1 Sum up of the previous lecture

Previously, we studied the diagrammatic expansion for the GREEN function and we showed that for $d \in \{0;1;2\}$ the contribution of diffusion modes (Cooperons and diffusons) to the conductivity is infinite. To get rid of this infinite contribution we will show in this part how these effective modes come from a nonlinear supersymmetric σ-model.

2 Supermathematics in a nutshell

In this part we introduce general mathematical tools for supersymmetry. We will give practical definitions instead of rigorous ones.

2.1 Grassmann variables

We introduce GRASSMANN (or anticommuting) variables χ_i, $i = 1, \ldots, N$ through their algebra: $\{\chi_i, \chi_j\} = 0$. In particular this definition implies $\chi_i^2 = 0$. Generalizing operations on c-numbers, we define also "complex conjugate" of Grassmann variables in such a way that $(\chi_i^*)^* = -\chi_i$; a "hermitian conjugate": $\chi^\dagger = (\chi_1^* \chi_2^* \ldots \chi_N^*)$; and a scalar product $(\chi^\dagger, \chi) = \chi_1^*\chi_1 + \ldots + \chi_N^*\chi_N$.

Following BEREZIN(1961) we define integration over GRASSMANN variables defining first of all the symbol $\mathrm{d}\chi_i$ which verifies $\{\chi_i, \mathrm{d}\chi_j\} = 0$. Then we put: $\int \chi_i \,\mathrm{d}\chi_i = 1$ and $\int \mathrm{d}\chi_i = 0$. All other integrals can be calculated with with help of these last relations, for example: $\int \chi_1\chi_2 \;\mathrm{d}\chi_1 \mathrm{d}\chi_2 = -\int \chi_1 \;\mathrm{d}\chi_1 \int \chi_2 \;\mathrm{d}\chi_2 = -1$. Most important integrals are of course Gaussian ones $\int \mathrm{e}^{-\chi^\dagger A \chi} \left(\prod_{i=1}^N \mathrm{d}\chi_i^* \,\mathrm{d}\chi_i\right) = \det A$ where A is an hermitian matrix. We notice that the result is different from conventional Gaussian integrals $\int \mathrm{e}^{-S^\dagger B S} \;\left(\prod_{i=1}^N \mathrm{dRe}S_i \,\mathrm{dIm}S_i\right) = \pi^N (\det B)^{-1}$ where B is a complex matrix with postitive definite hermitian part.

2.2 Superalgebra

Now with GRASSMANN variables we can generalize algebraic structures which will be composed of c-numbers and GRASSMANN variables: supervectors $\Psi = \begin{pmatrix} \chi \\ S \end{pmatrix}$; and supermatrix: $q = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}$ where a, b are conventional matrices and ρ, σ are matrices which elements are GRASSMANN variables. For these new objects we can define define a generalization of trace: $\operatorname{Str} q = \operatorname{tr} a - \operatorname{tr} b$; and a generalization of transpose: $q^T = \begin{pmatrix} a^t & -\rho^t \\ \sigma^t & b^t \end{pmatrix}$ where t stands for the usual transpose. Finally we can define a superdeterminant: $\int \mathrm{e}^{-\Psi^\dagger Q \Psi} \left(\prod_i \Psi_i^* \Psi_i\right) = \operatorname{Sdet} Q$, in particular if Q is diagonal then $\operatorname{Sdet} Q = 1$.

3 Application of super-mathematics

3.1 A new expression for Green function

We are interested in calculating such integrals:

$$\frac{\int S_i S_j^* \, \mathrm{e}^{-\sum_{i,j} S_i^* B_{ij} S_j} \, \left(\prod_i \mathrm{d}S_i^* \mathrm{d}S_i\right)}{\int \mathrm{e}^{-\sum_{i,j} S_i^* B_{ij} S_j} \, \left(\prod_i \mathrm{d}S_i^* \mathrm{d}S_i\right)} = (B^{-1})_{ij}. \tag{1}$$

This integral can be expressed more simply with supervectors:

$$\begin{aligned} (B^{-1})_{ij} &= \int S_i S_j^* \, \mathrm{e}^{-\sum_{i,j}(S_i^* B_{ij} S_j + \chi_i^* B_{ij} \chi_j)} \, \Big(\prod_i \mathrm{d}S_i^* \mathrm{d}S_i \mathrm{d}\chi_i^* \mathrm{d}\chi_i\Big) \\ &= \int \Psi_i^2 \Psi_j^{2*} \mathrm{e}^{-\Psi^\dagger B \Psi} \, \Big(\prod_i \mathrm{d}\Psi_i^* \mathrm{d}\Psi_i\Big). \end{aligned} \tag{2}$$

