Supersymmetry in Disorder and Chaos

(Random matrices, physics of compound nuclei, mathematics of random processes)

Literature:

K.B. Efetov, *Supersymmetry in Disorder and Chaos,* Cambridge University Press (1997,1999)

Supersymmetry and Trace Formulae, I.V. Lerner, J.P. Keating, D.E. Khmelnitskii, Kluwer, NATO ASI Series(1999)

K.B. Efetov, *Anderson Localization and Supersymmetry in 50 Years of Anderson Localization*, World Scientific (2010)

Conferences and workshops in Paris in 2012

Workshop "Supersymmetry and Random Matrices"
Insitute Henri Poincare, April 3-5
Workshop "Disordered Quantum Systems'',
Insitute Henri Poincare, May-July.

$$H = -\frac{1}{2m}\frac{d^2}{dr^2} + U(r)$$

U(r)-random (<U(r)>=0)

$$(\varepsilon - H)G_\varepsilon^{R,A}(r,r') = \delta(r - r')$$

$$G_\varepsilon^{R,A}(r,r') = \sum_n \frac{\phi_n(r)\phi_n^*(r')}{\varepsilon - \varepsilon_n \pm i\delta}$$

$$H\phi_n(r) = \varepsilon_n \phi_n(r)$$

Density of states: $$\rho(\varepsilon) = \pi^{-1}\left\langle \mathrm{Im}\, G_{\varepsilon}^{R}(r,r)\right\rangle$$

Density-density correlation function:

$$K(r,r',\omega) = \left\langle G_{\varepsilon}^{R}(r,r')G_{\varepsilon-\omega}^{A}(r',r)\right\rangle$$

Non-linear σ-model

Replica (Wegner 1979), Supermatrix (Efetov 1982)

Random Matrices

The beginning of the Random Matrix Theory (RMT):

Statistical theory of Complex Nuclei (E. Wigner (1951), F. Dyson (1962)

The main assumption: Matrix elements H_{mn} of a Hamiltonian H of a complex system are random.

The probability distribution P(H):

H_{mn} are independent

$$P(H) = A\exp\left(-\frac{\sum_{m,n}|H_{mn}|^2}{a^2}\right)$$

Three classes of universality: orthogonal, unitary and symplectic

1. Orthogonal: time reversal and central inversion invariance.
2. Unitary: the time reversal invariance is broken.
3. Symplectic: the central inversion invariance is broken but the time reversal one is not.

Level-level correlation function $R(\omega)$

$$R_{ort}(x) = 1 - \frac{\sin^2 x}{x^2} - \frac{d}{dx}\left(\frac{\sin x}{x}\right)\int_1^\infty \frac{\sin xt}{t}dt$$

$$x = \frac{\pi\omega}{\Delta}$$

$$R_{unit}(x) = 1 - \frac{\sin^2 x}{x^2}$$

Δ is the mean energy level spacing

$$R_{sympl}(x) = 1 - \frac{\sin^2 x}{x^2} + \frac{d}{dx}\left(\frac{\sin x}{x}\right)\int_0^1 \frac{\sin xt}{t}dt$$

Disordered systems:

Schrodinger Equation: $H\psi = E\psi$

$$H = H_0 + H_{dis}$$

$$H_{dis} = U(r) + U_s(r) + U_{so}(r)$$

$$H_0 = \frac{\left(-i\nabla - \frac{eA}{c}\right)^2}{2m}$$

U(r) is a potential describing scattering by impurities, the others are scatterings by magnetic and spin-orbit impurities.

Averaging over impurities:

$$\langle U(r) \rangle = 0 \qquad \langle U(r)U(r') \rangle = \frac{1}{2\pi\nu\tau}\delta(r - r')$$

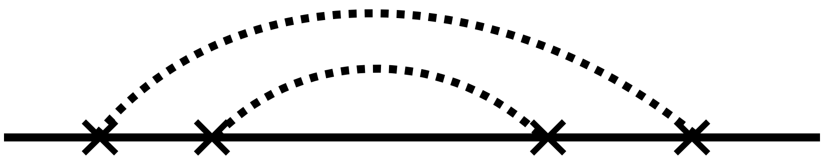
ν is the density of states, τ is the mean free time

No possibility to solve the equation exactly for an arbitrary disorder.

