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Cours de Physique Théorique: Black Holes in String Theory

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Abstract

DISCLAIMER: This is an unfinished version of the lecture notes, as written down by Bert. It is a rough text, lacking some subtleties and references. Please use the material modulo typos and small mistakes. The final version will follow later.

Contents

1 Why black holes?

This is a lecture series about black holes, but that does not mean that every little detail about what a black hole is will be explained. The purpose is to give tools and background for the contruction of black holes in string theory, and to address the problems associated to black holes with these tools. We start these notes in this section with placing the black hole into context and stating the main questions are raised in black hole physics. For more details on the (GR) aspects of black holes, see for instance the Cours de Physique at IPhT by Nathalie Deruelle in 2009 [1] and references therein.

Black holes are *classical* solutions that appear naturally in GR. The first black hole metric was written down for the first time almost a century ago by Karl Schwarzschild (although at that point it was only used to model the geometry outside of a spherically symmetric object as the Sun or the Earth). It is a solution to the Einstein equations determined by one parameter, the mass.

1.1 Classical black holes

Figure 1: *A* classical *black hole is the ultimate solution for those smelly diapers of your one-year-old daughter, nagging mother-in-laws or obnoxious TV stars: you can throw anything in, but nothing comes out.*

Very crudely, we can picture such a black hole as a region of spacetime in which things can fall, or be thrown in, but nothing comes out, see figure 1 for a cartoon. The boundary from which no round-trip tickets are available any more, is called the event horizon. The name "black hole" fits very well: classically, a black holedoes not emit anything, not even light.

We can say more than just drawing cartoons. In GR, there is a very well-defined picture one can make of a spacetime that showcases its causal properties, while it still fits on a page: the Penrose diagram. It can be obtained by performing a conformal transformation (scaling) on the metric. The Penrose diagram is then a two-dimensional picture of the conformal metric. The key feature is that time-like surfaces (light-rays) are still at 45° angles and we can therefore easily infer the causal structure of the spacetime. The Penrose diagram for the Schwarzschild black hole is shown in figure 2.

Any object travels on a causal curve: it has to stay within its future lightcone. We see that once something falls into the horizon, it can never get out again. From the Penrose diagram, we also see that anything that falls in will further collapse and eventually hit the singularity.

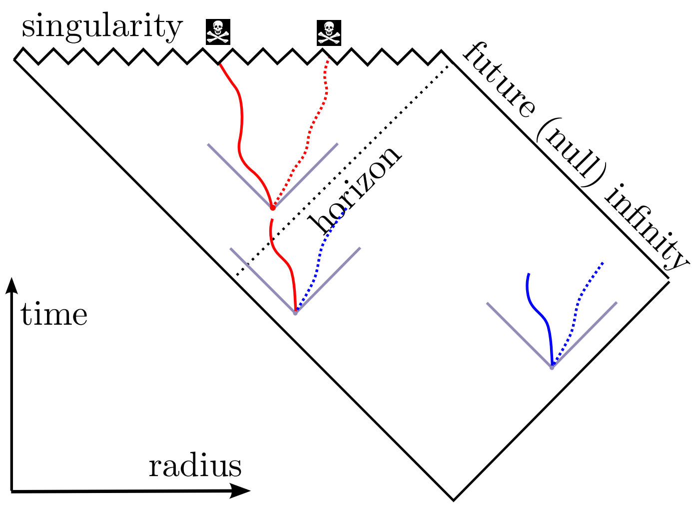


Figure 2: *The Penrose diagram for the Schwardschild metric. Some lightcones and particle trajectories are drawn outside and inside the black hole horizon. Note that the singularity (sawtooth line) is in the causal future of any object that falls behind the horizon.*

Two important observations where made by Carter, Hawking, Penrose... from the 60s onwards:

- No memory in horizon region of what the black hole is made of, this region is smooth and has no special features:
  "Black holes have no hair"

- Black hole uniqueness theorems ('60s–'70s):
  A static black hole is fully characterized by its mass.[1]

A black hole of a certain mass could thus be made up out of anything: ipods, elephants, grad students... from the outside it will look the same.

1.2 A little bit of quantum mechanics

What happens if we add quantum mechanics to the game? The region of spacetime around the horizon of a black hole has a curvature and hence a certain energy density. We know that in

[1] The more general time-independent solution, a stationary black hole, is fully determined by its mass ánd angular momentum. When GR is coupled to an electromagnetic field, a black hole can have an electric and a magnetic charge as well. However, there is no additional memory of what formed the black hole: there are no higher multipole moments etc.

QFT, energy can decay into a particle-antiparticle pair. This idea has led Hawking to perform a semiclassical analysis of QFT in a black hole background. Through the Hawking process, pairs will be created and once in a while one of the two falls into the black hole horizon, while the other escapes off to spatial infinity. The net result is that the black hole mass is lowered and energy, under the form of thermal radiation, escapes to infinity, see Figure 3.

The black hole behaves as a black body, with a temperature proportional to the strength of the gravitational field at the horizon. One finds this temperature is inversely proportional to the black hole mass:

$$T = \frac{\hbar c^3}{k_B} \frac{1}{8\pi G_4 M} \simeq 6 \times 10^{-8} \left(\frac{M}{M_\odot}\right) \text{Kelvin} \tag{1}$$

where $M_\odot \simeq 2 \times 10^{30}$ kg is the mass of the sun. How bigger the black hole is (more mass), how lower its gravitational field a the horizon and hence how lower its temperature. For a typical astrophysical black hole, ranging from several to several millions of solar masses, this is a very small temperature.

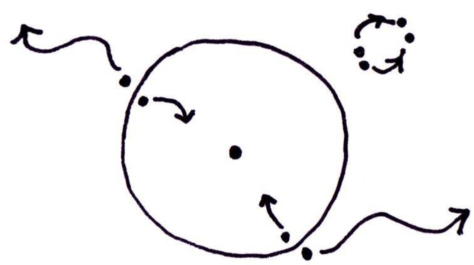


Figure 3: *A cartoon of the Hawking process. The black hole geometry is pictured as a point, the singularity, surrounded by a horizon. A QFT calculation in the black hole spacetime leads to pairwise particle creation such that close to the horizon, one of these particles can fall into the horizon, the other escaping to infinity.*

By the laws of black hole thermodynamics, a black hole also has an entropy. it was first conjectured by Bekenstein and later proven by Hawking that this entropy is proportional to the area of the black hole horizon:

$$S_{BH} = \frac{A_H}{4G_N}\,, \tag{2}$$

where G_N is Newton's constant, related to the Planck length as $G_N \sim l_P^2$. In Planck units, we thus have $S_{BH} = A_H/4l_P^2$ with $l_P \simeq 1.6 \times 10^{-35}$ m. The entropy of a typical black hole will thus be very large. For a Schwarzschild black hole, we find that the Bekenstein-Hawking entropy is proportional to the square of the black hole mass:

$$S_{Schw} \simeq 10^{77} \times \left(\frac{M}{M_\odot}\right)^2 \text{Joule/Kelvin}\,. \tag{3}$$

This is a huge entropy! For a solar mass black hole (which would have a radius of about 3 km) we find 10^{77}, for the black hole in the center of our galaxy of several million solar masses, we find about $S_{Gal} \simeq 10^{90}$.

How should we understand this entropy? Boltzmann has taught us that the entropy is related to a number N of microstates, microscopic configurations with the same macroscopic properties:

$$S \sim \log(N)\,. \tag{4}$$

We would hence conclude that the quantum mechanics of black holes leads to an incredibly large amounts of microstates: $N_{QM} \sim e^{10^{90}}$. However, in the classical GR picture we do not understand this number, as there is only one stationary solution with the black hole mass (the macroscopic parameter of the configuration): $N_{GR} = 1$. This numerical discrepancy is maybe the largest unexplained number in theoretical physics.[2]

1.3 Problems

- **Where are the microstates?** Maybe the N_{QM} states live in the region of the singularity, and GR is just does not see them? Recent arguments by Mathur and others show that this would not solve the information paradox (second point), and black hole microstates should differ from the black hole significantly also at horizon scales. Such 'microstate geometries' do not exist within general relativity.

- **Information paradox.** The Hawking radiation process has positive feedback: as a black hole radiates, it loses mass, increasing its temperature, which increases the rate of radiation. If we wait long enough, by the Hawking process a black hole will continue radiating until all of its mass is radiated away and we are left with only thermal radiation. This leads to a problem: where has the information of the initial state gone? Note that a black hole we start from that goes to a universe without black hole, but filled with thermal radiation cannot be obtained by unitary evolution. People have come up with many ideas to solve this problem: maybe physics is not unitary, or the black hole does not evaporate completely and there is a remnant with high entropy, and other explanations. Not one has proven satisfactory.[3]

We would like to solve these problems. The solution is in the study of black holes in a quantum gravity theory, that can unify classical GR with quantum mechanics.

String theory is a powerful mathematical framework that does exactly this. We do not have to believe that this theory describes the real world. As a quantum gravity theory, string theory can be tested by its answers to the issues related to black holes (information problem, entropy problem). If it does not pass this test, and cannot solve these problems, we throw it to the garbage as a quantum theory of gravity. If it does, we can start thinking about other tests and problems to attack – and maybe start believing it describes the real world after all.

[2]For comparison, the famous cosmological constant problem is the large ratio $\Lambda_{QFT}/\Lambda_{obs} \sim 10^{120}$ between the "expected" value Λ_{QFT} and the observed value Λ_{obs}. This number is peanuts compared to the required number of black hole microstates!

[3]Note: even if Hawking's calculation is not correct at the end of the evaporation process (when the black hole is small, curvature at the horizon is large and the semiclassical approximation is no longer valid), the entropy cannot all appear from this small end state.

2 Building blocks

In this section, we provide the tools to construct black hole solutions of string theory. It is not our intention to give a lecture series on string theory: *We will not tell you how to build the computer, but how to programme.*

2.1 Caught in the web

String theory is a framework that has grown dynamically over the past thirty or so years. Various limits of this theory have been studied, see Figure 4. Historically all the corners of this diagram where constructed as different theories and only about 15 years ago it was realized that they were all related through various dualities, and can be seen as limits of one theory. We reserve the term "string theory" for the encompassing framework.[4]

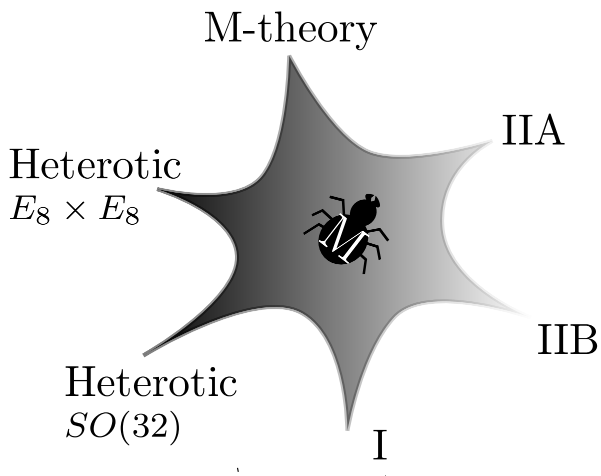


Figure 4: *We should view string theory as a web, of which we understand several corners, where perturbative and other techniques can be used. In these lectures, we will only consider M-theory, type IIA and type IIB string theory.*

In these lectures, we will only consider M-theory, type IIA and type IIB string theory. M-theory is 11-dimensional, while the type II strings live in ten dimensions. We will mainly study the low energy limits of string theory. "Low energy" is relative. We mean that we stick to the zero mass fields of the string spectrum. The low-energy limits of string theories are supergravity theories: gravity theories that are extensions of general relativity with other fields, fixed by the requirement of supersymmetry. See Table 1.

2.2 An analogy for M theory

To get a grip on the field content of these higher-dimensional beasts, we first make an analogy with Maxwell theory in four dimensions.

[4]Often people refer to the entire framework as "M-theory". We like to view this eleven-dimensional theory as one of the corners of the string web instead.

Theory	Low-energy limit
M	11d supergravity
IIA	10d IIA supergravity
IIB	10d IIB supergravity

Table 1: *The theories we work in.*

Maxwell theory. The action for Maxwell theory coupled to gravity is:

$$\mathcal{L} = \sqrt{-g}(R + F_{\mu\nu}F^{\mu\nu})\,, \tag{5}$$

where $F_{\mu\nu}$ is the electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu (= 2\partial_{[\mu}A_{\nu]})$.

What are the fundamental objects in this theory?

- **Electrons.** An electrically charged particle with electric charge e couples to the electric field as

$$S_{el} = e\int \left[A_\mu \frac{dx^\mu}{d\tau}\right] d\tau\,, \tag{6}$$

where τ parameterizes the worldline of the particle. A particle that sits stil, couples to the time component of the electric field as $e\int A_0 dx^0$ with $x^0 = t$. The electric field profile sourced by such a field is (in D spacetime dimensions)

$$A_0 = \frac{e}{r^{D-3}}\,, \qquad \vec{E} = \vec{\nabla} A_0 = -\frac{e}{r^{D-2}}\vec{u}_r\,, \tag{7}$$

where $\vec{u}_r$ is a unit vector in the radial direction. Note that a moving electron couples to magnetic components A_i of the gauge field as well.

- **Magnetic monopoles.** There also exist magnetically charged particles in four dimensions. These are monopole sources of the magnetic field. The charge of these particles can be measured by integrating the magnetic field lines over a two-sphere surrounding the charge (see Figure 5):

$$g_M = \frac{1}{8\pi}\int_{S^2} F_{\mu\nu}dx^\mu dx^\nu\,. \tag{8}$$

The magnetic monopole sources a profile for the magnetic field

$$F_{ij} = \epsilon_{ijk}B^k\,. \tag{9}$$

The coupling to the electromagnetic field is found in an indirect way. Just as the electron couples to the gauge field, the magnetic monopole couples to the (Hodge) *dual* electric field:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,, \tag{10}$$

and

$$S_{mag} = g_M\int \left[\tilde{A}_\mu \frac{dx^\mu}{d\tau}\right] d\tau\,, \tag{11}$$

The dual field sourced by a static magnetic monopole is then

$$\tilde{A}_0 = \frac{g_M}{r^{D-3}} \,, \tag{12}$$

and in flat space, this gives the magnetic field in polar coordinates (using 10)

$$F_{\theta\phi} = g_M \sin\theta \,, \qquad \text{or} \qquad \vec{B} = -\frac{g_M}{r^{D-2}} \vec{u}_r \,. \tag{13}$$

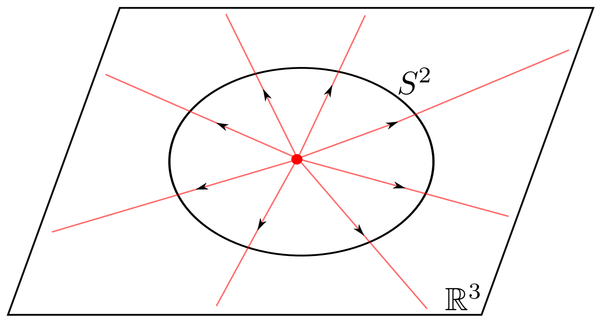


Figure 5: *Magnetic field lines from a magnetic monopole. The total charge is measure by integrating the flux over a surface (e.g. a two-sphere) surrounding the source.*

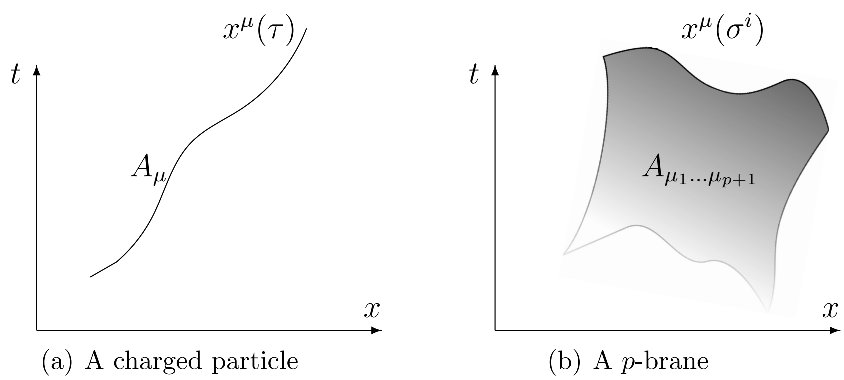


Figure 6: *A charged particle traces out a one-dimensional worldline, its higher dimensional analogue (a p-brane) traces out a $(p+1)$-dimensional worldvolume, sourcing a $p+1$-form potential. For a p-brane, we parametrize the worldvolume in terms of σ^i $(i = 0 \dots p)$.*

Eleven-dimensional supergravity. The features of eleven-dimensional supergravity (the low energy limit of M-theory) are very similar to those of four-dimensional Einstein-Maxwell theory. The bosonic fields are again the metric and a gauge field, which is now a two four-form field strength $F_{\mu\nu\rho\sigma}$, instead of the two-form of maxwell theory. These fields and there couplings are dictated by supersymmetry: supergravity theories are theories of gravity that are (locally) supersymmetric, and due to this extra symmetry, the possible fields and their couplings are constrained.

One finds the Lagrangian for eleven-dimensional supergravity (which is by the way the highest spacetime dimensions that can accomodate a supergravity theory) is

$$\mathcal{L} = \sqrt{-g}\,(R + F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma})\ , \tag{14}$$

where $F_{\mu\nu\rho\sigma}$ are the components of a four-form gauge field

$$F_{\mu\nu\rho\sigma} = 4!\partial_{[\mu}C_{\nu\rho\sigma]}\ , \tag{15}$$

and $C_{\mu\nu\rho}$ is its three-form potential.

What are the fundamental charged objects of this theory?

- **Electric object: M2 brane.** The equivalent of the electron (which couples to the gauge field component A_0) is an object that couples to the electric component of the three-form potential C_{0ij}. Because of the additional directions, this potential couples naturally to a two-dimensional extended object or membrane, with a three-dimensional worldvolume Σ (cf. the particle with a one-dimensional worldvolume). For a membrane in the directions x^1, x^2 we have:

  $$S_{M2} = N_{M2}\int_\Sigma C_{012}dx^0dx^1dx^2\,. \tag{16}$$

  The charge can be interpreted by the number of membranes. This membrane of M-theory is also called M2-brane.

- **Magnetic object: M5 brane.** In analogy with the magnetic particle, we can also consider a magnetic monpole charge for the field strength $F_{\mu\nu\rho\sigma}$. To measure its charge, we have to integrate the field strength now over a four-sphere, see Figure 7:

  $$g_M = \int_{S^4} F_{\mu\nu\rho\sigma}dx^\mu dx^\nu dx^\rho dx^\sigma\ , \tag{17}$$

From Figure 7 we can also find the dimensionality of the magnetic monopole of M-theory. The field lines run in a five-dimensional transverse plane (directions 1,2,3,4,5) and the magnetic monopole takes up the remaining five dimensions (6,7,8,9,11).[5] This object is called the M5-brane.

2.3 Type II string theory

We relate ten-dimensional string theories and M-theory.

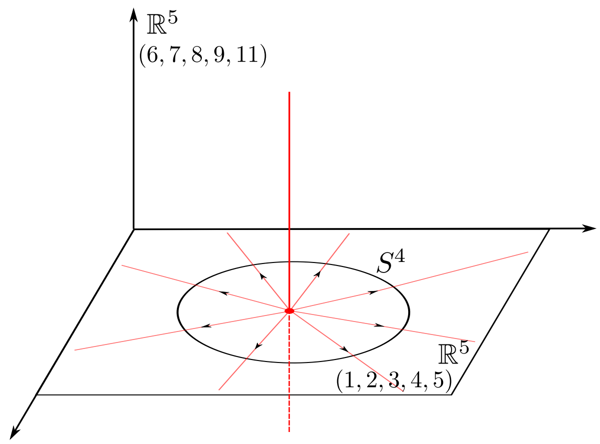


Figure 7: *Magnetic field lines from the M-theory magnetic are integrated over an S^4 in the transverse $\mathbb{R}^5(x^1 \ldots x^5)$. Hence, this magnetic monopole is a membrane extending in five space dimensions $(x^6 \ldots x^{10})$.*

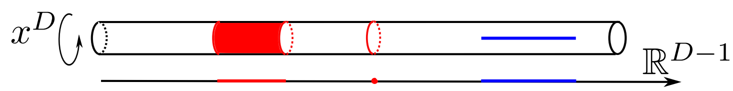


Figure 8: *Curling up one out of D dimensions makes a space look essentially $(D-1)$-dimensional. An object that is wrapped on the compact dimension has a worldvolume of one dimensions lower (a membrane becomes a string, a string becomes a point etc.), an unwrapped object remains of the same dimension.*

2.3.1 Type IIA supergravity from dimensional reduction

We imagine making the direction x^{11} small and 'compactifying' it on a circle. See Figure 8. What happens to the objects of M-theory? There are two distinct cases for each fundamental object: either the worldvolume of the object is wrapped on x^{11}, meaning that one of its directions shrinks away, or the worldvolume is completely inside the ten large dimensions of spacetime. We summarize in Table 2.

M-Theory		IIA supergravity	
Object	Directions	Object	Directions
M2	$0, 1, 11$	String	$0, 1$
	$0, 1, 2$	membrane	$0, 1, 2$
M5	$0, 1, 2, 3, 4, 11$	4d membrane	$0, 1, 2, 3, 4$
	$0, 1, 2, 3, 4, 5$	5d membrane	$0, 1, 2, 3, 4, 5$
Mom. wave	$0, 11$	particle	0

Table 2: *Objects in IIA after compactifying M-theory on a circle.*

An important new object is the momentum wave. Because we compactify on a circle, mo-

[5]We choose to write the time directions as x^0 and spacetime directions $x^1, x^2, \ldots$. However, we choose the 'eleventh' dimensions to be x^{11} and skip x^{10}.

mentum along x^{11} is quantized and momentum waves excitations have a dicrete mass spectrum:

$$m = \frac{1}{\ell_{11}}, \frac{2}{\ell_{11}}, \frac{3}{\ell_{11}} \dots \tag{18}$$

where ℓ_{11} is the radius of the circle. Upon compactification, the momentum waves become particles.

We interpret all these new objects after compactification. The resulting ten-dimensional theory is called IIA string theory. Its low energy limit is IIA supergravity. It was found independently in the 80ies and only in the mid-90ies people realized its connection to eleven-dimensional supergravity through compactification. The objects of IIA string theory, which were found earlier through quantization of the IIA string, correspond exactly to what we found above from compactifying M-theory. These are:

- F1: the fundamental quantized string of IIA string theory
- D2-brane: people have known Dirchlet-branes, or D-branes for short, from quantizing strings. They arise from possible Dirichlet boundary conditions one can put on the string. One finds that, depending on the type of string theory, only certain dimensionalities of submanifolds of spacetime can provide such Dirichlet-boundary conditions while remaining stable objects. These are the allowed D-branes. Surprisingly, one finds that these D-branes describing boundary conditions, also have a dynamics of their own. We will expand on this as we go on. In the same way we have other D-branes of even dimensions:
- D4-brane
- D0-brane or D-particle.
- D6-brane: There is also another D-brane in the string spectrum. It descends from a certain smooth type of geometry in M-theory known as the Kaluza-Klein monopole.
- NS5-brane: this is not a D-brane, but is in fact the 'magnetic monopole' associated to the 'electric' F1.
- Dilaton: the size of the M-theory circle becomes a dynamical scalar field in ten dimensions. We will not consider it further.

All of these branes couple to the appropriate higher-form gauge field, as for M-theory. The exact forms follow from the compactification procedure. For example, the gauge field three-form potential C of M-theory with on compactified direction:

$$B_{\mu\nu} \equiv C_{\mu\nu 11} \tag{19}$$

is the Neveu-Schwarz (NS) B-field of string theory. This field couples electrically to the F1 string and magnetically to the NS5-brane. The components of the gauge field in ten dimensions $C_{\mu\nu\rho}$ define the Ramond-Ramond three-form gauge field and they couple electrically to the D2 branes and magnetically to the D4-branes.[6] The gauge field for the D0-brane comes from the metric $C_\mu = g^{(11)}_{\mu 11}$. In Table 3, we organize the different branes through their electric and magnetic coupling to the various potentials

Note that historically, first all these higher-form gauge fields were found in the spectrum of string theory, but people had at that point (the 80'ies of Madonna) no idea what objects they coupled to. It took until almost fifteen years ago before it was realized that most of these objects are in fact the Dirichlet-branes.

In a similar way, IIB string theory has a plethora of higher-dimensional objects. The NS-sector (including the F1 string and the NS5 brane) also appear, but IIB has only stable branes of uneven dimensionality, versus the even branes of IIA. See Table 3.

	IIA			IIB			
Potential	B	C_1	C_3	B	C_0	C_2	C_4
Electric	F1	D0	D2	F1	D(-1)	D1	D3
Magnetic	NS5	D6	D4	NS5	D7	D5	D3

Table 3: *Coupling of branes to n-form potentials. In ten dimensions, an $n = (p+1)$-form potential couples to a p-brane through an electric coupling and to a $(6-p)$ through a magnetic coupling. We give the brane couplings of the NS-NS sector (F1 stands for fundamental string, NS5 for the magnetically dual NS5 brane) and R-R sector of type IIA and type IIB string theory. (We do not consider the IIA (magnetic) D8 brane and its electric counterpart. The D(-1) brane should be seen as an instanton.)*

2.3.2 Dualities

One may wonder how to relate IIB to IIA and M-theory, since at this point we wrote down the fields in a rather ad hoc way. The clue lies in several dualities of the string spectrum.

S-duality. We first focus on a symmetry of the spectrum of the IIB string. We observe that the spectrum can be organised in pairs: F1 – D1, NS5 – D5 (we also have NS7 – D7, but that example is a little special so we ignore it further). This corresponds to the pairing of the B-field $B_{\mu\nu}$ with the RR two-form $C_{\mu\nu}$ and the same for their magnetic dual fields $\tilde{B}$ and $\tilde{C}$ (which are in fact 6-forms as Exercise 2.1 asks you to show).

