursive equations in the bipartite case

In the bipartite case, setting $d = 2b$, Eq. (5.2) translates into

$$U_k^{(d)} = z\delta_{k,b-1} + \sum_{m=1}^{k} U_m^{(d)} U_{k-m}^{(d)} + U_{k+1}^{(d)} , \qquad 0 \le k \le b-1 \tag{5.9}$$

which determines $U_k^{(d)}$ for all $0 \le k \le b-1$ from the data of $U_b^{(d)}$. The latter is itself determined from (3.15) by

$$U_b^{(d)} = \sum_{j \ge b+1} x_{2j} \binom{2j-1}{j+b} (R^{(d)})^{j+b} \tag{5.10}$$

with moreover the relation $U_0^{(d)} = R^{(d)} - 1$. By a simple extension of the argument leading to (5.6), we find also in the special case $z = 0$

$$\begin{aligned} U_b^{(d)}(0; x_{d+2}, \ldots) = G_{d+2}(x_{d+2}, \ldots) + \sum_{m=1}^{b} U_m^{(d)}(0; x_{d+2}, \ldots) U_{b-m}^{(d)}(0; x_{d+2}, \ldots) \\ + U_{b+1}^{(d)}(0; x_{d+2}, \ldots) \end{aligned} \tag{5.11}$$

with $U_{b+1}^{(d)} = \sum_{j \ge b+2} x_{2j} \binom{2j-1}{j+b+1} (R^{(d)})^{j+b+1}$. This allows to deduce the generating function $G_{d+2}(x_{d+2}, \ldots)$ of maps of outer degree $d+2$ and girth $d+2$ [8].

It turns out that it is possible to obtain a single algebraic equation for $R^{(d)}$ as follows: we observe that the system (5.9) is triangular in the sense that the $(k+1)$-th (with $k \geq 0$) equation allows to express $U_{k+1}^{(d)}$ in terms of the $U_\ell^{(d)}$ with $\ell \leq k$, therefore in terms of $U_0^{(d)}$. Therefore, we may consider the semi-infinite triangular system of equations

$$\tilde{U}_k = \sum_{m=1}^{k} \tilde{U}_m \tilde{U}_{k-m} + \tilde{U}_{k+1}, \quad k \geq 0 \tag{5.12}$$

determining the $\tilde{U}_k$, $k > 0$, in terms of $\tilde{U}_0$ and then (5.9) is recovered by identifying $\tilde{U}_0 = U_0^{(d)} = R^{(d)} - 1$ and taking formally $\tilde{U}_b = z + U_b^{(d)}$. Introducing the generating function

$$\tilde{U}(t) = \sum_{k \geq 1} \tilde{U}_k t^k \ , \tag{5.13}$$

Eq. (5.12) translates immediately into

$$\tilde{U}_0 + \tilde{U} = \tilde{U} \left(\tilde{U}_0 + \tilde{U} + \frac{1}{t} \right) \tag{5.14}$$

hence

$$t = \frac{\tilde{U}}{(1 - \tilde{U})(\tilde{U}_0 + \tilde{U})} \ . \tag{5.15}$$

The Lagrange inversion formula [25-27] states that, for $k > 0$,

$$\tilde{U}_k = \frac{1}{k} [\tilde{U}^{k-1}] \left((1 - \tilde{U})(\tilde{U}_0 + \tilde{U}) \right)^k = -\frac{1}{k} \sum_{p=1}^{k} \binom{k}{p} \binom{k}{p-1} (-\tilde{U}_0)^p \ . \tag{5.16}$$

In particular, taking $k = b$, the above identification leads to the algebraic equation for $R^{(d)}$:

$$-\frac{1}{b} \sum_{p=1}^{b} \binom{b}{p} \binom{b}{p-1} (1 - R^{(d)})^p = z + \sum_{j \geq b+1} x_{2j} \binom{2j-1}{j+b} (R^{(d)})^{j+b} \ . \tag{5.17}$$

We let the reader verify that this coincides with (3.22) via yet another hypergeometric identity. Recall that the generating functions $F_{2n}^{(d)}$ are related to $R^{(d)}$ by the pointing formula (3.26).

Let us now give a few examples. Irreducible quadrangular dissections are obtained for $b = 2$ and $x_{2j} = 0$ for all $j \geq 3$, hence Eq. (5.17) reads

$$(r^{(4)} - 1) - (r^{(4)} - 1)^2 = z \ , \tag{5.18}$$

which matches (2.3). The case of irreducible hexangular dissections corresponds to $b = 3$ and $x_{2j} = 0$ for all $j \geq 4$, hence Eq. (5.17) now reads

$$(r^{(6)} - 1) - 3(r^{(6)} - 1)^2 + (r^{(6)} - 1)^3 = z \ , \tag{5.19}$$

which matches (3.30). Finally, the case of 2-irreducible quadrangulations, i.e. quadrangulations without multiple edges corresponds to $b = 1$, $z = 0$ and $x_{2j} = 0$ for all $j \geq 3$, in which case Eq. (5.17) gives

$$(R^{(2)} - 1) = x_4 \left(R^{(2)}\right)^3 \ , \tag{5.20}$$

in agreement with [28]. From (5.11), we then deduce that the generating function $G_4(x_4)$ of quadrangulations of outer degree 4 and girth 4 (i.e. without multiple edges) is $G_4(x_4) = (R^{(2)} - 1)(2 - R^{(2)})$.

5.5. Solution of the recursive equations in the general case

Let us now see how to extend to the general (non necessarily bipartite) case the strategy of Sect. 5.4. Our goal is again to obtain a system of algebraic equations for $R^{(d)}$ and $S^{(d)}$ without recourse to substitution, i.e. using the system (5.2) as starting point. Again we observe that this system is triangular, with its $(k+2)$-th (with $k \geq -1$) equation allowing to express $V_{k+2}^{(d)}$ in terms of the $V_\ell^{(d)}$ with $\ell \leq k$, hence eventually in terms of $V_{-1}^{(d)}$ and $V_0^{(d)}$ only. Again we introduce the semi-infinite triangular system of equations

$$\tilde{V}_k = \sum_{m=1}^{k+1} \tilde{V}_m \tilde{V}_{k-m} + \tilde{V}_{k+2}, \quad k \geq -1 \tag{5.21}$$

which determines the $\tilde{V}_k$, $k > 0$, in terms of $\tilde{V}_{-1}$ and $\tilde{V}_0$. Then we may obtain algebraic equations for $R^{(d)}$ and $S^{(d)}$ by simply identifying $\tilde{V}_{-1} = V_{-1}^{(d)} = S^{(d)}$, $\tilde{V}_0 = V_0^{(d)} = R^{(d)} - 1$ and by taking formally $\tilde{V}_{d-1} = V_{d-1}^{(d)}$ and $\tilde{V}_d = z + V_d^{(d)}$ so as to fulfill (5.2) at $k = d - 3$ and $k = d - 2$. If we now introduce the generating function

$$\tilde{V}(t) = \sum_{k \geq 1} \tilde{V}_k t^k \ , \tag{5.22}$$

Eq. (5.2) translates into

$$\frac{\tilde{V}_{-1}}{t} + \tilde{V}_0 + \tilde{V} = \tilde{V} \left(\frac{\tilde{V}_{-1}}{t} + \tilde{V}_0 + \tilde{V} + \frac{1}{t^2} \right) \tag{5.23}$$

hence

$$t\,\tilde{V}_{-1} + t^2(\tilde{V}_0 + \tilde{V}) = \frac{\tilde{V}}{1 - \tilde{V}} \ . \tag{5.24}$$

For fixed $\tilde{V}_{-1}$ and $\tilde{V}_0$, this determines t as a function of $\tilde{V}$, hence in principle all $\tilde{V}_k$ for $k > 0$ in terms of $\tilde{V}_{-1}$ and $\tilde{V}_0$ via a Lagrange inversion $\tilde{V}_k = \frac{1}{k}[\tilde{V}^{k-1}](\tilde{V}/t)^k$. We have not been able to perform the computation but, by inspection of the first terms, we conjecture that

$$\begin{aligned}
\tilde{V}_{2j-1} &= -\sum_{\substack{k\geq 0, m\geq 0\\ k+m\leq j-1}} \binom{j+m-1}{k+2m}\binom{j+m}{k+2m}\frac{\binom{k+2m}{2m}}{2m+1}(-\tilde{V}_{-1})^{2m+1}(-\tilde{V}_0)^k \\
\tilde{V}_{2j} &= -\sum_{\substack{k\geq 0, m\geq 0\\ 1\leq k+m\leq j}} \binom{j+m-1}{k+2m-1}\binom{j+m}{k+2m-1}\frac{\binom{k+2m-1}{2m-1}}{2m}(-\tilde{V}_{-1})^{2m}(-\tilde{V}_0)^k
\end{aligned} \tag{5.25}$$

for $j \geq 1$ (in the second equation, the coefficient $\binom{k+2m-1}{2m-1}/(2m)$ for $m = 0$ and $k \geq 1$, should be understood as its $m \to 0$ limit $1/k$). Assuming that these expressions are valid, we deduce, upon making the identifications mentioned above for $\tilde{V}_{d-1}$ and $\tilde{V}_d$, the equations

$$\begin{aligned}
&\sum_{\substack{k\geq 0, m\geq 0\\ k+m\leq \frac{d}{2}-1}} \binom{\frac{d}{2}+m-1}{k+2m}\binom{\frac{d}{2}+m}{k+2m}\frac{\binom{k+2m}{2m}}{2m+1}(-S^{(d)})^{2m+1}(1-R^{(d)})^k \\
&\qquad\qquad + \sum_{j\geq d+1} x_j P_{-d}(j-1; R^{(d)}, S^{(d)}) = 0 \\
&z + \sum_{\substack{k\geq 0, m\geq 0\\ 1\leq k+m\leq \frac{d}{2}}} \binom{\frac{d}{2}+m-1}{k+2m-1}\binom{\frac{d}{2}+m}{k+2m-1}\frac{\binom{k+2m-1}{2m-1}}{2m}(-S^{(d)})^{2m}(1-R^{(d)})^k \\
&\qquad\qquad + \sum_{j\geq d+2} x_j P_{-d-1}(j-1; R^{(d)}, S^{(d)}) = 0
\end{aligned} \tag{5.26}$$

if d is even and

$$\begin{aligned}
&\sum_{\substack{k\geq 0, m\geq 0\\ 1\leq k+m\leq \frac{d-1}{2}}} \binom{\frac{d-1}{2}+m-1}{k+2m-1}\binom{\frac{d-1}{2}+m}{k+2m-1}\frac{\binom{k+2m-1}{2m-1}}{2m}(-S^{(d)})^{2m}(1-R^{(d)})^k \\
&\qquad\qquad + \sum_{j\geq d+1} x_j P_{-d}(j-1; R^{(d)}, S^{(d)}) = 0 \\
&z + \sum_{\substack{k\geq 0, m\geq 0\\ k+m\leq \frac{d+1}{2}-1}} \binom{\frac{d+1}{2}+m-1}{k+2m}\binom{\frac{d+1}{2}+m}{k+2m}\frac{\binom{k+2m}{2m}}{2m+1}(-S^{(d)})^{2m+1}(1-R^{(d)})^k \\
&\qquad\qquad + \sum_{j\geq d+2} x_j P_{-d-1}(j-1; R^{(d)}, S^{(d)}) = 0
\end{aligned} \tag{5.27}$$

if d is odd. These equations determine $R^{(d)}$ and $S^{(d)}$ and are an alternative to the system (3.41).

6. Bijection between slices and trees

In this section, we exhibit a bijection between slices and some trees, which enjoy the same decomposition structure. For the sake of simplicity, we restrict ourselves to irreducible d-angular dissections, with $d \geq 3$. There seems to be no conceptual difficulty in extending the forthcoming discussion to general d-irreducible maps but the corresponding trees would become more complicated.

6.1. d-oriented k-trees

Recall that, by the iterated decomposition of Sect. 5.2, the generating functions of irreducible d-angular k-slices $v_k^{(d)}$ satisfy

$$v_k^{(d)} = z\delta_{k,d-2} + \sum_{q\geq 1} \sum_{\substack{1\leq m_1,\cdots,m_q\leq d-2 \\ m_1+\cdots+m_q=k+2}} \prod_{i=1}^{q} v_{m_i}^{(d)}, \qquad -1 \leq k \leq d-2 \tag{6.1}$$

as seen by specializing (5.3), and noting that $v_k^{(d)} = 0$ for $k \geq d-1$ by (3.38) at $x_i = 0$. Note that we may restrict the range to $1 \leq k \leq d-2$ and still obtain a closed system for $v_1^{(d)}, \ldots, v_{d-2}^{(d)}$. This system clearly specifies some trees, which we now describe.

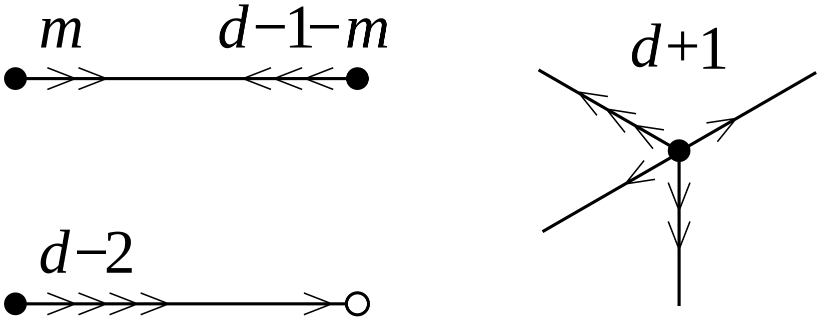

Fig. 11: Rules for the construction of d-oriented trees (see text). Black (resp. white) dots represent inner or root vertices (resp. non-root leaves).

