SUSY in Condensed Matter — K. Efetov
— 4 —

1 Mean field theory

We don't know how to calculate correlation functions with the interacting term. The first step is to simplify this term by keeping only slow fluctuations. In Fourier space we obtain:

$$(\overline{\Psi}\Psi)^2 = \sum_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4=0} (\overline{\Psi}_{\mathbf{p}_1}\Psi_{\mathbf{p}_2})(\overline{\Psi}_{\mathbf{p}_3}\Psi_{\mathbf{p}_4}) \tag{1}$$

$$\simeq \sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}<\mathbf{q}_0} \big[(\overline{\Psi}_{\mathbf{p}_1}\Psi_{-\mathbf{p}_1+\mathbf{q}})(\overline{\Psi}_{\mathbf{p}_2}\Psi_{-\mathbf{p}_2-\mathbf{q}}) + (\overline{\Psi}_{\mathbf{p}_1}\Psi_{\mathbf{p}_2})(\overline{\Psi}_{-\mathbf{p}_2-\mathbf{q}}\Psi_{-\mathbf{p}_1+\mathbf{q}}) + \tag{2}$$

$$(\overline{\Psi}_{\mathbf{p}_1}\Psi_{\mathbf{p}_2})(\overline{\Psi}_{-\mathbf{p}_1+\mathbf{q}}\Psi_{-\mathbf{p}_2-\mathbf{q}})\big] . \tag{3}$$

The first term is not important because we can show that it does not lead to nontrivial phenomena in the limit $\varepsilon\tau \gg 1$. The two other terms are equal. Finally the interacting term become

$$2 \sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}<\mathbf{q}_0} \sum_{\alpha,\beta} \Psi^{\alpha}_{\mathbf{p}_1} \overline{\Psi}^{\beta}_{-\mathbf{p}_1+\mathbf{q}} \Psi^{\alpha}_{\mathbf{p}_2} \overline{\Psi}^{\beta}_{-\mathbf{p}_2-\mathbf{q}} \tag{4}$$

where greek index α, β is used for the component of the supervector Ψ. With help of a Hubbard–Stratonovich transformation, we can simplify the interacting part of the action (implicit sum over α and β)

$$\exp\left[-\frac{1}{2\pi\nu\tau} \sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}<\mathbf{q}_0} \Psi^{\alpha}_{\mathbf{p}_1} \overline{\Psi}^{\beta}_{-\mathbf{p}_1+\mathbf{q}} \Psi^{\alpha}_{\mathbf{p}_2} \overline{\Psi}^{\beta}_{-\mathbf{p}_2-\mathbf{q}}\right] = \tag{5}$$

$$\int \exp\left[-\frac{1}{2\tau} \sum_{\mathbf{p}_1,\mathbf{q}} Q^{\beta\alpha}_{\mathbf{q}} \Psi^{\alpha}_{\mathbf{p}_1+\mathbf{q}} \overline{\Psi}^{\beta}_{-\mathbf{p}_1} - \frac{\pi\nu}{4} \sum_{\mathbf{q}} Q^{\alpha\beta}_{\mathbf{q}} Q^{\alpha\beta}_{-\mathbf{q}}]\right] \mathcal{D}Q \tag{6}$$

$$= \int \exp\left[-\frac{1}{2\tau} \int (\overline{\Psi} Q \Psi + \frac{\pi\nu}{4} \operatorname{Str} Q^2)\, \mathrm{d}\mathbf{r}\right] \mathcal{D}Q. \tag{7}$$

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We can calculate the term with Ψ field coming from the free part of the Hamiltonian and the interacting term, which is just a Gaussian integral

$$\int \exp\left[-\mathrm{i}\int \overline{\Psi}(H_0 + \frac{\mathrm{i}}{2\tau}Q)\Psi\,\mathrm{d}\mathbf{r}\right]\mathcal{D}\Psi = \exp\left[\mathrm{Str}\left(\int \ln(H_0 + \frac{\mathrm{i}}{2\tau}Q)\,\mathrm{d}\mathbf{r}\right)\right]. \tag{8}$$

