Supersymmetry in Disorder and Chaos
(Random matrices, physics of compound nuclei, mathematics of random processes)

**Literature:**


Conferences and workshops in Paris in 2012

Workshop “Supersymmetry and Random Matrices”
Insitute Henri Poincare, April 3-5

Workshop “Disordered Quantum Systems”,
Insitute Henri Poincare, May-July.
$$H = -\frac{1}{2m} \frac{d^2}{dr^2} + U(r)$$

U(r)-random ($<U(r)> = 0$)

$$(\varepsilon - H)G^{R,A}_{\varepsilon}(r, r') = \delta(r - r')$$

$$G^{R,A}_{\varepsilon}(r, r') = \sum_n \frac{\phi_n(r)\phi_n^*(r')}{\varepsilon - \varepsilon_n \pm i\delta}$$

$$H\phi_n(r) = \varepsilon_n \phi_n(r)$$
Density of states:

\[ \rho(\varepsilon) = \pi^{-1} \langle \text{Im} G^R_{\varepsilon} (r, r) \rangle \]

Density-density correlation function:

\[ K(r, r', \omega) = \langle G^R_{\varepsilon} (r, r') G^A_{\varepsilon-\omega} (r', r) \rangle \]

Non-linear $\sigma$-model

Replica (Wegner 1979), Supermatrix (Efetov 1982)
Random Matrices


The main assumption: Matrix elements $H_{mn}$ of a Hamiltonian $H$ of a complex system are random.

The probability distribution $P(H)$:

$H_{mn}$ are independent

$$P(H) = A \exp \left( - \frac{\sum_{m,n} |H_{mn}|^2}{a^2} \right)$$
Three classes of universality: orthogonal, unitary and symplectic

1. Orthogonal: time reversal and central inversion invariance.
2. Unitary: the time reversal invariance is broken.
3. Symplectic: the central inversion invariance is broken but the time reversal one is not.

Level-level correlation function $R(\omega)$

$$R_{ort}(x) = 1 - \frac{\sin^2 x}{x^2} - \frac{d}{dx}\left(\frac{\sin x}{x}\right)\int_1^\infty \frac{\sin xt}{t} dt$$

$$R_{unit}(x) = 1 - \frac{\sin^2 x}{x^2}$$

$$R_{sympl}(x) = 1 - \frac{\sin^2 x}{x^2} + \frac{d}{dx}\left(\frac{\sin x}{x}\right)\int_0^1 \frac{\sin xt}{t} dt$$

$x = \frac{\pi \omega}{\Delta}$

$\Delta$ is the mean energy level spacing
Disordered systems:

**Schroedinger Equation:**

\[ H\psi = E\psi \]

\[ H = H_0 + H_{\text{dis}} \]

\[ H_{\text{dis}} = U(r) + U_s(r) + U_{so}(r) \]

\[ H_0 = \left( -i\nabla - \frac{eA}{c} \right)^2 \frac{2m}{2m} \]

\[ \langle U(r) \rangle = 0 \quad \langle U(r)U(r') \rangle = \frac{1}{2\pi\nu\tau} \delta(r-r') \]

\[ \nu \quad \text{is the density of states,} \quad \tau \quad \text{is the mean free time} \]

No possibility to solve the equation exactly for an arbitrary disorder.

\[ U(r) \quad \text{is a potential describing scattering by impurities, the others are scatterings by magnetic and spin-orbit impurities.} \]
Diagrammatic expansions for the Green functions $G$ and subsequent averages over the random potential (Abrikosov, Gorkov, Dzyaloshinskii (1961))

$$(H_0 + H_1)G(r, r') = \delta(r - r')$$  \hspace{1cm} \text{(Expansion in $H_1$)}

Summation of non-crossing diagrams:

$$G_{\varepsilon}^{R,A}(p) = \frac{1}{\varepsilon - \varepsilon(p) \pm i/2\tau}$$

Where are the Wigner-Dyson formulae and random matrix theory?
Only singularities might help to find something non-trivial (if existed).