Exercise 2.1: *Generalize the dualization rule* (10) *for two-forms in four spacetime dimensions to arbitrary dimensions D and arbitrary p-forms (you need the inverse metric to raise indices). Use this to write down which form couples to which brane in both IIA and IIB theory.*

What about the D3 brane? What does it pair up with? In fact, in IIB supergravity, the five-form field strength obeys the property

$$F_{\mu_1 \dots \mu_5} = \frac{1}{5!}\sqrt{-g}\,\epsilon_{\mu_1 \dots \mu_5\ \mu_6 \dots \mu_{10}} F^{\mu_6 \dots \mu_{10}} \qquad (20)$$

[6]Different boundary conditions for the ferminonic fields living on the worldvolume of the type II string give different possible fields in the string spectrum. In the massless spectrum we observe that Neveu-Schwarz-boundary conditions (antiperiodic) give the NS-fields: metric $g_{\mu\nu}$, B-field $B_{\mu\nu}$, and dilaton ϕ. Ramond boundary conditions (periodic) give RR fields $C^{(0)}, C^{(2)}, C^{(4)}$.

and therefore, using Exercise 2.1, the five-form field strength that couples to the D3 brane is self-dual $F^{(5)} = \tilde{F}^{(5)}$, and hence the D3 brane pairs up with itself: the D3-brane is a dyon, it is both an electrically charged brane and a magnetic monopole! We will see below that this dyonic nature sperates the D3-brane from the other branes.

There exists a clean symmetry interchanging the fields $B^{(2)}$ with $C^{(2)}$, while leaving $F^{(5)}$ unaltered. This transformations ia called S-duality and it interchanges F1's with D1's, D5's with NS5's and leaves the D3 brane as it is. It is a very useful transformation in navigating through the zoo of brane solutions.[7]

T-duality. There is another symmetry, that maps the string spectra of IIA and IIB unto each other. Imagine wrapping the IIA string on a circle of radius R. A string wrapped on the compact dimension has a mass proportional to its tension T_{F1} times the length of the string. The string length ℓ_s is related to the string tensrion as $T_{F1} = 1/2\pi\ell_s$, so this mass comes in fundamental units of R/ℓ_s^2. The number of units is a topological number and describes how many times the string winds along the compactified dimensions. We call them (string) winding modes.

We can also put momentum modes on the string. These momentum modes should be viewed as oscillations travelling on the string. Again, these fundamental string excitations come in quanta, now of $1/R$. We can play the same game for IIB string theory compactified on a circle of radius $\tilde{R}$. See Table 4 and figure 9.

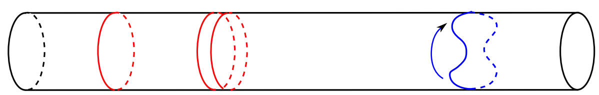


Figure 9: *Left (in red): a string winding one or several times around the compact dimensions, right (blue): a vibrational or momentum mode of the string.*

IIA	**Winding**	**Momentum**
$m =$	$\frac{R}{(\ell_s)^2}$	$\frac{1}{R}$
	$\frac{2R}{(\ell_s)^2}$	$\frac{2}{R}$
	$\frac{3R}{(\ell_s)^2}$	$\frac{3}{R}$
	$\dots$	$\dots$

IIB	**Winding**	**Momentum**
$m =$	$\frac{\tilde{R}}{(\ell_s)^2}$	$\frac{1}{\tilde{R}}$
	$\frac{2\tilde{R}}{(\ell_s)^2}$	$\frac{2}{\tilde{R}}$
	$\frac{3\tilde{R}}{(\ell_s)^2}$	$\frac{3}{\tilde{R}}$
	$\dots$	$\dots$

Table 4: *The mass of winding and momentum modes of IIA string theory compactified on a circle of radius R and IIB theory compactified on a circle of radius $\tilde{R}$.*

The spectra of IIA and IIB compactified on such circles are exactly mapped into each other under T-duality, interchanging momentum modes and winding modes. We reserve p for the units of momentum charge and F1 for the amount of string winding. Schematically, T-duality thus acts as:

[7]In the near-horizon geometry of a D3 brane, which is $AdS_5 \times S^5$ as we wil see below, S-duality becomes the strong-weak coupling duality of $N = 4$ super Yang-mills that is dual to the $AdS_5 \times S^5$ background through the AdS/CFT duality.

IIA		IIB
F1	$\longleftrightarrow$	p
p	$\longleftrightarrow$	F1

The symmetry of the string spectra in these two different string theories opens up a huge portion of parameter space where we can actually do perturbative string theory calculations. Say we consider type IIA string theory. As long as R is large compared to the string scale, we have a pretty good control. However, when the size of the circle is small compared to the string length scale, corrections due to the stringy nature are huge and we lose this control. Then T-duality makes it possible to go to type IIB theory with $\tilde{R} \gg l_s$. (Note that for circle radius $R \simeq l_s$, we still cannot say too much.)

Dualities for D-branes. Consider the setup of Figure 10. We compactify string theory on a circle. A brane that is wrapped on this circle, will no longer extend along this direction after T-duality. Conversely, a D-brane that does not wrap the T-duality circle, will become after T-duality a D-brane of one dimension higher that wraps the circle.

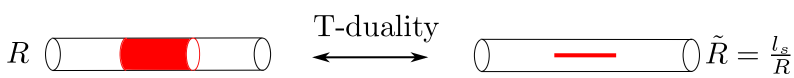


Figure 10: *Under T-duality, a D-brane wrapping the circle is mapped to a D-brane of one dimension lower and vice versa.*

To get the gist of it, we apply T-duality on the (supersymmetric) intersection of two species of D-branes. Let us start from a D3-D3 brane intersection in type IIB

IIB:	D3	0	1	2	3		
	D3	0			3	4	5

Say we compactify the 3-direction. Under a T-duality to IIA, we get the branes:

IIA	D2	0	1	2			
	D2	0				4	5

We can continue on this, see Exercise 2.2.

Exercise 2.2: *Show that three additional T-dualities on the two orthongal D2-branes, along directions 2, 1 and 3 give the D1-D5 brane intersection:*

IIB:	*D1*	*0*			*3*		
	D5	*0*	*1*	*2*	*3*	*4*	*5*

We will use this brane setup ('D1-D5 system') a lot in the study of black holes and their entropy.

Similarly, we can consider S-dualities. For instance, the D1-D5 setup of exercise 2.2 becomes after S-duality:

IIB:	F1	0			3		
	NS	05	1	2	3	4	5

We give an extra exercise with S-and T-dualities.

Exercise 2.3: *Consider the D3-D3 configuration. Perform a series of T-dualities to D1-D3, an S-duality to F1-D3. Find a set of T-dualities from the lattter configuration that links this to a D1-D5 system of IIB, with the D5 brane along say* $0,2,4,5,6,7$*. Note: you will need that the T-dual of F1 (winding mode) is p (momentum mode).*

We see that an entire zoo of complicated D-branes becomes really simple. On the level of supergravity, this is a solution generating tool (see the next subsection). We can interpret all these two-brane intersections as really one solution, which takes on different forms in different 'duality frames". We can get the supergravity solution in any frame in no time from the T-duality rules. This applies equally well to any other brane solution.

We will make extensive use of T- and S-dualities on black hole solutions. This will map to black holes which may look a bit different, but all have the same physical properties (entropy, temperature...). We will always work in the duality frame most adapted to the questions we are asking at that moment. In particular, we will often work in the D1-D5 duality frame, see Exercise 2.2.

2.4 D-brane supergravity solutions

D-brane solutions. Let us consider some actual D-brane solutions. We restrict to the solution of the low-energy approximation of string theory, supergravity.

For concreteness, we take the D2-brane of IIA supergravity, extending along directions 0,1,2 (time and two space directions). As in the analogy with electromagnetism, this brane sources a three-form potential C_{012}. It has a non-zero tension (mass density) and hence it also couples to the metric. (There is a third field it sources, the dilaton, but we will not consider that in detail).

The exact way the D2 brane source affects those fields, is through one function of the space-time coordinates. We call that function Z. One finds:

$$\begin{aligned} C_{012} &= Z^{-1}\,, \qquad e^{\phi} = Z^{1/4} \\ g &= Z^{-1/2}(-dx_0^2 + dx_1^2 + dx_2^2) + Z^{1/2}(dx_3^2 + \ldots + dx_9^2)\,. \end{aligned} \tag{21}$$

We will not consider the dilaton ϕ any further. Concentrating on the other fields, we see that the solution has Lorentz invariance along the D2 brane directions $0,1,2$ and rotational symmetry in the transverse directions.

The D2 brane behaves as a point particle in the transverse $\mathbb{R}^7$. The function Z plays the role of the Maxwell potential in the transverse $\mathbb{R}^7$. From the supergravity equations of motion, one finds that it obeys the Laplace equation on $\mathbb{R}^7$:

$$\Delta_7 Z = 0\,. \tag{22}$$

In the presence of sources, this is modified to

$$\Delta_7 Z = \rho_{D2}\,. \tag{23}$$

For a stack of N_{D2} D2-branes sitting at the origin of our coordinate system, the source is a delta function $\rho_{D2} = N_{D2}\delta(\vec{r}_7)$ and we find the solution

$$Z = c + \frac{N_{D2}}{r^5}\,, \tag{24}$$

where r is the radius of the transverse space $r^2 \equiv x_3^2 + \dots x_9^2$. The constant c can always be set to one by a constant rescaling of the coordinates and we will do so.

As $r \to 0$, we approach the D2-brane source and $Z \to \infty$. From the expression for the metric, we see that the $\mathbb{R}^{1,2}$ factor shrinks, while the $\mathbb{R}^7$ blows up. Therefore $r = 0$ is a singular locus in spacetime. The three-form potential C_{012} goes to zero at $r = 0$. However, the energy of the C-field

$$E = F_{\mu\nu\rho\sigma}F_{\mu'\nu'\rho'\sigma'}g^{\mu\mu'}g^{\nu\nu'}g^{\rho\rho'}g^{\sigma\sigma'} \tag{25}$$

blows up as $r \to 0$ and we conclude that there really the D2 brane geomertry contains a singularity, which is not shielded by a horizon.

Note that the equations of motion are linear in the sense that we can add multiple (singular) D2-brane sources :

$$\Delta_7 Z = N_a\delta(\vec{r} - r_a) + N_b\delta(\vec{r} - r_b) + \dots\,. \tag{26}$$

We consider all D2-branes of the same 'species', with the worldvolume along the $0, 1, 2$ directions.

The only thing that changes is that the function Z becomes a sum of harmonic functions, sourced at different locations:

$$Z = 1 + \frac{N_b}{|\vec{r} - \vec{r}_b|^5} + \frac{N_a}{|\vec{r} - \vec{r}_b|^5} + \dots\,. \tag{27}$$

We see that this solution can really describe any density of D2-branes, even a continuous one.

D3-brane from T-duality. We take up this idea to see what happens after T-duality. Start from a continuous distribution of D2 branes along a line in the transverse space (this is also called 'smearing' the D-brane charge). Say that we put this smeared D2-branes along the x_7 direction, see Figure 11.

The solution for such a continuous distribution of D2-brane charge on a line goes as:

$$Z \sim 1 + \frac{1}{r^4}\,. \tag{28}$$

Because the solution is now homogeneous in x_7, we can compactify this direction and perform a T-duality along x_7 to a D3-brane solution of type IIB string theory. What does this solution look like? Remember that the size of the compact circle is inverted after this duality transformation $g_{77} \to (g_{77})^{-1}$, and hence the metric of the resulting solution is:

$$g = Z^{-1/2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_7^2) + Z^{1/2}(g(6d))\,. \tag{29}$$

The three-form gets an additional leg:

$$C_{0127} = Z^{-1}\,. \tag{30}$$

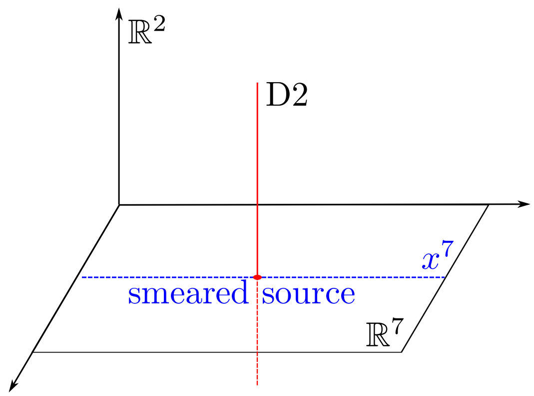


Figure 11: *A D2 brane smeared along x_7.*

and the solution for the function Z is

$$Z = 1 + \frac{N_{D3}}{r^4}\,. \tag{31}$$

The geometry before and after this operation are different. We see that T-duality interchanges D-branes as

smeared $\leftrightarrow$ localized (on top of each other)

Near-solution and brane throat. What does the geometry look like close to the D3-brane? We approach the D3-brane as we take $r \to 0$. This means that in the function Z, we can effectively drop the constant and write $Z = N_{D3}/r^4$ as $r \to 0$.

To reinstate the correct dimensions, we write $Z = R^4/r^4$. First write the transverse six-dimensional space in terms of polar coordinates as

$$g(6d) = dr^2 + r^2 d\Omega_5^2\,, \tag{32}$$

Then the near-geometry of the D3-brane is

$$g_{\text{near}} \;=\; \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_7^2) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2\,. \tag{33}$$

What has the D3-brane done? It has opened up a “throat”: as we approach $r \to 0$ from infinity, the S^5 will get smaller and smaller. But near the D3 brane it attains a finite size, set by the radius R, and the size of the S^5 stays constant. Note that the metric distance to $r = 0$ is actually infinite and the D3-brane throat is infinitely deep.

Physically, the D3-brane solution *forces* the $AdS_5 \times S^5$ geometry to appear.[8] This is a special feature of the D3-brane that the other D-branes do not possess (in fact, all the D0,D1…D6-branes represent a naked singularity). The origin lies in the dyonic nature of the D3 brane: it is both an electric and a magnetic charge for the four-form potential C_{0127}.

[8] For the afficionados: this is the same mechanism that forces the extremal Reissner-Nordstrom black hole to have a near-horizon region of the form $AdS_2 \times S^2$.

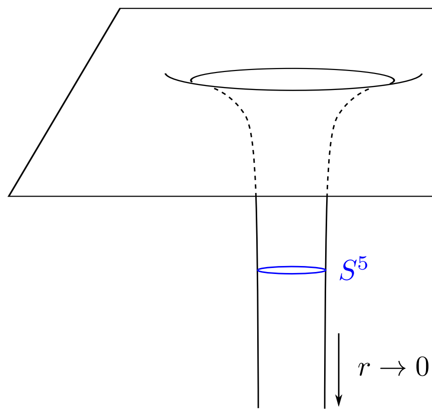


Figure 12: *A cartoon of the D3-brane geometry. As we approach the D3 brane, an infinity throat opens with constant transverse S^5 size.*

The $AdS_5 \times S^5$ geometry is the riding horse of holography and classical gravitational physics on this background is dual, through the AdS/CFT correspondence, to strongly coupled conformal field theory in $N = 4$ super-Yang-Mills. We will come back to this later.

BPS property: mass = charge. The charge of a D3 brane is given by integrating the gauge field that couples magnetically to it over a surface surrounding the brane (as for the magnetic monopole of electromagnetism)

$$Q = \frac{1}{5!} \int_{S^5} F_{ijklm}\, dx^i dx^j dx^k dx^l dx^m \,. \tag{34}$$

So far, we have only given the electric component of the gauge field C_{0123}. Exercise 2.4 asks you to derive the magnetic component of the field strength.

Exercise 2.4: *Derive, using the duality*

$$F_{ijklm} = \sqrt{-g}\epsilon_{ijklm\mu_1\mu_2\mu_3\mu_4} g^{\mu_1\mu_1'} g^{\mu_2\mu_2'} g^{\mu_3\mu_3'} g^{\mu_4\mu_4'} g^{\mu_5\mu_5'} F_{\mu_1'\mu_2'\mu_3'\mu_4'\mu_5'} \,. \tag{35}$$

and the expression for the field strength

$$F_{0123r} = \partial_r Z^{-1} \,. \tag{36}$$

the form of the dual field strength F_{45678}.

With this result, we find that from integrating over an S^5 at $r \to \infty$ to cover the entire flux emanating from the D3 brane, that the charge of the D-brane is

$$Q = N \,. \tag{37}$$

The mass of the D3-brane can be derived from the component g_{tt} of the metric, following the prescription of Arnowit, Deser and Misner (ADM). In particular, when expanding this component for large r, the leading terms are:

$$g_{tt} = 1 - \frac{2G_N M}{r} \,, \tag{38}$$

where G_N is Newton's constant and we see from (29) that M is proportional to the number of D-branes. We have not been too careful about prefactors in the expression for the metric, so we just state the dependence on g_s:

$$M = \frac{N}{g_s}\,, \tag{39}$$

where g_s ("g-string") is the string coupling constant. This is an interesting feature: the masses of all D-branes are inversely proportional to the string coupling constant. This should be contrasted with electromagnetism. The mass of the electron, the fundamental object, is independent of the coupling, let's call it g. On the other hand, the mass of a soliton in field theory goes as $1/g^2$. The magnetic monopole's mass has this behaviour. So we see that the D-brane is neither a fundamental object nor a soliton of string theory.

The mass of the fundamental string, the fundamental object of string theory, is independent of g_s (we have seen that the string tension, or mass density, is $T_{F1} = 1/2\pi\ell_s^2$. The mass of the NS5 brane goes as

$$M_{NS5} \sim \frac{1}{g_s^2}\,. \tag{40}$$

and we see that the NS5 brane is really a soliton of string theory. The different dependence on g_s of the masses of all these objects shows up in the 'warp factor' Z of the supergravity solutions. We track the dependence on g_s and drop other proportionality factors (such as the string length ℓ_s etc.) Newton's constant G_N goes as $G_N \sim g_s^2$ (this follows from the low-energy (supergravity) action of ten-dimensional string theories). In general we have

$$Z = 1 + \frac{G_N M}{r^{\#}}\,. \tag{41}$$

For a D-brane, this gives

$$Z_{\mathrm{D-brane}} = 1 + \frac{N_{\mathrm{D}} g_s}{r^{\#}}\,, \tag{42}$$

for an NS5 and a string we have

$$\begin{aligned} Z_{\mathrm{NS5}} &= 1 + \frac{N_{\mathrm{NS5}}}{r^{\#}}\,, \\ Z_{\mathrm{F1}} &= 1 + \frac{N_{\mathrm{F1}} g_s^2}{r^{\#}}\,. \end{aligned} \tag{43}$$

Going back to the D3 brane, we find in dimensionless units that

$$Q = M\,. \tag{44}$$

We interpret this as: "the mass (density) of a D3-brane is equal to its charge (density)".

What does this mean physically? The gravitational attraction and the electric repulsion are exactly balanced, even though both forces are huge. This is why we can have D-brane solutions with sources at many many points and still remain stable. This is different in electromagnetism, where two electrons would fly apart; the electric repulsoin always takes the upper hand and we cannot build multi-center electron-solutions.

Note that there is an underlying physical bound $M \geq Q$ for any charged object. When the mass is smaller than the charge, then the solution is unphysical (more on this in the next section). This bound is called BPS bound (after Bogomol'nyi, Prasad and Sommerfield), and we call the equal mass and charge of the D3 brane a BPS property. The BPS'ness of the D3-brane (and all the other D-branes) is a consequence of supersymmetry. All D-brane solutions (and the F1 and NS5) are invariant under a set of supersymmetry transformations, and the mass of any supersymmetric object is equal to its charge.

3 Black hole solutions

We discuss how to obtain black hole solutions from D-branes that are wrapped on compact spaces. For completeness, we first show how to make a black non-extremal, D-brane. We will later focus on supersymmetric (and hence extremal) black holes, because these are easier to construct and study.

3.1 Non-extremal black holes

Let us forget about supersymmetry for a moment, and see if we can make a black hole with $M > Q$. We do not try to make a multi-D-brane solution or anything like that, but just want to make a black hole, or a black object, with more mass than charge.

The easiest such solution is to make a black D3-brane. Its metric is given by

$$ds^2 = -Z^{-1/2}(-f(r)dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + Z^{1/2}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right) . \tag{45}$$

and the gauge field takes the same form as for the ordinary D3-brane

$$C_{0123} = Z^{-1} . \tag{46}$$

When the function $f(r) = 1$, this is just the supersymmetric D3 brane we have encountered before. By adding the function $f(r)$, the D3-brane is turned into a "non-extremal black brane". The function f obeys the same Laplace equation in transverse space as Z:

$$\Delta f = 0 . \tag{47}$$

Typically, one considers the solution

$$f(r) = 1 - \frac{\Delta M}{r^4} . \tag{48}$$

The charge for this solution is still given as for the normal D3 brane

$$Q = \int F_5 = N . \tag{49}$$

The mass (obtained from g_{tt} as before) is now

$$M = Q + \Delta M \tag{50}$$

We make two remarks. First note that when $\Delta M < 0$, this describes a singular solution with a naked singularity. Hence we consider $\Delta M > 0$ for physical reasons. Also, we see that unlike the supersymmetric D3 brane, two (or more) of these objects are not in equilibrium any more. Two black branes will attract and eventually collapse to a single black object, because the gravitational attraction is larger than the electrostatic repulsion.

A black hole, or black brane, that obeys $M = Q$ is also called extremal. Such a black object has zero Hawking temperature and does not emit radiation. When $M > Q$, the black object is non-extremal and has a non-zero temperature. For small ΔM, the temperature is proportional to the mass excess:

$$T \sim \Delta M\,. \tag{51}$$

We see that by the $f(r)$ "black deformation", we can create a solution with non-trivial mass, charge and temperature.

This solution is very useful for holography. In the near-brane solution $r \to 0$, $Z \sim 1/r^4$ and the black brane metric describes a black hole in $AdS_5 \times S^5$. Following the AdS/CFT correspondence, this maps to turning on a temperature in $N = 4$ super-Yang Mills theory in four dimensions, see Table 5. So a black hole corresponds to warming up the field theory. Conversely, a temperature in field theory gives a black hole in AdS_5.

Table 5: *Near-horizon geometry of black D3-brane and thermal physics in field theory.*

String Theory in $AdS_5 \times S^5$	$N = 4$ **Super-Yang-Mills (4d)**
Weak coupling	Strong coupling
Black hole in $AdS_5 \times S^5$ at temperature T	$N = 4$ SYM at temperature T

A question from the audience:

- What about quantum effects? Quantum effects are controlled by g_s, the string coupling. In the limit we consider, which is A_H and Q very large in planck units (for instance $A_H \gg \ell_P^2$), we expect quantum effects not to destroy the geometry. Of course, when we only consider one D-brane, this limit does not hold and the question of quantum corrections becomes really important. More on this in section 4.1 Another way to state the question is the following. We know that the D3-brane geometry is controlled by

$$Z_{\mathrm{D3}} = 1 + \frac{N g_s}{r^4}\,. \tag{52}$$

  If Z_{D3} is describing and AdS_5 geometry, then $Ng_s \gg 1$, which maps to the planar limit in field theory. When $Ng_s \ll 1$, we do not trust the geometry.

In conclusion, we see that by warming up the D3 brane with $f(r)$, we can study the field theory and its properties (conductivity, transport coefficients ...) from weakly coupled strings in the $AdS_5 \times S^5$ black hole background. We could call this field "applied string theory". A lot of people nowadays thing of string theory no longer as a theory that describes the real world, but as a sort of calculator; a legitimate opinion.

We make two more remarks.

- The warp factor $Z = c + N/r^4$, where c is a constant. When $c = 1$ the metric describes a black *membrane* in ten dimensions, with flat asymptotics. When $c = 0$, the metric describes a black *hole* in AdS_5.

- All D-branes have such a non-extremal version. We can get for instance a black D2-brane very easily by T-duality. See exercise 3.1. Black Dp-branes all have

$$M > Q \qquad T_{\text{BH}} > 0\,, \tag{53}$$

and they are found from a deformation of the metric by one function $f(r)$ determined by

$$\Delta_d f = 0\,, \tag{54}$$

where d is the number of transverse dimensions.

Exercise 3.1: *T-dualize the metric* (45) *and four-form potential* (46) *of the black D3 brane along direction* x^3*. Show that this becomes a black D2-brane. In particular, repeat the mass calculation from the* g_{tt} *metric component and show that* $M = Q + \Delta M$*.*

3.2 Supersymmetric black holes in four dimensions

For the largest part of these lectures, we will concentrate on supersymmetric black holes. The reason is that when a black hole solution preserves supersymmetry, it can be constructed more easily and even be understood microscopically.

Consider again the orthogonal D2 brane system

IIA:	D2	0	1	2		
	D2	0			3	4

Normally any two objects that we put together would either attract or repel. However, this combination of D2-branes is supersymmetric and feels a flat potential: supersymmetry exactly amounts to cancelling forces and we can put them together in a stable fashion.

We can even do more. It is possible to add an extra D2 brane and even a D6 brane, while still preserving supersymmetry, in the following configuration:

IIA:	D2_1	0	1	2	–	–	–	–
	D2_2	0	–	–	3	4	–	–
	D2_3	0	–	–	–	–	5	6
	D6_4	0	1	2	3	4	5	6

This combination of branes experiences a flat potential and is stable. This follows from the supersymmetry it preserves. To show this, we would need to check the supersymmetry algebra; we will not do this in these lectures. For the black hole discussion, we smear the D2 branes on their transverse directions inside $x^1 \dots x^6$, which we denote by "–" and we number the branes from 1 to 4 for practical reasons.

The solution for the D2 – D6 system can actually be written down! The metric takes a very intuitive form:

$$\begin{aligned} ds^2 \quad &= -(Z_1Z_2Z_3Z_4)^{-1/2}dt^2 + (Z_1Z_2Z_3Z_4)^{1/2}(dx_7^2 + dx_8^2 + dx_9^2) \\ &+ \frac{(Z_2Z_3)^{1/2}}{(Z_1Z_4)^{1/2}}(dx_1^2 + dx_2^2) + \frac{(Z_1Z_3)^{1/2}}{(Z_2Z_4)^{1/2}}(dx_3^2 + dx_4^2) + \frac{(Z_1Z_2)^{1/2}}{(Z_3Z_4)^{1/2}}(dx_5^2 + dx_6^2) \, . \end{aligned} \tag{55}$$

This solution reduces to any single brane solution when only one of the four branes is present (for say Z_1 non-trivial, and the other ones constant, $Z_2 = Z_3 = Z_4 = 1$, we retrieve the metric of the D2-brane). Amazingly, this D2-D2-D2-D6 solution, which is constructed from the same "harmonic function rule" ($Z^{-1/2}$ metric component when the brane worldvolume is along that direction, $Z^{1/2}$ when it is orthogonal) applies to all of the Z_i individually, regardless of the presence of the other branes. This is a very non-trivial feature and would not happen for a generic solution; it is only for such a special class of supersymmetric solutions, that we get such a nice structure at the end of the day.