For $1 \leq k \leq d-2$, we define a *d-oriented k-tree* as a planted (with a marked univalent root vertex) plane tree such that:

- the edges of the tree carry arrows and are of two types (see Fig. 11):
  - inner edges of type $m/(d-1-m)$, with $1 \leq m \leq d-2$, whose two half-edges carry respectively m and $d-1-m$ arrows pointing away from the associated edge extremity. These edges connect only inner vertices or the root vertex.
  - leaf edges with one half-edge carrying $d-2$ arrows pointing away from the associated edge extremity and the other half-edge carrying a single arrow pointing toward the associated extremity (this arrow plays no role in the following but we decided to introduce it so as to recover trees similar to those of [14]). The

first extremity of the edge is necessarily an inner vertex or the root vertex, and the second extremity is a leaf of the tree.

- the out-degree of any inner vertex, defined as the total number of arrows pointing away from that vertex among all the incident half-edges, is $d+1$.
- the out-degree of the root vertex is k.

In a d-oriented k-tree, the edge emerging from the root vertex can be a leaf edge only if $k = d-2$, resulting in a tree with a single edge and a single leaf. If it is instead an inner edge, it must be of type $k/(d-1-k)$ linking the root vertex to some inner vertex. The descending subtrees attached to this vertex form a sequence of a number $q \geq 1$ of trees, the i-th one being a d-oriented m_i-tree for some m_i between 1 and $d-2$. From the out-degree condition on inner vertices, we deduce that $\sum_i m_i = d+1-(d-1-k) = k+2$. We immediately deduce that the generating function of d-oriented k-trees, counted with a weight z per leaf, is equal to $v_k^{(d)}$ since it satisfies the same Eq. (6.1). Otherwise stated, d-irreducible k-slices with a given number n of inner faces are in one-to-one correspondence with d-oriented k-trees with n leaves.

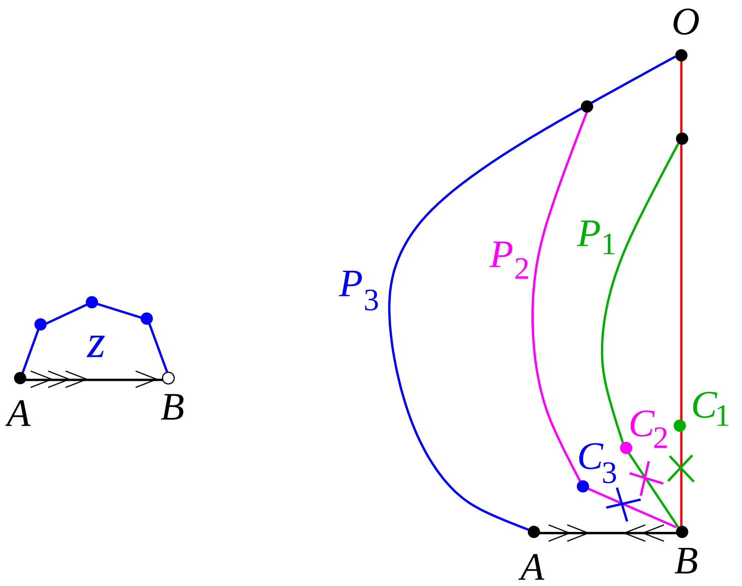

Fig. 12: Inductive definition of the bijection between d-irreducible k-slices and d-oriented k-trees. Left: the $(d-2)$-slice reduced to a single inner face is associated with the $(d-2)$-tree reduced to a single leaf edge. Right: given a k-slice of size > 1, we perform its iterated decomposition: the corresponding k-tree is obtained by keeping the root edge AB, which we decorate into an edge of type $k/(d-1-k)$, and then inductively constructing the trees associated with the subslices delimited by $P_1, P_2, \ldots$ (thus the edges $BC_1, BC_2, \ldots$ will be kept in the tree at the next stage).

Obviously, an explicit bijection between these two sets can be defined through their recursive decompositions. Recall from Sect. 5.2 that the size of a d-irreducible k-slice is defined as the number of its non-left edges (rather than that of its inner faces).

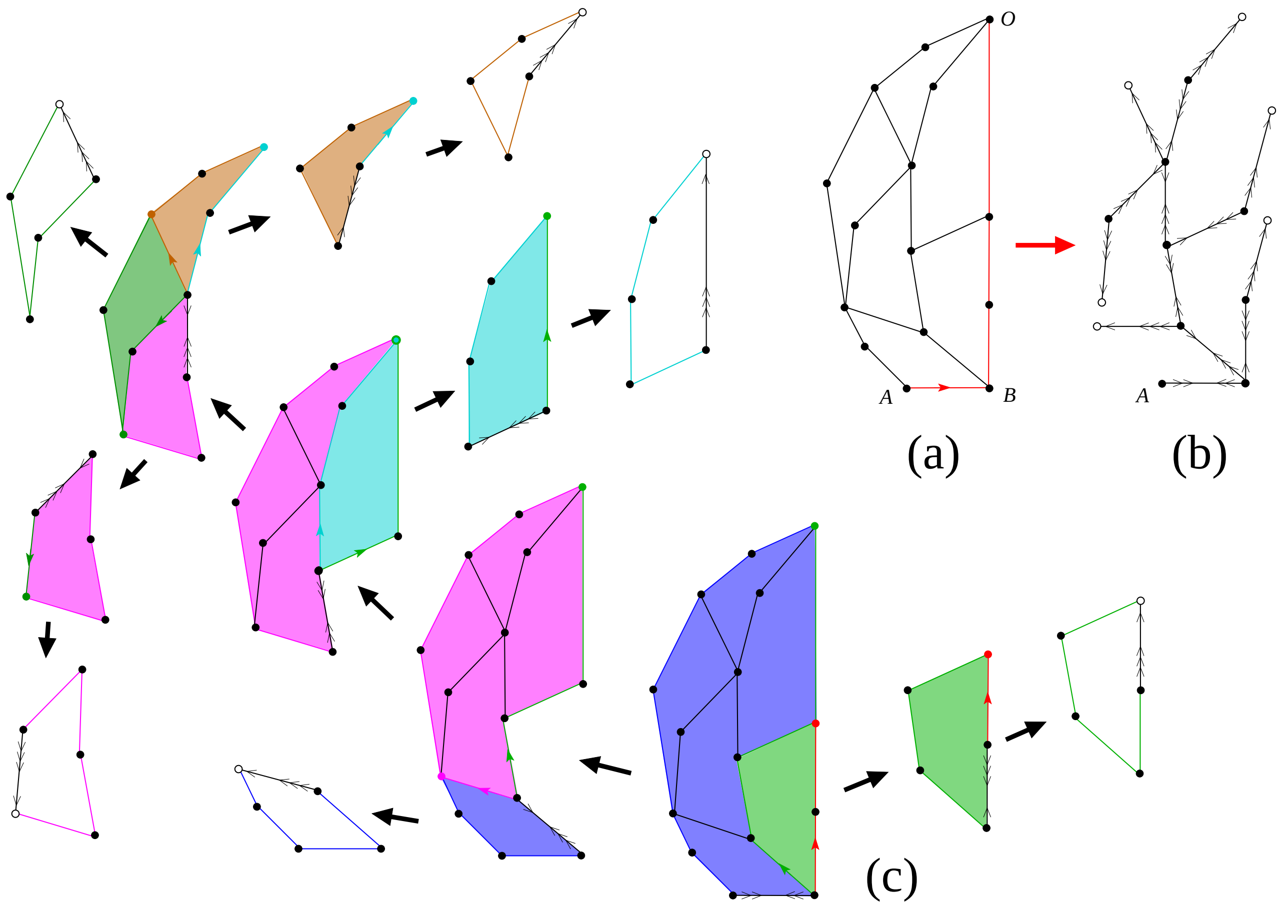

Fig. 13: Example of the full construction (c) of a d-oriented k-tree (b) from a d-irreducible k-slice (a) (here $d = 5$ and $k = 2$).

Similarly, we define the size of a d-oriented k-tree as its number of edges (rather than that of leaves). The only k-slice of size 1 is the $(d-2)$-slice reduced to a single d-valent face, and we associate it with the d-oriented $(d-2)$-tree reduced to a single leaf edge. Suppose now that we have defined the bijection up to size N. Given a k-slice of size $N+1$, we perform its iterated decomposition to obtain a sequence of $q \geq 1$ of subslices whose sizes add up to N. By induction hypothesis we may associate each subslice with a d-oriented tree, and we merge the root vertices of these trees together, and add an inner edge of type $k/(d-1-k)$ to obtain the k-tree of size $N+1$ corresponding to the k-slice at hand. Interestingly, the k-tree may be represented as a subgraph of the k-slice by identifying at each step the added tree edge with the root edge of the slice at hand, see Fig. 12. Fig. 13 displays an example of the construction of the 5-oriented 2-tree associated with a 5-irreducible 2-slice.

To conclude this section, let us discuss the particular case $d = 3$. Taking the general rules of Fig. 11 at $d = 3$, we see that the edges of 3-oriented trees are of the type displayed in Fig. 14, with an out-degree 1 for each half-edge leaving an inner vertex. The constraint of out-degree $d+1 = 4$ at each inner vertex implies that all inner vertices have

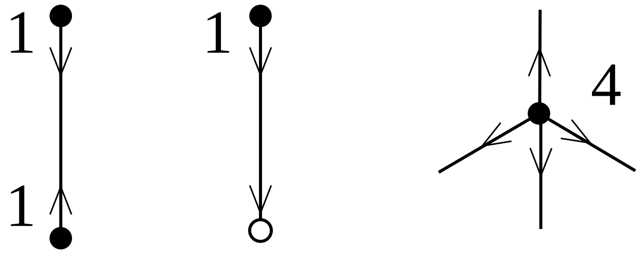

Fig. 14: The rules of Fig. 11 in the particular case $d = 3$. The out-degree of all inner vertices is always 1, so that inner vertices have necessarily degree 4. We deduce that 3-oriented 1-trees reduce to ternary trees.

degree 4, hence the tree is a ternary tree. In other words, irreducible triangular slices are in bijection with ternary trees, and we recover the bijection of [13]. The generating function $v_1^{(3)}$ may be identified with the generating function of planted ternary trees with a weight z per leaf, and satisfies

$$v_1^{(3)} = z + (v_1^{(3)})^3 \tag{6.2}$$

accordingly. This is nothing but Eq. (6.1) at $k = 1$ while at $k = -1$ and $k = 0$, this equation yields $v_{-1}^{(3)} = v_1^{(3)}$ and $v_0^{(3)} = v_{-1}^{(3)} v_1^{(3)}$. Setting $s^{(3)} = v_{-1}^{(3)} = v_1^{(3)}$ and $r^{(3)} = 1 + v_0^{(3)}$, we recover Eq. (2.15). Since the number of leaves in a planted ternary tree is twice the number of inner vertices plus one, we see that, upon setting $s^{(3)} = zT$ as in Sect. 2.2, T may be interpreted as the generating function for ternary trees with a weight z^2 per inner vertex, and satisfies (2.19) accordingly.

6.2. Alternative description of the mapping from trees to slices

While the k-slice associated with a k-tree may be obtained inductively by following the above construction backwards (we decompose the k-tree into subtrees, construct inductively their associated slices and glue them together), we find it worthwile to describe an alternative (but equivalent) construction. It is a closing procedure similar to that of [14], which consists in going counterclockwise around the tree from its root vertex and, whenever a leaf is followed by at least d edge-sides before encountering a new leaf, connecting it to the corner following immediately the d-th encountered edge-side. The procedure must in general be repeated several times, and a left boundary must be restored, before the slice is recovered.

Let us now give a more precise definition of this procedure: given a d-oriented k-tree, we consider its *contour walk* obtained by going counterclockwise around the tree from its root vertex, thus visiting all corners of the tree successively upon following edge sides. Each corner receives a height equal to the height of the preceding corner minus 1 unless the two corners are separated by an edge side that belongs to a leaf edge and corresponds to the second visit of this edge, i.e. while going away from the leaf: in this case the height of the second corner is incremented by $d - 1$ instead (see Fig. 15-left).

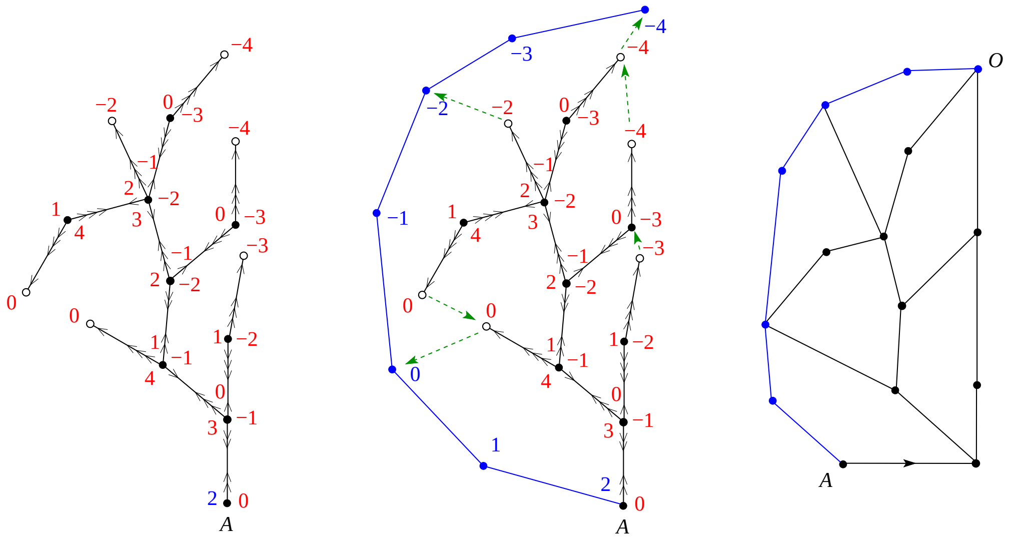

Fig. 15: The reconstruction of the d-irreducible k-slice of Fig. 13 from its associated d-oriented k-tree (with $d = 5$, $k = 2$). Left: we first label the corners of the tree according to the rules explained in the text and call $-p-1$ the minimal label (here $p = 3$). Middle: we then add $k+p+1$ new edges (blue edges), thus creating new corners which we label $k, k-1, \cdots, -p-1$. Right: the slice is obtained by connecting each leaf to the first corner with same label encountered counterclockwise around the tree. This slice is of type $p/k+p+1$.