Diffusion modes (cooperons and diffusons),
Gorkov, Larkin and Khmelnitskii (1979)

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{Dk^2 - i\omega}
\]

D is the classical diffusion coefficient

As \( \omega \to 0 \) divergence for \( d=2 \) (films), 1 (wires), 0 small metal particles (random matrices).
Non-linear supermatrix $\sigma$-model for describing localization (and not only) effects

For any correlation function $O$ (expressed in terms of the Green functions)

Due to supersymmetry

$$\langle O \rangle = \int O(\hat{Q}) \exp \left( -F \left[ \hat{Q} \right] \right) \mathcal{D}\hat{Q}$$

$$1 = \int \exp \left( -F \left[ \hat{Q} \right] \right) \mathcal{D}\hat{Q}$$

$$Q^2 = 1$$

$$F = \frac{\pi \nu}{8} \int \text{Str} [ D(\nabla Q)^2 + 2i(\omega + i\delta) \Lambda Q ] dr$$
The main ideas

Grassmann anticommuting variables $\chi$:

\[ \{\chi_i, \chi_j\} = 0 \quad \chi_i^2 = 0 \]

Integrals (Berezin 1961):

\[ \int \chi_i d\chi_i = 1 \quad \int d\chi_i = 0 \]

All other integrals are repetitions of these two.
The most important integrals (the basis of the method)

\[ \int \exp(-\chi^* A \chi) d\chi^* d\chi = \det A \]

Not \((\det A)^{-1}\) as for conventional complex numbers!

Supervector:
\[ \psi = (\chi, S) \]

Supermatrix:
\[ q = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \]
\[ \chi, \sigma, \rho \quad \text{- anticommuting} \]
\[ S, a, b \quad \text{- conventional} \]
\[ \text{Str}q = a - b \]

\[ \text{Str}(P_1 P_2) = \text{Str}(P_2 P_1) \]
\[ \text{Str}(P_1 P_2 P_3) = \text{Str}(P_3 P_1 P_2) \]
Scalar product
\[ \bar{\psi} = (\chi^*, S^*) \]
\[ \psi = \begin{pmatrix} \chi \\ S \end{pmatrix} \]
\[ \bar{\psi}\psi = \chi^* \chi + S^* S \]

Transposition
\[ \bar{\psi}_1 P \psi_2 = \bar{\psi}_2 P^T \psi_1 \]
\[ P = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \]
\[ P^T = \begin{pmatrix} a^T & -\rho^T \\ \sigma^T & b^T \end{pmatrix} \]

All rules for the "superobjects" are the same!

No weight denominator!
The basis of the method.

\[ A^{-1} = \int \bar{\psi} \psi \exp(-\bar{\psi} A \psi) d\psi \]
\[ G_\varepsilon^{R,A}(r, r') = (\varepsilon - H \pm i\delta)^{-1} \]
\[ = \mp i \int \psi(r) \bar{\psi}(r) \exp[-\int (\mp i\bar{\psi}(r)(\varepsilon - H_0 - U(r) \pm i\delta)\psi(r)dr)]D\psi \]
A possibility to average immediately over the random potential $U(r)$!

$$
\langle G_{\varepsilon}^{R,A}(r, r') \rangle = \mp i \int \psi(r) \bar{\psi}(r') \exp(-\int [\mp i \bar{\psi}(r)(\varepsilon - H_0 \pm i \delta)\psi(r) + \gamma (\bar{\psi}(r)\psi(r))^2 ]dr)D\psi
$$

$$
H = H_0 + U(r) \quad \langle U(r)U(r') \rangle = \gamma \delta(r-r')
$$

The disorder is avoided but an effective interaction appears instead.