Exercise 3.2: *Consider the directions $x^1 \dots x^6$ to be compact (they become a six-torus, or T^6). T-dualize the D2-D2-D2-D6 metric 6 times along $x_1, \dots, x_6$. Write down the resulting D4-D4-D4-D0 metric.*

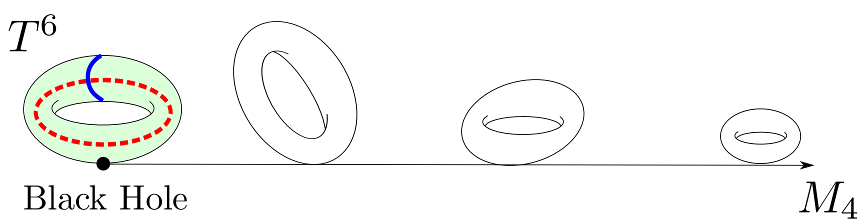


Figure 13: *Four-dimensional black hole from compactification on a six-torus. The T^6 has a different size and shape at each point of four-dimensional spacetime M_4. At the position of the black hole, there are branes present that are wrapped on T^6.*

What is the physics of this solution? In order to write down the solution, we need to smear the D2 branes uniformly along the transverse directions in T^6 (the compact directions $x^1 \dots x^6$).[9] This means we have to smear $D2_1$ along 3456, $D2_2$ along 1256 and $D2_3$ along 1234. Then all branes are points in three-dimensionsl space spanned by x^7, x^8, x^9. Therefore the warp factors Z_i obey:

$$\Delta_3 Z_i = 0 \qquad \rightarrow \qquad Z_i = 1 + \frac{Q_i}{r} \, . \tag{56}$$

We will show in the next subsection how the dimensionful charges Q_i are related to the integers N_i counting the number of D-branes.

[9] In order to build a black hole, we need to build all D-brane objects. If you T-dualize one D-brane, then the others become smeared in the other direction. To get a black hole, which looks similar in all duality frames, we need to work in a duality frame where the branes are smeared on orthogonal compact directions (we never get 'nice' D-branes).

Exercise 3.3: *Convince yourself that smearing a D-brane along a spacelike direction changes the radial dependence of Z in the correct way. For example, take a D2 brane along directions $1, 2$ and smear it along the circular dimension x^3 with a homogeneous denstity $\rho_{\text{smear}} \sim 1/R_3$, where R_3 is the radius of the 3-circle. Show that in this process, the solution to the Laplace equation becomes $Z = 1 + Q/r^3$ rather than $Z = 1 + Q/r^4$.*

What is our solution? We study the asymptotic limits.

- At $r \to \infty$: The metric becomes that of four-dimensional Minkowski spacetime times a flat torus fixed radii:

$$ds^2_{r\to\infty} = -dt^2 + ds^2(\mathbb{R}^3) + ds^2(T^6) \tag{57}$$

  This means we have compactified string theory on a six-torus to a four-dimensional theory. The next-to-leading term of the g_{tt} metric component is

$$g_{tt} = 1 - \frac{1}{2}\frac{Q_1 + Q_2 + Q_3 + Q_4}{r}\,. \tag{58}$$

  and we see that the mass of this solution is (up to factors we don't care about at this point)

$$M = Q_1 + Q_2 + Q_3 + Q_4\,. \tag{59}$$

  Again, this solution saturates the BPS bound and is extremal (the minimal amount of gravitational mass for the given charges): its mass is the sum of the charges of the individual branes; there is no binding energy. This is a consequence of supersymmetry.

- At $r \to 0$: all the 1's drop out of the warp factors Z_i and the metric becomes

$$\begin{aligned} ds^2_{r\to 0} \;&=\; -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}(dx_7^2 + dx_8^2 + dx_9^2) + \left(\frac{Q_2 Q_3}{Q_1 Q_4}\right)^{1/2}(dx_1^2 + dx_2^2) \\ &\quad + \left(\frac{Q_1 Q_3}{Q_2 Q_4}\right)^{1/2}(dx_3^2 + dx_4^2) + \left(\frac{Q_1 Q_2}{Q_3 Q_4}\right)^{1/2}(dx_5^2 + dx_6^2)\,. \end{aligned} \tag{60}$$

  The six-torus spanned by the directions $x_1 \ldots x_6$ has constant radii. If we go to polar coordinates in $\mathbb{R}^3$ spanned by x_7, x_8, x_9, then the metric is

$$ds^2 = \frac{r^2}{R^2}dt^2 + R^2\frac{dr^2}{r^2} + R^2 d\Omega_2^2 + ds^2(T^6)\,. \tag{61}$$

  The first two terms are the metric of AdS_2. The other terms describe an S^2 and T^6 of constant radii. Therefore, the $r \to 0$ limit of the D2-D2-D2-D6 spacetime describes a compactification of string theory on T^6 to the four-dimensional $AdS_2 \times S^2$ geometry.

  We also observe that $g_{tt} \to 0$ as $r \to 0$ and hence $r \to 0$ describes an event horizon.

From these facts we conclude that the metric of this D2-D2-D2-D6 brane system describes a real black hole in four dimensions.

Note that all the Q_i appearing in the metric are positive. Only the gauge fields (which we have not given) are sensitive to the sign of the charges, the gravitational field sourced by a positive or a negative charge is exactly the same. An anti-D2 brane would have the same metric and mass as a D2-brane, but opposite electric field.

To understand the full spacetime, we make our lives a bit easier and restrict to all charges equal:

$$Q_i \equiv Q\,, \qquad Z_i \equiv Z\,. \tag{62}$$

The black hole metric (55) becomes

$$ds^2 = -Z^{-2}dt^2 + Z^2 dr^2 + Z^2 r^2 d\Omega_2^2 + ds^2(T^6)\,. \tag{63}$$

The T^6 has a constant metric and does not play a role in the physics. We will not consider it further.

Our claim is that this metric represents a black hole. A very special one even, with $M = Q$. Let us go over the textbook black hole teachings to see if our claim is valid.

1. The first black hole you learn about in classical GR, is the Schwarzschild black hole. It is a solution to vacuum gravity, described by the metric

$$ds_{\mathrm{S}}^2 = -\left(1 - \frac{M}{\rho}\right) dt^2 + \left(1 - \frac{M}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\Omega_2^2\,. \tag{64}$$

   This is clearly not the same as our solution. We need to include a charge for the black hole.

2. Luckily there is also the second black hole you encounter in your favourite classical GR course. It is the Reissner-Nordström black hole. This black hole is a solution to the Einstein-Maxwell theory (see the Lagrangian (5)). It is given by

$$ds_{\mathrm{RN}}^2 = -\left(1 - \frac{2M}{\rho} + \frac{Q}{\rho}\right)^{-1} dt^2 + \left(1 - \frac{2M}{\rho} + \frac{Q}{\rho}\right) d\rho^2 + \rho^2 d\Omega_2^2\,. \tag{65}$$

   It has a very interesting limit

$$M = Q\,. \tag{66}$$

   Then the metric becomes

$$ds_{\mathrm{RN}}^2 = -\left(1 - \frac{Q}{\rho}\right)^{-2} dt^2 + \left(1 - \frac{Q}{\rho}\right)^{2} d\rho^2 + \rho^2 d\Omega_2^2\,. \tag{67}$$

   What does this have to do with our black hole metric, which has $g_{tt} = Z^{-2}dt^2 = 1/(1 + Q/r)^2$? Well, redefine

$$r = \rho - Q\,. \tag{68}$$

   Then we find the D-brane black hole solution on the nose!

These "$M = Q$" black holes are the ones we can construct most easily. They are extremal and have zero temperature:

$$T_{BH} = 0\,, \tag{69}$$

but have a non-zero mass M and entropy S_{BH}. The Bekenstein-Hawking entropy is given by the horizon area (we ignore numerical factors)

$$S_{BH} = A_H = 4\pi R^4 = \sqrt{Q_1 Q_2 Q_3 Q_4}\,, \tag{70}$$

or

$$S_{BH} = Q^2 \tag{71}$$

when all $Q_i = Q$.

All supersymmetric configurations are "frozen" to $T = 0$. We believe this entropy comes from some microscopic states. Who makes this entropy? – we will answer this below.

In ρ-coordinates ("isotropic coordinates"), this is clearly a black hole. The horizon is at $\rho = Q$ and the radial coordinate can be extended into the horizon all the way to the singularity at $\rho = 0$ without any problem. The coordinate r we used for the string theory metric is only well-defined outside the horizon ($r > 0$). Note that in the single D2-brane solution, the coordinate r is a measure for the distance from the brane and the brane was located at $r = 0$ (the same goes for D0, D1, D4, D5 and D6 brane solutions). For the supersymmetric black hole, space is "created" behind the $r = 0$ point and a large $AdS_2 \times S^2$ throat develops. The way to see this is through the ρ-coordinate, for which the metric extends from the horizon to the point $\rho = 0$, where the D-branes sit.

We come back to the regime of validity of supergravity. The curvature of a number N of D-brane goes as

$$\frac{1}{N g_s}\,. \tag{72}$$

Therefore the solutions we have considered are only valid when $N g_s \gg 1$ (small curvature, we can trust classical physics). Otherwise the solution is singular from the perspective of (super)gravity. This does not mean there is no D-brane solution. Think of classical electromagnetism. The electron is also a singular solution, but this gets resolved in the quantum theory. Similarly, string theory takes over for the quantum description of D-branes when $N g_s \sim 1$: string loops are suppressd by $g_s N$, rather than g_s. We will go into that in a later lecture.

We will refer to this four-dimensional black hole as the "four-charge black hole".

3.3 Supersymmetric black holes in five dimensions

For phenomenological and other existential reasons, we like four dimensions. Nonetheless, we make the switch to five dimensions, because five-dimensional black holes are easier to construct and analyze. Consider eleven-dimensional supergravity, with three orthogonal M2 branes as:

$M2_1$	0	1	2	–	–	–	–
$M2_2$	0	–	–	3	4	–	–
$M2_3$	0	–	–	–	–	5	6

We consider the directions $x^1 \ldots x^6$ to be compactified such that they again form a T^6. As for the four-dimensional black hole, the branes are smeared on their transvere directions on T^6, denoted by "–" in the table. Therefore the M2-branes are all pointlike in the transverse $\mathbb{R}^4$ spanned by x^7, x^8, x^9, x^{10} and the solution is determined by functions:

$$Z_1 = 1 + \frac{Q_1}{r^2}\,, \qquad Z_2 = 1 + \frac{Q_2}{r^2}\,, \qquad Z_3 = 1 + \frac{Q_3}{r^2}\,. \tag{73}$$

It turns out that for an eleven-dimensional supergravity solution, we can play the same game with the harmonic functions. The only difference is that different powers appear in the metric. When a brane is wrapped along a direction, we add a factor $Z^{-2/3}$ to the corresponding metric component, when the brane is transverse, we add $Z^{1/3}$. In particular, the supergravity solution for the (supersymmetric) M2-M2-M2 brane system is

$$\begin{aligned} ds^2 &= -(Z_1Z_2Z_3)^{-2/3}dt^2 + (Z_1Z_2Z_3)^{1/3}(dx_7^2 + dx_8^2 + dx_8^2 + dx_{10}^2) \\ &\quad + \frac{(Z_2Z_3)^{1/3}}{Z_1^{1/3}}(dx_1^2 + dx_2^2) + \frac{(Z_1Z_3)^{1/3}}{Z_2^{1/3}}(dx_3^2 + dx_4^2) + \frac{(Z_1Z_2)^{1/3}}{Z_3^{1/3}}(dx_5^2 + dx_6^2)\,. \end{aligned} \tag{74}$$

This solution describes a black hole in five spacetime dimensions. This can be seen from the limits

- $r \to \infty$: The metric describes a direct product of five-dimensional flat Minkowski spacetime with a six-torus with constant radii. This is a compactification of flat eleven-dimension spacetime to five dimensions.

- $r \to 0$: This is the horizon of the black hole. Write the transverse $\mathbb{R}^4$ metric in polar coordinates $dx_{78910}^2 = dr^2 + r^2 d\Omega_3^2$. Then for $r \to 0$, the metric looks like

  $$ds^2 = -\frac{r^4}{(Q_1Q_2Q_3)^{2/3}}dt^2 + (Q_1Q_2Q_3)^{1/3}\frac{dr^2}{r^2} + (Q_1Q_2Q_3)^{1/3}d\Omega_3^2 + ds^2(T^6)\,, \tag{75}$$

  where the last term describes a torus of constant radii. By the coordinate redefinition $\rho = r^2$, we see that the first two terms form the metric of AdS_2 ($g_{tt} \to 0$ and $g_{rr} \to \infty$ in such a way to give AdS_2) and the S^3 has constant radius. Hence the near-horizon geometry is $AdS_2 \times S^3 \times T^6$.

  (Note that we have encountered many examples of $AdS_p \times S^q$ geometries from D-branes: $AdS_5 \times S^5$ from the D3 brane, $AdS_2 \times S^2 \times T^6$ from D2-D2-D2-D6, $AdS_2 \times S^3 \times T^6$ from M2-M2-M2. We will later also encounter $AdS_3 \times S^3 \times T^4$.)

Entropy in gory detail. We want to give the exact expression for the Bekenstein-Hawking entropy of the black hole. This entropy is proportional to the horizon area in Planck units, or more precisely

$$S_{BH} = \frac{A_H}{4G_N}\,. \tag{76}$$

Note that this looks independent of the dimension. However, if we use the horizon area in D dimensions, we should also use Newton's constant in D dimensions.

Let's get our hands dirty and derive this beast. At $r \to 0$, the horizon area of the eleven-dimensional metric is really the area of $S^3 \times T^6$ at $r = 0$ (area of constant r in the eleven-dimensional metric):

$$A_H = \int_{S^3 \times T^6} \sqrt{g} = \int_{S^3} \sqrt{g_{S^3}} \int_{T^6} \sqrt{g_{T^6}}\,. \tag{77}$$

The area of the S^3 in the metric (75) is:

$$\int_{S^3} \sqrt{g_{S^3}} = (\sqrt{(Q_1 Q_2 Q_3)^{1/3}})^3 \Omega_3 = 2\pi^2 \sqrt{Q_1 Q_2 Q_3}\,, \tag{78}$$

where $\Omega_3 = 2\pi^2$ is the area of a three-sphere with unit radius. The area of the T^6 for the metric (75) is

$$\begin{aligned}\int_{T^6} \sqrt{g_{T^6}} &= \int dx_1 \dots dx_6 \sqrt{\frac{(Z_2 Z_3)^{1/3}}{Z_1^{1/3}} \frac{(Z_1 Z_3)^{1/3}}{Z_2^{1/3}} \frac{(Z_1 Z_2)^{1/3}}{Z_3^{1/3}}} \\ &= \prod_{i=1}^{6} 2\pi L_i\,,\end{aligned} \tag{79}$$

where L_i are the radii of the x_i circles.[10]

We want to express the entropy in terms of a dimensionless number, that can be related to a number of microstates. Before we can continue, we thus have to find the exact relation of the supergravity charges Q_i to the actual integer numbers that count the number of M2 branes of type i that source the supergravity solution. So far, we have been sloppy with the distinction between the supergravity charges Q_i (with dimensions of length squared and appearing in the functions Z_i) and the actual brane numbers N_i. All numerical factors in the exact relation $Q_i = (\dots)N_i$ are extremally important: these will become prefactors in the entropy, which is exponentiated to get the number of black hole microstates. A mistake of a factor of 2 in a number as e^N or e^{2N} has huge consequences!

To find the relation between Q_i and N_i, we first consider the gauge fields of the solution. These are given by

$$C_{012} = Z_1^{-1}\,, \qquad C_{034} = Z_2^{-1}\,, \qquad C_{056} = Z_3^{-1}\,. \tag{80}$$

Remember that Q_i represent *densities* of M2-branes, smeared on some directions. For instance, Q_1 describes the density of N_1 M2-branes smeared on the directions x^3, x^4, x^5, x^6. Hence on general grounds, we expect that such a density should scale as

$$Q_1 = \frac{N_1}{L_3 L_4 L_5 L_6} (\dots)\,. \tag{81}$$

The exact coefficient $(\dots)$ is left to as a homework problem in Exercise 3.4.

[10] The T^6 radii L_i are defined by identifying the x_i periodically as $x_i = x_i + 2\pi L_i$.

Exercise 3.4: *The number of M2-branes can be read of by integrating its magnetic gauge field strength over a surface that surrounds the M2-branes as:*[11]

$$(2\pi\ell_P)^6 N_{M2} = \int_{\Sigma_7} F^7 \,, \tag{82}$$

where ℓ_P is Planck's constant (in eleven dimensions), Σ_7 is the surface surrounding the M2-branes:

$$\Sigma^7 = T^4_{3456} \times S^3 \,, \tag{83}$$

and the magnetic seven-form gauge field is found from the Hodge dualization relation $F_7 = star_{11} F_4$, which is written out as

$$F_{i_1 \dots i_7} = \frac{1}{4!}\sqrt{-g}\,\epsilon_{i_1 \dots i_7; j_8 j_9 j_{10} j_{11}} g^{j_8 j_8'} g^{j_9 j_9'} g^{j_{10} j_{10}'} g^{j_{11} j_{11}'} F_{j_8 j_9 j_{10} j_{11}} \,. \tag{84}$$

Take the metric (75) *and the four-form field strength $F_4 = dC$ with components*

$$F_{012r} = \partial_r C_{012} = \partial_r Z_1^{-1} \,, \tag{85}$$

and analog for F_{034r} and F_{056r}. Calculate the dual seven-form and use (82) *to express the charges Q_i in terms of the integers N_i that count the number of branes. Show that the exact relation is*

$$Q_1 = \frac{N_1 (\ell_P)^6}{L_3 L_4 L_5 L_6} \,, \qquad Q_2 = \frac{N_2 (\ell_P)^6}{L_1 L_2 L_5 L_6} \,, \qquad Q_3 = \frac{N_3 (\ell_P)^6}{L_1 L_2 L_3 L_4} \,, \tag{86}$$

where L_i are the radii of the circles at infinity.

We continue with the horizon area. It is given in terms of the charges as

$$A_H = 2\pi^2 \sqrt{Q_1 Q_2 Q_3} \prod_{i=1}^{6} (2\pi L_i) \,. \tag{87}$$

By subsituting the result from Exercise 3.4, equation (86), we get

$$A_H = 2\pi^2 (2\pi)^6 (\ell_P)^9 \sqrt{N_1 N_2 N_3} \,. \tag{88}$$

Before we evaluate the entropy $S_{BH} = A_H / 4G_N$, we give the definition of the Planck length in terms of Newton's constant for any dimension D:

$$16\pi G_N = (2\pi)^{D-3} \ell_P^{D-2} \,. \tag{89}$$

Plugging this into $S_{BH} = A_H / 4G_N$, we conclude that the entropy of the black hole is

$$\boxed{S_{BH} = 2\pi \sqrt{N_1 N_2 N_3}} \,. \tag{90}$$

[11]Remember the analogy with electromagnetism, for a magnetic monopole we find $N \sim \int_{S^2} F^2$.

A few remarks are in order. Note that we began with D-branes on a torus. As the torus gets smaller or larger (by changing L_i), the solution changes drastically. We get different black holes because the Q_i's change. But: the entropy does not care whether the black hole is of size 1 mm or 1 km long. This is a very interesting fact: S_{BH} does not change as you change the torus. For a non-supersymmetric solution, you would expect that the entropy depends on the parameters of the torus; the invariance of S_{BH} under variations of the torus radii is a feature due to supersymmetry.

We can use this feature to do dualities on the internal torus. The five-dimensional black hole will have the same entropy, but the black hole can be made up out of different branes in some other stirng theory. Take the duality chain: (1) reduce along x_6 to IIA, (2) two T-dualities along x_1, x_2, (3) a T-duality along x_5:

			IIA:									**IIA:**									**IIB:**	
D2	0	1	2	–	–	–	T	D0	0	–	–	–	–	–	T	D1	0	–	–	–	–	5
D2	0	–	–	3	4	–	$\longrightarrow$	D4	0	1	2	3	4	–	$\longrightarrow$	D5	0	1	2	3	4	5
F1	0	–	–	–	–	5	$x_{1,2}$	F1	0	–	–	–	–	5	x_5	p	0	–	–	–	–	5

(For this T-dualization, you need to know that for F1's that do not wrap the T-duality circle, nothing happens at all: they remain F1's.) The end result of this little exercise is an intersection of D5 branes with D1 branes and momentum along the common direction. This is the celebrated D1-D5-P system. We will mainly study the three-charge black hole in five-dimensions in this duality frame.

Exercise 3.5: *Use the T-duality chain from the M2-M2-M2 system to the D1-D5-P frame to show that the metric becomes*

$$ds^2 = -(Z_1Z_5)^{-1/2}dt^2 + (Z_1Z_5)^{-1/2}Z_p^{-1}dx_5^2 + (Z_1Z_5)^{1/2}dx_{78910}^2 + (Z_1Z_5)^{-1/2}dx_{1234}^2 \, . \tag{91}$$

When $p = 0$, there is no momentum charge. The metric only depends on the functions

$$Z_{1,5} = 1 + \frac{g_s N_{1,5}}{r^2} \, . \tag{92}$$

In the limit $r \to 0$, we can drop the 1's in the harmonic functions and the metric becomes $AdS_3 \times S^3 \times T^4$, see Exercise 3.6. Also this geometry is very useful for holography. String theory on this background is dual to a $1+1$ dimensional CFT.

Exercise 3.6: *Show that for $r \to 0$, the metric* (91) *with $p = 0$ ($Z_p = 1$), the metric becomes*

$$ds^2 = r^2(-dt^2 + dx_5^2) + \frac{dr^2}{r^2} + d\Omega_3^2 + ds^2(T^4) \, , \tag{93}$$

where the last term describes the metric on a T^4 with constant radii.

But ... there is a "but": the horizon area is zero. This can be seen from our earlier result that $S_{BH} \sim \sqrt{N_1 N_2 N_3}$, or from the metric (93). One can show that the Ricci scalar in five-dimensional spacetime blows up at $r = 0$ and the horizon coincides with the a curvature singularity.

When $p \neq 0$, the entropy is

$$S_{BH} = 2\pi \sqrt{N_1 N_5 N_p}\,. \tag{94}$$

String theory on the near-horizon region is dual to the same $1+1$ dimensional CFT as for the $p = 0$ solution, but now there is a non-trivial momentum in the game. See Table 6.

Table 6: *Near-horizon geometry and dual CFT of D1-D5 system*

Near horizon of D1-D5	**Dual CFT**
with $p = 0$	$1+1$ dim. CFT
with $p \neq 0$	$1+1$ dim. CFT with some momentum

The D1-D5-P black hole entropy comes from the many ways in the which the CFT can carry this momentum p. This result is proven in the next section. It is *most amazing*: the entropy of a black hole is recovered from a $1+1$ dimensional CFT!

4 Black Hole Microscopics

To properly account for the entropy of the black hole we first have to learn some very basic string theory. In the spirit of the rest of the lectures we'll eschew any details we don't need and ask the reader to trust us since we're supposed to be experts.

We explain how to derive the black hole entropy from a microscopic counting of states for:

1. **D1-D5-P black hole** (also "three-charge black hole") with entropy:

$$S_{BH} = 2\pi\sqrt{N_1 N_5 N_p}\,. \tag{95}$$

2. **D6-D2-D2-D2 black hole** (also: "four-charge black hole") with entropy:

$$S_{BH} \sim \sqrt{N_{\mathrm{D2}} N_{\mathrm{D2}} N_{\mathrm{D2}} N_{\mathrm{D6}}}\,. \tag{96}$$

   When all the charges above are equal this black hole has a very nice interpretation as the extremal Reissner-Nordtström black hole in four dimensions.

We will discuss the three-charge black hole first. Historically, this was the first black hole for which a microscopic counting was done that could explain the entropy (by Strominger and Vafa). We will treat the four-charge black hole in four dimensions afterwards. It is the latter one which may have more appeal, as it describes the extremal black hole of Eisntein-Maxwell theory in four dimensions ('extremal Reissner-Nordström black hole').[12]

4.1 A Brief Review of Open and Closed String theory

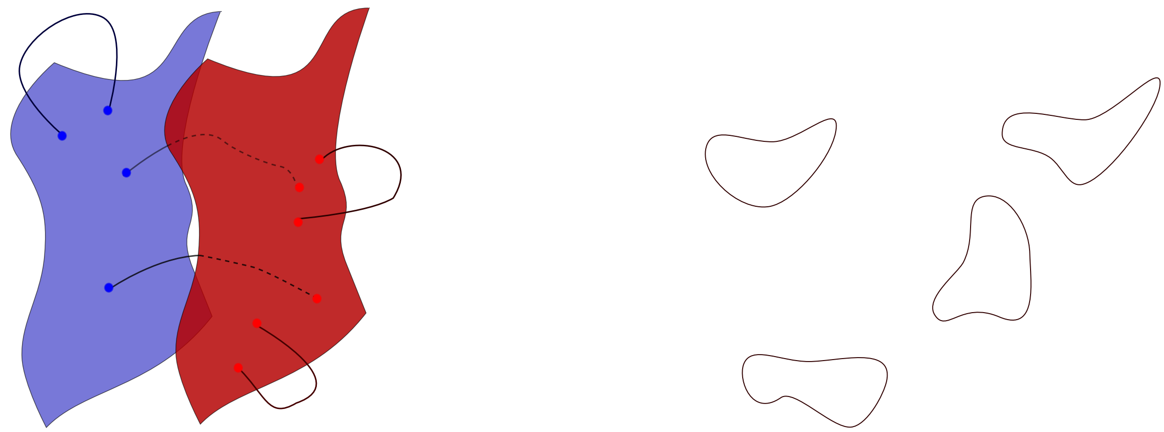
String theory is a theory of (surprise, surprise:) strings. Strings come in two types: closed strings form closed loops in spacetime with no end-points (imagine rubber bands floating around in spacetime), while open strings have two ends (imagine a strand of rope streched between. . . between what?), see Figure 14.