Starting from height 0, we reach after a complete exploration of the tree the height k. Indeed, it is easily seen by induction that any subtree whose root vertex has out-degree m separates two corners whose heights differ by m. Calling $-p-1$ (with $p \geq 0$) the minimum height obtained along the contour walk, necessarily attained on a leaf, we complete the tree by a sequence of $k+p+1$ new edge sides creating new corners with respective heights $k, k-1, \cdots, -p-1$ and accordingly extend the contour walk so as to end at the minimal height $-p-1$ (see Fig. 15-middle). We define the *closure* of the tree as the slice obtained by simply connecting each leaf with height m to its successor, which is the first corner with the same height m encountered along the contour walk (see Fig. 15-right). The apex of the slice is the (unique after connection) vertex with smallest height $-p-1$. We leave it as an exercice to the reader to check that the closure of the tree indeed coincides with the k-slice obtained by the inductive bijection.

Two remarks are in order: first, let us observe that our construction (in particular the fact that we add new edges to obtain the left boundary) is somewhat reminiscent of the construction of the "discrete map with geodesic boundaries" associated with a labeled tree [11] (actually slices and DMGBs are essentially the same objects). It is however unclear that the two constructions could be unified: the involved trees (d-oriented vs labeled) are quite different and, in the DMGB construction, the slice is obtained by closing the tree on its right boundary rather than the left one here . Second, our construction differs from those of [13,14] on the fact that constructing a slice requires only a "partial closure" (using the terminology of [13,14]) whereas constructing a true dissection requires instead a "complete closure" performed on some unrooted tree. While it seems possible to define a complete closure of d-oriented trees, we do not explore this direction here.

6.3. Trees in the bipartite case

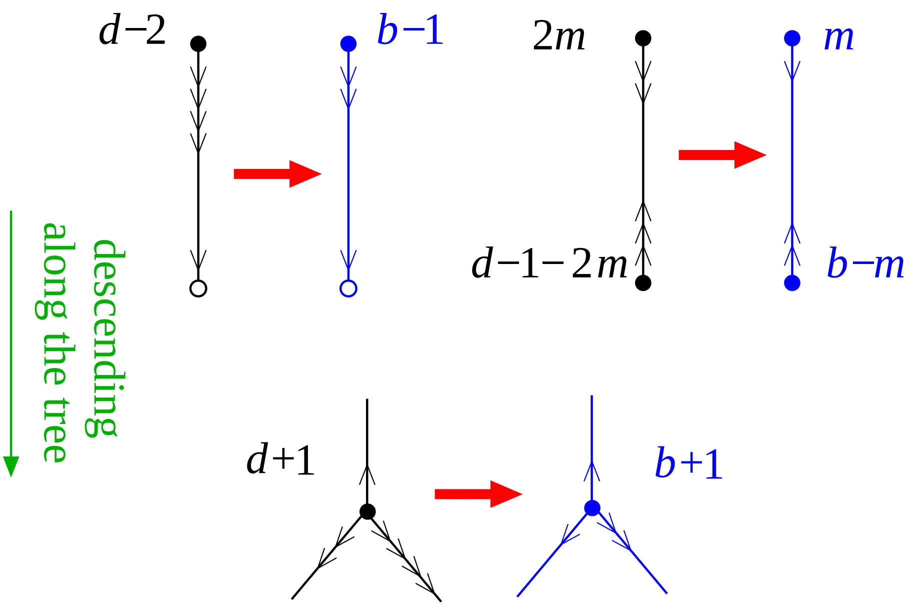

Fig. 16: Simplification of the rules of Fig. 11 for d-oriented trees in the case $d = 2b$. Upon descending along the tree, the original edges (black) are converted into simpler edges (blue) as shown. The original rule of total out-degree $d + 1$ for inner vertices translates after conversion into a constraint of total out-degree $b + 1$ (bottom).

We now consider the case of irreducible d-angulations when d is even. As before, we then set $b = d/2$. Let us first look at d-oriented $(2k + 1)$-trees ($k \geq 0$), i.e trees with an odd root out-degree. Then the root edge is necessarily an inner edge connecting the root vertex to an inner vertex B. The constraint $\sum_i m_i = 2k + 3$ for the sum of the out-degrees of the descending subtrees at B implies that there is an odd number of such subtrees with odd root out-degree m_i. In particular, there is at least one such subtree with odd root out-degree. By the same reasoning, this subtree has itself at least a descending subtree with odd root out-degree and, by iteration, the d-oriented $(2k+1)$-tree cannot be finite. This is consistent with $v_{2k+1}^{(d)} = 0$ as generating functions count finite trees. As for d-oriented $2k$-trees, they are for the same reason build only out of leaf edges or inner edges of type $2m/(d - 1 - 2m)$, i.e. having an even number of arrows $2m$ followed by the complementary odd number of arrows $2b - 1 - 2m$ when descending along the tree. We may then simplify the trees by converting these edges of type $2m/(d - 1 - 2m)$ into simpler edges of type $m/(b - m)$ (see Fig. 16). Consistently we convert the leaf edges into edges with one half-edge carrying $b-1$ arrows pointing away from the associated edge extremity and the other half-edge carrying a single arrow pointing toward the associated extremity. Once this conversion is done, the out-degree of an inner vertex dangling from an edge of type $m/b-m$ becomes $\sum_i m_i+b-m=b+1$ (since before conversion, we had $\sum_i 2m_i+2b-1-2m=2b+1$). The generating functions $u_k^{(d)}=v_{2k}^{(d)}$, viewed as generating functions of d-oriented $2k$-trees, may thus alternatively be interpreted as counting trees, with a weight z per leaf, such that:

- the edges of the tree carry arrows and are of two types:
  - inner edges of type $m/(b-m)$, with $1\leq m\leq b-1$, connecting only inner vertices or the root vertex.
  - leaf edges with one half-edge, connected to an inner vertex or to the root vertex, carrying $b-1$ arrows pointing away from this vertex and the other half-edge carrying a single arrow pointing toward a leaf of the tree.
- the out-degree of any inner vertex is $b+1$.
- the out-degree of the root vertex is k.

As such, the $u_k^{(d)}$ satisfy

$$u_k^{(d)}=z\delta_{k,b-1}+\sum_{q\geq 1}\sum_{\substack{1\leq m_1,\ldots,m_q\leq b-1\\ m_1+\cdots+m_q=k+1}}\prod_{i=1}^{q}u_{m_i}^{(d)}\qquad 1\leq k\leq b-1, \tag{6.3}$$

with the convention $u_m^{(d)}=0$ for $m\geq b$. This system of equations is easily seen to follow from (5.9) in the same way that (6.1) follows from (5.2).

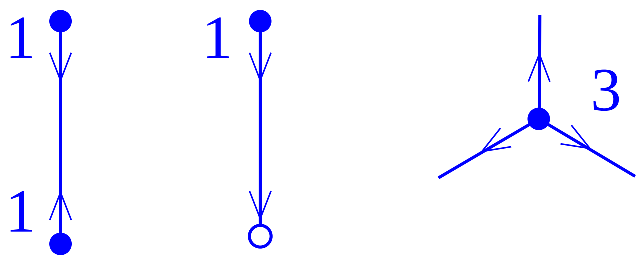

Fig. 17: The rules of Fig. 16 (after conversion) in the particular case $d=4$ ($b=2$). The out-degree of all inner vertices is always 1, so that inner vertices have necessarily degree 3. We deduce that 4-oriented 2-trees reduce to binary trees.

To conclude this section, let us discuss the particular case $d=4$, i.e. $b=2$. After conversion, taking the general rules of Fig. 16 at $b=2$, we see that the edges of the obtained trees are of the type displayed in Fig. 17, with an out-degree 1 for each half-edge leaving an inner vertex. The constraint of out-degree $b+1=3$ at each inner vertex implies that all inner vertices have degree 3, hence the tree is a binary tree. In other words, irreducible quadrangular slices are in bijection with binary trees, and we recover the bijection of [14]. The generating function $u_1^{(4)}$ is nothing but the generating function of planted binary trees with a weight z per leaf, and satisfies

$$u_1^{(4)} = z + (u_1^{(4)})^2 \tag{6.4}$$

accordingly. This is nothing but Eq. (6.3) at $k = 1$ while, from (5.3) at $k = 0$, we deduce $u_0^{(4)} = v_0^{(4)} = v_2^{(4)} = u_1^{(4)}$. Using $r^{(4)} = 1 + u_0^{(4)} = 1 + u_1^{(4)}$, we then recover Eq. (2.3). Setting $r^{(4)} = 1 + zT$ as in Sect. 2.1, i.e. $u_1^{(4)} = zT$, we see immediately that T may now be interpreted as the generating function of binary trees with a weight z per inner vertex, and satisfies (2.7) accordingly.

7. Bijective proof of the pointing formulas

The purpose of this section is to give a combinatorial proof of the general pointing formula (3.42) which, by specialization, implies the bipartite pointing formula (3.26), alternatively obtained as a consequence of the computations of Sect. 3.3. Observe that the l.h.s. of (3.42), $\partial F_n^{(d)}/\partial z$, is the generating function of d-irreducible maps with outer degree n and a marked inner face of degree d (which does not receive a weight z). Such a map is said *annular* and its marked face is called the *central face*. Our purpose is then to show that annular d-irreducible maps with outer degree n are in bijection with the objects naturally counted by $P_d(n; R^{(d)}, S^{(d)})$, namely three-step paths from $(0, 0)$ to (n, d) endowed with the data of:

- for each down-step, a slice of type $p/p+1$ for some arbitrary p (i.e. either a 0-slice or the map reduced to a single root edge if $p = 0$),
- for each level-step, a slice of type p/p (i.e. a -1 slice).

This bijection, which should preserve the total number of (non central) inner faces of each degree, will of course be a variant of the slice decomposition introduced in [10] and already encountered in Sect. 4.

7.1. From paths to annular maps

Here, it is simpler to first describe the mapping from slice-decorated three-step paths to annular maps, see Fig. 18. It again consists in gluing the slices together but we shall be precise about the procedure. Let us denote by $h_0 = 0, h_1, \ldots, h_n = d$ the successive heights of the three-step path and, for $1 \le i \le n$, let $\mathcal{S}_i$ be the slice attached to the i-th step $h_{i-1} \to h_i$ (for an up-step, $\mathcal{S}_i$ is by convention the map reduced to a single edge).

We first define the *partial gluing* $\tilde{\mathcal{S}}_n$ of $\mathcal{S}_1, \ldots, \mathcal{S}_n$ inductively. We take $\tilde{\mathcal{S}}_0$ to be the vertex-map (the map reduced to a single vertex with no edge, which we view as a rooted map of outer degree 0). Let us now assume by induction that we have defined $\tilde{\mathcal{S}}_i$, the partial gluing of $\mathcal{S}_1, \ldots, \mathcal{S}_i$, as a d-irreducible map of outer degree $i + h_i + 2q_i$ for some $q_i \ge \max(0, -h_i)$, satisfying the following constraint. When turning around $\tilde{\mathcal{S}}_i$ in counterclockwise direction starting from the root edge, we divide its outer boundary into three parts:

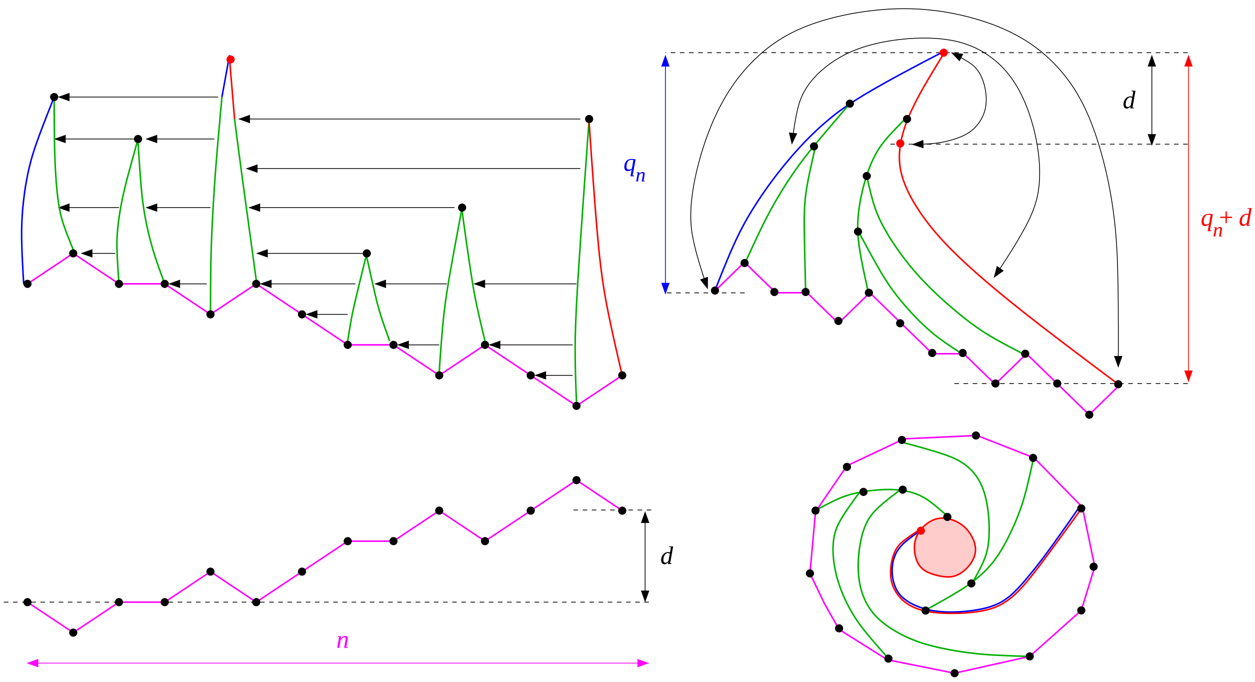

Fig. 18: Illustation of the gluing procedure. Left: the partial gluing of a slice-decorated three-step path from $(0,0)$ to (n,d). Right: the folding of the left boundary onto the right boundary yielding an annular map. (The heights of the path represent the relative distance of the lower boundary vertices to the apex in the partial gluing, hence with the convention of representing the slices pointing upwards the path appears vertically reflected in the map.)