Diagrammatic expansions for the Green functions G and subsequent averages over the random potential (Abrikosov, Gorkov, Dzyaloshinskii (1961))

$$\left(H_0 + H_1\right)G(r, r') = \delta(r - r')$$

(Expansion in H_1)



Summation of non-crossing diagrams:

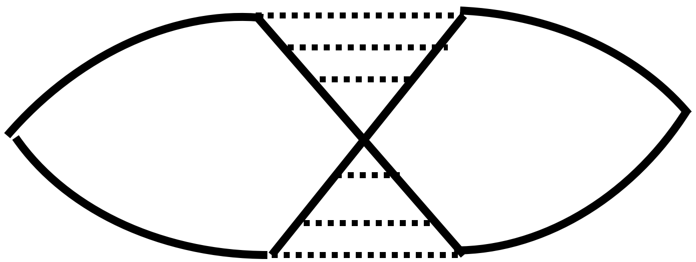
$$G_{\varepsilon}^{R,A}(p) = \frac{1}{\varepsilon - \varepsilon(p) \pm i/2\tau}$$

Where are the Wigner-Dyson formulae and random matrix theory?

Only singularities might help to find something non-trivial (if existed).

Diffusion modes (cooperons and diffusons),

Gorkov, Larkin and Khmelnitskii (1979)



$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{Dk^2 - i\omega}$$

D is the classical diffusion coefficient

As $\omega \to 0$ ⟹ divergence for d=2 (films), 1 (wires), 0 small metal particles (random matrices).

Non-linear supermatrix σ-model for describing localization (and not only) effects

$$\langle O \rangle = \int O(\hat{Q}) \exp\left(-F\left[\hat{Q}\right]\right) \mathcal{D}\hat{Q}$$

For any correlation function O (expressed in terms of the Green functions)

$$1 = \int \exp\left(-F\left[\hat{Q}\right]\right) \mathcal{D}\hat{Q}$$

Due to supersymmetry

$$Q^2 = 1$$

$$F = \frac{\pi\nu}{8} \int Str[D(\nabla Q)^2 + 2i(\omega + i\delta)\Lambda Q]dr$$

The main ideas

Grassmann anticommuting variables χ :

$$\{\chi_i, \chi_j\} = 0$$

$$\chi_i^2 = 0$$

Integrals (Berezin 1961):

$$\int \chi_i d\chi_i = 1$$

$$\int d\chi_i = 0$$

All other integrals are repetitions of these two.

The most important integrals (the basis of the method)

$$\int \exp(-\vec{\chi}^* A\vec{\chi})\,d\chi^* d\chi = \det A$$

Not $(\det A)^{-1}$ as for conventional complex numbers!

Supervector:

$$\psi = (\chi, S)$$

Supermatrix:

$$q = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}$$

χ, σ, ρ - anticommuting

S, a, b -conventional

$$Str\,q = a - b$$

⇓

$$Str(P_1 P_2) = Str(P_2 P_1)$$

$$Str(P_1 P_2 P_3) = Str(P_3 P_1 P_2)$$

Scalar product

$$\overline{\psi} = \begin{pmatrix} \chi^* & S^* \end{pmatrix}$$

$$\psi = \begin{pmatrix} \chi \\ S \end{pmatrix}$$

$$\overline{\psi}\psi = \chi^*\chi + S^*S$$

Transposition

$$\overline{\psi}_1 P \psi_2 = \overline{\psi}_2 P^T \psi_1$$

$$P = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}$$

$$P^T = \begin{pmatrix} a^T & -\rho^T \\ \sigma^T & b^T \end{pmatrix}$$

All rules for the “superobjects” are the same!