In general the ends of open strings are not free to move in all directions of spacetime but are constrained to lie along higher dimensional "membranes". It turns out that these membranes are nothing more that the D-branes we found before as solutions to supergravity! Although it is hard to see why this is the case we will try to argue it briefly later.

Scales and limits. One of the nice features of string theory is that it very naturally introduces a new length scale, ℓ_s, the string length. Thi sis becuase fundemental strings (like all strings) have a tension, τ_{F1}, and this can be defined in terms of the string length, ℓ_s, a new fundemental length scale defined by this tension,

$$\tau_{F1} = \frac{1}{\ell_s^2}\,. \tag{97}$$

[12]This does not mean that it is a realistic astrophysical black hole. In nature, black holes will shed (almost) all there charge and be charge neutral. Supersymmetric black holes are extremal; they have the maximum amount of charge allowed for their mass and are hence not the black holes we observe in the sky.



(a) Open strings end on D-branes.

(b) Closed strings propagating in spacetime.

Figure 14: *Strings can be of two types, depending on the boundary conditions we put on the string: closed or open strings. The end-point of open strings are confined to D-branes.*

Note that the length dimension of τ_{F1} is defined so that integrating the tension over a one-dimensional volume yields a unit of mass, namely the mass of a string.

Oscillations on a world-volume of a string have an energy cost dependent on the string tension just like a regular guitar string. The mass of the harmonic modes is quantized in units of the string mass

$$M_s \propto \frac{1}{\ell_s}\,. \tag{98}$$

When this value is large then stringy modes are very massive and we can, to a good approxi-

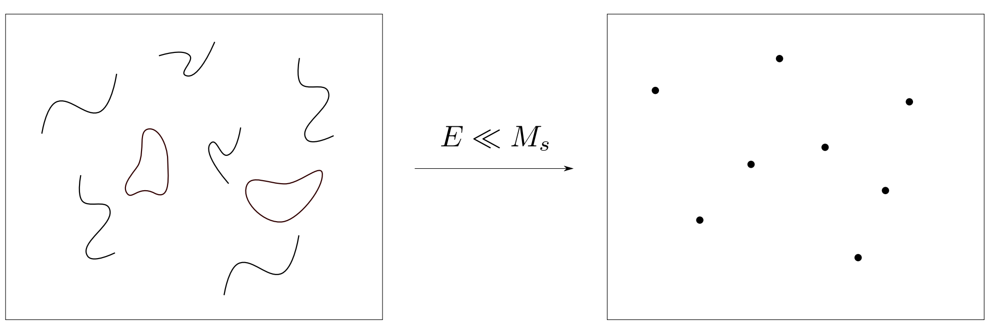


Figure 15: *When the string scale is large than the ernergy of excitations $M_2 \gg E$, those stringy excitations look like point particles.*

mation, restrict ourselves to only the lowest lying sector corresponding to massless strings, see Figure 15. In this limit when the string mass is very large and only a few modes remain strings essentially look like point particles and (owing to the various possible massless oscillations possible) generate a spectrum of fields in spacetime, see Table 7.

String type	Spacetime fields generated
closed	$g_{\mu\nu}, B_{\mu\nu}, C^{(3)}_{\mu_1\mu_2\mu_3}, \psi^{\alpha}_{\mu}, \ldots$
open	$A_\mu, \phi, \psi_\alpha, \ldots$

Table 7: *The massless spectrum of closed and open strings.*

Even though we will not generally need the details of this spectrum, it is important to realize that the closed string spectrum generates supergravity with the associated fields. Open strings, on the other hand, are described at low energy by a gauge theory since A_μ has the degrees of freedom to be a gauge field (coupled to matter and fermions). This theory however does not live on all of spacetime but only on the D-branes on which the open strings are restricted to end.

In any gravity theory, including string theory, there is a fundamental length scale related to Newton's constant and the strength of gravitational interactions: the Planck scale. This is set by the Planck length ℓ_P, through the relation with Newton's constant (in D dimensions of spacetime):

$$G_N = (2\pi)^{D-3}(\ell_P)^{D-2}\,. \tag{99}$$

The introduction of a second fundamental length scale in string theory, ℓ_s, means that string theory has an associated dimensionless constant, the string coupling or g-string

$$g_s = f\left(\frac{\ell_s}{\ell_P}\right)\,. \tag{100}$$

The exact dependence can be extracted from the definition of G_N

$$G_N \propto \ell_P^{D-2} \propto g_s^2\,\ell_s^{D-2}\,, \tag{101}$$

and we find

$$g_s \propto \frac{\ell_s}{\ell_P}\,. \tag{102}$$

From this we see that g_s controls the hiarachy of scales in string theory. When $g_s \ll 1$ we have $\ell_s \ll \ell_P$ so stringy excitations are much less massive than the plank scale and we can do "classical string theory". On the other hand when $g_s \gg 1$ then any stringy excitations is more massive than the plank scale and thus highly quantum. Therefore g_s acts as a *dimensionless* coupling in string theory telling us when the theory can be treated classically versus when it is necessarily strongly coupled.

While tuning g_s puts us in a theory with a certain string and plank scale we have a further freedom to choose the energy scale at which we probe this theory. In any given physical process there is an associated dimensionalful energy scale such as e.g. the mass of the heaviest particle we consider, the energy of a scattering process, etc.... Thus even if we choose $g_s \ll 1$ we have the further freedom to consider only processes with E restricted to

$$E \ll M_s \ll M_P\,, \tag{103}$$

which means that the scale of our physics is smaller than the string scale (which in turn is lower than the plank scale). The limit $M_s \ll M_P$ (or $g_s \ll 1$) means that string perturbation theory is valid and that we can look at classical string theory. The limit $E \ll M_s$ means that strings effectively look like point particles (we only look at those excitations that our very low energy compared to the scale set by the string length and we cannot distinguish the stringy nature of the string). This is the limit in which we will often find ourselves.

String perturbation theory. Above we motivated g_s as a dimensionless coupling emerging from comparing the dimensionful ℓ_s and ℓ_P but within string theory this can actually be derived. String perturbation theory is described in terms of the mathematical "genus" of the string world-sheet (the two dimensional submanifold describing the strings path in spacetime). Let's take a look at the loop expansion of a string process. As a Feynman diagram represents the worldlines of in- and outgoing particles and intermediate processes (propagators, loops), a string diagram represents the worldvolume of a string.

We represent perturbation theory for an ingoing closed string to an outgoing closed string in Figure 16, which explaines visually the genus expansion. For every number of loops, there is exactly one type (topology) of string worldvolume.

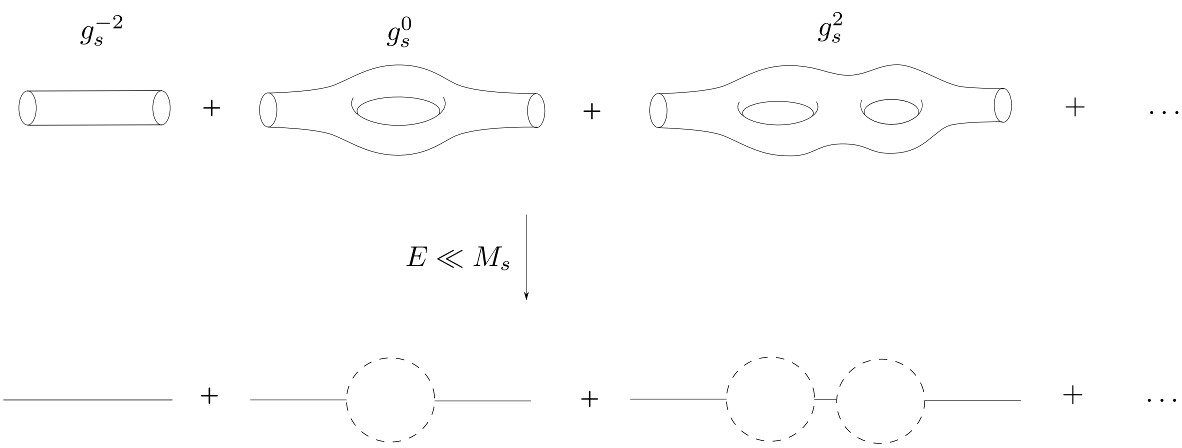


Figure 16: *String perturbation theory is a genus (# holes) expansion of string world-sheets. For closed strings, every hole introduces a factor of $(g_s)^2$ in the expansion. For excitations well below the string scale, strings behave like particles and we recover ordinary Feynman diagrams.*

Every loop in a closed string diagram introduces an extra factor of $(g_s)^2$. The limit where $g_s \to 0$ suppressed the loops and hence also quantum effects: this is the classical limit. If we further impose the extra "low-energy limit" $E \ll M_s$, such that the strings look like particles then the string diagrams reduce to standard Feynman diagrams because in this limit we send $\ell_s \to 0$ so the worldsheet compresses down to a world-line (see also Figure 16).

We will generally work in this regime and keep only the zero-mass excitations of the string[13]. Then the closed string gives exactly the fields of supergravity, see Table 7: the metric, dilaton and B-field and the gauge potentials that we have seen when discussing D-branes (Ramond-Ramond

[13]These are not the lowest-energy modes of the string. Those are tachyonic (negative energy) modes, that can be consistently projected out of the spectrum of string theory.

fields). Thus one can think of supergravity as the *low energy limit* of *weakly coupled* string theory (and indeed this is where we will mostly be working).

What about open string perturbation theory? Open strings strech between D-branes and their end-points are labelled by the branes they end on. Thus open string perturbation theory gains an additional factor, N, the number of D-branes, from the degeneracy of open string considered in any scattering process (see Figure 17). Thus the perturbation series is a power series in $g_s N$. This is similar to the expansion in a gauge theory with N_c colors, where we get an expansion in powers of gN_c, and indeed as we will see below this resemblance is no accident.

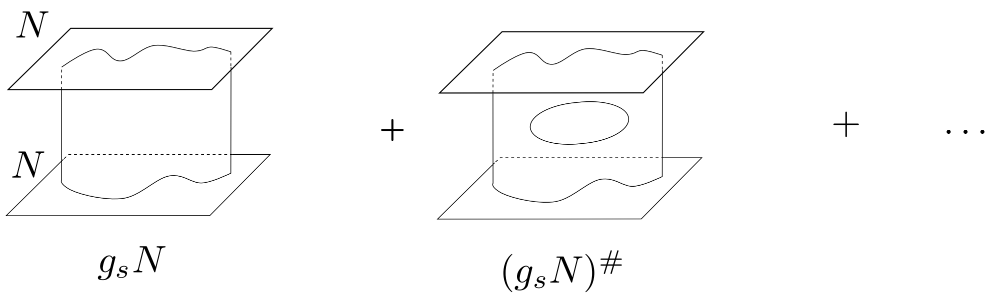


Figure 17: *Open string perturbation theory is an expansion in $g_s N$, where N is the number of branes.*

The low lying (massless) sector of the open strings are a vector field A_μ, a number of spinors ψ^α (fermions) and scalar fields ϕ^i, see Table 7. These fields are bound to the brane, because the open string endpoints are. They can be interpreted as describing the D-brane dynamics: the scalars describe the transverse motion of the brane (there is one scalar for every direction transverse to the brane worldvolume), the vector (which has only directions on the worldvolume) describes a gauge theory living on the brane and the fermions are needed for supersymmetry. Note that if we only consider open strings, we cannot get a metric: a metric (gravitons) sits only in the closed string spectrum.

Questions from the audience:

☞ *Have we not introduced an cut-off E by restricting our energies to $E \ll M_s$.* No because what we mean by $E \ll M_s$ is that we consider only massless excitations so the cuttoff is actually $E \sim 0$. We assume that any dynamics or additional scales we introduce will be small with respect to M_s unless we explicitly state otherwise. The number of massless excitations can be very large: for open strings on N D-branes, we get a $U(N)$ gauge theory, which has many (massless) fields.

☞ *Why and how do open strings leave a D-brane?* We have not yet said what closed strings do with a D-brane. Figure 19 shows the process by which a closed string leaves a D-brane.

The gauge/gravity duality we mentioned before, is really an open/closed string duality. The theory living on the worldvolume of a string (the so-called worldsheet theory) which describes the propagation of a string in spacetime has a symmetry allowing us to interchange proper time (τ) and proper length (σ) on the worldsheet (it is a symmetry of the string itself). Then a loop diagram in open string theory, looks like a tree level diagram describing the exchange of closed strings between two D-branes, see Figure 18. We will return to this later.

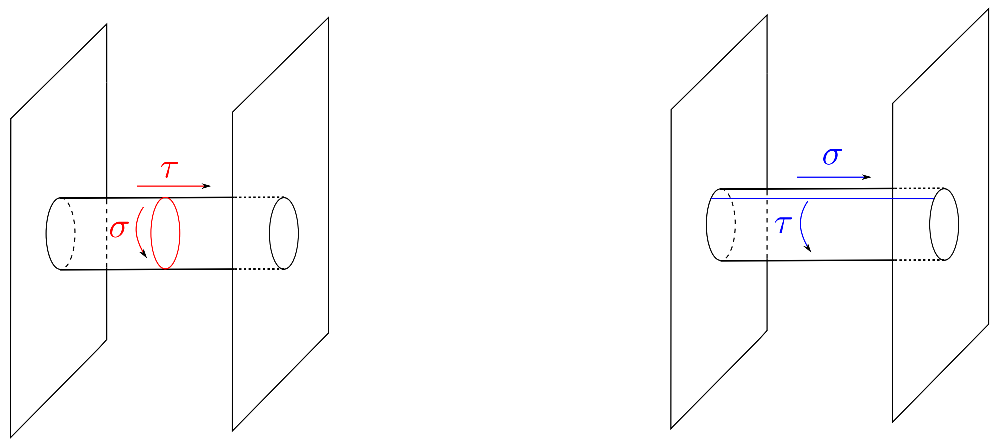


Figure 18: *By exchanging the role of string time (τ) and length σ, we can interpret this diagram as an exchange of closed string between D-branes (left), or a loop diagram in open string theory (right).*

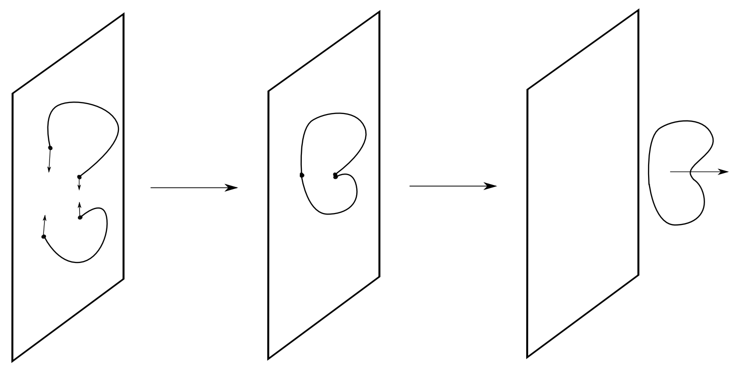


Figure 19: *Interpretation of a closed string leaving from a D-brane from open string interaction. Note that each interaction (each pair of end points joining) introduces a factor of g_s in the amplitude of this process.*

From Figure 19, we see that a process of a closed string interacting with a D-brane has a factor of $(g_s)^2$: we can see this as a graviton exchange. This explains why $G_N = (g_s)^2$.

For the discussion of the black hole entropy, we will take the limit $g_s \to 0$. In this limit, open and closed strings naively decouple, since their interaction (Figure 19) is suppressed. Note however that the open-closed diagram receives an enhancement from the degeneracy of open strings so the final effective coupling controlling the interaction of closed and open strings will be $g_s N$, the same coupling that governs interactions between open strings. Thus by taking $g_s \to 0$ but with $g_s N$ fixed we can supress quantum gravity effects but still allow open-strings, or D-branes, to source closed strings (yielding the supergravity solutions described in previous sections). We will return to this later.

The stringy D1-D5-P black hole. We consider the D1-D5-P system along the following directions. The D5 branes are on compact directions in spacetime, the D1 and the momentum are along one of the directions of the D5:

		T^4				S^1	$\mathbb{R}^4$			
	0	1	2	3	4	5	6	7	8	9
$D5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$				
$D1$	$\times$					$\times$				
P	$\times$					$\times$				

We can picture this as in Figure 20.

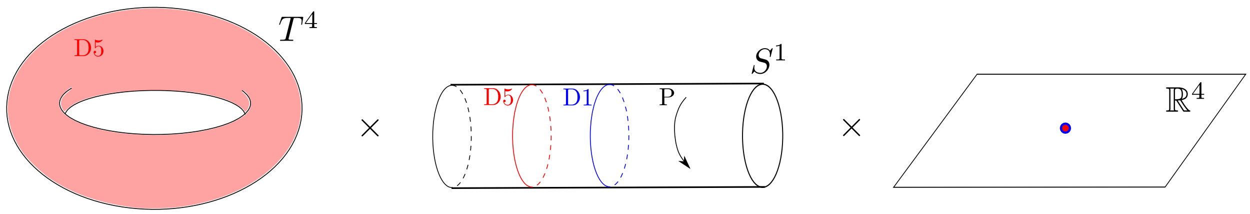


Figure 20: *The D1-D5-P system. The D5's are wrapped on $T^4 \times S^1$, along the S^1 we also wrap D1's and we put gravitational waves (momentum), denoted P.*

Question from the audience:

☞ *What is "P", the momentum, exactly?* We can see this by a manipulation of the metric (91). By changing coordinates, $x_- \to x_5 - t$, the metric looks like

$$\begin{aligned} ds^2 &= -(Z_1 Z_5)^{-1/2} dt^2 + (Z_1 Z_5)^{1/2} dx_-^2 + Z_p^{-1} dt dx_- \\ &\quad + (Z_1 Z_5)^{1/2} (dr^2 + r^2 d\Omega_3^2) + ds^2(T^4) \,, \qquad Z_i = 1 + \frac{Q_i}{r^2} \,. \end{aligned} \tag{104}$$

The angular momentum of this solution is related to the mixed time-space components of the metric, in this case $p \sim \partial/\partial x^5$ is given by the $1/r^2$ term $Z_p^{-1} = 1 - Q_p/r^2 + \ldots$, so Q_p is indeed the momentum charge.

Remember that the supergravity charges are actually charge densities (we omit numerical factors):

$$Q_1 \sim g_s (\ell_s)^2 N_1 \qquad Q_5 \sim g_s (\ell_s)^2 N_5 \,, \qquad Q_p \sim g_s^2 N_p \,. \tag{105}$$

The horizon area depends on the string length and the string coupling:

$$A_H \sim \sqrt{Q_1 Q_5 Q_p} \sim g_s^2 (\ell_s)^3 \sqrt{N_1 N_5 N_p} \,, \tag{106}$$

but the Bekenstein-Hawking entropy is independent of the coupling and length scales:

$$S_{BH} = \frac{A_H}{4 G_N} = 2\pi \sqrt{N_1 N_5 N_p} \,. \tag{107}$$

The scale of the supergravity solution is set by the charges Q_i in the warp factors. Remember that the supergravity charges appear as $Z = 1 + Q_i/r^2$ and they determine the size of the solution.

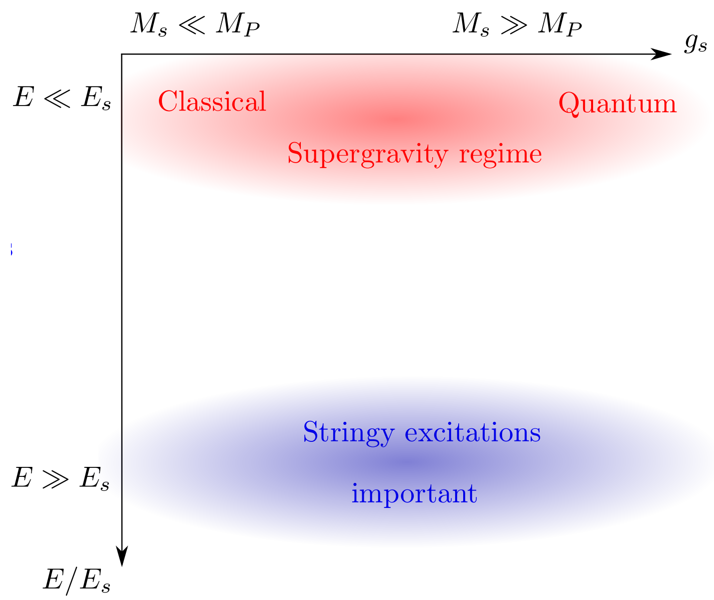


Figure 21: *Tuning the coupling in closed string perturbation theory. Kepping only the low eenergy (zero mass) modes, we have a theory of particles, supergravity. We restrict to small g_s, and only consider* classical *supergravity.*

In general, we have $Q_i = G_N M_i$, see Equations (41)–(43). For a D-brane, we have $M_D = N/g_s$ and hence $Z \sim g_s N_D$, for the momentum excitations, we have $M_p = N_p$ (just an excitation) and hence $Z \sim g_s^2 N_p$. If $g_s N$ is small, the area of the black hole is small in string units. Hence we cannot use supergravity to describe it: massive string modes become important, and supergravity only describes the *massless* modes. This violates our earlier physical requirement $E \ll M_s$ (put another way such black holes would involve curvature at the string length and thus probing them would involve energies at this scale). We see that we need the horizon to be large in string units to describe (super)gravity black holes and we require:

$$g_s N \gg 1 \,. \tag{108}$$

We subsequently impose $g_s \to 0$. Closed string theory is non-interacting in this regime, with $M_s \ll M_p$. This suppresses quantum gravity corrections (whether you are in string theory or in gravity), at low energy a closed string loop looks like a gravtion loop, see Figure 21.

For a large black hole, we demand that A_H is large in string units, $A_H \gg (\ell_s)^3$, or through (106), $N_1 N_5 N_p$ should be very large. By dialing the number $g_s N$, we can interpolate between the near-horizon region of the black hole $AdS_2 \times S^3$ (large $g_s N$) and a picture where the open string theory is perturbative ($g_s N \ll 1$), this is the setup of Figure 22

Because the entropy is independent of the coupling, we expect to be able to reproduce the entropy from a counting of supersymmetric states in the weakly coupled open string picture. Note that we keep $g_s \to 0$ throughout this diagram, also in the open string picture, such that there is no interaction between the closed and open string sector (since closed string perturbation theory goes with powers of g_s rather than $g_s N$, D-branes do not source gravitons in this limit). Therefore we can consider open strings on D-branes in flat spacetime.

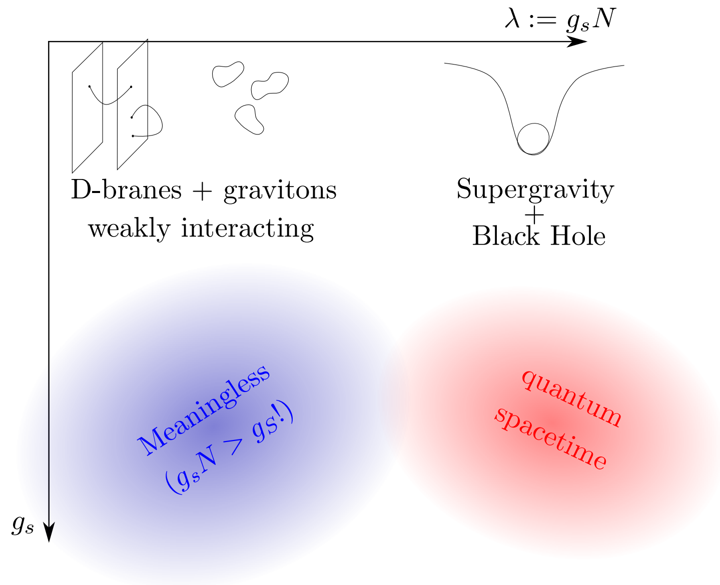


Figure 22: *By dialling the coupling $g_s N$ (while keeping g_s small), we can interpret the D1-D5-P system as a black hole or as open strings stretching between D-branes. Since the torus volume goes as $V_{T^4} \sim Q_1/Q_5$ in string units, it disappears from the picture and we only retain the five-dimensional geometry.*

We summarize:

- If $g_s \to 0$, you alway suppress closed string loop effects (quantum gravity effects)
- $g_s N$ tells you how much strings feel the source. Tuning $g_s N$ is turning open string loop effects on/off.
- If $g_S N \ll 1$, you can count the number of states of these strings stretching between the D-branes, because essentially we get a free theory (loop effects suppressed). This is reminiscent of holography, where we have $g_s N \ll 1$ giving yang-Mills weakly coupled, no gravity, and $g_s N \gg 1$ giving Yang-Mills strongly coupled, or AdS_5 gravity.

Note that if the entropy would depend on g_s, then none of this would make sense. A toy model will follow with a rigourous proof that it is g_s is independent.[14]

A question from the audience:

☞ *Can we get the gravity solution from open string calculations?* Yes you can, but it's a pain. Say we want to find the metric. You can expand the solution in the open string coupling $g_s N$

$$g_{tt} = (Z_1 Z_5)^{-1/2} \sim 1 + g_s N + (g_s N)^2 + g_s^3 + \dots \tag{109}$$

The one-loop computation is doable and has been done (Stefano Giusto, a former postdoc at IPhT is doing this). Higher loops are extremely tough; solving supergravity equations of motion is much simpler.

[14] Extrapolating from toy models is many a string theoriest's idea of a mathematical proof of complicated string theory effects.

4.2 Supersymmetric indices

We have just argued that the D1-D5-P system looks like a black hole for $g_s N \gg 1$, and like a system of very weakly coupled strings for $g_s N \ll 1$. We want to count the states that make up the entropy in the weakly coupled theory. Why can we trust such a computation? We answer that through a toy model.