We see that with the help of supermathematics we get rid of denominator. This trick is useful in order to write differently the Green function which is the most important quantity in Anderson localization

$$G_\varepsilon^{R,A}(\mathbf{r},\mathbf{r}') = \mp \mathrm{i} \int \Psi(\mathbf{r}) \overline{\Psi}(\mathbf{r}') \\ \exp\left[\pm \mathrm{i} \int \overline{\Psi}(\mathbf{r}) \left\{\varepsilon - H_0 - U(\mathbf{r}) \pm \mathrm{i}\delta\right\} \Psi(\mathbf{r}) \, \mathrm{d}\mathbf{r}\right] \, \mathcal{D}\Psi^* \mathcal{D}\Psi \tag{3}$$

We introduced i in order to have convergent integrals for the part with δ.

Without denominator it becomes easier to calculate the average of Green function

$$\langle G_\varepsilon^{R,A}(\mathbf{r},\mathbf{r}') \rangle = \mp \mathrm{i} \int \Psi(\mathbf{r}) \overline{\Psi}(\mathbf{r}') \\ \exp\left[\pm \int \left(\mathrm{i}\overline{\Psi}(\mathbf{r}) \left\{\varepsilon - H_0 \pm \mathrm{i}\delta\right\} \Psi(\mathbf{r}) - \gamma(\overline{\Psi}(\mathbf{r})\Psi(\mathbf{r}))^2\right) \mathrm{d}\mathbf{r}\right] \, \mathcal{D}\Psi^* \mathcal{D}\Psi.$$

(4)

After averaging there is no more disordered potential but an interaction term $\gamma(\overline{\Psi}(\mathbf{r})\Psi(\mathbf{r}))^2$ appears instead.

3.2 Nonlinear supersymmetric σ-model

In this section we want to derive a supersymmetric field theory for the ANDERSON model. Diffusions modes (Cooperon and diffuson) will appear in this theory. More precisely, we will see that they are GOLDSTONE modes which appear because of a spontaneous breaking symmetry. It is possible to see this already in the mean Green function $\langle G \rangle = \frac{1}{\varepsilon + H_0 \pm \frac{\mathrm{i}}{2\tau}}$, where the term $\frac{1}{2\tau}$ is like an order parameter even if it is always nonzero.

First of all it is interesting to look at the mean field. As for the BCS theory, we substitute pairs of the field by an average value:

$$\Psi_\alpha(\mathbf{r})\Psi_\beta(\mathbf{r}) \rightarrow \langle \Psi_\alpha(\mathbf{r})\Psi_\beta(\mathbf{r}) \rangle \doteq Q_{\alpha\beta} \tag{5}$$

where Q is a 8×8 supermatrix. Here, in the Lagrangian we have the term $\overline{\Psi}_\alpha \Psi_\alpha \overline{\Psi}_\beta \Psi_\beta$ which can be written as a sum of six pairs. Only four of them are physically interesting, the others $(\overline{\Psi}_\alpha \Psi_\alpha)$ just represent a renormalization of the FERMI energy. Finally, the interacting term becomes $4\overline{\Psi}_\alpha Q_{\alpha\beta} \Psi_\beta$ and the Lagrangian is $\mathcal{L}_{\text{eff}} = -\mathrm{i} \int \overline{\Psi}(\varepsilon - H_0 - \frac{\omega}{2} + \mathrm{i}\frac{Q}{2\tau})\Psi$. The self-consistent equation is $\langle \Psi_\alpha \overline{\Psi}_\beta \rangle_{\text{eff}} = Q$ which solution is $Q = \frac{1}{\pi\nu} \int g(\mathbf{p}) \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d}$ where $g(\mathbf{r}, \mathbf{r}) = \mathrm{i}\left[\varepsilon(\mathbf{p}) + \frac{\omega + \mathrm{i}\delta}{2} - \varepsilon + \frac{1}{2}(\omega + \mathrm{i}\delta) + \Lambda \mathrm{i}\frac{Q}{2\tau}\right]^{-1}$. When $\omega \rightarrow 0$ the equation becomes $Q^2 = 1$ which implies that there is a unitary supermatrix V —i.e. $V\overline{V} = \mathbb{1}$— such that $Q = V\Lambda\overline{V}$. In this limit the ground state is infinitely degenerate leading to the formation of GOLDSTONE modes.