- a first part of length i, called the lower boundary,
- a second part of length $h_i + q_i$, called the right boundary, which is the unique shortest path between its endpoints,
- a third part of length q_i, called the left boundary, which is a (non necessarily unique) shortest path between its endpoints.

Recall that the outer boundary of the slice $\mathcal{S}_{i+1}$ (assuming that it corresponds to a level- or down-step) is also split into three parts:

- a first path of length 1, the root edge,
- a right boundary of length p_i, for some $p_i \geq 0$, which is the unique shortest path between its endpoints,
- a left boundary of length $p_i + h_i - h_{i+1}$ (i.e. p_i or $p_i + 1$ depending on whether the i-step is a level- or down-step), which is a shortest path between its endpoints.

For an up-step, we view $\mathcal{S}_{i+1}$ as having a right boundary of length 1 and a left boundary of length 0. We then naturally define $\tilde{\mathcal{S}}_{i+1}$ by gluing the right boundary of $\tilde{\mathcal{S}}_i$ to the left boundary of $\mathcal{S}_{i+1}$, identifying the origin of the root edge of $\mathcal{S}_{i+1}$ with the first vertex of the right boundary of $\tilde{\mathcal{S}}_i$ (in particular, $\tilde{\mathcal{S}}_1 = \mathcal{S}_1$) . Note that these boundaries do not necessarily have the same length: for instance if $q_i > p_i - h_{i+1}$ then there remain some unmatched edges on the right boundary of $\tilde{\mathcal{S}}_i$, which become part of the right boundary of $\tilde{\mathcal{S}}_{i+1}$. It is easily seen that $\tilde{\mathcal{S}}_{i+1}$ has outer degree $i+1+h_{i+1}+2q_{i+1}$, with $q_{i+1} = \max(q_i, p_i - h_{i+1})$, and we take its root edge to be that of $\tilde{\mathcal{S}}_i$. Furthermore, using the data that $\tilde{\mathcal{S}}_i$ and $\mathcal{S}_{i+1}$ are d-irreducible and that their left/right boundaries are shortest paths, it is not difficult to check that the same properties hold for $\tilde{\mathcal{S}}_{i+1}$. Thus, we fulfill the induction hypothesis.

Having defined the partial gluing $\tilde{\mathcal{S}}_n$, we finish by gluing its right boundary (of length $d+q_n$) to its left boundary (of length q_n), identifying the origin of the root edge to the first vertex of the right boundary. The d unmatched edges of the right boundary yield a central inner face of degree d, and the lower boundary yields an outer face of degree n, thus we obtain an annular map, which is easily shown to be d-irreducible. In particular, to see that the only cycle of length $\leq d$ winding around the central face is its boundary, we use the fact that the right boundary is the unique shortest path between its endpoints, so that each vertex of the left boundary of $\tilde{\mathcal{S}}_n$ except the first one is identified with a vertex of the right boundary at distance $> d$ in $\tilde{\mathcal{S}}_n$.

Finally, let us note the cyclic invariance of the construction. More precisely, we let $\epsilon_i = h_i - h_{i-1} \in \{-1, 0, 1\}$, $1 \leq i \leq n$, denote the i-th *increment* of our three-step path. Then, for any m between 1 and n, we consider the circularly shifted sequence $\epsilon_m, \epsilon_{m+1}, \ldots, \epsilon_n, \epsilon_1, \ldots, \epsilon_{m-1}$, which is the sequence of increments of another three-step path from $(0,0)$ to (n,d). Attaching to this path the slice sequence $\mathcal{S}_m, \mathcal{S}_{m+1}, \ldots, \mathcal{S}_n, \mathcal{S}_1, \ldots, \mathcal{S}_{m-1}$, we obtain another slice-decorated three-step path to which we may apply the gluing procedure. It is not difficult to see that the resulting annular map is the same, except for the position of the root edge which is moved by $m-1$ steps along the outer face.

7.2. Slice decomposition of annular maps

We now explicit the inverse mapping from annular maps to slice-decorated three-step paths. Because of the cyclic invariance noted above, it is desirable to have a construction in which the root edge does not play a specific role. Such a construction turns out to be naturally described on the *lift* of the annular map which we define as follows. Let $\mathcal{M}$ be a d-irreducible annular map of outer degree n, which we think of as being drawn on the complex plane, the origin being in the interior of the central face. Then, we define its lift $\tilde{\mathcal{M}}$ as the preimage of $\mathcal{M}$ by the mapping $z \mapsto \exp(2i\pi z)$, i.e. the map whose vertices and edges are the preimages of those of $\mathcal{M}$ (it is not difficult to convince oneself that the notion of lift is well behaved with respect to continuous deformation). The map $\tilde{\mathcal{M}}$ is infinite but locally finite, i.e. each vertex has finite degree (equal to the degree of its image-vertex in $\mathcal{M}$). Furthermore, the preimage of the central (resp. outer) face of $\mathcal{M}$ forms a single face of infinite degree, the *upper* (resp. *lower*) face of $\tilde{\mathcal{M}}$, while each inner non central face of $\mathcal{M}$ yields infinitely many faces of finite degree. The translation $z \mapsto z+1$ induces a natural automorphism T of $\tilde{\mathcal{M}}$, and we endow $\tilde{\mathcal{M}}$ with the graph distance $\tilde{D}(\cdot,\cdot)$.

Since we are in the plane, the notion of leftmost shortest path from a given vertex of $\tilde{\mathcal{M}}$ to another is defined in an obvious manner. Let us denote by

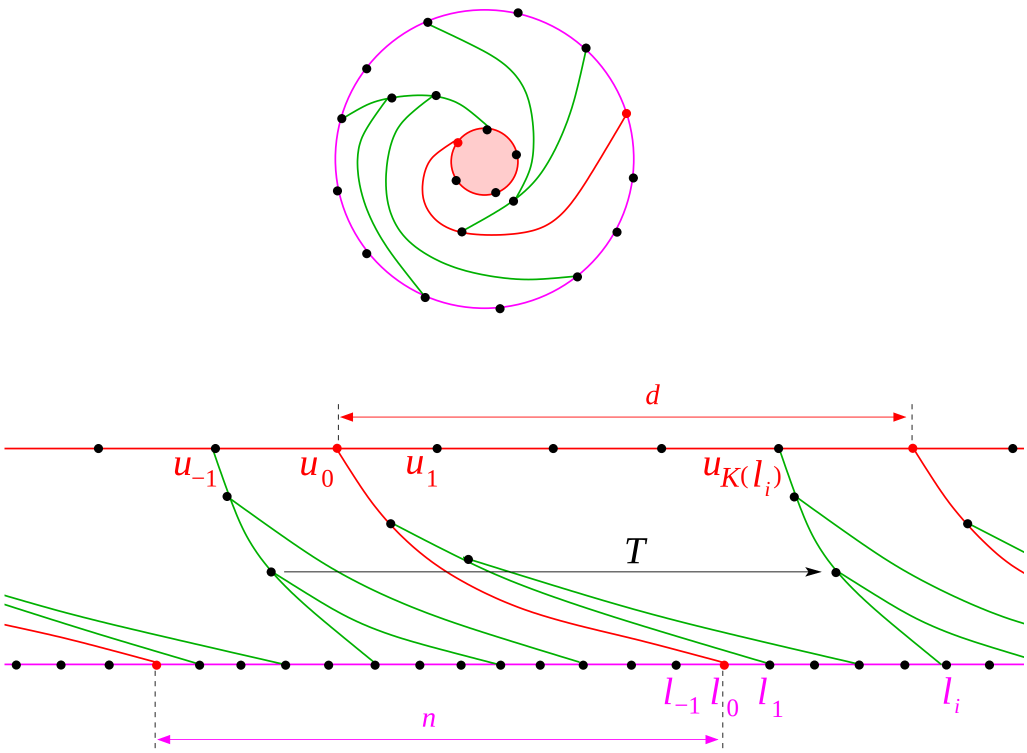

Fig. 19: Illustration of the slice decomposition of an annular map $\mathcal{M}$ (top). We pass to its lift $\tilde{\mathcal{M}}$ (bottom) and for each vertex ℓ_i of the lower boundary, we draw the leftmost shortest path from ℓ_i to $u_{K(\ell_i)}$, where $K(\ell_i)$ is defined by the coalescence lemma (intuitively speaking, this path is the "leftmost shortest path from ℓ_i to $-\infty$"). Clearly the figure is invariant by the translation T and we obtain the wanted slice decomposition of $\mathcal{M}$.

$\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots$ the vertices incident to the upper face, read in successive order by following the boundary from $-\infty$ to $+\infty$ (the choice of u_0 will turn out to be irrelevant). Note that $T(u_k) = u_{k+d}$ for all k. We then have the following:

> **Coalescence lemma:** for each vertex v of $\tilde{\mathcal{M}}$, there exists a unique integer $K(v)$ such that, for all $k \leq K(v)$, any shortest path from v to u_k passes through $u_{K(v)}$ (and hence follows the upper boundary from $u_{K(v)}$ to u_k), but some of the shortest paths do not pass through $u_{K(v)+1}$. In particular we have $\tilde{D}(v, u_k) - (K(v) - k) = \tilde{D}(v, u_{K(v)}) \leq \tilde{D}(v, u_{K(v)+1})$.

In other words, for $k \to -\infty$, all shortest paths to u_k eventually coalesce with the upper boundary. The proof of the lemma, which relies crucially on the d-irreducibility of $\mathcal{M}$, is postponed to the end of this section. Note that $K(T(v)) = K(v) + d$ by translation invariance.

We now denote by $\ldots, \ell_{-1}, \ell_0, \ell_1, \ell_2, \ldots$ the successive vertices incident to the lower face, again read from $-\infty$ to $+\infty$ so that $T(\ell_i) = \ell_{i+n}$ for all i (see Fig. 19). Here we pick ℓ_0 as an arbitrary preimage of the origin of the root edge of $\mathcal{M}$, and, at the price of relabeling the vertices of the upper boundary $u_k \to u_{k-K(\ell_0)}$, we may assume that $K(\ell_0) = 0$. We then let

$$\epsilon_i = \tilde{D}(\ell_i, u_{K(\ell_{i-1})}) - \tilde{D}(\ell_{i-1}, u_{K(\ell_{i-1})}) \tag{7.1}$$

which clearly belongs to $\{-1, 0, 1\}$ by the triangular inequality. By translation invariance we have $\epsilon_i = \epsilon_{i+n}$. Furthermore, it is easily seen that by planarity $K(\ell_{i-1}) \leq K(\ell_i)$ so that, using the coalescence lemma,

$$\begin{aligned} \sum_{i=1}^{n} \epsilon_i &= \sum_{i=1}^{n} \left(\tilde{D}(\ell_i, u_{K(\ell_i)}) + K(\ell_i) - K(\ell_{i-1}) - \tilde{D}(\ell_{i-1}, u_{K(\ell_{i-1})}) \right) \\ &= \tilde{D}(\ell_n, u_{K(\ell_n)}) - \tilde{D}(\ell_0, u_{K(\ell_0)}) + K(\ell_n) - K(\ell_0) = d \end{aligned} \tag{7.2}$$

since $T(\ell_0) = \ell_n$ so that $K(\ell_n) = K(\ell_0) + d$ and $\tilde{D}(\ell_0, u_{K(\ell_0)}) = \tilde{D}(\ell_n, u_{K(\ell_n)})$. Hence the sequence $\epsilon_1, \ldots, \epsilon_n$ forms the increments of a three-step path from $(0, 0)$ to (n, d). Let us now consider the bounded region delimited by the edge (ℓ_{i-1}, ℓ_i) and the two leftmost shortest paths from ℓ_{i-1} and ℓ_i to $u_{K(\ell_{i-1})}$. When $K(\ell_{i-1}) = K(\ell_i)$ these two paths may merge before their endpoint and we then remove their common part, letting p_i be the length of the proper part of the path with starting point ℓ_i. Clearly, the submap enclosed within this region is nothing but a slice of type $p_i/p_i - \epsilon_i$, which we denote by $\mathcal{S}_i$ (when $\epsilon_i = +1$ we obtain a slice reduced to a single edge as wanted). Note that $\mathcal{S}_{i+n}$ is simply a translate of $\mathcal{S}_i$, so no information is lost by restricting to the interval $\{1, \ldots, n\}$.