Consider supersymmetric quantum mechanics. It is defined by the Hamiltonian

$$H = \{Q, Q^\dagger\} \equiv Q^\dagger Q + Q Q^\dagger \, . \tag{110}$$

The operator Q is fermionic, and anticommutes with itself:

$$\{Q, Q\} = 2Q^2 = 0 \tag{111}$$

We define BPS states (or "supersymmetric states") as states that are annihilated by Q, but are not given by acting with Q on another state (Q-closed but not Q-exact):

$$|\psi\rangle_{\text{BPS}} : \qquad Q|\psi\rangle_{\text{BPS}} = 0 \, , \quad |\psi\rangle_{\text{BPS}} \neq Q|\psi'\rangle \, . \tag{112}$$

Exercise 4.1: *Prove the following properties:*

1. *The Hamiltonian H has only positive eigenvalues. Show that BPS states are states of minimal (zero) energy:*

$$H|\psi\rangle_{\text{BPS}} = 0 \, . \tag{113}$$

2. *Let $|\phi\rangle$ be a non-BPS state. Prove that ϕ is degenerate to*

$$|\phi'\rangle = Q|\phi\rangle \, , \qquad E_\phi = E_{\phi'} \, . \tag{114}$$

Introduce the operator $(-1)^F$, defined through its action on bosonic and fermionic states as:

$$(-1)^F|\text{boson}\rangle = |\text{boson}\rangle \, , \qquad (-1)^F|\text{fermion}\rangle = -|\text{fermion}\rangle \, . \tag{115}$$

This operator $\mathbb{Z}_2$-grades the Hilbert space. Note that it anticommutes with the operator Q:

$$\{(-1)^F, Q\} = 0 \, . \tag{116}$$

Define the Witten index

$$Z = \text{Tr}\,[(-1)^F e^{-\beta H}] \, , \tag{117}$$

where β is a number.

3. *Show that*

$$Z = (\#\text{ bosonic BPS states}) - (\#\text{ fermionic BPS states}) \tag{118}$$

4. . Show that

$$\frac{\partial Z}{\partial \beta} = 0\,. \tag{119}$$

5. Redo the calculation with the Hamilatonian

$$H = H_0 + gH_1\,, \tag{120}$$

where both the original Hamiltonian H_0 and the perturbed Hamiltonian H obey the supersymmetry property

$$H_0 = \{Q_0, Q_0^\dagger\}\,, \qquad H = \{Q, Q^\dagger\} \tag{121}$$

for two different fermionic operators Q_0, Q. Show that the function Z is independent of g.

In this exercise, you have proven that the Witten index, which counts the difference in the number of bosonic and fermionic ground states, is independent of the coupling g. The key thing to note is that at strong coupling, the total number of ground states at strong coupling is equal to the Witten index. By its independence on the coupling g, we can calculate the Witten index at weak coupling to count the number of ground states at strong coupling.

We rephrase that in a more mathematical language. Define the trace over the BPS Hilbert space:

$$Z_{\mathrm{BPS}} = \mathrm{Tr}_{\mathrm{BPS}}\,(e^{-\beta H}) = \mathrm{Tr})\mathrm{BPS}1 = \#\mathrm{bosons} + \#\mathrm{fermions}\,. \tag{122}$$

This counts the total number of ground states (in the exercise you have proven that the BPS states are exactly the ground states of the Hamiltonian). Note that this is always larger than the Witten index:

$$Z_{\mathrm{BPS}} > Z_{\mathrm{Witten}}\,. \tag{123}$$

At weak coupling, we expect that this is much larger. But at large values of the coupling, you expect that the number of BPS states will match the index because "Anything that can lift, will lift"; i.e. perturbing the system enough will lift degenerate boson/fermions pairs until we have only one species or the other left. Thus at strong coupling we expect the number of states to match the Witten index. Since the latter is independent of the value of the coupling, we can calculate it at weak coupling and use it to know the number of BPS states at strong coupling.

Question:

☞ *Are there any restrictions on the validity of the extrapolation to strong coupling?* One way it could break down, is because of a phase transition or discontinuity. There are no walls of marginal stability for this index, so that does not pose a problem. Howeverm for extended-supersymmetry theories, where you have several operators Q_i:

$$H = \sum_{i,j=1}^{N} \epsilon^{ij}\{Q_i, Q_j^\dagger\}\,, \tag{124}$$

the counting of $1/N$ BPS states (that are only annihilated by 1 of the N operators Q_i), is a lot more subtle. And the black hole states are exactly of this form – but we will not go into the details.

4.3 Counting states for the three-charge black hole

We study the D1-D5-P system of Figure 20 in the limit $R^4 \gg V_{T^4}$, which means in terms of the charges

$$\frac{Q_p}{Q_1 Q_5} \gg 1\,. \tag{125}$$

Effectively, we can think of the system in this limit as a theory living on the S^1 with radius R, see Figure 23. The rotation (momentum along x^5) of the D1 and D5 will translate in rotation of the open strings, so we put momentum on the strings to account for Q_p.[15]

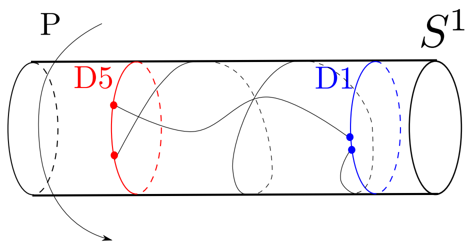


Figure 23: *In the limit $R^4 \gg V_{T^4}$, the system is effectively a theory of string streching between D1- and D-5 type branes on a circle, with added momentum.*

In the regime where $g_s \ll 1$ and $M_s \gg E$, strings are point particles moving around the circle. On top of that, we consider the open string coupling $g_s N \ll 1$ for the entropy calculation, and hence the gauge theory describing the open strings is a free theory. Therefore the wave function for momentum on the open strings has the form

$$\psi(x_5) = \sum_n e^{-\frac{2\pi n}{R} x_5}\,. \tag{126}$$

The wave function has to be single valued as we go around the circle. Look at a string with two endpoint going around the circle several times. Take for example a D1 brane that wraps the circle twice, and a D5 brane that wrap the circle three times.[16] If we unwrap the circle, this configuration looks like Figure 23.

The open string wave function depends on the string coordinate x_5 and has two labels, determined by the D1-branes and D5-branes:

$$\psi(x^{(1)}, x^{(5)})\,. \tag{127}$$

Depending on the label, we have the periodicities:

$$x^{(1)} \sim x^{(1)} + 2R\,, \qquad x^{(5)} \sim x^{(5)} + 3R\,. \tag{128}$$

[15] We will motivate everything from the open string picture. It is not easy to show that D1/D5 momentum follows from F1 with momentum, so you will have to take our word for it. In principle, you can divide momentum over all possibilities: open F1, closed F1, D1's, D5's one or several branes, combinations, single wrapping, multiple wrapping etc: everything can cary momentum. We are interested in the typical, dominant contributions. We will find that most entropy takes all of the momentum from the open string sector.

[16] Note that for "2 D-branes' on a compact circle, we have either 2 distinct D-branes or a D-brane wrapping the circle 2 times.

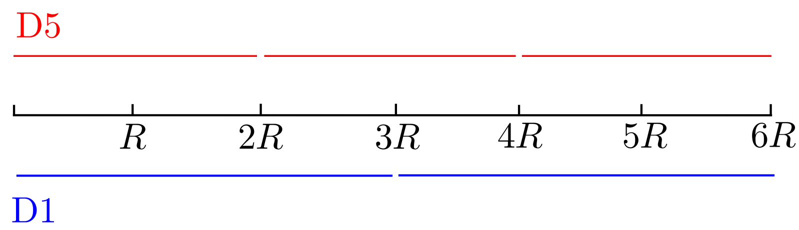


Figure 24: *A D1 brane wrapping the circle twice and a D5 brane wrapping the circle three times. We need to go six times around the circle before we reach the same point again.*

The wave function of the string is not periodic in R, but rather we find a periodicity in $6R$:

$$\psi(x^{(1)}, x^{(5)}) = \psi(x^{(1)} + 6R, x^{(5)} + 6R)\,. \tag{129}$$

For a general number of branes, we conclude that the string wave function is periodic in N_1N_5R (at least if N_1 and N_5 are coprime), we write:

$$\psi(x_5) \sim e^{2\pi \frac{n}{N_1 N_5 R} x_5} \tag{130}$$

The number n denotes the number of momentum units; momentum on such D1-D5-string is quantized in units of $1/N_1N_5R$ rather than $1/R$.

What about non-coprime N_1, N_5? We can always consider the nearest-coprime number by subtracting a small number $m \ll N_{1,5}$ such that $N_1 - m$ and N_5 are coprime. Then the leading contribution to the entropy is still $N_1N_5B_p$. Note that we get spacetime momentum from the momentum of the wave function, because we are building a wave function in a completely free theory.

We want to put N_p units of momentum on the D1-D5-string system. This finally leads us to the counting problem:

In how many ways can we get the momentum $p = N_p/R$ from partitioning the momentum over the D1-D5 open strings (with wave function (130)*?*

We can translate this to counting the number of partitionings

$$\sum_{m=1}^{\infty} \frac{n_m m}{N_1 N_5 R} = \frac{N_p}{R}\,. \tag{131}$$

The number m counts the momentum in units of $1/N_1N_5R$ added by n_m strings of this type. For instance, the easiest way to get such a partitioning is to take one string with $m = N_pN_1N_5$.

We count the number of different ways to form free strings (free excitations)

$$M \equiv N_1N_5N_p = \sum_{m=1}^{\infty} n_m m\,. \tag{132}$$

This is a counting of partitions of integers. We claim that this is counted by the partition function

$$Z = (1 + q + q^2 + \ldots)(1 + q^2 + q^4 + \ldots)(1 + q^3 + q^6 + \ldots)(\ldots)\,. \tag{133}$$

The first contributions are

$$Z = 1 + q + 2q^2 + 3q^3 + \ldots \tag{134}$$

and the coefficients of q^n indeed count the partitionings of the numbers n: one partitioning of 1, two of the number 2 ($1+1$ and 2), three for 3 ($1+2$, $2+1$ and 3) and so on. If we write the partition function as

$$Z = \sum_{n=0}^{\infty} d_n q^n \,, \tag{135}$$

then d_n counts the number of partitionings of the integer n.

Using our knowledge of a geometric series for $q < 1$:

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \,. \tag{136}$$

we see that the partition function can be written as the product

$$Z = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \,. \tag{137}$$

How to evaluate this partition function? We perform a calculation in the canonical ensemble, and we will go to a "high temperature"-limit. First we write q as

$$q = e^{-\beta} \,. \tag{138}$$

We calculate the average occupation number

$$\langle n \rangle = \frac{1}{Z} \sum_n n \, d_n e^{-\beta n} = \frac{\partial}{\partial \beta} \log Z \,. \tag{139}$$

This number will give us the leading contribution to the entropy.

First we evaluate the logarithm of the partition function:

$$\begin{aligned} \log Z &= -\sum_{n=1}^{\infty} \log(1-q^n) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(q^n)^m}{m} \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} (q^m)^n \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} \left(\frac{1}{1-q^m} - 1 \right) \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \frac{q^m}{1-q^m} \,. \end{aligned} \tag{140}$$

In the second to last line we used (136), and compensated for the over counting for $n = 0$.

Now we take a "high temperature"-limit, by taking $\beta \ll 1$:

$$q \lesssim 1\,. \tag{141}$$

Then $\langle n \rangle$ will be large because we get the large n contributions of the sum $Z = \sum_n d_n q^n$. The leading terms in this limit are

$$q = 1 - \beta + \mathcal{O}(\beta^2)\,, \qquad q^m = 1 - m\beta + \mathcal{O}(\beta^2)\,. \tag{142}$$

The logarithm of the partition function becomes

$$\log Z = \frac{1}{\beta} \sum_m m^{-2} + \mathcal{O}(\beta^0)\,. \tag{143}$$

We can rewrite this in terms of $\zeta(n)$, Riemann's ζ function, which gives for n an integer:

$$\zeta(n) \equiv \sum_{m=1}^{\infty} \frac{1}{m^n}\,. \tag{144}$$

Then the average particle number is

$$\langle n \rangle = \frac{\zeta(2)}{\beta^2}\,. \tag{145}$$

Standard thermodynamics gives us that the entropy in the canonical ensemble is

$$S \;=\; \log Z + \beta \langle n \rangle\,, \tag{146}$$

and this gives

$$S = \frac{2}{\beta}\zeta(2)\,. \tag{147}$$

To express the entropy in terms of the number $M \equiv \langle n \rangle$, we invert the relation (145), $\beta = \sqrt{\zeta(2)/M}$, and we use that $\zeta(z) = \pi^2/6$. This gives the entropy:

$$S = 2\pi\sqrt{\frac{M}{6}} = 2\pi\sqrt{\frac{N_1 N_5 N_p}{6}}\,. \tag{148}$$

There is a factor of 6 off in the square root compared to the supergravity result! Did we make a counting mistake?

Some remarks:

☞ What about the counting in microcanonical ensemble (calculating d_n) vs. the canonical one ($\langle n \rangle$) we used here? Compare to the energy in the canonical ensemble for standard statistical mechanics: E in the canonical ensemble is replaced by $\langle H \rangle$, the expectation value of the Hamiltonian.

☞ We assumed $\beta \to 0$. We need to check this was a valid assumption. By

$$\beta = \sqrt{\frac{\zeta(2)}{M}}\,, \tag{149}$$

this gives $M \to \infty$: this is exactly the regime we are interested in from the validity of the supergravity solution.

Let us get back to this factor of 6. With the results of Exercise 4.2, we find that the entropy for a "supersymmetric system" (equal number of fermionic and bosonic excitations) is

$$S = 2\pi\sqrt{\frac{cM}{4}}\,, \tag{150}$$

with c the number of bosons. We count the number of massless modes on S^1, but the entropically dominant strings are those stretching between the D1 and the D5-branes. Those 1-5 strings have four bosonic degrees of freedom, from their freedom of moving around in T^4.[17]

Therefore, the D1-D5-P system has $c = 4$ and we reproduce the black hole entropy on the nose:

$$S = 2\pi\sqrt{N_1 N_5 N_p}\,. \tag{151}$$

Hooray to string theory!

Exercise 4.2: *Prove the following statements:*

- *For the partition function*

$$Z_c = \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right)^c\,, \qquad c \in \mathbb{N}\,, \tag{152}$$

  the entropy in the large temperature limit is

$$S = 2\pi\sqrt{\frac{cN}{6}}\,. \tag{153}$$

  In a free theory, this formula is easy to show. This partition function is for one type of object (as the open string).

- *Z_c was the partition function for c bosons. For fermions, which have either occupation number 0 or 1, we need to put in something extra. Using a similar reason as for $c = 1$ boson partition function, show that the partition function for fermions is*

$$Z_{\text{fermions}} = \prod_{n=1} (1+q^n)\,. \tag{154}$$

[17] There are also contributions from 1-1 and 5-5 strings, but these go as $p \sim 1/N_1$ and $- \sim 1/N_5$ and are hence subleading.

Show that for the partition function for c bosonic and fermionic string excitations is

$$Z = \left[\prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right) \right] \tag{155}$$

and that in the high temperature limit, this gives the entropy

$$S = 2\pi \sqrt{\frac{cN}{4}} \,. \tag{156}$$

5 AdS/CFT

In this section, we want to formalize the counting arguments of the previous section. We have seen that g_s, the string coupling, and the number of D-branes N allow to interpolate between different regimes, see Figures 21 and 22.

The coupling g_s sets the 'quantum' nature of closed string interactions. When $g_s \ll 1$: we have $M_s \ll M_P$ and string theory is classical. String excitations do not go into the regime of quantum gravity. When $g_s \gg 1$, any excited string is in the quantum gravity regime and we cannot use supergravity.

In this section, we wish to go on with the continuous description that interpolates between regimes in open string theory. We fix $E \ll M_s$ (we only keep massless excitations, such that we cansee only point particles) and $g_s \ll 1$, and we want to connect the regimes:

- $g_s N \ll 1$: There is a weakly coupled field theory describing the system, but no gravity.
- $g_s N \gg 1$: The same field theory, now strongly coupled, describes the physics, there is no gravity in this picture. This is a dual description of supergravity on the D-brane background.

We want to motivate this duality between the gravity, or closed string, picture and the field theory picture (open strings). For small energies, this is known as the AdS/CFT correspondence.

5.1 'Deriving' AdS/CFT

Let's start from closed string theory. Take the metric of the D1-D5-P system

$$ds^2 = \frac{1}{\sqrt{Z_1 Z_5}}(-dt^2 + dx_5^2 + Z_p dx_-^2) + \sqrt{Z_1 Z_5}d\vec{x}_4^2 \,, \tag{157}$$

with the light cone coordinate

$$x_- = t - x_5 \,. \tag{158}$$

This metric describes a momentum excitation along one direction, because the light cone coordinate x_+ is absent.

Remember that this metric has the following asympotics:

- $r \to \infty$: asymptotically flat $\mathbb{R}^{1,4} \times S^1 \times T^4$.
- $r \to 0$: interior. There is an $(AdS_2 \times S^3) \times (S^1 \times T^4)$ throat, the black hole sits at the bottom.

We will take $N_1 = N_5 = N$. We define

$$\lambda = g_s N \,. \tag{159}$$

We represent the small λ and large λ system in Figure 22. From the metric, we know that the charges Q in the harmonic functions $Z = 1 + Q/r^2$ go as $Q \sim g_s N(\ell_s)^2 =: \lambda(\ell_s)^2$. Therefore the scale of the throat is set by

$$L \sim \sqrt{\lambda}\ell_s \,. \tag{160}$$

Low energy excitations. We want to work with "low energy excitations". But: what is energy in this setup? We start with the energy E^∞ measured by an observer at infinity in the black hole spacetime, we want that:[18]

$$E^\infty \ell_s \ll 1\,. \tag{161}$$

This means that no strings are excited and we only see gravity modes. Asymptotically, string theory reduces to just (super)gravity. On the other hand, the energy of the throat is set by $E_{\text{throat}} = 1/L \sim 1/\sqrt{\lambda}\ell_s$. An excitation with an energy lower than

$$E^\infty \ell_s < \frac{1}{\sqrt{\lambda}} \tag{162}$$

decouples: its wave length is larger than the scale of the throat and any such mode you shoot in will fly by. Another way to put it is that any excitation in the thoat you push out of the black hole, will have an asymptotic energy larger than the scale of the throat: $E^\infty \ell_s > 1/\sqrt{\lambda}$. Therefore, asymptotically low energies are set by the the scales ℓ_s and λ:

$$E^\infty \ell_s \ll \min\left(1, 1/\sqrt{\lambda}\right)\,. \tag{163}$$

So much for asymptotics. What's happening inside the geometry? Energy is a local statement. The redshift relates the energy between two observers at r_1 and r_2 as $\int_{r_1}^{r_2} \sqrt{g_{tt}}$. Approximating this integral by its value down the throat, the energy E_0 of a local observer at say $r = 1$ in the throat is related to the asymptotically measured energy as

$$E^\infty \sim (Z_1 Z_5)^{1/4} E_0 = \sqrt{\lambda} E_0\,. \tag{164}$$

What can we say about the energy of excitations down the throat? Consider the two possibilities:

1. $\lambda > 1$: Then by (163) we have $\sqrt{\lambda} E_0 \ell_s \ll 1/\sqrt{\lambda}$, which can be written as:

   $$E_0 \ell_s \ll 1/\lambda < 1\,. \tag{165}$$

   There are no stringy excitations down the throat.

2. $\lambda \ll 1$; Then by (163) we have $\sqrt{\lambda} E_0 \ell_s \ll 1$, or:

   $$E_0 \ell_s \ll 1/\sqrt{\lambda}\,, \tag{166}$$

   but we also have $1/\sqrt{\lambda} \gg 1$ and thus we can have stringy modes down the throat. At infinity in this regime, excitations at infinity decouple from in the throat and vice versa because of redshift.

We conclude that there *can* be stringy excitations down the throat only when $\lambda \ll 1$. These are decoupled from the asymptotic region due to redshift.

[18] In natural units $\hbar = c = 1$, energy is measured in dimensions of inverse length $[E] = L^{-1}$.

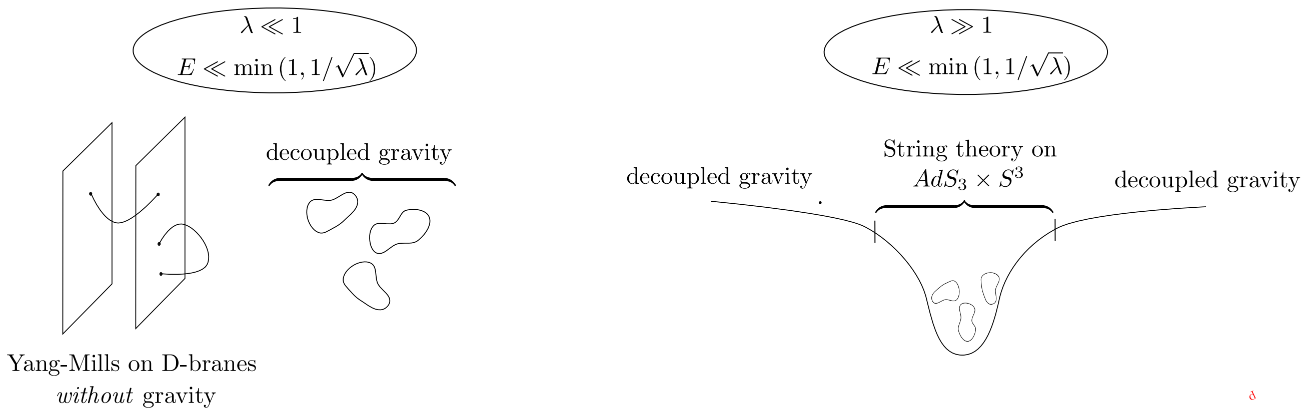


Figure 25: *We consider low-energy excitations $E^{\infty}\ell_s < \min(1, 1/\sqrt{\lambda})$. Left: In the regime $\lambda \gg 1$, we have a field theory describing open string theory, right: for $\lambda \ll 1$, we can have a "stringy" black hole, with (open) string excitations and gravitons down the throat, which decouple from the asymptotic geometry.*

Throat geometry. We discuss the geometry of the throat. First we put $Z_p = 1$. We can later add the momentum as excitations on the throat geometry. Deep in the throat we have $r \ll \sqrt{\lambda}\ell_s$ and hence

$$Z_{1,5} \sim \frac{\lambda(\ell_s)^2}{r^2}\,. \tag{167}$$

The geometry becomes

$$ds^2 = \frac{r^2}{\lambda\ell_s^2}(-dt^2 + dx_5^2 + \ldots) + (\lambda\ell_s^2)\frac{dr^2}{r^2} + \lambda\ell_s^2 d\Omega_3^2 + ds^2(T^4)\,. \tag{168}$$

This is the geometry of $AdS_3 \times S^3$ (times a constant volume T^4). The radius of anti-de Sitter space and the three-sphere are equal and set by λ in string units:

$$R_{AdS} = R_S = \sqrt{\lambda}\ell_s\,. \tag{169}$$

Note that the geometry $AdS_3 \times S^3 \times T^4$ is a solution to the equations of motion itself, essentially because the equations for the warp factors

$$\Delta Z_i = 0\,, \tag{170}$$

do not care for the integration constant $Z = cst + Q/r^2$.

Open and closed string effects together. Now we can combine the open string and closed string pictures. We take the same low energy limit (163) for the $g_sN \ll 1$ regime as well. Then we have the flat space geometry at $g_sN \ll 1$:

$$M_{1,10} = \mathbb{R}^{1,4} \times S^1 \times T^4\,. \tag{171}$$

In flat space, the energies are equal for both observers:

$$E_0 = E_\infty\,. \tag{172}$$

How does a process where open strings interact with closed strings depend on the energy? Such a process was depicted in Figure 19. At low energy, we have gravitons leaving the brane. The amplitude for such a process is proportional to:

$$g_s^2 \ell_s^{D-2} N^2 = G_N N^2 \,. \tag{173}$$

There is one factor of $g_s N$ for each open string endpoint on N D-branes. For instance, for the D1-D5 system, we have to sum over all the ways we can get this process, and there are $N_1 N_5$ possible ways of making 1–5 strings, see Figure 26.

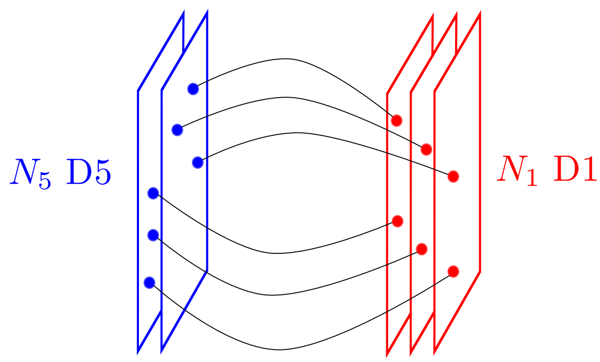


Figure 26: *Strings stretching between N_1 D1 branes and N_5 D5 branes. There are $N_1 N_5$ ways of making D1-D5 strings.*

We will look at the $D = 6$ geometry $\mathbb{R}^{1,4} \times S^1$ (dropping the T^4 part of the geometry). The dimensionless rate depends on the energy E of the process as

$$E^4 G_N = (g_s N)^2 \ell_s^{D-2} E^4 = \lambda^2 (E\ell_s)^4 \,. \tag{174}$$

By our working assumption (163), we find that this rate is small:

$$E^4 G_N \ll 1 \,. \tag{175}$$

And hence open strings interact only very weakly with closed strings.

This shows that we have made a self-consistent argument for $E\ell_s \ll \min\,(1, 1/\sqrt{\lambda})$:

- We assumed furst that branes decouple and do not source the geometry (everything is flat)
- In the decoupling limit, we showed that D-branes *cannot* source the gravitational field

In this regime, we find open/closed string decoupling. On top of that, we have taken a low-energy limit: we only consider massles states of open string theory. Therefore, this system is described by a field theory (no gravity), in particular by Yang-Mills theory (the gauge theory living on the D-branes). These are supersymmetric and conformal theories, but we will not dwell on that.

5.2 AdS/CFT dictionary

We have alluded to the equivalences of Table 8. When the field theory is weakly coupled, the AdS space has a very small radius L and string theory corrections are important (strongly coupled string theory on AdS). When the field theory is strongly coupled, the AdS space is large and well described by classical supergravity. For small λ, we have a good control on the gauge theory, for large λ, we have a good control on the gravitation anti-de Sitter physics, see Table 8.