$$
\langle G_{\varepsilon-\omega/2}^{R}(r, r)G_{\varepsilon+\omega/2}^{A}(r', r') \rangle = \int \psi^2(r)\bar{\psi}^2(r)\psi^1(r')\bar{\psi}^1(r') \exp[-\int (-i \bar{\psi}(r)(\varepsilon - H_0 - \frac{\omega}{2})\psi(r) + \gamma (\bar{\psi}(r)\psi(r))^2 )dr]D\psi
$$

The next idea: Spontaneous breaking of the (super)symmetry $\rightarrow$ existence of Goldstone modes.
A spontaneous average appears!

\[ \langle \psi_{\alpha}(r)\bar{\psi}_{\beta}(r) \rangle = Q_{\alpha\beta}(r) \]

\( Q_{\alpha\beta}(r) \) -is an 8x8 supermatrix

A self-consistent solution for \( Q \) leads to \( Q^2(r) = 1 \)

A general structure for \( Q \): \( Q = V\Lambda\bar{V}, \quad V\bar{V} = 1 \)

Degeneracy of the ground state, gapless (Goldstone) modes.

The free energy functional \( F \) can be obtained expanding in small gradients of \( Q \) (the frequency is assumed small).
This is a way how one comes to a non-linear supermatrix \( \sigma \) -model.
Physical quantities as integrals over the supermatrices

\[ F = \frac{\pi V}{8} \int Str[D(\nabla Q)^2 + 2i(\omega + i\delta)\Lambda Q]dr \]

\[ \int B(Q) \exp(-F[Q])DQ \]

Adding magnetic or spin-orbit interactions one changes the symmetry of the supermatrices Q (orthogonal, unitary and symplectic).

Depending on the dimensionality (geometry of the sample) one can study different problems (localization in wires and films, Anderson metal-insulator transition, etc.)

Everything that can be written in terms of products of Green functions can be expressed in terms of an integral over the supermatrices with the \( \sigma \)-model.
The explicit structure of $Q$

$Q = U Q_0 \overline{U}$

$U = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$

$u,v$ contain all Grassmann variables

All essential structure is in $Q_0$

$Q_0 = \begin{pmatrix} \cos \hat{\theta} & i e^{i\phi} \sin \hat{\theta} \\ -i e^{-i\phi} \sin \hat{\theta} & -\cos \hat{\theta} \end{pmatrix}$

$\hat{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & i\theta \end{pmatrix}$

$\phi = \begin{pmatrix} \varphi & 0 \\ 0 & \chi \end{pmatrix}$

( unitary ensemble)

Mixture of both compact and non-compact symmetries rotations: rotations on a sphere and hyperboloid glued by the anticommuting variables.
Explicit form

**Orthogonal**
\[
\hat{\theta}_{11} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}
\]
\[
\hat{\theta}_{22} = i \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & \theta_1 \end{pmatrix}
\]
\[
0 < \theta < \pi, \theta_1 > 0, \theta_2 > 0
\]

**Unitary**
\[
\hat{\theta}_{11} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}
\]
\[
\hat{\theta}_{22} = i \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_1 \end{pmatrix}
\]
\[
0 < \theta < \pi, \theta_1 > 0
\]

**Symplectic**
\[
\hat{\theta}_{11} = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & \theta_1 \end{pmatrix}
\]
\[
\hat{\theta}_{22} = i \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}
\]
\[
\theta > 0, 0 < \theta_1 < \pi, 0 < \theta_2 < \pi / 2
\]
What is zero dimensionality?

In a finite volume one comes to the space quantization of the diffusion modes:

\[-D\nabla^2 + 2i\omega\Phi_n(r) = E_n \Phi_n\]

\[E_n = n^2 \frac{D}{L^2}\]

\[n = 0, \pm 1, \pm 2, \pm 3, \ldots\]

\[\frac{D}{L^2}\] is the Thouless Energy

Zero dimensionality (0D) \[\iff\] \[D/L^2 \geq \omega\]