To summarize, starting from a d-irreducible annular map of outer degree n, we have constructed a slice-decorated three-step path from $(0, 0)$ to (n, d). It is clear that applying then the gluing procedure of the previous section restores the original annular map. To establish that we have a bijection, it remains to check that a slice-decorated three-step path is indeed recovered as the slice decomposition of its gluing.

Consider such a path and its partial gluing $\tilde{\mathcal{S}}_n$ as defined in the previous section. The lift $\tilde{\mathcal{M}}$ of the corresponding annular map $\mathcal{M}$ is obtained by gluing infinitely many copies $\ldots, \tilde{\mathcal{S}}_n^{(-1)}, \tilde{\mathcal{S}}_n^{(0)}, \tilde{\mathcal{S}}_n^{(1)}, \ldots$ of $\tilde{\mathcal{S}}_n$ to each other along their left/right boundaries, see Fig. 19. Consistently with the previous notations, let $\ell_{ni}, \ell_{ni+1}, \ldots, \ell_{n(i+1)}$ be the vertices of the lower boundary of $\tilde{\mathcal{S}}_n^{(i)}$, and $u_{di}, u_{di+1}, \ldots, u_{d(i+1)}$ be the vertices of the upper boundary, i.e. the last $d + 1$ vertices of the right boundary in counterclockwise direction (note indeed that ℓ_{ni} and u_{di} are identified with vertices of $\tilde{\mathcal{S}}_n^{(i-1)}$ in the gluing procedure). Using the data that the left and right boundary of $\tilde{\mathcal{S}}_n$ are leftmost shortest paths, it is not difficult to check that the leftmost shortest path in $\tilde{\mathcal{M}}$ from ℓ_{ni} to u_k follows the left boundary of $\tilde{\mathcal{S}}_n^{(i)}$ for all $k \leq di$, in particular it passes through u_{di} but not u_{di+1}, so that $K(\ell_{ni}) = di$ where $K(\cdot)$ is defined as in the coalescence lemma. This implies that the slice decomposition of $\tilde{\mathcal{M}}$ cuts precisely along the boundaries of the $\tilde{\mathcal{S}}_n^{(i)}$ for all i. Furthermore, within each $\tilde{\mathcal{S}}_n^{(i)}$, the leftmost shortest paths from ℓ_{ni+k}, $0 \leq k \leq n$, to u_{di} precisely delimit the same slices as those attached to the original path, so the slice decomposition is indeed the inverse mapping of the gluing procedure.

It remains to prove the coalescence lemma. We start with the intermediate:

Wrapping lemma: for v a vertex of $\tilde{\mathcal{M}}$ and m an integer, we have $\tilde{D}(v, T^m(v)) \geq |m|\, d$. For $m \neq 0$, equality holds if and only if v is incident to the upper face, and in this case the unique shortest path from v to $T^m(v)$ follows the upper boundary.

Proof: we may assume $m > 0$ without loss of generality, upon exchanging the role of v and $T^m(v)$. Let us consider a path from v to $T^m(v)$: in $\mathcal{M}$ its image is a closed path of the same length whose winding number around the central face is m. The case $m = 1$ is simpler: by possibly removing some edges we obtain a cycle of $\mathcal{M}$ which, by d-irreducibility, has length $\geq d$ with equality iff it coincides with the boundary of the central face. For $m > 1$, note that the closed path cannot be simple (since the possible winding numbers for a cycle are 0 or ± 1), so it has a multiple vertex. Splitting at this vertex, we obtain two closed paths of smaller length, whose winding numbers (around the central face) add up to m. If one of these subpaths has winding number > 1, we further split it, and so on until we are left with a collection of closed paths of winding numbers ≤ 1. In particular we have at least m subpaths of winding number 1, each of them of length $\geq d$. Thus the original path has length $\geq md$, and equality holds iff all subpaths coincide with the boundary of the central face, Q.E.D.

A corollary of the wrapping lemma is that, for all integers i, j, we have $\tilde{D}(u_i, u_j) = |j - i|$ and the upper boundary is the unique shortest path from u_i to u_j: consider a path of length L between them and, assuming without loss of generality that $i < j$, pick $m > 0$ such that $i + md \geq j$. By appending the upper boundary from u_j to $u_{i+md} = T^m(u_i)$ we obtain a path of length $L + i + md - j$ from u_i to $T^m(u_i)$, so that $L + i + md - j \geq md$, i.e. $L \geq j - i$ and equality holds iff the path follows the upper boundary.

Proof of the coalescence lemma: let v be again an arbitrary vertex of $\tilde{\mathcal{M}}$. By the triangular inequality $\tilde{D}(v, u_k) \geq |k| - \tilde{D}(v, u_0) \to +\infty$ for $k \to \pm\infty$, so in particular there exists an integer $K'(v)$ such that, for all $k \leq K'(v)$, $\tilde{D}(v, u_k)$ is larger than the number M of vertices of $\mathcal{M}$. By the pigeonhole principle, any path in $\mathcal{M}$ of length $\geq M$ passes necessarily twice through the same vertex. We apply this principle to the image of a shortest path γ from v to u_k, and conclude that γ necessarily passes successively through two distinct vertices v' and v'' of $\tilde{\mathcal{M}}$ having the same image in $\mathcal{M}$, i.e. $v'' = T^m(v')$ for some $m \neq 0$. Actually, we may assume that $\tilde{D}(v, v') < M$ by applying the principle to the prefix of length M of γ. Cutting γ at v' and v'', we obtain three subpaths γ', γ'' and γ'''. Then, by concatenating γ' with $T^{-m}(\gamma''')$ and then with the upper boundary from u_{k-md} to u_k, we obtain another path from v to u_k which has length $\tilde{D}(v, u_k) - \tilde{D}(v', v'') + |m|d$. Its length should be not lesser than $\tilde{D}(v, u_k)$ and we deduce from the wrapping lemma that v' is incident to the upper face, i.e. $v' = u_{k'}$ for some k'. From $\tilde{D}(v, v') < M$ we deduce that $k' > K'(v)$, and since the concatenation of γ'' and γ''' is a shortest path between $u_{k'}$ and u_k we deduce from the corollary of the wrapping lemma that it passes through $u_{K'(v)}$, hence $u_{K'(v)}$ belongs to any shortest path from v to u_k for all $k \leq K'(v)$. This shows that the set of the integers K such that, for all $k \leq K$, u_K belongs to the leftmost shortest path from v to u_k is nonempty. It is bounded from above since $\tilde{D}(v, u_k) \to +\infty$ for $k \to +\infty$ thus contains a maximal element $K(v)$ which clearly satisfies the conditions of the coalescence lemma, Q.E.D.

8. Discrete integrable equations

It has been noted in several occasions that maps are related to integrable systems, for instance, map generating functions are tau-functions of the KP hierarchy [29]. In the planar setting, a slightly different connection has been uncovered: generating functions of maps with marked points at a prescribed distance have been shown to satisfy a hierarchy of "discrete integrable equations" [17-19], whose combinatorial meaning is now quite understood [10]. Remarkably, the integrability phenomenon subsists in the context of irreducible maps.

Integrable equations are obtained by adding a new parameter in the game: at the combinatorial level it consists in controlling the maximal length of the slices. Both the substitution and the slice decomposition approaches still work in this case. Here, we choose to emphasize the cases of irreducible quadrangular and triangular dissections, as they are related to naturally embedded trees. For simplicity, we follow the substitution approach, with an analysis parallel to that of Sect. 2. The general case, and the slice decomposition, are then discussed in the last subsection.

8.1. Integrable equations from irreducible quadrangular dissections

In the case of quadrangular dissections, the starting point is some refined version of Eq. (2.1) of Sect. 2.1, which states that [18,10]

$$\begin{aligned} F_2 &= R_i - (x_4 R_{i-1} R_i R_{i+1})\,, \\ F_4 &= R_i(R_i + R_{i+1}) - (R_i + R_{i+1} + R_{i+2})\,(x_4 R_{i-1} R_i R_{i+1})\,, \qquad i \geq 1, \end{aligned} \tag{8.1}$$

where R_i is the solution of the equation

$$R_i = 1 + x_2 R_i + x_4 R_i(R_{i-1} + R_i + R_{i+1}), \qquad i \geq 1, \tag{8.2}$$

with $R_0 = 0$ (here, besides squares weighted by x_4, we also possibly allow bivalent faces with weight x_2). More precisely, whenever (8.2) is satisfied, the expressions in the r.h.s. of (8.1) are *conserved quantities*, i.e. their value does not depend on i and may moreover be identified with F_2 and F_4 respectively. This is a sign of integrability for Eq. (8.2), which indeed admits the explicit solution [17]

$$R_i = R\frac{(1-y^i)(1-y^{i+3})}{(1-y)^{i+1}(1-y)^{i+2}}, \qquad i \geq 0, \qquad \text{where } y + \frac{1}{y} + 1 = \frac{1}{x_4 R^2}, \tag{8.3}$$

with R solution of the homogeneous (i.e. without indices) version of (8.2), $R = 1+x_2R+3x_4R^2$. Performing the same specialization $(x_2, x_4) \to (X_2(z), X_4(z))$ as in Sect. 2.1, we deduce that the generating functions $f_2^{(4)}(z)$ and $f_4^{(4)}(z)$ of irreducible quadrangulations with a boundary of length 2 satisfy a generalization of (2.2), namely

$$\begin{aligned} f_2^{(4)} &= 1 = r_i^{(4)} - \left(X_4(z) r_{i-1}^{(4)} r_i^{(4)} r_{i+1}^{(4)}\right), \\ f_4^{(4)} &= 2 + z = r_i^{(4)}(r_i^{(4)} + r_{i+1}^{(4)}) - (r_i^{(4)} + r_{i+1}^{(4)} + r_{i+2}^{(4)}) \left(X_4(z) r_{i-1}^{(4)} r_i^{(4)} r_{i+1}^{(4)}\right), \qquad i \geq 1, \end{aligned} \tag{8.4}$$

where $r_i^{(4)} = R_i(X_2(z), X_4(z))$. Eliminating $X_4(z)$, we see that $r_i^{(4)}$ is now fully determined by the equation

$$z + r_i^{(4)} r_{i+2}^{(4)} - (r_i^{(4)} + r_{i+1}^{(4)} + r_{i+2}^{(4)}) + 2 = 0, \quad i \geq 1, \tag{8.5}$$

with initial conditions $r_1^{(4)} = 1$ and $r_2^{(4)} = 1 + z$ (from (8.4) at $i = 1$ with $r_0^{(4)} = 0$). In practice, we may equivalently extend Eq. (8.5) to include the case $i = 0$ and use as initial conditions $r_0^{(4)} = 0$ and $r_1^{(4)} = 1$. By substituting $(x_2, x_4) \to (X_2(z), X_4(z))$ in (8.3), we readily obtain the explicit expression

$$r_i^{(4)} = r^{(4)} \frac{(1 - y^i)(1 - y^{i+3})}{(1 - y^{i+1})(1 - y^{i+2})}, \qquad i \geq 0, \qquad \text{where } y + \frac{1}{y} = \frac{1}{r^{(4)} - 1}, \tag{8.6}$$

with $r^{(4)}$ solution of (2.3). Here, we simplified the equation for y, upon using the relation $X_4(z)(r^{(4)})^2 = (r^{(4)} - 1)/r^{(4)}$ read off (2.2). We observe that the same expression appears in [30, Proposition 4.5], where it is interpreted as a generating function of symmetric irreducible quadrangular dissections (such objects are indeed obtained by gluing several copies of a same slice counted by $r_i^{(4)}$).

As in Sect. 2.1, Eq. (8.5) is made more transparent upon setting

$$r_i^{(4)}(z) = 1 + z\, T_{i-1}(z), \qquad i \geq 1, \tag{8.7}$$

as it then reads

$$T_i = 1 + z\, T_{i-1} T_{i+1}, \qquad i \geq 1, \tag{8.8}$$

with initial condition $T_0 = 0$. Here we recognize the equation determining the generating function of "naturally embedded binary trees" (NEBT) introduced in [21], whose integrability remained so far quite mysterious. Now, from (8.6) and the first line of (8.4), we simply deduce

$$\begin{aligned} T_i = \frac{r_{i+1}^{(4)} - 1}{z} = \frac{X_4(z)}{z} r_i^{(4)} r_{i+1}^{(4)} r_{i+2}^{(4)} = T \frac{(1 - y^i)(1 - y^{i+5})}{(1 - y^{i+2})(1 - y^{i+3})}, \qquad i \geq 0, \\ \text{where } y + \frac{1}{y} = \frac{1}{zT} \end{aligned} \tag{8.9}$$

and $T = (r^{(4)} - 1)/z$ is solution of (2.7). We recover the particular form of the solution found in [21, Prop. 25]. Let us summarize the combinatorial steps of our derivation.

First, we note that Eq. (8.8), which is easily interpreted in the language of NEBT, also follows from slice decomposition, see Sect. 8.3 below. Second, generating functions of irreducible and arbitrary slices are simply related by a change of variables. Thus, the explicit form (8.3), combinatorially explained in [10, Sect. 6.2], directly translates into (8.6). Finally, the fact that $T_i = (r_{i+1}^{(4)} - 1)/z$ also has a nice factorized form is a consequence of the first line of (8.1), combinatorially explained in [10, Sect. 3.3].