Table 8: *Equivalence between open string and closed string theory by varying λ.*

Yang-Mills on a D-brane	**Closed string theory on AdS**
Decoupled sector: gauge theory on a brane	Closed strings down the throat
No strings/no gravity	*Full, closed string theory*
λ: gauge ('t Hooft) coupling constant	$\lambda = L/\ell_s$: size of AdS in string units
N: rank of gauge group	$N = L/\ell_p$: size of AdS in Planck units
λ small: weakly coupled **Control**	AdS small $\to$ stringy *No Control*
λ large: strongly coupled *No Control*	AdS large $\to$ Supergravity **Control**

There is in fact a second parameter that plays a role in the duality. The system is characterized by the string coupling g_s and the number of D-branes N, or, equivalently, by N and $\lambda = g_s N$. In gauge theory, N is the rank of the gauge group, in gravity, N is the size of the AdS space in Planck units (while λ is the size of AdS in string units).

Exercise 5.1: *Show that the AdS length (size of the D1-D5 black hole throat) in string units is set by λ, and in Planck length by N:*

$$\lambda = L/\ell_s\,, \qquad N = (L/\ell_P)^{\#} \tag{176}$$

for some number #.

Because supergravity is only valid at large N, we only understand large N gauge groups from supergravity. On the other hand, we could invert this to maybe learn quantum gravity from small N gauge groups. For instance, for $N = 2, 3$ the size of AdS space is a few Planck units and gravity is very quantum.

Note that the AdS/CFT correspondence is a conjecture. We haven't proven anything, we have just given motivation! It is very hard to prove: a proof would require a detailed knowledge of strongly coupled field theories. However, it is very well established and most string theorist hold it to be true: it is nothing more than the low energy limit of the much more poweful open/closed string duality, remember Figure 18. The closed string exchange between D-branes, which can be interpreted as a tree level open string diagram, has all the massive modes implicit. For AdS/CFT, we only consider the massless, non-oscillatory modes.

Formal AdS/CFT duality. The correspondence can be formalized by equating the path integrals of the two theories:

$$\boxed{Z_{\mathrm{CFT}}(\lambda, N) = Z_{\mathrm{IIB}}^{\mathrm{string}}(\lambda, N)|_{\text{on asympt. AdS space}}} \,. \tag{177}$$

This equality summarizes the AdS/CFT conjecture.

We often restrict to λ very large, and then we get an equivalence between large 't Hooft coupling CFT and IIB supergravity on an asymptotically AdS space:

$$Z_{\mathrm{CFT}}(\lambda \to \infty, N) = Z_{\mathrm{IIB}}^{\mathrm{sugra}}(N)|_{\mathrm{AdS}} \,, \tag{178}$$

where sugra stands for supergravity. Schematically, we can write the supergravity path integral as

$$Z_{\mathrm{IIB}}^{\mathrm{sugra}}(N) = \int \mathcal{D}g \, e^{-\int \sqrt{-g}(\text{gravitons}+\ldots)} \,, \tag{179}$$

there are other fields besides the metric g, but let's just forget about them for the sake of the argument. When N is large, we are doing classical supergravity: at fixed $\lambda = g_s N$, loops are suppressed because g_s is small. Then we can perform a saddle point approximation around the minima of the action (the classical solutions to the equations of motion), and the large N approximation is

$$Z_{\mathrm{IIB}}^{\mathrm{sugra}}(N \to \infty) = \int \mathcal{D}g \sum_i e^{-S_i} \,, \tag{180}$$

The sum runs over solutions to the equations of motion (saddle points) and it is actually possible to calculate its main contributions. In the limit $\lambda \to \infty$ (large 't Hooft coupling) and $N \to \infty$ (planar limit), states in the CFT are hence related to classical solutions in AdS.

The left-hand side of these equations is always well-defined, because it deals with a CFT. By the AdS/CFT correspondence, this means that also the right-hand side is well defined: quantum gravity on AdS spaces is hence better defined than a generic QFT!

5.3 Entropy counting

A black hole solution has an entropy. In CFT a state with an entropy is an ensemble with one quantum number (the mass). Such an ensemble is a density matrix that looks like:

$$\rho_{BH} = \sum_{\psi} e^{-\beta H} |\psi\rangle\langle\psi| \,. \tag{181}$$

For a large temperature, there is no difference between the microcanonical and the canonical ensemble. Therefore we can work with the temperature, the thermodynamic dual of the mass, rather than with the mass itself.

Remember the set-up of the D1-D5-P system wrapped on $T^4 \times S^1$ of Figure 20. The CFT that describes this system lives on the two-dimensional spacetime formed by the common circle on which the branes are wrapped and the time direction: $S^1 \times \mathbb{R}_t$. (This is the CFT dual to the AdS_3 near-horizon geometry of the D1-D5 black hole.)

Cardy gave us a formula for the entropy in a CFT at high temperature, irrespective of the coupling:

$$S \sim \sqrt{\frac{cL_0}{6}}\,, \tag{182}$$

where L_0 is the momentum along one direction, and c is the central charge. This is a formalization of what we had earlier. Then c was the "entropy density". For a boson in a free teory, $c = 1$, for a free fermion one has $c = 1/2$.

For gravity on an AdS space, the central charge of the dual CFT is the AdS length in Planck units:

$$c = \frac{L}{\ell_P}\,, \tag{183}$$

which is independent of the coupling for the D1-D5 background:

$$c = N_1 N_5\,. \tag{184}$$

Note that this would not be the case if the central charge was the AdS length in string units, because then c would be equal to $\sqrt{g_s}\sqrt{N_1 N_5}$ and hence coupling-dependent. The fact that c is independent of the string coupling g_s is very important, because it assures that the entropy (through the Cardy formula) is independent of the coupling as well.

If we put the momentum excitations on the D1-D5 AdS_3 throat to account for the D1-D5-P black hole entropy, we are adding momentum to the dual CFT as well, in one direction (L_0).[19] The Cardy formula gives the entropy

$$S \sim \sqrt{\frac{N_1 N_5 N_p}{6}}\,. \tag{186}$$

This is not relying on weak coupling, but is valid for any value of g_s.

Through the AdS/CFT correspondence, we have learnt that a strongly coupled CFT is dual to a black hole geometry, and that we can use the Cardy formula to count the entropy. We give some more motivation for the fact that the central charge is

$$c = N_1 N_5\,. \tag{187}$$

In a CFT, the partition function at high temperature goes as

$$Z_{\text{CFT}} \sim e^{cT}\,, \tag{188}$$

and hence the entropy goes as

$$S \sim \log Z_{\text{CFT}} \sim cT\,. \tag{189}$$

[19] Remember that the metric of the D1-D5-P system looks like

$$ds^2 = -dt^2 + dx_5^2 + Z_p dx_-^2\,, \tag{185}$$

with $dx_- = dt - dx_5$. This fixes a particular chirality of the plane wave.

This also shows why we can interpret c as the entropy density.

In supergravity, the partition function goes as

$$Z_{\text{sugra}} = e^{\frac{1}{G_N}S[g]} = e^{(L/\ell_p)T} \tag{190}$$

and hence $c \sim L/\ell_p$.

Questions from the audience:

☞ *The black hole is extremal. How can there be a (CFT) temperature?* In CFT, there is a left and a right temperature, related to the total amount of left- and right moving excitations. Using the null circle x_- (or x_+ if we would have that coordinate in the metric), gives a length of this thermal circle that gives a temperature T_L (T_R for x_+). The total temperature of a state is related to those temperatures as

$$\frac{1}{T} = \frac{1}{T_R} + \frac{1}{T_L} \,. \tag{191}$$

In the extremal D1-D5 setup, we only have left-moving excitations and hence $T_L \neq 0$, but still $T_R = 0$. Therefore the BH temperature T is zero, even though there is a CFT temperature T_L.

☞ *We have treated AdS/CFT. Here we had AdS_3 of the near-horizon plus the dual CFT. What happens if you insert a black hole inside an asymptotically AdS space?* Consider AdS with a black hole inside it. This corresponds to a CFT at a non-zero temperature T (so both T_L and T_R are non-zero), see Table 5.

5.4 Non-supersymmetric black holes

For supersymmetric black holes, we have seen that the microscopic entropy matches the macroscopic one as in Table 9.

$S_{\text{BH}}^{\text{micro}}$	$=$	$S_{\text{BH}}^{\text{macro}}$
$\downarrow$		$\downarrow$
$\log(N)$		$A_H/4G_N$
weak coupling		**strong coupling**

Table 9: *For supersymmetric black holes, we can match the Bekenstein-Hawking entropy from a weak coupling computation.*

We have seen two arguments why the weak-coupling, microscopic calculation gives the correct result for the entropy of the black hole at strong coupling:

- An index which is protected by supersymmetry: it can be calculated at weak coupling an trusted at strong coupling.
- AdS/CFT correspondence. The result for the entropy uses the Cardy formula and can be calculated regardless of the coupling, as long as we only consider low energy excitations.

Both these arguments hinge on supersymmetry. What about non-supersymmetric solutions in asymptotically flat spacetime? The index will no longer be protected, and we cannot rely on the AdS/CFT correspondence anymore, because the near-horizon solution of a non-extremal black hole does not have an AdS factor.

We can at least make a statement for non-supersymmetric black holes we put in an asymptotic AdS spacetime. We put a non-extremal black hole (black hole with a non-zero temperature) in $AdS_5 \times S^5$. Without the black hole, the geometry is dual to a conformal field theory, namely $N = 4$ super-Yang Mills theory. It is a supersymmetric and conformal (there is no dimensionful scale) field theory that is very similar to QCD.

When we put a black hole in spacetime, this is dual by the AdS/CFT correspondence to heating up the CFT, and hence introducing a scale. This is the setup of Table 5. A high temperature excites the many states of this field theory (gluons, fermions. . .), and therefore you get an entropy, a number of states that are excited at a given temperature. The temperature breaks both conformal invariance and supersymmetry in the field theory and we get a non-supersymmetric black hole.

We can repeat the counting of the previous section and find the entropy, both in the field theory (a non-trivial calculation involving fermions, an SU gauge group and so on) and in gravity (an easy calculation using the horizon entropy). One finds:

$$\begin{array}{ll} N = 4 \text{ SYM} & \left\{ \begin{array}{l} S^{\text{micro}} = aT^3 \\ \\ S^{\text{macro}} = \frac{3}{4} aT^3 \end{array} \right. \\ \text{Supergravity} & \end{array}$$

The supergravity entropy only sees three quarters of the entropy of the microscopic counting. We can interpret this as the degrees of freedom that are changing. At strong string coupling, we don't know what the degrees of freedom are, we only observe that we get a smaller entropy from them. See Figure 27.

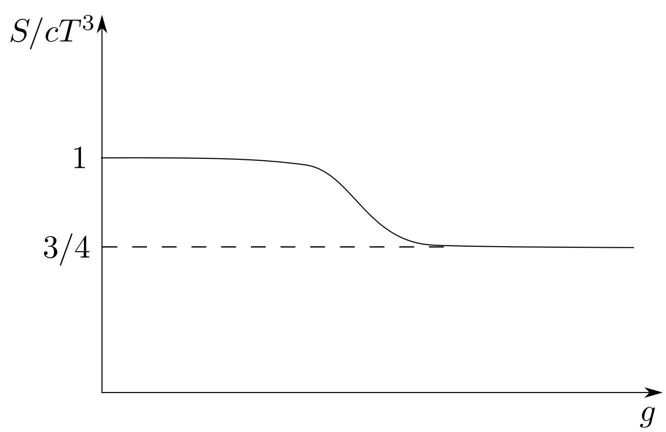


Figure 27: *The entropy as a function of the coupling for the blcak hole in $AdS_5 \times S^5$.*

This is a common feature. Think about QCD. At low coupling, the degrees of freedom are quarks. At strong coupling, which is the regime of our interest, the degrees of freedom are neutrons, protons, pions etc.: the building block are very different.

The nice observation to make is that all such field theories, whether they are $N = 4$ SYM, QCD, or some field theory with 18 bosons and 27 fermions, all have similar properties when we put them at a high temperature that is larger than all scales of the theory. As an example, take two fundamental properties of fluids in such theories: the entropy density s and the viscosity η. The entropy to viscosity ratio η/s for the quark gluon plasma of QCD can be observed experimentally. The value of η/s can be found exactly in $N = 4$ SYM, from a weakly coupled gravity computation, and this value is of the same order as the observed value in the RHIC collider, see Table 10. Moreover, any calculation in the string theory ballpark allways gives the same value of $\eta/s = 1/4\pi$. This is all the more intriguing because existing QCD theories find a number which is off by an order of magnitude.

Table 10: *Entropy to viscosity ratio.*

η/s	Theory/Experiment
$1/4\pi \cong 0.796$	$N = 4$ SYM
$0.12 \pm \ldots$	QCD (Experiment)
$\mathcal{O}(1)$	QCD (Theory)

For this reason, people use AdS/CFT to describe strongly coupled QCD, and also strongly coupled condensed matter theories (so-called AdS/CMT, for instance for superconductors at strong coupling). In fact, this has been the main use of the AdS/CFT correspondence so far and this entire field can be put under the name “holography”. There are many articles which can lead you in this direction, see for instance the previous courses on holography at IPhT [2, 3] (see also [4]).

6 The fuzzball proposal and black hole hair

In this section, we elucidate the idea that black hole entropy is explained by the existence of a large number of 'black hole microstate' solutions. These are geometries that are solutions to the equations of motion of string theory, have no horizon themselves, but come in large enough numbers to account for the black hole entropy.

Let us get back to the main problem. We have a microscopic and a microscopic entropy, which agree numerically, but both are valid in different regimes:

Entropy	S^{micro}	S^{macro}
Valid when		$g_s N \ll 1$

As an example, think about the air in a room. It is made up out of many molecules. Still, we can extract the entropy without reference to the microscopic state of the molecules through equations of state:

$$\begin{aligned} pV &= nRT\,, \\ dE &= TdS + pdV\,. \end{aligned} \tag{192}$$

We can determine the entropy S without knowing what the air is made of – thermodynamically, the entropy is a measure of the energy change in a system on which we have no control or understanding (in contrast to the work term pdV).

So much for thermodynamics, on to statistical mechanics. Boltzmann has taught us that for a given energy E and temperature T, all N different states of the molecules in the room make up the entropy as:

$$S^{\text{micro}} = \log(N)\,. \tag{193}$$

This connection between statistical mechanics and thermodynamics is already 150 years old. Does it work for a black hole too?

At this point, it does not. The microscopic calculation ("statistical mechanics") takes place in one regime, but this statistical description is not valid when $g_s N \ll 1$. We have the following question:

☞ Say you take a state that makes up the entropy in the microscopic calculation. What happens if you follow such states one by one and bring them over to strong coupling?

People believed for a long time that as gravity grows stronger, a horizon forms around the D-branes and the objects end up "being" the black hole, see Figure 28. Take for instance a neutron star. If we scale up G_N, r_H will become larger until the neutron star collapses into a black hole. This intuition caused people to think for a long time that whatever state you take out of the $\exp(2\pi\sqrt{N_1 N_2 N_3})$ possible ones, all of them become a black hole with a singularity in the middle.

Because gravity is always attractive, you expect that as you make g_s (and hence $G_N \sim g_s^2$) larger, objects only becomes smaller. On the other hand, the horizon radius for a (Schwarzschild) black hole scales with Newton's constant as

$$r_H \sim G_N M\,, \tag{194}$$

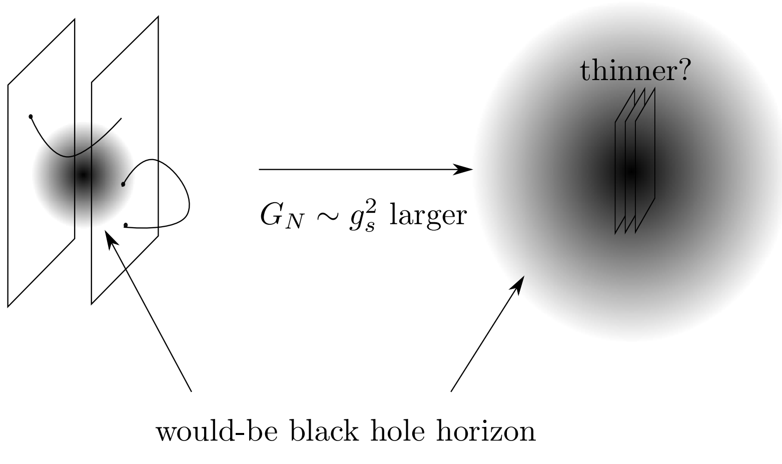


Figure 28: *At low G_N ($g_s \ll 1$), the would-be black hole horizon is of smaller or equal size as the brane system. For large G_N, the black hole horizon is much bigger than the size of the D-brane system at weak coupling.*

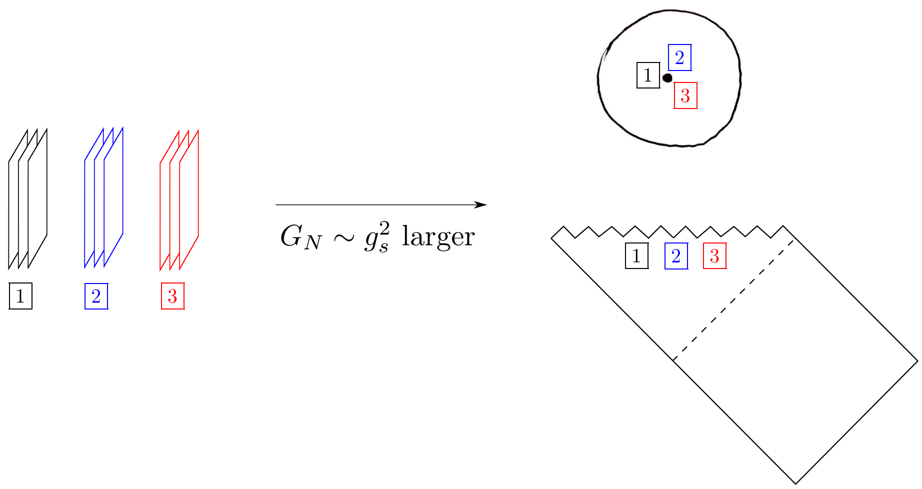


Figure 29: *In the naive picture, cranking up G_N puts the information of the microstate 1,2 or 3 into the garbage near the singularity.*

with M the mass of the black hole. Thus the horizon actually *grows* when you make gravity stronger.

We can represent this pictorially. Say we have three microstates made up out of open strings on D1-D5 branes in the decoupled regime, as in picture 29. As we make gravity stronger, all of these would seem to fall behind the horizon and the information of the state making up the black hole is in the region near the singularity.

We discussed earlier the information paradox: We can throw anything in to the black hole, but in GR, this information gets lost and never comes out, as the black hole evaporates into thermal radiation. Since the Hawking radiation process deals with the region around the black hole horizon, the intuitive picture of what happens to a brane microstate does not solve the problem. The horizon region is in the causal past of the singularity and physics in this region has no idea of what happens at the singularity. All information still sits near the singularity and the information paradox is still there.[20]

Through the D1-D5-P black hole and the AdS/CFT duality, we should be able to find the CFT process dual to Hawking radiation. In CFT, we can actually address this problem. The proposal goes as follows. Look at a microstate. As g_s grows large, they actually become *bigger* and will be of the *same size* as the would-be black hole horizon, see Figure 30.

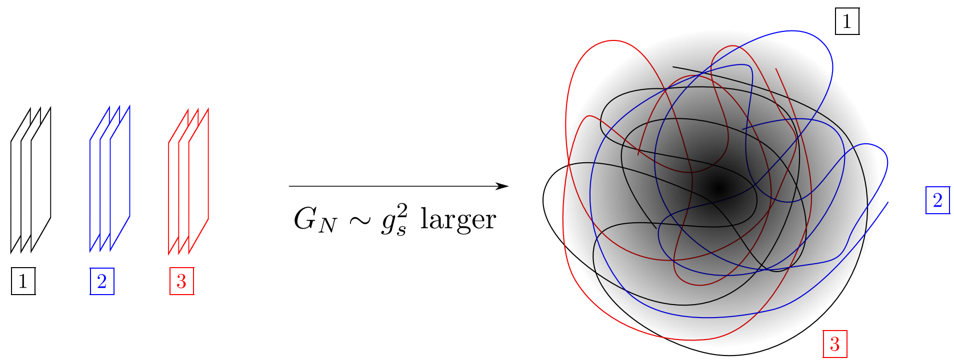


Figure 30: *The 'fuzzball proposal': cranking up g_s gives a complicated state of strings and branes of horizon size.*

Hawking evaporation will know about what information made the black hole, and it is no longer stuck at the horizon. The main problem with this proposal is that you need 'microstates' of the same size as the black hole horizon. The black hole horizon grows as G_N, but most things get smaller for increasing G_N. We need some special objects. We will show how to build such growing states that correspond to the CFT we counted at g_s small. These will not have a horizon at large g_s.

Supersymmetric black holes radiate, and there is no comparison of the Hawking process. For non-supersymmetric black holes, some large G_N microstates ('microstate geometries') have been constructed. They radiate and the Hawking radiation rate of the black hole agrees nicely with the decay of these states.

Note that we have come at the frontier of research: we have some hints about it, but people do not know yet if the proposal is generally true or not. In the next section we will show how to build (certain) fuzzball solutions for the supersymmetric 3-charge black hole.

[20]The information paradox leads to a breakdown of unitarity in quantum theory and hence a breakdown of quantum mechanics itself. If we want to save quantum mechanics, we need to make sure there is no information loss.

7 Multicenter solutions

In this section, we show how to construct five-dimensional multicenter solutions that generalize the string theory black holes we have seen earlier. The microstate geometries, or fuzzballs, for the black hole will be in this class.

7.1 Preliminaries

We will use differential form notation throughout. Therefore we first quickly discuss the necessary basics. We then give some exercises that will illustrate a new term in the supergravity Lagrangian as opposed to the Einstein-Maxwell one, that allows for solutions with 'charge dissolved in fluxes'. This is crucial for the construction of microstate geometries.

7.1.1 Forms, Einstein-maxwell, sources

We review the following notions:

- An intro on forms and form notation and the definition of the Hodge star operator $\star$.
- Monopole vs "moving electron" (curl free) magnetic sources: e.g. $F = dC + \Theta$.
- Sourced electromagnetism. $d \star F = \delta(\vec{x})$ and $F = dA$ can be solved by thinking of A_0 as harmonic $\nabla^2 A_0 = \delta(\vec{x})$. This equation has solutions of the form $A_0 = 1 + \frac{q}{r}$ (actually there is a larger class of solutions contructed of polynomials of the coordinates but the latter are not normalizable).
- Sourced Einstein-Maxwell. In Einstein-Maxwell the equation $d \star F = \delta(\vec{x})$ involved the metric via $\star$ so the solution is more complicated so convince ourself for metrics of the D-brane type we have the solution looks like $A_0 \sim H^{-1}$ where H is some harmonic function in metric.

Consider electromagnetism. The anti-symmetric two-form is

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_z & -B_x & 0 \end{pmatrix} . \tag{195}$$

In terms of this matrix, the Einstein equations in vacuum are:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0 \,, \\ \partial_{[\mu} F_{\mu\nu]} &= 0 \,. \end{aligned} \tag{196}$$

In form notation they are equivalent to

$$\begin{aligned} d \star F &= 0 \,, \\ dF &= 0 \,. \end{aligned} \tag{197}$$

The first exspression is the equation of motion that follows from the Lagrangian of electromagnetism. The second equation is the Bianchi identity, which just says that locally F is the exterior derivative of a potential $F = dA$.

Exercise 7.1: *If you are not familiar with the expressions* (197) *(exterior derivative, Hodge star operator $\star$), read up on it in a book on differential geometry and show that the equations* (196) *and* (197) *are equivalent. In particular, in d dimensions the Hodge star $\star$ takes an m-form to a $n = d - m$ form as follows*

$$(\star\lambda)_{\mu_1\ldots\mu_n} := \frac{1}{m!}\sqrt{g}\,\epsilon_{\mu_1\ldots\mu_n\nu_1\ldots\nu_m}\,g^{\nu_1\rho_1}\ldots g^{\nu_m\rho_m}\lambda_{\rho_1\ldots\rho_m}\,. \tag{198}$$

Here ϵ is the totoally antisymmetric tensor.

Recall that in electromagnetism we can generate a magnetic field by accelerating an electron. However, while a speeding electron generates a magnetic field it does not generate a *magnetic charge.* This is because electric charge only appears in the equation

$$d\star F = q\,\delta(x)\,, \tag{199}$$

whereas the magnetic charge sources the bianchi identity

$$dF = m\,\delta(x)\,. \tag{200}$$

The difference between these two is the following. If $m = 0$ then $dF = 0$ everywhere and so in flat space this implies there exists a globally defined one-form, $A = A_\mu dx^\mu$, the vector potential, such that $F = dA$. If on the other hand $m \neq 0$ then at the origin F is not closed so there is no globally defined object A such that $F = dA$. We can still define an object A everywhere away from the origin (or define it patchwise). As a side note one might object by arguing that solving the electric equation requires e.g. $A_0 \sim \frac{q}{r}$ which is singular at the origin but the fact is we can always smooth this singular source by allowing a charge distribution (e.g. replace $q\delta(x)$ with e.g. a gaussian qe^{-qr^2}). The same trick will not work for m because this equation has d^2A which always equals 0 if A is globally defined.

To write a general field strength that includes both electric and magnetic charge we can do the following. We write

$$F = dA + \Theta\,, \tag{201}$$

with A a global one form encoding the electric charge (and perhaps some magnetic field) via $d\star dA = q\delta(x)$. The two-form Θ on the other hand is *not* of the form $d($*something*$)$ but rather satisfies $d\Theta = m\delta(x)$ and hence encodes the part of the field strength coming from the magnetic charge. To see this recall that the definition of the magnetic charge is the integral of the field strength on an S^2 around the origin.

$$m = \int_{S^2} F = \int_{S^2}(dA + \Theta) = \int_{S^2}\Theta \tag{202}$$

where the last equality follows because S^2 is a compact manifold without boundary and dA is a total derivative of a globally defined object.

Exercise 7.2: *Write $\Theta = dB$ where B is only locally defined and show that the above works. (Find the form of B first).*

7.1.2 Important exercises

We show how the appearance of new terms in the supergravity Lagrangians (compared to electromagnetism) allow for 'fuzzball' solution.