In 0D only the mode with \(n=0\) is important \[\iff\] the \(\sigma\)-model is zero dimensional.
The level-level correlation function \( R(x) \)

\[
R(\omega) = \frac{1}{16\pi^2 V^2} \left\langle \sum_{\sigma, \sigma'} \int \left( G_{\epsilon-\omega}^{\text{A}}(y, y) - G_{\epsilon-\omega}^{\text{R}}(y, y) \right) \left( G_{\epsilon}^{\text{A}}(y', y') - G_{\epsilon}^{\text{R}}(y', y') \right) dr dr' \right\rangle
\]

\[
y = (r, \sigma), \quad \sigma \text{ is the spin variable}
\]

\[
x = \frac{\pi \omega}{\Delta} \quad \Delta = (\nu V)^{-1} \quad \Delta \text{ is the mean level spacing, } V \text{ is volume.}
\]

\[
R(x) = -\int (Q^{11}Q^{22}) \exp(-F_0[Q]) dQ
\]

\[
F_0[Q] = \frac{\pi i(\omega + i \delta)}{4\Delta} \text{Str}(\Lambda Q)
\]

Definite integral over the elements of \( Q \)

Everything is applicable if \( \omega \leq D / L^2 \) is necessary. This is possible for weak disorder in thick wires, 2D and 3D.
The integral over the supermatrices $Q$ in a more “human” form:

$$R_{orth}(\omega) = 1 + \text{Re} \int_{1}^{\infty} \int_{1}^{\infty} \int_{-1}^{1} \frac{(\lambda_{1} \lambda_{2} - \lambda)^2 (1 - \lambda^2) \exp[i(x + i\delta)(\lambda_{1} \lambda_{2} - \lambda)] d\lambda_{1} d\lambda_{2} d\lambda}{(\lambda_{1}^2 + \lambda_{2}^2 + \lambda^2 - 2\lambda\lambda_{1}\lambda_{2} - 1)^2}$$

$$R_{unit}(\omega) = 1 + \frac{1}{2} \text{Re} \int_{1}^{\infty} \int_{-1}^{1} \exp[i(x + i\delta)(\lambda_{1} - \lambda)] d\lambda_{1} d\lambda$$

$$R_{sympl}(\omega) = 1 + \text{Re} \int_{1}^{\infty} \int_{0}^{1} \int_{-1}^{1} \frac{(\lambda - \lambda_{1} \lambda_{2})^2 (\lambda^2 - 1) \exp[i(x + i\delta)(\lambda - \lambda_{1} \lambda_{2})] d\lambda_{1} d\lambda_{2} d\lambda}{(\lambda_{1}^2 + \lambda_{2}^2 + \lambda^2 - 2\lambda\lambda_{1}\lambda_{2} - 1)^2}$$

Calculation of the integrals gives the corresponding formulae for the Wigner-Dyson ensembles:
proof of the relevance of the RMT for disordered metal particles (Efetov 1982).

**The main assumption:** Matrix elements $H_{mn}$ of a Hamiltonian $H$ of a complex system are random.

The probability distribution $P(H)$:

$$P(H) = A \exp \left( - \frac{\sum_{m,n} |H_{mn}|^2}{\alpha^2} \right)$$

$H_{mn}$ are independent
Three classes of universality: orthogonal, unitary and symplectic

1. Orthogonal: time reversal and central inversion invariance.
2. Unitary: the time reversal invariance is broken.
3. Symplectic: the central inversion invariance is broken but the time reversal one is not.

Level-level correlation function $R(\omega)$

$$R_{ort}(x) = 1 - \frac{\sin^2 x}{x^2} - \frac{d}{dx} \left( \frac{\sin x}{x} \right) \int_1^\infty \frac{\sin xt}{t} dt$$

$$x = \frac{\pi \omega}{\Delta}$$

$$R_{unit}(x) = 1 - \frac{\sin^2 x}{x^2}$$

$$R_{sympl}(x) = 1 - \frac{\sin^2 x}{x^2} + \frac{d}{dx} \left( \frac{\sin x}{x} \right) \int_0^1 \frac{\sin xt}{t} dt$$

$\Delta$ is the mean energy level spacing