8.2. Integrable equations from irreducible triangular dissections

If we now play the same game for triangular dissections, we may use the following expressions [10], valid for $i \geq 1$:

$$\begin{aligned} F_1 &= S_{i-1} - (x_3 R_i R_{i-1}), \\ F_2 &= (S_{i-1})^2 + R_i - (S_{i-1} + S_i)(x_3 R_i R_{i-1}), \\ F_3 &= (S_{i-1})^3 + R_i(2S_{i-1} + S_i) - ((S_{i-1})^2 + S_{i-1}S_i + (S_i)^2 + R_i + R_{i+1})(x_3 R_i R_{i-1}), \end{aligned} \tag{8.10}$$

where R_i and S_{i-1} ($i \geq 1$) are determined via

$$R_i = 1 + x_2 R_i + x_3 R_i (S_{i-1} + S_i), \qquad S_{i-1} = x_1 + x_2 S_{i-1} + x_3((S_{i-1})^2 + R_{i-1} + R_i) \tag{8.11}$$

with $R_0 = 0$. These equations are again integrable, with explicit solution

$$\begin{aligned} S_{i-1} = S - x_3 R^2 y^{i-1} \frac{(1-y)(1-y^2)}{(1-y^i)(1-y^{i+1})}, \qquad R_i = R \frac{(1-y^i)(1-y^{i+2})}{(1-y^{i+1})^2} \\ i \geq 1 \qquad \text{where} \quad y + \frac{1}{y} + 2 = \frac{1}{x_3^2 R^3} \end{aligned} \tag{8.12}$$

with R and S solutions of the homogeneous version of (8.11): $R = 1 + x_2 R + 2x_3 RS$, $S = x_1 + x_2 S + x_3(S^2 + 2R)$.

Specializing these expressions at the particular renormalized values $x_1 = X_1(z)$, $x_2 = X_2(z)$, $x_3 = X_3(z)$ of Sect. 2.2, we deduce expressions for the generating functions of irreducible triangulations with a boundary of length 1, 2 and 3:

$$\begin{aligned} f_1^{(3)} &= 0 = s_{i-1}^{(3)} - \left(X_3(z) r_{i-1}^{(3)} r_i^{(3)}\right), \\ f_2^{(3)} &= 1 = (s_{i-1}^{(3)})^2 + r_i^{(3)} - (s_{i-1}^{(3)} + s_i^{(3)}) \left(X_3(z) r_{i-1}^{(3)} r_i^{(3)}\right), \\ f_3^{(3)} &= z = (s_{i-1}^{(3)})^3 + r_i^{(3)}(2s_{i-1}^{(3)} + s_i^{(3)}) \\ &\quad - ((s_{i-1}^{(3)})^2 + s_{i-1}^{(3)} s_i^{(3)} + (s_i^{(3)})^2 + r_i^{(3)} + r_{i+1}^{(3)}) \left(X_3(z) r_{i-1}^{(3)} r_i^{(3)}\right), \end{aligned} \tag{8.13}$$

in terms of $r_i^{(3)}(z) = R_i(X_1(z), X_2(z), X_3(z))$ and $s_i^{(3)}(z) = S_i(X_1(z), X_2(z), X_3(z))$. Eliminating X_3, $r_i^{(3)}$ and $s_i^{(3)}$ are fully determined in terms of z via:

$$r_i^{(3)} = 1 + s_{i-1}^{(3)} s_i^{(3)}, \qquad z + s_{i-1}^{(3)} s_i^{(3)} s_{i+1}^{(3)} - s_i^{(3)} = 0, \qquad i \geq 1 \tag{8.14}$$

with initial condition $s_0^{(3)} = 0$. Using $s^{(3)} = X_3(z)(r^{(3)})^2$ from the first line of (2.14), formulas (8.12) specialize to

$$\begin{aligned} s_{i-1}^{(3)} &= s^{(3)}\left(1 - y^{i-1}\frac{(1-y)(1-y^2)}{(1-y^i)(1-y^{i+1})}\right) = s^{(3)}\frac{(1-y^{i-1})(1-y^{i+2})}{(1-y^i)(1-y^{i+1})} \\ r_i^{(3)} &= r^{(3)}\frac{(1-y^i)(1-y^{i+2})}{(1-y^{i+1})^2}, \qquad i \geq 1 \qquad \text{where} \quad y + \frac{1}{y} + 1 = \frac{1}{(s^{(3)})^2} \end{aligned} \tag{8.15}$$

with $r^{(3)}$ and $s^{(3)}$ solutions of the system (2.15). Here we used $(X_3(z))^2(r^{(3)})^3 = (s^{(3)})^2/r^{(3)} = (s^{(3)})^2/((s^{(3)})^2 - 1)$. See also [30, Proposition 5.5] for equivalent expressions interpreted as generating functions of symmetric irreducible triangular dissections.

Again the equation for $s_i^{(3)}$ in (8.14) is more transparent upon setting

$$s_i^{(3)}(z) = z\, T_i(z), \qquad i \geq 0 \tag{8.16}$$

as it reads

$$T_i = 1 + z^2\, T_{i-1}T_iT_{i+1}, \qquad i \geq 1 \tag{8.17}$$

with $T_0 = 0$, allowing to identify T_i with the generating function of some "naturally embedded" ternary trees [20] (with a weight z^2 per inner vertex). Eq. (8.17) was already encountered in the context of quadrangulations without multiple edges [28], and this explained its integrability. It is remarkable to find it here in the different context of irreducible triangulations, hence providing a new explanation for integrability. Let us mention that a direct bijection between irreducible triangulations and non-separable maps (themselves in bijection with quadrangulations without multiple edges) was given by Fusy [31].

8.3. Slices of controlled boundary lengths

Eqs. (8.1) and (8.10) are particular examples of the general formula [10]

$$F_n = Z_{i-1,i-1}^+(n; \{R_{m+1}, S_m\}_{m\geq 0}) - \sum_{k\geq 1} Z_{i-1,i-1+k}^+(n; \{R_{m+1}, S_m\}_{m\geq 0}))V_{k;i-2}, \tag{8.18}$$

for $i \geq 1$, where $V_{k;p}$ is defined by

$$V_{k;p} = \sum_{j\geq k+2} x_j Z_{p+k+1;p}(j-1; \{R_{m+1}, S_m\}_{m\geq 0})), \qquad k, p \geq -1. \tag{8.19}$$

Here $Z_{p,p'}(n; \{R_{m+1}, S_m\}_{m\geq 0})$ (with $p, p' \geq -1$) denotes the generating function of three-step paths of length n, starting at height p, ending at height p', where each level-step at height m receives a weight S_m while each down-step from height $m+1$ to height m receives a weight R_{m+1}. The quantity $Z_{p,p'}^+$ (with $p' \geq p \geq -1$) denotes the same generating function limited to paths whose heights remain larger than or equal to p.

It will always be assumed that $R_0 = 0$ so in practice the paths which dip below 0 do not contribute and $Z_{p,p'}$ and $Z^+_{p,p'}$ depend only on R_{m+1} and S_m for $m \geq 0$. This assumption also implies that $V_{k,-1} = 0$.

In the above expressions, R_m and S_m must be taken as the solutions of

$$R_{m+1} = 1 + V_{0,m}, \qquad S_m = V_{-1,m}, \qquad m \geq 0. \tag{8.20}$$

The interpretation of R_{m+1} (respectively S_m) is that it is the generating functions of slices of type $m'/m'+1$ (respectively of type m'/m') with $0 \leq m' \leq m$. By an argument similar to that of Sect. 4.1, the reader will be convinced that $V_{k;p}$ for $k \geq -1$ and $p \geq 0$ may then be understood as the generating function of k-slices of type $p'/p'+k+1$ with $0 \leq p' \leq p$ and that (8.19) simply translates the decomposition of Fig. 7.

Applying the general substitution relation (3.6), we get

$$F_n^{(d)} = Z^+_{i-1,i-1}(n;\{R^{(d)}_{m+1}, S^{(d)}_m\}_{m\geq 0}) - \sum_{k\geq 1} Z^+_{i-1,i-1+k}(n;\{R^{(d)}_{m+1}, S^{(d)}_m\}_{m\geq 0}) V^{(d)}_{k,i-2} \tag{8.21}$$

for $i \geq 1$, where $R^{(d)}_m$, $S^{(d)}_m$ and $V^{(d)}_{k,p}$ are the series obtained by substituting, for all j between 1 and d, the formal variable x_j by the series $X^{(d)}_j$ in R_m, S_m and $V_{k,p}$. As before, the $V^{(d)}_{k,p}$ for $k \geq d-1$ are easily determined in terms of $R^{(d)}_m$, $S^{(d)}_m$ via (8.19), the x_j for $j \geq d+1$ being kept un-substituted. In particular, in the specialized case of irreducible d-angulations, we have $v^{(d)}_{k,p} = 0$ for $k \geq d-1$. As for the $V^{(d)}_{k,p}$ for $-1 \leq k \leq d-2$, a first way to determining them is to use again the conditions (1.1): (8.21) yields a linear system for the $V^{(d)}_{k,p}$ with $1 \leq k \leq d$, which we may solve. To that end, we note that the inverse of the semi-infinite unitriangular matrix $(Z^+_{i-1,i-1+k}(n;\{r_{m+1}, s_m\}_{m\geq 0}))_{n,k\geq 0}$ admits an explicit expression in terms of monomer-dimers [32], which reduces to (3.39) when $r_m = r$ and $s_m = s$ for all m. This means that we may repeat the same strategy as that of Sect. 3.4, by simply adding a new parameter in the game, and derive a closed system of nonlinear recurrence equations for $R^{(d)}_m$ and $S^{(d)}_m$, generalizing (3.41). However, we find the combinatorial meaning of this approach to be quite unclear.

A more transparent approach is to understand $V^{(d)}_{k,p}$ as the generating functions of d-irreducible k-slices of type $p'/p'+k+1$ with $0 \leq p' \leq p$ (recall that the substitution does not modify the length of the boundaries of a slice). As such, they satisfy new recursive equations which provide an alternative route to determine them. Indeed, the binary decomposition of Sect. 5.1 yields immediately

$$V^{(d)}_{k,p} = z\delta_{k,d-2} + (1-\delta_{p,0}) \sum_{m=1}^{k+1} V^{(d)}_{m,p-1} V^{(d)}_{k-m,p+m} + V^{(d)}_{k+2,p-1}\ , \quad -1 \leq k \leq d-2,\ p \geq 0, \tag{8.22}$$

which is a refined version of (5.2). For instance, in the case of irreducible triangulations

($d=3$), these equations read

$$\begin{aligned} v_{1,p}^{(3)} &= z + v_{1,p-1}^{(3)} v_{0,p+1}^{(3)} \\ v_{0,p}^{(3)} &= v_{1,p-1}^{(3)} v_{-1,p+1}^{(3)} \\ v_{-1,p}^{(3)} &= v_{1,p-1}^{(3)}, \qquad p \geq 0, \end{aligned} \tag{8.23}$$

with the convention $v_{1,-1}^{(3)} = 0$. These equations reproduce precisely (8.14) upon identifying, $v_{0,p}^{(3)} = r_{p+1}^{(3)} - 1$ and $v_{-1,p}^{(3)} = s_p^{(3)}$.

Using the slice/tree bijection in the specialized case of irreducible d-angulations, the $v_{k,p}^{(d)}$ for $1 \leq k \leq d-2$ may as well be understood as generating functions of d-oriented k-trees with *depth* at most p. We say that the tree has depth p ($p \geq 0$) if the minimal height assigned to a corner in the closure procedure of Sect. 6.3 is $-p-1$. As such, they satisfy the recursive equations:

$$v_{k,p}^{(d)} = z\delta_{k,d-2} + (1-\delta_{p,0}) \sum_{q \geq 1} \sum_{\substack{m_i \geq 1, i=1,\cdots,q \\ m_1+\cdots+m_q = k+2}} \prod_{i=1}^{q} v_{m_i, p-1+\sum_{j=1}^{i-1} m_j}^{(d)} \qquad 1 \leq k \leq d-2, \ p \geq 0 \tag{8.24}$$

(recall that the height increases by m_i when going around a subtree whose root vertex has out-degree m_i), while $v_{k,p}^{(d)} = 0$ for $k \geq d-1$.

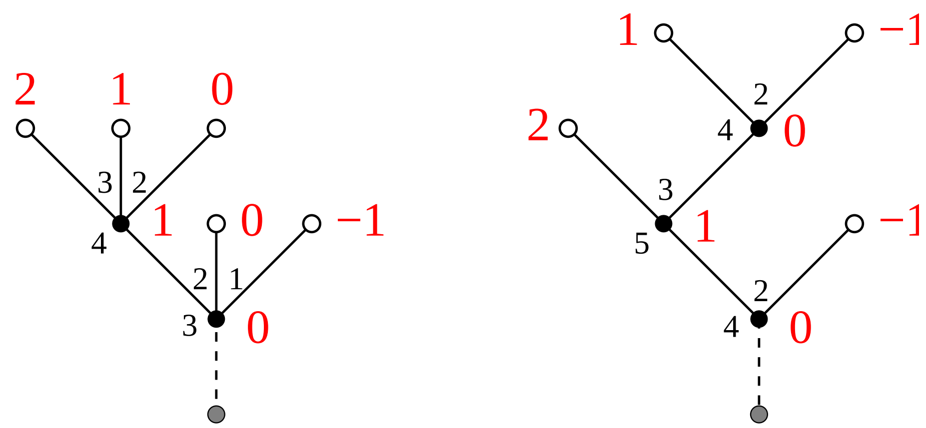

Fig. 20: An example of naturally embedded ternary (left) and binary (right) tree with corners labeled according to the rules of Sect. 6.2 when $d=3$ and $d=4$ respectively (starting with label 0 at the first encountered inner corner). We see the first corner encountered at each vertex receives a label (in red) equal to (minus) its horizontal position.