The Lagrangian of electromagnetism in four dimensions is

$$\begin{aligned} \mathcal{L}_4 &= \tfrac{1}{4}\sqrt{-g}F_{\mu\nu}F_{\mu\nu} \\ &= \tfrac{1}{2}F\wedge\star F\,. \end{aligned} \tag{203}$$

This is the gauge and Lorentz invariant action for the Maxwell field A_μ. In five dimensions, an extra term is possible:

$$\begin{aligned} \mathcal{L}_5 &= \tfrac{1}{4}\sqrt{-g}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma\tau}A_\mu F_{\nu\rho}F_{\sigma\tau} \\ &= \tfrac{1}{4}F\wedge\star F + A\wedge F\wedge F\,. \end{aligned} \tag{204}$$

This new terms seems to be breaking gauge invariance. Consider the gauge transformation:

$$A_\mu \to A_\mu + \partial_\mu\lambda\,, \tag{205}$$

with λ a function. The field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is clearly gauge invariant. The second term in the five-dimensional Lagrangian has a "naked" A_μ and seems to be breaking gauge invariance.

Exercise 7.3: *Show that in five dimensions, the term*

$$\int \epsilon^{\mu\nu\rho\sigma\tau}A_\mu F_{\nu\rho}F_{\sigma\tau}\,. \tag{206}$$

is invariant under gauge transformations (205).

Everything based on string theory (in particular supergravity) has such a term, it comes naturally in string theory. So it is important to study its physical consequences. (It is also important for confinement in gauge theories.)

Remember that a static electron couples to the gauge field as

$$\int A_0 dt\,. \tag{207}$$

Because of the term (206), a non-trivial A_0 is sourced by magnetic terms $F_{12}F_{34}$ (see Exercise 7.4). Even if you don't have electrons, but just magnetic fields of two different kinds, you can have electric fields!

Exercise 7.4: *Derive the equations of motion for A_μ following from the action* (204). *Show that you can source electric fields with magnetic fields along different directions.*

We have two ways of getting electric fields and we will use this kind of solutions for microstate geometries/fuzzballs. In fact, this mechanism is crucial for the existence of microstate geometries. The absence of such a term in regular electromagnetism is also the reason people haven't found such geometries before the advent of string theory.

Note that also for the AdS/CFT dual geometries to confining gauge theories, you make use of such solutions.

7.2 Building General Solutions

We discuss how to obtain new solutions with 'charge dissolved in fluxes'. We do this in a stepwise fashion: first we discuss the five-dimensional black hole (without and with rotation), and then we show how to put in magnetic charges.

7.2.1 M2-M2-M2 black hole

Let us write down a five dimensional electrically charged black hole by starting in M-theory (11-dimensions) and writing a solution down that involves a compact six-torus. Recall, in particular, the supergravity solution for the (supersymmetric) M2-M2-M2 brane system is

$$\begin{aligned} ds^2 &= -(Z_1Z_2Z_3)^{-2/3}dt^2 + (Z_1Z_2Z_3)^{1/3}\underbrace{(dx_7^2+dx_8^2+dx_8^2+dx_{10}^2)}_{\mathbb{R}^4} \\ &\quad + \frac{(Z_2Z_3)^{1/3}}{Z_1^{1/3}}(dx_1^2+dx_2^2) + \frac{(Z_1Z_3)^{1/3}}{Z_2^{1/3}}(dx_3^2+dx_4^2) + \frac{(Z_1Z_2)^{1/3}}{Z_3^{1/3}}(dx_5^2+dx_6^2)\,. \end{aligned} \tag{208}$$

This solution describes a black hole in five spacetime dimensions because we actually take the coordinates $x_1, \dots, x_6$ to be compact ($x_i \sim x_i + 2\pi$ for $i = 1, \dots, 6$) so they describe a six-torus T^6 which we wrote as the product of three two-tori T^2.

The M2-branes are all pointlike in the transverse $\mathbb{R}^4$ spanned by x^7, x^8, x^9, x^{10} which we can write in radial coordinates

$$ds_4^2 = d\rho^2 + \rho^2 d\Omega_3^2 \tag{209}$$

and the solution is determined by the functions:

$$Z_1 = 1 + \frac{Q_1}{\rho^2}\,, \qquad Z_2 = 1 + \frac{Q_2}{\rho^2}\,, \qquad Z_3 = 1 + \frac{Q_3}{\rho^2}\,. \tag{210}$$

These functions are actually defined simply by requiring them to solve the equation:

$$\Box_4 Z_I(x) = Q_I \delta(\rho) \tag{211}$$

where $\Box_4 = \nabla^2$ is defined with respect to the 4-d flat metric in the solution above (on $\mathbb{R}^4$). This equation just says that we have M2 sources sitting at $r = 0$ with charges Q_I. The 1 in the equation above is simply a homogenous solution we are free to add to any given solution to the the equation (211). Since this equation is linear we are free to add solutions so actually the most general solution corresponds to an arbitrary number of M2 sources at various positions $\vec{\rho}_p \in \mathbb{R}^4$ and p just labels a "center":

$$Z_I = \text{const} + \sum_p \frac{Q_p}{|\rho - \rho_p|^2} \tag{212}$$

We make two remarks: the unusual power 2 rather than 1 in the denominator above is because we are solving this equtaion in 4d rather than 3d. The fact that we have three functions labelled by $I = 1, 2, 3$ corresponds to the three T^2 factors in the metric which is simply part of our ansatz, see Figure 31.

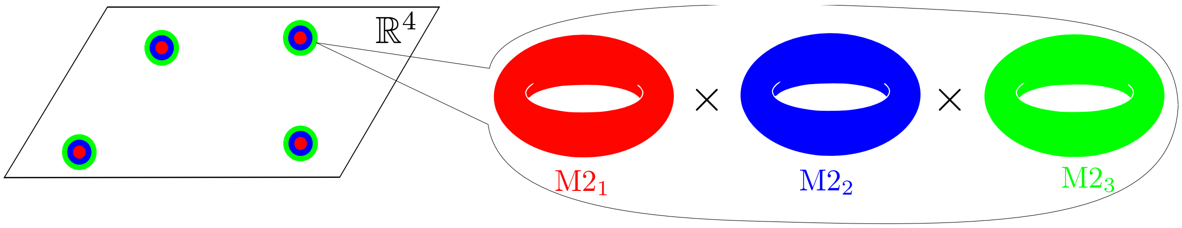


Figure 31: *Multiple M2-brane sources in $\mathbb{R}^4$. Each source can correspond to three types of M2-branes wrapped on a T^2 inside T^6.*

Recall that in M-theory we have a 3-form gauge potential and for the solution above it has the following form

$$C_{012} = Z_1^{-1}\,, \qquad C_{034} = Z_2^{-1}\,, \qquad C_{056} = Z_3^{-1}\,. \tag{213}$$

By "compactifying" on the $x_1, \ldots, x_6$ directions we can think of this as a five dimensional solution times T^6 and one can show that this six-torus is actually small (i.e. the length of each cyle is order 1 in string units) so at low energies this spacetime looks five-dimensional. In this case the different components of the three form C_{0ij} reduce to three independent gauge fields A_μ^I in five dimensions:

$$A_\mu^{(1)} = C_{\mu 12}, \qquad A_\mu^{(2)} = C_{\mu 34}, \qquad A_\mu^{(3)} = C_{\mu 56} \tag{214}$$

And likewise there are three field-strengths, $F^{(I)} = dA^{(I)}$ with $I = 1, 2, 3$.

In five dimensions this looks like a spherically symmetric, electrically charged black hole in $\mathbb{R}^{1,4}$.

We can generalize this solution in three ways

- ☞ Adding angular momentum
- ☞ Adding magnetic charge
- ☞ Adding a more complicated base space (instead of $\mathbb{R}^4$)

7.2.2 Adding Angular Momentum

The first generalization is to add angular momentum to this solution by replacing dt in the metric *and the gauge field* with $dt + k$ where $k = k_i dx^i$ is a one-form in the four-dimensional base space. Since the gauge field and metric are coupled via the equations of motion, adding angular momentum to the metric modifies the gauge field as well.

By adding a $k_i dx^i$ term to the metric we get non-vanishing g_{ti} cross-terms in the metric (from $(dt + k)^2$) and such terms imply that the spacetime itself carries angular momentum. This is *not* to be confused with being time-dependent. None of the fields above, including the metric, contains any explicit dependence on the time coordinate. Rather a good analogy is to consider a *featureless* spinning ring in for instance $\mathbb{R}^3$, see Firgure 32. Since the ring is featureless nothing changes in time: the ring is always just sitting there spinning but from one instance to the next everything looks identical. Nonetheless, this solution carries angular momentum. In GR, such solutions with mixed g_{ti} components but no time-dependence are referred to as *stationary*. Solutions with no time dependence and $g_{ti} = 0$ are *static*.

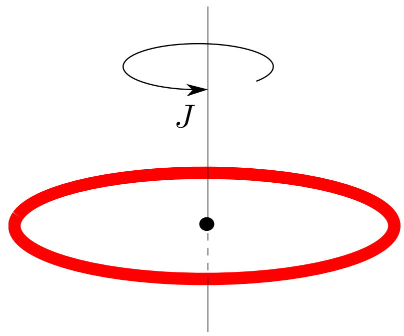


Figure 32: *A uniformly spinning ring with angular momentum J around its symmetry axis.*

In $\mathbb{R}^4$ there are two independent angular momentum, because we can think of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$: we have two independent angular momenta in each plane. Supersymmetry, however, is only preserved if we force these two angular momenta to be equal (unless we turn on additional feels as we will see later). This condition can be written as

$$(1 + \star_4)dk = 0 \tag{215}$$

which implies k is self-dual. Here $\star_4$ is the Hodge dual defined on the flat $\mathbb{R}^4$ given by $x_7, \dots, x_{10}$. Note that acting on this with d we find $d \star dk = 0$, meaning k is a harmonic one-form.

Exercise 7.5: *Angular momentum. . . still incomplete*

After turning on angular momentum we find that the field-strength, which used to be $F^{(I)}_{0\rho} = \partial_\rho A^{(I)}_0 = \partial_\rho Z^{-1}_I$, or in form notation

$$F = d(Z^{-1}dt)\,, \tag{216}$$

now get the more complicated form

$$F = d(Z^{-1}) \wedge (dt + k) + Z^{-1}dk \tag{217}$$

Note this field strength has magnetic F_{ij} components (from $\partial_i k_j$) now because we have a moving charge (this does not represent a genuine magnetic monopole charge).

Recall that without k we had the entropy $S_{BH} \sim \sqrt{Q_1Q_2Q_3}$. It turns out that when one turns out k we get an asymptotic angular momentum J. It can be read off from the asymptotic expansion of k in terms of the angles ϕ_1 and ϕ_2 in the two orthogonal $\mathbb{R}^2$-planes. If we write the metric on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ as

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi_1^2 + \cos^2\theta d\phi_2^2)\,, \tag{218}$$

the asymptotically leading terms of the momentum one-form k are

$$k = \frac{J}{r^2}\sin^2\theta d\phi_1 + \frac{J}{r^2}\cos^2\theta d\phi_2\,. \tag{219}$$

One can compute the horizon area to be

$$S \sim \sqrt{Q_1 Q_2 Q_3 - J^2}\,. \tag{220}$$

We see that angular momentum reduces the entropy. From a macroscopic point of view this is not hard to understand as the horizon is spinning very fast and this causes it to Lorentz contract and shrink. If we try to spin it up too fast, to the point that $J^2 = Q_1 Q_2 Q_3$, the horizon shrinks to zero size and indeed we cannot go further (at least not with this ansatz). Although we will not say much about it it is possible to reproduce this entropy using techniques quite similar to those of Section 4 (Lecture 3) (and indeed this was done shortly after the $J = 0$ entropy was first reproduced).

7.2.3 Magnetic Charges

Above we added angular momentum to the metric which did source magnetic components of the field strength but they did so in much the same way as a moving electron generates a magnetic field. While a speeding electron generates a magnetic field it does not generate a *magnetic charge* as discussed in the preliminaries of Section 7.1.

If we want magnetic charges we need to add a closed but not exact term to F which we denote by Θ^I. The field strengths becomes

$$F^{(I)} = d\left(\frac{(dt+k)}{Z^I}\right) + \Theta^I\,. \tag{221}$$

Of course this would not be consistent without modifying the form of the metric as well but it turns out this modification is rather straightforward. Recall that in the original metrics the Z_I were potentials sourced by delta-function sources at the locations of the M2's:

$$\Box_4 Z_I(x) = Q_I \delta(x) \tag{222}$$

This source naturally corresponds to an electric field which must satisfy

$$d \star F = \delta(x) \qquad dF = 0 \tag{223}$$

But recall that in string theory we have peculiar terms in the action such as

$$\int F \wedge \star F + \int A \wedge F \wedge F\,, \tag{224}$$

which implies that the magnetic part of the field-strength couples to the gauge potential like a source (magnetic fields F_{12}, F_{34} source electric components A_0). Put another way, magnetic flux in this theory can source electric charge via the equation

$$d \star F = F \wedge F\,. \tag{225}$$

This equation translates, in this setting, into a constraint on the functions Z_I which now are no longer simply sourced by a delta-function but now look like

$$\Box_4 Z_I(x) = \delta(x) + \star_4 \Theta_J \wedge \Theta_K \tag{226}$$

It is important to realize that what is happening here is that if we have two pairs of magnetic charges in this theory they can induce *electric* charge. Thus even if our solution has no explicit electric source (i.e. no delta function on the RHS above) there can be non-trivial electric charge carried by the field F itself. Note that this phenomenon, and even the equation above, should look very familiar from non-abelian gauge theories where the gauge field sources itself and carries electric charge (think of glueballs in QCD). The difference is that here we are dealing with an **abelian** theory, and the non-linear interactions arise because of the strange second term in the action (224).

While it is obvious that Θ must be closed away from sources this is not the only constraint it must satisfy. It is harder to show but it turns out that supersymmetry also imposes that the Θ's appearing above are self-dual so that

$$\Theta = \star_4 \Theta \,. \tag{227}$$

Angular Momentum from Crossed Fields Recall from electromagnetism that when the electromagnetic field has both an electric and magnetic component it carries angular momentum in the form of a Poynting vector

$$\vec{J} = \vec{E} \times \vec{B} \tag{228}$$

While the original solution given above had angular momentum coming from themetric encoded in the mixed metric components $g_{ti} \sim k_i$, the addition of a magnetic field changes the angular momentum. This is come from the supergravity equation

$$(1 + \star)dk = Z_I \Theta^I \tag{229}$$

which modifies (215) in a way that is essentially analogous to (228) with Z_I encoding the electric field and Θ^I the magnetic.

Exercise 7.6: *For a flavour of why a constraint like this might follow from SUSY consider the action for electromagnetism*

$$S = \int F \wedge \star F \tag{230}$$

and decompose $F = F^+ + F^-$ into self-dual and anti-self-dual parts $F^\pm = (1 \pm \star)F$. Rewriting the action in terms of $F^\pm$ show that it takes the form

$$S = \int F^+ \wedge F^+ - F^- \wedge F^- \tag{231}$$

so we see that if we put $F = F^+$ (or put otherwise $F^- = 0$) then the action is a postive definite perfect square. This is related, morally, to supersymmetry because the latter has a Hamiltonian $H = \{Q^\dagger, Q\}$ which is also a sum of squares implying in both cases that the energy is always greater an zero. In both cases solving the quadratic equations can be reduced to solving linear ones:

$$F^+ = 0, \qquad vs. \qquad Q|\phi\rangle = 0 \,, \tag{232}$$

and the solution are minimal action/minimal energy configurations.

Overview before continuing. We have derived the following system of equations that describes a solution with 3 electric charges, 3 magnetic charges and angular momentum:

$$\begin{aligned} \Theta^{(I)} &= \star_4 \Theta^{(I)}\,, \\ \Box_4 Z_I(x) &= \delta(x) + \star_4 \Theta_J \wedge \Theta_K\,, \\ (1+\star)dk &= Z_I \Theta^I\,. \end{aligned} \tag{233}$$

We wrote it in a suggestive form. To solve these equations, first find a set of self-dual two-forms $\Theta^{(I)}$ on $\mathbb{R}^4$. Then solve the functions Z_I in terms of those two-forms. Finally, find the momentum k from Z_I and $\Theta^{(I)}$.

Before we solve this system in that order, we extend the four-dimensional space $\mathbb{R}^4$ to a non-trivial base space.

7.2.4 Non-trivial base space

So far we have taken the metric along the $x_7, \ldots, x_{10}$ to be flat (up to some overall function). However, it turns out that supersymmetry does not require this space to be trivial but rather a more general condition known as hyper-Kähler.

An interesting and pretty general class of four-dimensional metrics that are hyperkahler are the so called Taub-NUT or Gibbons-Hawking metrics which take the form

$$ds_4^2 = V^{-1}(d\psi + A)^2 + V\underbrace{(dy_1^2 + dy_2^2 + dy_3^2)}_{\mathbb{R}^3}\,, \tag{234}$$

with

$$\nabla \times A = \nabla V\,, \tag{235}$$

and we have $\psi \sim \psi + 4\pi$.

Exercise 7.7: *Show that if we choose $V = \frac{1}{r}$ (with r the radial distance in the $\mathbb{R}^3$ given by y_i above) we recover the trivial metric on $\mathbb{R}^4$. In so doing you show that the ψ circle shrinks to zero size* smoothly *at the location of any pole in V since, whatever the form of V, near a pole it looks like $V \sim \frac{1}{r}$ so the spacetime near a pole always looks like a smooth $\mathbb{R}^4$ and* ***the spacetime is smooth at the location of the poles.***

The harmonic V on this space has the general form $V = \epsilon_0 + \sum_i \frac{q_i}{r_i}$ where now $r_i = |\vec{r}_i|$ and $\vec{r}_i \in \mathbb{R}^3$. When working on this space instead of $\mathbb{R}^4$ we will use $\star_3$ and r in place of $\star_4$ and ρ since this space only has three flat non-compact directions.

Exercise 7.8: *If we set $Z_I = 1$ and $V = 1 + \frac{n}{r}$ and we take the product of the spacetime (7.2.4) with $\mathbb{R}^{1,6}$ then we get an 11-d metric that is a solution of M-theory. As shown in the previous exercise this metric is* ***smooth*** *since the poles in V actually do not give any singularities in spacetime. Now check that we can reduce on ψ and get a 10-dimensional solution corresponding to a D6-brane in IIA supergravity (***hint:** *see Sec. 2.4 of Amanda Peet's lecture notes hep-th/0008241 or Polchinski Ch. 8 to see*

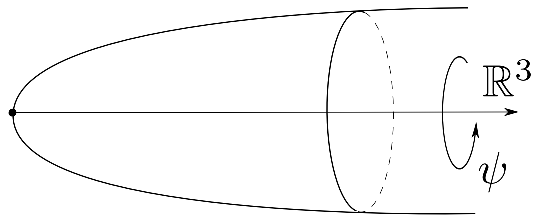


Figure 33: *Taub-NUT space (metric (7.2.4) with harmonic function $V = 1+n/r$) looks like a cigar. Near $r \to 0$, the ψ circle shrinks. Asymptotically, the ψ circle is of constant radius and spacetime asymptotes to $\mathbb{R}^3 \times S^1$.*

how to do the dimensional reduction). As a consequence, D6-branes in M-theory lift to smooth geometries in M-theory since the D6-brane poles correspondn to poles in the V function which are smooth in 11-dimensional spacetime.

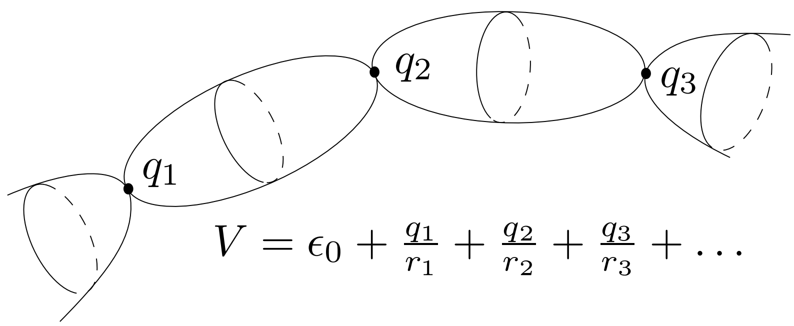


Figure 34: *Multicenter taub-NUT space ($V = \epsilon_0 + \sum_i q_i/r_i$) is a "bubbled geometry". At each center, the size of the ψ circle goes to zero and the geometry looks like smooth$\mathbb{R}^4$. Asymptotically, the geometry is $\mathbb{R}^3 \times S^1$.*

On this space, like $\mathbb{R}^4$, it is not hard to solve $\Theta = \star_4\Theta$. We first define the vielbeins

$$e^0 = V^{-1/2}(d\psi + A) \quad e^i = V^{1/2}dy_i\,, \tag{236}$$

such that the four-dimensional Taub-NUT metric (7.2.4) is written as a sum of squares:

$$ds^2 = (e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^3)^2\,. \tag{237}$$

Then one can check that

$$\Omega = (e^0 \wedge e^1 + e^2 \wedge e^3)$$

is self-dual (i.e. $\Omega = \star_4\Omega$). Note of course that there are actually three such self-dual Ω's (and three anti-self-dual ones) we can construct by permuting the first term $e^0 \wedge e^2$, $e^0 \wedge e^3$ (the second is fixed by (anti-)self-duality).

Exercise 7.9: *Check the above statement. First prove that*

$$\star_4\,(e^A \wedge e^B) = \epsilon^{ABCD}(e^C \wedge e^D) \tag{238}$$

for A, B, C, D all run from 0 to 3. and then prove that the three Ω^a defined as

$$\Omega^1 = e^0 \wedge e^1 + e^2 \wedge e^3, \quad \Omega^2 = e^0 \wedge e^2 + e^3 \wedge e^1, , \quad \Omega^3 = e^0 \wedge e^3 + e^1 \wedge e^2, , \quad (239)$$

are self-dual two-forms under $\star_4$.

We are actually interested in constructing $\Theta's$ and these must not only be self-dual but also closed (and hence co-closed because they're harmonic). Thus we start with Ω^a above defined as $e^0 \wedge e^a$, $a = 1, 2, 3$ and construct a closed Θ as

$$\Theta^I = \partial_a \left(\frac{K^I}{V} \right) \Omega^a \quad (240)$$

Exercise 7.10: *Show that Θ^I so defined is closed if K^I is harmonic on $\mathbb{R}^3$ ($\nabla^2 K^I = 0$).*

7.3 Equations of Motion and Supersymmetry

We have introduced a number of complications and have not really carefully specified how this modifies the equations of motion and supersymmetry conditions. It turns out, however, that there's a nice way to repackage all of this)in a way that is relatively tractable.

We start by taking a more general class of metric given as follows but now with the Z-functions unspecified

$$\begin{aligned} ds^2 &= -(Z_1 Z_2 Z_3)^{-2/3}(dt + k)^2 + (Z_1 Z_2 Z_3)^{1/3} ds^2_{TN} \\ ds^2_{TN} &= V^{-1}(d\psi + A)^2 + V \underbrace{(dy_1^2 + dy_2^2 + dy_3^2)}_{\mathbb{R}^3} . \end{aligned} \quad (241)$$

For simplicity we work directly in five dimensions and neglect the non-compact part of the geometry (though that is easy to add in). The field strength now takes the form

$$F^{(I)} = d\left(\frac{(dt + k)}{Z^{(I)}} \right) + \Theta^{(I)} . \quad (242)$$

The solutions above involve unkowns $k = k_i dx^i$, Z_I and $\Theta^I = \Theta^I_{ij} dx^i \wedge dx^j$. We take the base space to be fixed but of course this means we should specify a V and then fix A via $\nabla \times A = \nabla V$.

Supersymmetry and the equations of motion can now be simply repackaged into the following conditions

$$\Theta^I = \star_4 \Theta^I , \quad (243)$$

$$\nabla^2 Z_I = \star_4 \frac{1}{2} C_{IJK} \Theta^J \wedge \Theta^K , \quad (244)$$

$$(1 + \star_4) dk = Z_I \Theta^I , \quad (245)$$

where C_{IJK} is the completey symmetric tensor.

We have already specified how to solve (243) on the Taub-NUT base space so we only need to solve the last two equations. It turns out that these system of equations are essentially *linear* if solved in the right order (there are no quadratic interactions or fields sourcing *themselves* quadratically). So once we have Θ we can plug it into (244) and solve for Z_I. The solution must be sourced by the RHS of (244) but can also include a homogenous countribution that solves the equation $\nabla^2 Z_I = 0$. Combining these we get

$$Z_I = \frac{C_{IJK} K^J K^K}{V} + L_I \,, \tag{246}$$

Where L_I is a harmonic function (on $\mathbb{R}^3$) satisfying $\Delta^2 L_I = 0$.

Exercise 7.11: *Check that Z_I given above satisfies (244).*

Note that the final equation (245) simply reproduces the (anti-)self-duality condition we mentioned above ($dk = - \star dk$) in the absense of explicit magnetic source ($\Theta = 0$). When such sources are turned on we solve this equation by taking the following ansatz:

$$k = \mu(d\psi + A) + \omega \,, \tag{247}$$

with $\omega = \omega_i dx^i$ a form on $\mathbb{R}^3$.

Exercise 7.12: *Show that plugging this into (245) yields an equation for ω*

$$\vec{\nabla} \times \vec{\omega} = (V\vec{\nabla}\mu - \mu\vec{\nabla}V) - V\, Z_I \vec{\nabla}\left(\frac{K^I}{V}\right) \tag{248}$$

where as always we sum over $I = 1, \ldots, 3$.