Finally the action is

$$S = \exp\left[\mathrm{Str}\left(\int (\ln(H_0 + \frac{\mathrm{i}}{2\tau}Q) - \frac{\pi\nu}{4\tau}Q^2)\,\mathrm{d}\mathbf{r}\right)\right] \tag{9}$$

Before we had a disordered term with large fluctuations and now we have an interacting term with slow variations which it is easier to deal with.

2 Saddle point

Now we can try to calculate this integral by a saddle point method. The saddle point equation is

$$Q(\mathbf{r}) = \frac{1}{\pi\nu}g(\mathbf{r},\mathbf{r}) \quad \text{with} \quad g = \left(\varepsilon(\mathbf{p}) - \varepsilon_0 + \frac{\mathrm{i}\omega}{2}\Lambda + \frac{\mathrm{i}}{2\tau}Q\right)^{-1} \tag{10}$$

We look first for a solution independent of $\mathbf{r}$. Q is solution of

$$Q = -\frac{\mathrm{i}}{\pi\nu}\int \frac{\nu}{\xi + \frac{\mathrm{i}\omega}{2}\Lambda + \frac{\mathrm{i}}{2}Q}\,\mathrm{d}\xi \quad \text{with} \quad \xi = \varepsilon(\mathbf{p}) - \varepsilon_0. \tag{11}$$

If $\omega \to 0$ then $Q_0 = V\Lambda\overline{V}$ where $V\overline{V} = \mathbb{1}$.

Now we are interested in fluctuations near this solution : $Q(\mathbf{r}) = Q_0 + \delta Q(\mathbf{r})$. These fluctuations can be decomposed in longitudinal $\delta_l Q$ and transverse $\delta_t Q$ fluctuations. $\delta_l Q$ commute with Q_0 while $\delta_t Q$ anti-commute.

$$\ln\left(H_0 + \frac{\mathrm{i}}{2\tau}Q_0 + \frac{\mathrm{i}}{2\tau}\delta Q\right) \simeq \tag{12}$$

$$-\frac{1}{(2\tau)^2}\int \frac{1}{2}\left(H_0 + \frac{\mathrm{i}}{2\tau}Q_0\right)^{-1}\delta Q(\mathbf{r}')\left(H_0 + \frac{\mathrm{i}}{2\tau}Q_0\right)^{-1}\delta Q(\mathbf{r})\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}' \tag{13}$$

$$= -\frac{1}{2}\int \left(H_0 + \frac{\mathrm{i}}{2\tau}Q_0\right)^{-1}\left(H_0 + \frac{\mathrm{i}}{2\tau}Q_0\right)^{-1}\delta_l Q(\mathbf{r})\delta_l Q(\mathbf{r}')\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}' \tag{14}$$

$$+\frac{1}{2}\int \left(H_0 + \frac{\mathrm{i}}{2\tau}Q_0\right)^{-1}\left(H_0 - \frac{\mathrm{i}}{2\tau}Q_0\right)^{-1}\delta_t Q(\mathbf{r})\delta_t Q(\mathbf{r}')\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}'. \tag{15}$$

If we take the fluctuations at the same point and integrate over $\mathbf{r}'$ then only transverse fluctuations give a non-zero contribution.

We do the same decomposition for the Q^2 term in the action and integrate over longitudinal fluctuations. Finally we do a gradient expansion of $\delta_t Q(\mathbf{r}')$ to obtain the non linear σ model

$$S = \frac{\pi\nu}{8}\mathrm{Str}\int \left[D_0(\nabla Q)^2 + 2\mathrm{i}(\omega + \mathrm{i}\delta)\Lambda Q\right]\,\mathrm{d}\mathbf{r}. \tag{16}$$