$$A^{-1} = \int \psi \overline{\psi} \exp(-\overline{\psi} A \psi) d\psi$$

No weight denominator! ➡

The basis of the method.

$$G_{\varepsilon}^{R,A}(r,r') = (\varepsilon - H \pm i\delta)^{-1}$$

$$= \mp i \int \psi(r) \overline{\psi}(r) \exp[-\int (\mp i \overline{\psi}(r)(\varepsilon - H_0 - U(r) \pm i\delta)\psi(r) dr)] D\psi$$

A possibility to average immediately over the random potential U(r)!

$$\left\langle G_{\varepsilon}^{R,A}(r,r')\right\rangle = \mp i\int \psi(r)\overline{\psi}(r')\exp(-\int[\mp i\overline{\psi}(r)(\varepsilon - H_0 \pm i\delta)\psi(r) + \gamma(\overline{\psi}(r)\psi(r))^2]dr)D\psi$$

$$H = H_0 + U(r)$$

$$\left\langle U(r)U(r')\right\rangle = \gamma\delta(r-r')$$

The disorder is avoided but an effective interaction appears instead.

$$\left\langle G_{\varepsilon-\omega/2}^{R}(r,r)G_{\varepsilon+\omega/2}^{A}(r',r')\right\rangle$$
$$= \int \psi^2(r)\overline{\psi}^2(r)\psi^1(r')\overline{\psi}^1(r')\exp[-\int(-i\overline{\psi}(r)(\varepsilon - H_0 - \frac{\omega}{2}\Lambda)\psi(r) + \gamma\left(\overline{\psi}(r)\psi(r)\right)^2)dr]D\psi$$

<u>The next idea:</u>

Spontaneous breaking of the (super)symmetry ⇨ existence of Goldstone modes.

A spontaneous average appears!

$$\left\langle \psi_\alpha(r)\bar{\psi}_\beta(r)\right\rangle = Q_{\alpha\beta}(r)$$

$Q_{\alpha\beta}(r)$ -is an 8x8 supermatrix

A self-consistent solution for Q leads to $Q^2(r)=1$

A general structure for Q: $Q = V\Lambda\bar{V},$ $V\bar{V}=1$

⇩

Degeneracy of the ground state, gapless (Goldstone) modes.

The free energy functional F can be obtained expanding in small gradients of Q (the frequency is assumed small).
This is a way how one comes to a non-linear supermatrix σ -model.

$$F = \frac{\pi\nu}{8}\int Str[D(\nabla Q)^2 + 2i(\omega + i\delta)\Lambda Q]dr$$

Physical quantities as integrals over the supermatrices

$$\int B(Q)\exp(-F[Q])DQ$$

Adding magnetic or spin-orbit interactions one changes the symmetry of the supermatrices Q (orthogonal , unitary and symplectic).

Depending on the dimensionality (geometry of the sample) one can study different problems (localization in wires and films, Anderson metal-insulator transition, etc.)

Everything that can be written in terms of products of Green functions can be expressed in terms of an integral over the supermatrices with the σ -model.

The explicit structure of Q

$$Q = U Q_0 \bar{U}$$

$$U = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

u,v contain all Grassmann variables

All essential structure is in Q_0

$$Q_0 = \begin{pmatrix} \cos\hat{\theta} & ie^{i\hat{\phi}}\sin\hat{\theta} \\ -ie^{-i\hat{\phi}}\sin\hat{\theta} & -\cos\hat{\theta} \end{pmatrix}$$

$$\hat{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & i\theta \end{pmatrix}$$

$$\hat{\phi} = \begin{pmatrix} \varphi & 0 \\ 0 & \chi \end{pmatrix}$$

(unitary ensemble)

Mixture of both compact and non-compact symmetries rotations: rotations on a sphere and hyperboloid glued by the anticommuting variables.