For instance, for $d=3$, we get

$$v_{1,p}^{(3)} = z + (1-\delta_{p,0})\, v_{1,p-1}^{(3)} v_{1,p}^{(3)} v_{1,p+1}^{(3)}, \qquad p \geq 0, \tag{8.25}$$

which matches (8.14) upon identifying $v_{1,p}^{(3)} = s_{p+1}^{(3)}$. The combinatorial meaning of this identification is transparent, upon observing that the corner labelling of a 3-oriented, i.e. ternary, tree essentially coincides with the natural embedding of its vertices, see Fig. 20. In the bipartite case (d even), it is easily seen along the same lines as Sect. 6.3 that $v_{k,p}^{(d)}$ vanishes for k odd. In the case $d = 4$, we find that $v_{2,p}^{(4)}$ satisfies

$$v_{2,p}^{(4)} = z + (1 - \delta_{p,0})\, v_{2,p-1}^{(4)} v_{2,p+1}^{(4)}, \qquad p \geq 0, \tag{8.26}$$

which matches (8.8) upon identifying $v_{2,p} = zT_{p+1}$ (consistenly with the relations $r_p^{(4)} - 1 = v_{0,p-1}^{(4)} = v_{2,p-2}^{(4)}$). Again the combinatorial meaning of the identification is transparent, see again Fig. 20.

Let us conclude by giving a few more integrable equations as obtained from this framework. If we consider irreducible hexangulations ($d = 6$) we find the system

$$\begin{aligned} v_{4,p}^{(6)} &= z + v_{2,p-1}^{(6)} v_{2,p+2}^{(6)} + v_{4,p-1}^{(6)} v_{0,p+4}^{(6)} \\ v_{2,p}^{(6)} &= v_{2,p-1}^{(6)} v_{0,p+2}^{(6)} + v_{4,p-1}^{(6)} \\ v_{0,p}^{(6)} &= v_{2,p-1}^{(6)}, \qquad\qquad p \geq 0, \end{aligned} \tag{8.27}$$

with the convention $v_{2,-1}^{(6)} = v_{4,-1}^{(6)} = 0$. This triangular system yields an equation for $v_{0,p}^{(6)}$ which, upon setting

$$v_{0,p}^{(6)} = r_{p+1}^{(6)} - 1 \tag{8.28}$$

reads

$$\begin{aligned} z - r_p^{(6)} r_{p+2}^{(6)} r_{p+4}^{(6)} + r_p^{(6)} r_{p+2}^{(6)} + r_p^{(6)} r_{p+3}^{(6)} + r_p^{(6)} r_{p+4}^{(6)} + r_{p+1}^{(6)} r_{p+3}^{(6)} + r_{p+1}^{(6)} r_{p+4}^{(6)} \\ + r_{p+2}^{(6)} r_{p+4}^{(6)} - 2(r_p^{(6)} + r_{p+1}^{(6)} + r_{p+2}^{(6)} + r_{p+3}^{(6)} + r_{p+4}^{(6)}) + 5 = 0, \quad p \geq 1, \end{aligned} \tag{8.29}$$

with $r_1^{(6)} = r_2^{(6)} = 1$, a refined version of (3.30) which, upon setting

$$r_p^{(6)} = 1 + zT_{p-2} \tag{8.30}$$

may itself be rewritten as

$$T_p = 1 + z(T_{p-2}T_{p+1} + T_{p-1}T_{p+1} + T_{p-1}T_{p+2}) - z^2 T_{p-2}T_pT_{p+2}, \quad p \geq 1 \tag{8.31}$$

with initial condition $T_{-1} = T_0 = 0$. This equation is integrable and an explicit determinantal formula for T_p may easily be obtained from the results of [10].

For $d = 8$, a similar calculation leads to the integrable equation

$$\begin{aligned} T_p = 1 &+ z(T_{p-3}T_{p+1} + T_{p-2}T_{p+1} + T_{p-1}T_{p+1} + T_{p-2}T_{p+2} + T_{p-1}T_{p+2} + T_{p-1}T_{p+3}) \\ &- z^2(T_{p-2}T_pT_{p+2} + T_{p-3}T_{p-1}T_{p+2} + T_{p-3}T_pT_{p+2} + T_{p-3}T_pT_{p+3} + T_{p-2}T_pT_{p+3} \\ &+ T_{p-2}T_{p+1}T_{p+3}) + z^3 T_{p-3}T_{p-1}T_{p+1}T_{p+3}, \qquad p \geq 1 \end{aligned} \tag{8.32}$$

with initial condition $T_{-2} = T_{-1} = T_0 = 0$ by setting $v_{0,p}^{(8)} = r_{p+1}^{(8)} - 1$ and $r_p^{(8)} = 1 + zT_{p-3}$.

9. Discussion

Let us end this paper by discussing some extensions of our work. The first one concerns d-irreducible maps with outer degree d, which are made nontrivial by slightly relaxing the notion of irreducibility. The second extension concerns d-irreducible maps with two marked faces of degree strictly larger than d. The third extension, suggested by one of the referees, concerns maps with two marked faces, subject to a control on two distinct girth parameters.

9.1. d-irreducible maps with outer degree d

In this section, we deal with d-irreducible maps with outer degree d, and with at least one inner face (i.e. maps not reduced to a tree). In the way we defined d-irreducibility so far, we have a unique such map, made of a single inner face of degree d glued to the external face: we shall call it the *trivial* map of outer degree d. Indeed the boundary of the external face (called *external boundary* in the following) forms a cycle of length d and, as such, has to be the boundary of an inner face of degree d. A weaker and somewhat more natural definition of d-irreducibility among maps with outer degree d consists in simply picking those maps of girth d such that all cycles of length d in the map are the boundary of an arbitrary (i.e. inner or external) face of degree d. With this definition, the external boundary needs not surround a single inner face any longer as it is already the boundary of the external face of degree d.

We shall call $H_d(z; x_{d+1}, \ldots)$ the generating function of such (weakly) d-irreducible maps with outer degree d counted, as before, with a weight z per inner face of degree d and weights x_j per face of degree $j \geq d+1$. We claim that we have the relation:

$$\begin{aligned} H_d(z; x_{d+1}, \ldots) &= 2z - X_d(z; x_{d+1}, \ldots), \qquad d \text{ odd} \\ H_d(z; x_{d+1}, \ldots) &= 2z + \frac{d}{2} \times \frac{z^3}{1+z} - X_d(z; x_{d+1} \ldots), \qquad d \text{ even} \end{aligned} \tag{9.1}$$

where $X_d(z; x_{d+1}, \ldots)$ is defined via the inversion (3.3). In practice, X_d may be obtained for instance from the knowledge of $V_{d-2}^{(d)}$ via the relation

$$X_d = \frac{1}{(R^{(d)})^{d-1}} \left(V_{d-2}^{(d)} - \sum_{j \geq d+1} x_j P_{-d+1}(j-1; R^{(d)}, S^{(d)}) \right) \tag{9.2}$$

obtained by taking (3.34) at $k = d-2$ and substituting the renormalized weight $x_d \to X_d = X_d^{(d)}(z; x_{d+1}, \ldots)$.

To prove (9.1), we use the same substitution approach as in Sect. 3.1. i.e. start from an arbitrary map with outer degree d and girth d, and obtain a d-irreducible one by replacing each outermost cycle of length d by a single d-valent face. As we shall now see, this transformation leads to a functional relation between H_d and the generating function $G_d(x_d, x_{d+1}, \ldots)$ (as defined in Sect. 3) of maps with outer degree d and girth

d, counted with weights x_j per inner face of degree $j \geq d$. It is important to note that, with our weaker notion of d-ireducibility, we now have to take the convention that, in the determination of the outmost cycles, the external boundary itself should *not* be considered as a cycle of length d. This leads to two caveats: (i) the trivial map of outer degree d (which is d-irreducible and contributes z to H_d) has a single antecedent which is the trivial map of outer degree d itself (contributing x_d to G_d) and (ii) some outermost cycles may now overlap so that some extra prescriptions are required for a well defined replacement. Let us analyze this overlapping issue more precisely: when repeating the arguments of Sect. 3, the only problem that we face is, with the notations of Fig. 3, that the cycle C_{12} may coincide with the external boundary so that it creates no contradiction with the fact that C_1 and C_2 were considered as outermost in the first place. Still, since C_{12} has length d, this implies that, necessarily, $n_1 = n_2 = d/2$ (recall that n_1 and n_2 are at least $d/2$) so that v and v' are antipodal around the external boundary while the two internal paths connecting them form mutually avoiding *diagonal paths* of length $d/2$. Since no paths of length $d/2$ exist for d odd, we conclude that outermost paths cannot overlap for d odd and the caveat (ii) never occurs. For odd d, we deduce the relation

$$G_d(x_d, x_{d+1}, \ldots) - x_d = H_d(G_d(x_d, x_{d+1}, \ldots); x_{d+1}, \ldots) - G_d(x_d, x_{d+1}, \ldots) \qquad d \text{ odd} \tag{9.3}$$

for the enumeration of nontrivial maps with outer degree d and girth d (the r.h.s. is obtained by substituting $z \to G_d$ in $H_d - z$, the generating function of nontrivial d-irreducible maps of outer degree d). Now, from (3.3), we may perform the substitution $G_d \to z$ by setting $x_d = X_d(z; x_{d+1}, \ldots)$ in (9.3).This yields the announced result (9.1) for odd d.

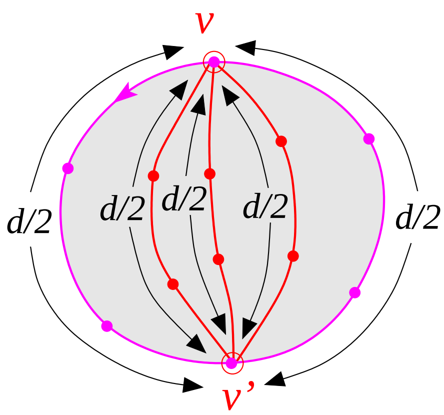

Fig. 21: Schematic picture of a configuration of map with outer degree d and girth d having overlapping outermost cycles of length d (by convention, the external boundary – in magenta – is not considered here as a cycle of length d). Two antipodal vertices v and v' along the external boundary are linked by $p \geq 2$ diagonal paths of length $d/2$.

For even d, we note that, in a map of girth d, there cannot be more than one pair of antipodal vertices connected by a diagonal path of length $d/2$ as otherwise, diagonals would have to cross and a cycle of length $< d$ would be created. Overlapping therefore appears only in situations where exactly one pair of antipodal vertices v and v' are connected by an arbitrary number $p \geq 2$ of mutually avoiding diagonal paths of length $d/2$ (see Fig. 21). We therefore decide to treat separately and remove from our original set of maps with girth d the configurations having two antipodal vertices connected by at least one diagonal path of length $d/2$ (we found simpler to also include in the removed set situations having a single diagonal although they do not give rise to overlapping). The set of these removed configurations clearly displays a $d/2$-fold symmetry by rotation around the external face. Once these configurations are removed, the $d/2$ d-irreducible maps with outer degree d made of two inner faces of degree d sharing a diagonal of length $d/2$ cannot be recovered any longer. Each of these configurations contributes z^2 to H_d, so we deduce

$$\begin{aligned} &G_d(x_d, x_{d+1}, \ldots) - x_d - \frac{d}{2} G_{d,D}(x_d, x_{d+1}, \ldots) \\ &\qquad = H_d(G_d(x_d, x_{d+1}, \ldots); x_{d+1}, \ldots) - G_d(x_d, x_{d+1}, \ldots) - \frac{d}{2}(G_d(x_d, x_{d+1}, \ldots))^2 \\ &\qquad\qquad d \text{ even} \end{aligned} \tag{9.4}$$

for the enumeration of nontrivial maps with outer degree d and girth d without a diagonal of length $d/2$ (the r.h.s. is obtained by substituting $z \to G_d$ in $H_d - z - \frac{d}{2} z^2$). In the l.h.s., $G_{d,D}$ enumerates maps with outer degree d and girth d with $p \geq 1$ diagonal paths of length $d/2$ connecting, say the origin of the root edge to its antipodal vertex. Cutting along all the diagonal paths of length $d/2$, any such map forms a sequence of $p + 1 \geq 2$ rooted maps (see Fig. 21 for an example with $p = 3$). These latter maps cannot have any more diagonal paths of length $d/2$ connecting the origin of the root edge to its antipodal vertex but are otherwise arbitrary maps of outer degree d and girth d (in particular they may have diagonal paths of length $d/2$ connecting other pairs of antipodal vertices). Their generating function is therefore $G_d - G_{d,D}$ and we have

$$G_{d,D} = \frac{(G_d - G_{d,D})^2}{1 - (G_d - G_{d,D})}, \tag{9.5}$$

or equivalently

$$G_{d,D} = \frac{(G_d)^2}{1 + G_d}, \tag{9.6}$$

so that (9.4) may eventually be written as

$$G_d = x_d - \frac{d}{2} \times \frac{G_d^3}{1 + G_d} + H_d(G_d; x_{d+1}, \ldots) - G_d \qquad d \text{ even} . \tag{9.7}$$

Setting $x_d = X_d(z; x_{d+1}, \ldots)$ in (9.7) yields the announced result (9.1) for even d.