Explicit form

Orthogonal

$$\hat{\theta}_{11} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}$$

$$\hat{\theta}_{22} = i\begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & \theta_1 \end{pmatrix}$$

$$0 < \theta < \pi, \theta_1 > 0, \theta_2 > 0$$

Unitary

$$\hat{\theta}_{11} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}$$

$$\hat{\theta}_{22} = i\begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_1 \end{pmatrix}$$

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Symplectic

$$\hat{\theta}_{11} = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & \theta_1 \end{pmatrix}$$

$$\hat{\theta}_{22} = i\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}$$

$$\theta > 0, 0 < \theta_1 < \pi, 0 < \theta_2 < \pi / 2$$

Random Matrices ⟺ Small Metal Particles ⟺ Zero Dimensionality.

What is zero dimensionality?

In a finite volume one comes to the space quantization of the diffusion modes:

$$(-D\nabla^2 + 2i\omega)\Phi_n(r) = E_n\Phi_n$$

$$E_n = n^2 D/L^2$$

$$n = 0, \pm 1, \pm 2, \pm 3, ...$$

D/L^2 is the Thouless Energy

Zero dimensionality (0D) ⟺ $D/L^2 \geq \omega$

In 0D only the mode with n=0 is important ⟹ the σ-model is zero dimensional.

The level-level correlation function R(x)

$$R(\omega)=\frac{1}{16\pi^2\nu^2V^2}\left\langle\sum_{\sigma,\sigma'}\int\left(G^A_{\varepsilon-\omega}(y,y)-G^R_{\varepsilon-\omega}(y,y)\right)\left(G^A_{\varepsilon}(y',y')-G^R_{\varepsilon}(y',y')\right)dr\,dr'\right\rangle$$

$y=(r,\sigma)$, σ is the spin variable

$$x=\frac{\pi\omega}{\Delta}$$

$\Delta=(\nu V)^{-1}$ Δ is the mean level spacing, V is volume.

$$R(x)=-\int(Q^{11}Q^{22}\exp(-F_0[Q])dQ$$

$$F_0[Q]=\frac{\pi i(\omega+i\delta)}{4\Delta}Str(\Lambda Q)$$

Definite integral over the elements of Q

Everything is applicable if $\omega\le D/L^2$ $\Rightarrow$ $\Delta\le D/L^2$ is necessary. This is possible for weak disorder in thick wires, 2D and 3D.

The integral over the supermatrices Q in a more “human” form:

$$R_{orth}(\omega)=1+\mathrm{Re}\int_{1}^{\infty}\int_{1}^{\infty}\int_{-1}^{1}\frac{(\lambda_1\lambda_2-\lambda)^2(1-\lambda^2)\exp[i(x+i\delta)(\lambda_1\lambda_2-\lambda)]d\lambda_1 d\lambda_2 d\lambda}{(\lambda_1^2+\lambda_2^2+\lambda^2-2\lambda\lambda_1\lambda_2-1)^2}$$

$$R_{unit}(\omega)=1+\frac{1}{2}\mathrm{Re}\int_{1}^{\infty}\int_{-1}^{1}\exp[i(x+i\delta)(\lambda_1-\lambda)]d\lambda_1 d\lambda$$

$$R_{sympl}(\omega)=1+\mathrm{Re}\int_{1}^{\infty}\int_{0}^{1}\int_{-1}^{1}\frac{(\lambda-\lambda_1\lambda_2)^2(\lambda^2-1)\exp[i(x+i\delta)(\lambda-\lambda_1\lambda_2)]d\lambda_1 d\lambda_2 d\lambda}{(\lambda_1^2+\lambda_2^2+\lambda^2-2\lambda\lambda_1\lambda_2-1)^2}$$

Calculation of the integrals gives the corresponding formulae for the Wigner-Dyson ensembles:
proof of the relevance of the RMT for disordered metal particles (Efetov 1982).

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