Let us end this section by listing expressions for the generating function $h_d(z) = H_d(z; 0, \ldots)$ of (weakly) irreducible d-angular dissections of the d-gon at $d = 3, 4, 5, 6,$

as obtained by a specialization at $x_j = 0$, $j \geq d+1$. Eq. (9.2) reduces to the simpler relation

$$X_d(z;0,\ldots) = \frac{v_{d-2}^{(d)}}{(r^{(d)})^{d-1}} \ . \tag{9.8}$$

For $d = 3$, using $v_1^{(3)} = s^{(3)}$, we obtain

$$\begin{aligned} h_3(z) &= 2z - \frac{s^{(3)}}{(r^{(3)})^2} \\ &= z + z^3 + z^7 + 3z^9 + 12z^{11} + 52z^{13} + 241z^{15} + 1173z^{17} + 5929z^{19} + 30880z^{21} \\ &\quad + 164796z^{23} + 897380z^{25} + 4970296z^{27} + 27930828z^{29} + O\left(z^{31}\right) \end{aligned} \tag{9.9}$$

in agreement with [1].
For $d = 4$, using $v_2^{(4)} = u_1^{(4)} = r^{(4)} - 1$, we obtain

$$\begin{aligned} h_4(z) &= 2z + 2\frac{z^3}{1+z} - \frac{r^{(4)} - 1}{(r^{(4)})^3} \\ &= z + 2z^2 + z^5 + 4z^7 + 6z^8 + 24z^9 + 66z^{10} + 214z^{11} + 676z^{12} + 2209z^{13} + 7296z^{14} \\ &\quad + 24460z^{15} + 82926z^{16} + 284068z^{17} + 981882z^{18} + 3421318z^{19} + O\left(z^{20}\right) \end{aligned} \tag{9.10}$$

in agreement with [12].
For $d = 5$, using $v_3^{(5)} = 3s^{(5)} + (s^{(5)})^3 - 2s^{(5)}r^{(5)}$, we obtain

$$\begin{aligned} h_5(z) &= 2z - \frac{3s^{(5)} + (s^{(5)})^3 - 2s^{(5)}r^{(5)}}{(r^{(5)})^4} \\ &= z + 5z^3 + 46z^5 + 1350z^7 + 52360z^9 + 2382508z^{11} + 119914425z^{13} \\ &\quad + 6470326059z^{15} + 367369835490z^{17} + 21686295649075z^{19} + O\left(z^{21}\right). \end{aligned} \tag{9.11}$$

For $d = 6$, using $v_4^{(6)} = u_2^{(6)} = -2 + 3r^{(6)} - (r^{(6)})^2$, we obtain

$$\begin{aligned} h_6(z) &= 2z + 3\frac{z^3}{1+z} - \frac{-2 + 3r^{(6)} - (r^{(6)})^2}{(r^{(6)})^5} \\ &= z + 3z^2 + 2z^3 + 5z^4 + 42z^5 + 266z^6 + 1986z^7 + 15552z^8 + 127738z^9 + 1086998z^{10} \\ &\quad + 9517362z^{11} + 85291440z^{12} + 779292490z^{13} + 7237661226z^{14} + O\left(z^{15}\right). \end{aligned} \tag{9.12}$$

9.2. d-irreducible maps with two marked faces of degree strictly larger than d

This section deals with bipartite maps for simplicity. So far we considered maps with a single marked face (the external face) of degree, say $2m$ and a marked oriented edge (the root edge) incident to the external face and oriented so that the external face lies on its right (in practice we marked the root edge first). We may instead consider maps with two marked distinct (and distinguished) faces of respective degrees $2m$ and $2m'$, and a marked oriented edge incident to each of these marked face (and having the marked face on its right). Again we may demand that these maps be d-irreducible, i.e. have girth at least d and be such that all cycles of length d are the boundary of an inner face (i.e. a face different from the marked faces) of degree d. Assuming $d = 2b$ and $m, m' > b$, we may simply obtain the generating function $F^{(d)}_{2m,2m'}(z; x_{d+1}, \ldots)$ of d-irreducible maps with two marked faces of respective degree $2m$ and $2m'$ via

$$F^{(d)}_{2m,2m'} = 2m' \frac{\partial F^{(d)}_{2m}}{\partial x_{2m'}} \tag{9.13}$$

since the desired maps are obtained from d-irreducible maps with a single marked face of degree $2m$ (as enumerated by $F^{(d)}_{2m}$) by marking a face of degree $2m'$ (via the action of $\partial \cdot / \partial x_{2m'}$) and then marking an incident oriented edge (among $2m'$ choices). Now we may use (3.21) to compute

$$\frac{\partial F^{(d)}_{2m}}{\partial x_{2m'}} = \binom{2m}{m-b} (R^{(d)})^{m-b} \left(K(m') - \frac{b+m'}{m+m'} \binom{2m'-1}{m'+b} (R^{(d)})^{b+m'} \right) \tag{9.14}$$

where the quantity

$$\begin{aligned} K(m') = \Bigg(&\sum_{\ell=0}^{b-1} (-1)^{b-\ell-1} (b-\ell) \binom{b+\ell}{2\ell} \mathrm{Cat}(\ell) (R^{(d)})^{b-\ell-1} \\ &- \sum_{k \geq b+1} (b+k) \binom{2k-1}{k+b} x_{2k} (R^{(d)})^{b+k-1} \Bigg) \frac{\partial R^{(d)}}{\partial x_{2m'}} \end{aligned} \tag{9.15}$$

is independent of m. Eq. (9.14) is valid also for $m = b$, which allows to determine $K(m')$ from the identity $\partial F^{(d)}_d / \partial x_{2m'} = 0$ (since $F^{(d)}_d = z + \mathrm{Cat}(d/2)$), namely

$$K(m') = \binom{2m'-1}{m'+b} (R^{(d)})^{b+m'} \ . \tag{9.16}$$

Plugging this value in (9.14), we arrive at

$$F^{(d)}_{2m,2m'} = (m-b)(m'-b) \binom{2m}{m-b} \binom{2m'}{m'-b} \frac{(R^{(d)})^{m+m'}}{m+m'} \qquad m, m' > b. \tag{9.17}$$

This formula may be viewed as a generalization of a similar formula [33] for general (non necessarily irreducible) maps with two marked faces, corresponding to $d = b = 0$ here. The more general formulas of [33] for maps with more than two marked faces can also be extended to the case of d-irreducible maps [34].

9.3. (d,d')-irreducible annular maps

We are indebted to the anonymous referee for pointing out this extension of our work. Maps with two marked faces are naturally endowed with two distinct girth parameters: the *separating girth* and the *non separating girth*, defined respectively as the minimum length of cycles separating and not separating the marked faces from one another. Maps subject to an independant control on both girth parameters were enumerated in [8], and a natural question is whether these results can be extended to the context of irreducible maps.

We are led to slightly generalize the notion of annular map, introduced in Section 7, as follows. For d, d' nonnegative integers, we define a (d,d')*-quasi-irreducible annular map* as a rooted map with a marked inner face of degree d' called the *central face*, subject to the following constraints:

- any non separating cycle (i.e. a cycle which does not separate the outer face and the central face) has length at least d, and if its length is equal to d then it is necessarily the boundary of an inner face of degree d,
- any separating cycle (i.e. a cycle which separates the outer face and the central face) has length at least d'.

If furthermore the only separating cycle of length d' is the boundary of the central face, then the map is called a (d,d')*-irreducible annular map*. In other words, a (d,d')-irreducible annular map has non separating girth d, separating girth d', and all its minimal non separating or separating cycles are "trivial" (in the case of (d,d')-quasi-irreducible annular map, the triviality condition for minimal separating cycles is relaxed). Let us denote by $I_n^{(d,d')} \equiv I_n^{(d,d')}(z; x_{d+1}, x_{d+2}, \ldots)$ [resp. $\tilde{I}_n^{(d,d')} \equiv \tilde{I}_n^{(d,d')}(z; x_{d+1}, x_{d+2}, \ldots)$] the generating function of (d,d')-irreducible [resp. (d,d')-quasi-irreducible] annular maps whose root face has degree n, counted with a weight z per inner non central face of degree d and, for all $i \geq d+1$, a weight x_i per inner non central i-valent face (by convention, the central face receives no weight). Note that, for $d = d'$, (d,d')-irreducible annular maps are precisely the annular maps considered in Section 7, and thus $I_n^{(d,d)} = \frac{\partial F_n^{(d)}}{\partial z} = P_d(n; R^{(d)}, S^{(d)})$ by virtue of (3.42). Note also that, for $z = 0$, $d' I_n^{(d,d')}(0; x_{d+1}, x_{d+2}, \ldots)$ [resp. $d' \tilde{I}_n^{(d,d')}(0; x_{d+1}, x_{d+2}, \ldots)$] coincides with the quantity denoted $\widehat{G}_{d+1}^{(d',n)}$ [resp. $G_{d+1,d'}^{(d',n)}$] in [8] (where the annular maps are doubly rooted, thus the extra factor d').

We then have the beautiful identities

$$I_n^{(d,d')} = P_{d'}(n; R^{(d)}, S^{(d)}) \tag{9.18}$$

$$\tilde{I}_n^{(d,d')} = P_{-d'}(n; R^{(d)}, S^{(d)}) \tag{9.19}$$

where we recall that $P_k(n; \cdot, \cdot)$ is a three-step path generating function, given by (3.32). These identities generalize results from [8, Section 6.2], in particular Equation (7) of that paper coincides up to notations with our second identity at $z = 0$. We may establish them bijectively along the same lines as in the proof of the pointing formula (3.42) in

Section 7. Let us simply mention which modifications have to be made, and leave the reader check the details.

The first identity (9.18) is obtained rather straightforwardly. Starting from a slice-decorated three-step path counted by $P_{d'}(n; R^{(d)}, S^{(d)})$, we define the partial gluing as in Section 7.1, but the gluing of the right and left boundaries now leaves d' unmatched edges, which form the central face. The slice decomposition of annular maps is still performed by passing to the lift and cutting along leftmost shortest paths to $-\infty$.

For the second identity (9.19), given a slice-decorated three-step path counted by $P_{-d'}(n; R^{(d)}, S^{(d)})$, its partial gluing now has a left boundary longer by d' edges than the right boundary, and thus the central face is formed by "wrapping" on the other side. The fact that the left boundary is a non necessarily unique shortest path between its endpoints explains why the resulting annular map is (d, d')-quasi-irreducible but not necessarily (d, d')-irreducible. Conversely, the slice decomposition must be adapted: informally speaking, we have to cut the lift along leftmost shortest paths to $+\infty$ instead of $-\infty$. Because there might be several minimal separating cycles in the annular map, it is no longer true that, in the lift, all shortest paths to $+\infty$ coalesce with the upper boundary, still the leftmost ones do.

As a corollary of (9.18), (9.19) and the path definition of $P_k(n; \cdot, \cdot)$, we have the relation

$$\tilde{I}_n^{(d,d')} = I_n^{(d,d')} \left(R^{(d)}\right)^{d'} \tag{9.20}$$

which may alternatively obtained by decomposing a (d, d')-quasi-annular map of outer degree n along its outermost minimal separating cycle. Using a similar decomposition, we may easily obtain an extension of Theorem 32 in [8], i.e. an expression for the generating function of (d, d')-quasi-irreducible annular maps where we no longer impose that the central face has degree d' (in other words, the theorem is still valid with the "extra variable z").

Finally, let us observe that $\tilde{I}_d^{(d,d)} = \left(R^{(d)}\right)^d$ is closely related to the generating function H_d discussed in Section 9.1. Indeed, a (d, d)-quasi-irreducible annular map with outer degree d is "almost" a weakly d-irreducible map with a marked inner face of degree d, except for the fact that there might exist nontrivial minimal separating cycles. More precisely, we have the relation

$$\begin{aligned} \left(R^{(d)}\right)^d &= \frac{1}{1 - \left(\frac{\partial H_d}{\partial z} - 1\right)}, \qquad d \text{ odd} \\ \left(R^{(d)}\right)^d &= \frac{1}{1 - \left(\frac{\partial H_d}{\partial z} - 1 - dz\right) - \frac{d}{2}\frac{2z+z^2}{1+2z+z^2}}, \qquad d \text{ even} \end{aligned} \tag{9.21}$$

which may be obtained by differentiating (9.1) with respect to z, and noting that, by (3.3), $\frac{\partial X_d}{\partial z} = \left(R^{(d)}\right)^{-d}$ since $\frac{\partial G_d}{\partial x_d} = \tilde{I}_d^{(d-1,d)}(0; x_d, x_{d+1}, \ldots) = \left(R^{(d-1)}(0; x_d, x_{d+1}, \ldots)\right)^d$ and $R^{(d-1)}(0; X_d(z, x_{d+1}, \ldots), x_{d+1}, \ldots) = R^{(d)}(z; x_{d+1}, \ldots)$. Alternatively, a combinatorial proof is obtained by decomposing a (d, d)-quasi-irreducible annular map along its minimal separating cycles. This decomposition works straightforwardly for d odd, but for d even there are some pathologies related to the possible existence of diagonal paths, already observed in Section 9.1. Working out the details is left as a pleasant exercise to the reader.

Acknowledgements: We are grateful to the referees for their numerous suggestions of improvements. The work of JB was partly supported by the ANR projects "Cartaplus" 12-JS02-001-01 and "IComb" ANR-08-JCJC-0011.

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