To solve the equation above for ω we take a further divergence and use $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\omega}) = 0$ to argue

$$V\nabla^2\mu = 2\vec{\nabla} \cdot \left(V\, Z_I \vec{\nabla}\left(\frac{K^I}{V}\right)\right) \,. \tag{249}$$

Exercise 7.13: *Show that this can be solved as*

$$\mu = \frac{1}{6} C_{IJK} \frac{K^I K^J K^K}{V^2} + \frac{K^I L_I}{2V} + M \,, \tag{250}$$

and that the corresponding solution for ω is

$$\vec{\nabla} \times \omega = V\vec{\nabla}M - M\vec{\nabla}V + \frac{1}{2}(K^I \vec{\nabla} L_I - L_I \vec{\nabla} K^I) \tag{251}$$

While it is possible to get an explicit form for ω in simple cases one generally has to resort to patches to specify the solution for ω given the harmonics V, K^I, L_I and M. Note that there is a nice and clean way of writing the solution for ω in terms of the harmonic functions. Write the harmonic functions as a 2 by 4 matrix

$$H \equiv \begin{pmatrix} V & M \\ L_1 & K^1 \\ L_2 & K^2 \\ L_3 & K^3 \end{pmatrix} \,. \tag{252}$$

Then the RHS of (251) defines a symplectic product of such matrices:

$$\vec{\nabla} \times \omega = \langle H, \vec{\nabla} H \rangle \,. \tag{253}$$

7.4 Physical Solution and Fuzzballs

Summary. Above we have given a general form of a solution in terms of some functions and forms. It turns out all of these can be specified in terms of eight harmonic functions V, K^I, L_I and M. We started with a black hole with harmonic functions $Z_I = L_I$ and angular momentum:

M2's: L_1, L_2, L_3	Angular Momentum: M

Now we have also some magnetic fields. The black hole charge can be dissolved in these fields. They are encoded in the four extra harmonic functions:

M5's: K^1, K^2, K^3	Taub-NUT: V

These harmonic function were found from solving the system (243–245) in four steps:

1. Choose the four-dimensional hyper-Kähler base space. We have chosen the multi-center Taub-NUT metric (7.2.4). This introduces one harmonic function on $\mathbb{R}^3$, V.
2. Find the three self-dual two-forms. We have done this in Exercise 7.10. This introduces three more harmonic functions K^I. Their poles are magnetic charges (M5 branes in 11d).
3. Solve the functiona Z_I. This introduces three harmonic functions we call L_I.
4. Find the momentum one-form k. This introduces one more harmonic function M.

Recall that a harmonic function in $\mathbb{R}^3$ satisfies $\nabla^2 H = 0$ which has the general solution

$$H = \text{const} + \sum_p \frac{Q_p}{|\vec{r} - \vec{r}_p|}\,. \tag{254}$$

where $\vec{r}_p$ are arbitary vectors in $\mathbb{R}^3$ at which H can be singular. In fact $\nabla^2 H = 0$ only away from $\vec{r}_p$ and is actually given as $\nabla^2 H = Q_p\,\delta(r - r_p)$. We see that our solution can have an arbitrary number of centers ('sources') on $\mathbb{R}^3$.

Physical requirements. At this point, getting the solution from harmonic function is like blindly using a computer. We still have many questions: Is this solution physical? Are there singularities? What are its properties? We will answer these questions now.

We start with the vector $\vec{\omega}$. To have it well-defined in spacetime, the divergence of (251) should be zero:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\omega}) = 0\,. \tag{255}$$

From the RHS of (251), the harmonic functions then need to satisfy:

$$0 = V\nabla^2 M - M\nabla^2 V + \frac{1}{2}(K^I \nabla^2 L_I - L_I \nabla^2 K^I)\,. \tag{256}$$

The leading terms are those at the positions of the centers. Writing the charges for the harmonic functions at each center as Q^i_H, we have

$$0 = \sum_i (V Q^i_M - M Q^i_V)\delta(\vec{r}_i) + \frac{1}{2}\sum_i (K^I Q^i_{L_I} - V_I Q^i_{K^I})\delta(\vec{r}_i)\,. \tag{257}$$

We can write this as the symplectic product of the vector of harmonic functions H

$$H = C + \sum_i \frac{Q_H^i}{r_i}\,, \tag{258}$$

with the vector of charges Q_H^i at each center:

$$\sum_i \langle H, Q_H^i \rangle \delta(\vec{r}_i) = 0\,. \tag{259}$$

The physical interpretation of this equation is that it assures there are no Dirac-Misner strings in the geometry (such that there is no source on the RHS of (255)).

For generic charges, you get an $\vec{\omega}$. Once the charges are fixed, the equations (259) then give constraints on the center positions $\vec{r}_i$: these equations tell you where the points are. We call these 'bubble equations' (r's in terms of Q's), because the resulting geometries have 'bubbles' (non-trivial two-cycles). Other names for these equations are 'integrability equations' (term coined by the original discoverer, Denef) and 'Denef equations'.

What are these relations the positions $\vec{r}_i$ of the centers have to satisfy such that $\vec{\omega}$ is well-defined? Take three centers for concreteness. The harmonic functions are

$$H = \frac{Q_H^1}{r_1} + \frac{Q_H^2}{r_2} + \frac{Q_H^3}{r_3} + C\,. \tag{260}$$

From (259), we get three equations, one at each center (from the $\delta(\vec{r}_i)$-contributions)

$$\begin{aligned}
\frac{\langle Q_H^1, Q_H^2\rangle}{r_{12}} + \frac{\langle Q_H^1, Q_H^3\rangle}{r_{13}} + \langle Q_H^1, C\rangle &= 0\,,\\
\frac{\langle Q_H^2, Q_H^1\rangle}{r_{12}} + \frac{\langle Q_H^2, Q_H^3\rangle}{r_{23}} + \langle Q_H^2, C\rangle &= 0\,,\\
\frac{\langle Q_H^1, Q_H^3\rangle}{r_{13}} + \frac{\langle Q_H^2, Q_H^3\rangle}{r_{23}} + \langle Q_H^3, C\rangle &= 0\,.
\end{aligned} \tag{261}$$

where r_{ij} is the distance between points 1 and 2:

$$r_{ij} = |\vec{r}_i - \vec{r}_j|\,. \tag{262}$$

Questions from the audience:

- *Is this some balance of forces?* The symplectic products pairs electric with magnetic charges (V, L_I are electic, K^I, M magnetic). You get a huge angular momentum forcing the points away from each other.
- *Why is this stable then?* Supersymmetry is the answer. All forces cancel.

What is the space of solutions of the bubble equations? For simplicity, we restrict to two centers first. Then there is only one equation:

$$\frac{\langle Q_1, Q_2\rangle}{r_{12}} + \langle Q_H^1, C\rangle = 0\,. \tag{263}$$

We should have $\langle Q_H^1, C\rangle < 0$ to find a solution. This fixes the distance r_{12}, we get 2 points fixed by a rigid rod. The system has two degrees of freedom: two points in spacetime have three degrees of freedom in $\mathbb{R}^3$ (three for each point, minus three for the centre of mass), and the bubble equation fixes one. The solutions space is the S^2 of possible positions of the second point at a distance r_{12} of the first one.

The constant C is the vector of constants in the harmonic functions:

$$H = C + \sum_i \frac{Q_H^i}{r_i}\,, \tag{264}$$

and it determines what the space looks like asymptotically (for instance $V = c_V + Q_V/r$). A very interesting thing can happen. Take a space with constants $C = (c_V, c_L, c_K, c_M)$ such that $\langle Q_H^1, C\rangle < 0$ and the bubble equations (263) have a solution. By tuning the asymptotic parameters, we could go from $\langle Q_H^1, C\rangle < 0$ to $\langle Q_H^1, C\rangle = 0$ and even $\langle Q_H^1, C\rangle > 0$: the solution disappears (it is no longer a valid physical solution). In the solution space, the boundary $\langle Q_H^1, C\rangle = 0$ is called a "wall of marginal stability". When crossing a wall of marginal stability ("wall-crossing"), these states just disappear. When $\langle Q_H^1, C\rangle < 0$, the solution is part of the solution space, and we have an entropy associate to them (the number of such states). When we cross the wall of marginal stability, the solution is gone and the entropy jumps.

Let's take a solutions with thee centers. Define

$$A_{ij} \equiv \langle Q_i, Q_j\rangle\,. \tag{265}$$

Note that the symplectic product is antisymmetric and hence so is the matrix A. By a cyclic permutation of charges at the different centers, we can always take

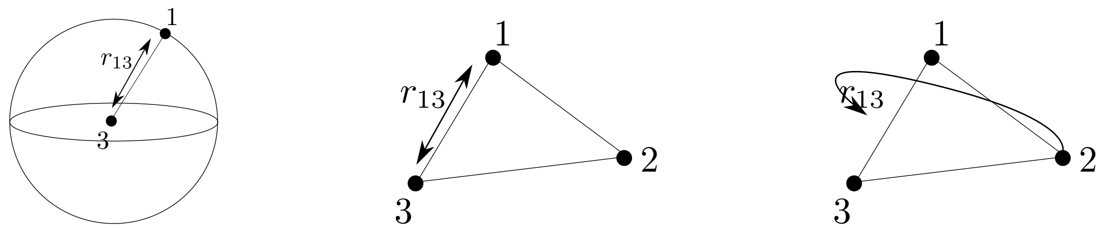
$$A_{12} > 0\,, \qquad A_{23} > 0\,, \qquad A_{31} > 0\,. \tag{266}$$

Then the bubble equations are

$$\begin{aligned}
\frac{A_{12}}{r_{12}} - \frac{A_{31}}{r_{13}} + C_1 &= 0\,,\\
-\frac{A_{12}}{r_{12}} + \frac{A_{23}}{r_{23}} + C_2 &= 0\,,\\
\frac{A_{12}}{r_{12}} - \frac{A_{23}}{r_{23}} + C_3 &= 0\,,
\end{aligned} \tag{267}$$

where the constants $C_i = \langle Q_H^i, C\rangle$. Only two of these equations are independent (for instance the sum of the first two gives the third one), and they leave only one of the distances r_{ij} unfixed. In total, three centers in $\mathbb{R}^3$ have 9 degrees of freedom. Only relative positions are relevant, that kills three (center of mass degrees of freedom). The bubble equations fix two more. We thus have 4 degrees of freedom left. We can take these to be

- The radius r_{13} (1 dof)
- The orientation of r_{13} (2 dofs)



(a) S^2 of orientations of r_{13}

(b) Scale of r_{13}

(c) $U(1)$ angle around r_{13}

Figure 35: *A three-center configuration has 4 free parameters by the bubble equations.*

- The $U(1)$ angle around r_{13}

For n points, the bubble equations allow for a $2(n-1)$-dimensional space of solutions.

One solution looks very interesting. If the triangle inequalities are satisfied:

$$A_{12} + A_2 3 \geq A_{31} \,, \tag{268}$$

(and cyclic), there is a limit where the radii go te zero:

$$\begin{aligned} r_{12} &= |A_{12}|\epsilon + \mathcal{O}(\epsilon^2)\,, \\ r_{13} &= |A_{13}|\epsilon + \mathcal{O}(\epsilon^2)\,, \\ r_{23} &= |A_{23}|\epsilon + \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{269}$$

As $\epsilon \to 0$, the bubble equations are satisfied up to first order, because the constants C_i are 'eaten up' by order $\mathcal{O}(\epsilon)$ terms, for instance:

$$\frac{A_{12}}{r_{12}} = \frac{1}{\epsilon} + \mathcal{O}(\epsilon)\,. \tag{270}$$

The r_{ij}'s are the lengths of the sides of a triangle and always satisfy triangle inequalities. The limit $\epsilon \to 0$ can only be done when also the $|A_{ij}|$ satisfy the triangle inequalities. We then have a limit where all radii go to zero. The points sit on a fixed triangle which gets smaller and smaller. If the triangle inequalities are not satisfied, we cannot have such a scaling limit.

Scaling solutions. What is so special about these solutions? We have stated before the idea to replace the black hole geometry with some other object. In this section, we have made this more concrete. We can find an object with the same (electric/M2) charges as the black hole, but which also has magnetic charges. The black hole is replaced by a solution with many centers and magnetic charges, by finding the solution from the harmonic functions V, K^I, L_I, M.[21]

The solutions can go scaling, such that the several centers can come closer and closer by sending some parameter $\epsilon \to 0$. When the guys are on top of each other, we recover the black hole.

[21] In fact, there are certain conditions the harmonic functions H have to obey such that the multicenter geometry is also smooth and horizonless at each center. We will not dwell on that.

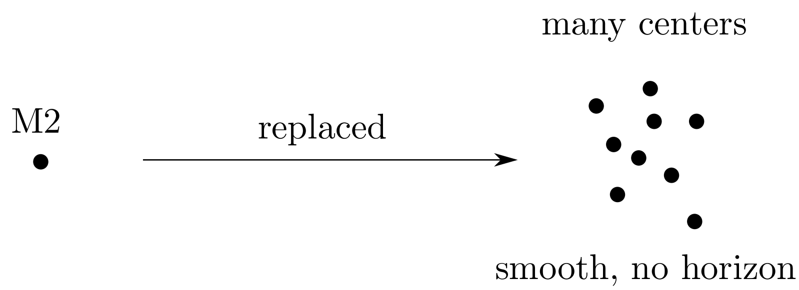


Figure 36: *Replacing the black hole with a multi-center configuration*

Remember that we were considering extremal black holes. These have an infinitely deep throat. A scaling solution with scaling size ϵ, has a throat of length $L \sim 1/\epsilon$. As $\epsilon \to 0$, you get a throat with a cap that gets longer and longer. These solutions form an infinite family, see Figure 37 for an illustration.

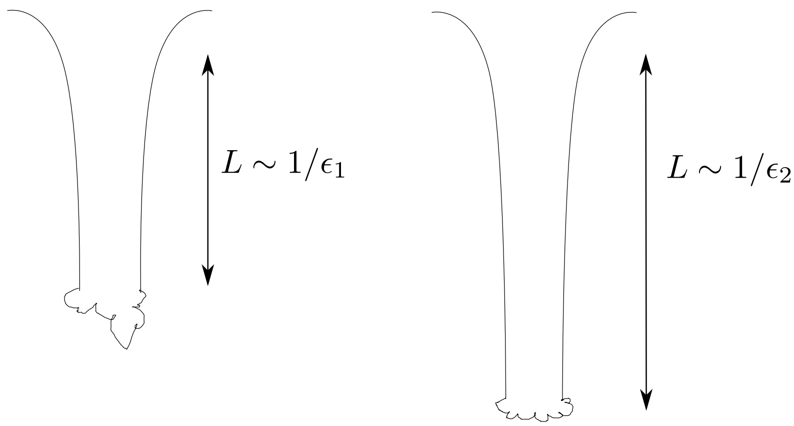


Figure 37: *For every value of ϵ we find a scaling solution with a deep throat. As $\epsilon \to 0$, we recover the infinitely deep black hole throat.*

The scaling solution form an infinite family: we can make ϵ smaller and smaller, we always find good solutions. But from AdS/CFT, we know that there is a finite entropy

$$S = \sqrt{Q_1 Q_2 Q_3}\,, \tag{271}$$

which tells us there is a finite number of states. This is a puzzle:

☞ $N_{\text{micro}} = e^{S_{\text{micro}}}$ is large but finite.

☞ $N_{\text{class. grav.}}$ (number of scaling solutions) is infinite.

How to reconcile these pictures? That's for the next section!

Note: only a subset of this multicenter solutions are actual fuzzballs. We need some more information to discuss them, we will leave it at this for the moment.

8 Quantizing geometries

8.1 Constraint equations and solution space

We start in eleven-dimensions from the metric and gauge field

$$\begin{aligned} ds^2 &= (Z_1Z_2Z_3)^{-2/3}dt^2 + (Z_1Z_2Z_3)^{1/3}ds_4^2 + ds^2(T^6)\,, \\ F_4 &= [d(Z_1^{-1}(dt+k)) + \Theta^I] \wedge dx_1 \wedge dx_2 + \ldots \end{aligned} \tag{272}$$

with the four-dimensional multi-center Taub-NUT metric

$$ds_4^2 = V^{-1}(d\psi + A) + V ds^2(\mathbb{R}^3)\,. \tag{273}$$

The functions Z_I, one-form k and two-forms Θ^I that determine the solution are found from the harmonic functions

$$H \equiv (V, K^I, L_I, M)\,, \tag{274}$$

as explained in the previous section.

The harmonic functions satisfy a sourced harmonic equation:

$$\nabla^2 H = \sum_i Q_p \delta(\vec{r} - \vec{r}_p)\,. \tag{275}$$

The solution is

$$H = \sum_{p=1}^{N} \frac{Q_p}{|\vec{r} - \vec{r}_p|} + h_0\,, \tag{276}$$

where $\vec{r}_p$ are the position vectors of the different centers in $\mathbb{R}^3$ and h_0 is a vector of constants for the different harmonic functions. The charges at each center give poles in the harmonic functions, corresponding to multiple sources, and each may or may not have a horizon.

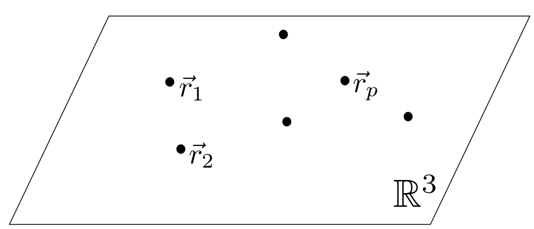


Figure 38: *The multi-center solutions are sourced on multiple positions in the $\mathbb{R}^3$ base of Taub-NUT space.*

Given a set of asymptotic charges $Q = \sum_p Q_p$, the solution space of all possible solutions with N centers, is given by the possible ways in which we are allowed to divide the charges over the centers. At first glance, we would think this space is $\mathbb{R}^{3N-3}$, the space of locations of N centers on $\mathbb{R}^3$.[22]

[22] Only the relative positions are of importance, hence the degrees of freedom of one of the centers do not count and we get $3N-3$ coordinates that specify a physical solution with N centers.

However, the positions of the centers are constrained in terms of the charges, by the bubble equations:

$$\forall p : \qquad \sum_{\substack{q=1 \\ q \neq p}}^{N} \frac{\langle Q_p, Q_q \rangle}{|\vec{r}_p - \vec{r}_q|} + \langle Q_p, h \rangle = 0 \,. \tag{277}$$

We write the harmonic functions and charges as symplectic vectors:

$$H = (\underbrace{V,\, K^I}_{\text{elec.}}, \underbrace{L_I,\, M}_{\text{magn.}}) \,, \qquad \Gamma = (q_V, q_K^I, q_{L,I}, q_M) \,. \tag{278}$$

For every point, we have eight charges.

Given two symplectic vectors of harmonic functions H and H', there is a symplectic inner product that we can define, that connects electric with magnetic components

$$\langle H, H' \rangle = VM' - MV' + K^I L'_I - L_I K'^I \,. \tag{279}$$

This product is antisymmetric. You should think of it as giving momentum from crossed electric and magnetic fields, similar to the Poynting vector in electromagnetism:

$$\vec{J} = \vec{E} \times \vec{B} \,. \tag{280}$$

What is the physical meaning of the constraints (277)? Each indivdual center breaks down the $N = 2$ supersymmetry of the supergravity theory to $N = 1$. There is a phase is associated with this supersymmetry breaking, one phase per center. Generically all the centers break supersymmetry to a different residual $N = 1$, each center has a different phase. When the distances between the centers are chosen such that we can continuously connect the phases from all centers, there is one overall (position-dependent) $N = 1$ supersymmetry that is left, and the solution is stable. The equations (277) exactly express that condition.

Say you have 2 centers. The Poynting vector gives an angular momentum "binding". For electromagnetism in flat space, we get for a magnetic charge m and an electric charge q that

$$J = \frac{qm}{2} \,, \tag{281}$$

no matter what the distance is between the two centers. With gravity, the angular momentum depends on the distance between the centers:

$$J = \frac{qm}{r} \,, \tag{282}$$

and there is a non-zero force. The constraint equations can be interpreted as the condition for all those forces to balance.

Exercise 8.1: *Show that the sum over p (from 1 tot N) of* (277) *is zero.*

From Exercise 8.1, we see that there are in fact only $N-1$ independent constraints. Therefore, the solution space is a $(2N-2)$ dimensional submanifold of $\mathbb{R}^{3N-3}$:

$$M_{2N-2} \subset \mathbb{R}^{3N-3} \,. \tag{283}$$

For instance, for two centers we get

$$M_2 \subset \mathbb{R}^3 \,. \tag{284}$$

The constraint fixes the distance $r_{12} = \vec{r}_1 - \vec{r}_2$ and corresponds to the possible rotations of the position $\vec{r}_2$ around $\vec{r}_1$, where both are connected by the fixed rod r_{12}. Therefore

$$M_2 = S^2 \,. \tag{285}$$

The constraint equations should be understood as follows. When we fix the asymptotic charges, there is still a continuous family of positions we can vary. Hence the solutions space is a function of the charges $M_{2N-2}(Q)$. We want to calculate the "number of states" in the solution space. For a given charge vector Q, this would hopefully reproduce the entropy of a single center black hole with total charge Q. To get the number of states, we have to quantize the solution space. Therefore we first give some basic quantum mechanics to see how to get a quantum space out of a classical solution space.

8.2 Basic quantum mechanics

8.2.1 Hamiltonian formulation

We start with the Hamiltonian formulation. This is non-covariant, as it separates out time, but it will make things clear.

Say we have a Lagrangian of a system with coordinates q:

$$L(q^i, \dot{q}^i) \,. \tag{286}$$

The generalized momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \,. \tag{287}$$

For ease of notation, we will mostly suppress indices on position and momentum vectors.

Given the Hamiltonian as a function of the generalized coordinates and momenta

$$H(q,p) = p\dot{q} - L \,, \tag{288}$$

the equations of motion are

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} \,, \\ \dot{q} &= \frac{\partial H}{\partial p} \,, \end{aligned} \tag{289}$$

The phase space is the space of q's and p's at a certain time. Because the q's are spatial positions at a fixed time, this is clearly non-covariant.

The Poisson bracket is defined as

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial q}\,. \tag{290}$$

One often has systems with $\{q,p\}=1$, but this is not necessary. In general, we have

$$\{q^i,p^j\} = \omega^{ij}\,. \tag{291}$$

It is very important that ω^{ij} is invertible. Then we can define the symplectic two-form:

$$\omega \equiv \omega_{ij} dq^i \wedge dp^j\,. \tag{292}$$

ω defines the complex structure of the phase space. In short, in the Hamiltonian formalism, (p,q,H,ω^{ij}) are *given*.

Consider a D-dimensional subspace M_D of the $2D$-dimensional phase space, as in Figure 39. Describe the D coordinates on M_D as x^i (these are p's and/or q's).

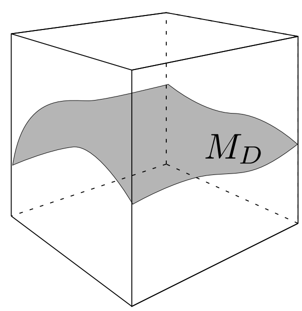


Figure 39: *A D-dimensional subspace M_D.*

If

$$\omega|_{M_d} = 0\,, \tag{293}$$

then one finds that

$$\{x^i,x^j\} = 0\,, \tag{294}$$

and M_D defines a Lagrangian subspace.

In quantum mechanics, the Poisson bracket is replaced by a commutator

$$[q,p] = i\hbar\,. \tag{295}$$

The p's and q's no longer describe $2N$ numbers but some of them are operators. Normally, we take the q's to be commuting numbers, and p are their derivatives

$$p = \frac{\hbar}{i}\frac{\partial}{\partial q}\,. \tag{296}$$

When we are on a Lagrangian submanifold, the coordinates on M_D all commute:

$$[x^i,x^j] = 0\,, \tag{297}$$

and the x^i define 'nice' coordinates (numbers). The states of the system are then functions of coordinates. The Hilbert space is the space of wave functions

$$\mathcal{H} = \{\psi(x) \in \mathcal{L}^2(M_D, \mathbb{C})\} . \tag{298}$$

In quantum mechanics, this is a good way to get the Hilbert space $\mathcal{H}$.Remark that in geometrric quantization, we actually need a foliation of Lagrangian submanifolds $M_D(t)$, where t is time. Mostly we have nice spaces and we do not need to go this far.

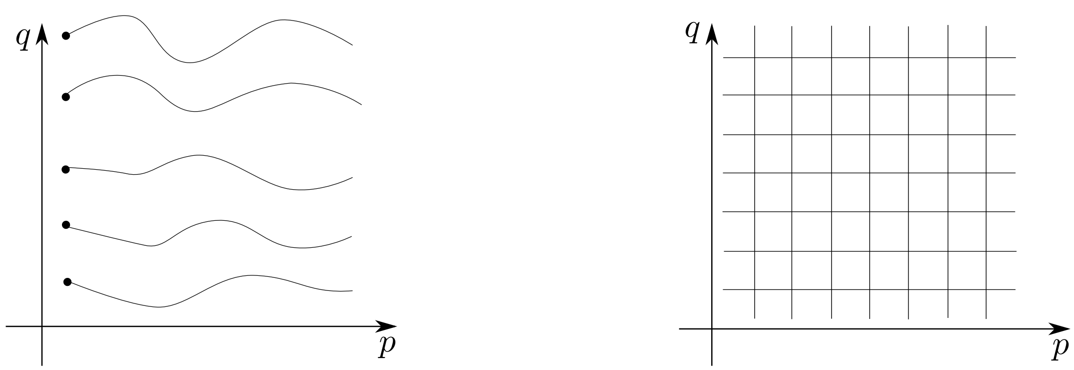
However, there is a different way to get the states, or at least the number of states:

$$\#\text{ states} = \int_{\text{phase space}} \omega . \tag{299}$$

Classically, this is not a number, because it is not integer quantized. In quantum mechanics, it is an integer and really counts the number of states in the Hilbert space. As a side remark, note that this is often related to the index of some operator D:

$$\operatorname{ind} D = \int (\ldots) . \tag{300}$$

In our setup, things are simple enough that the $\ldots$ are just ω^D. A hand-waving argument for the integer quantization is given in Figure 40.



(a) Classically, we can continuously integrate histories.

(b) In quantum mechanics, phase space is a discrete grid of points.

Figure 40: *Classical versus quantum phase space. The volume of classical phase space can be a real number, in quantum mechanics it is an integer.*

Remark by Iosif:

☞ *Classically, I expect an infinite number of states (everyting is continuous), I should be able to go anywhere in phase space and hence I have an infinite number of allowed states. But classically, $\int \omega^D$ should be finite? Is there a clash?* We will answer this question explicitly in an example below. Yes, classically the number of states is infinite, but the *volume* of phase space is finite. Only in quantum mechanics, the volume is the number of states.

Exercise 8.2: *Consider a particle in a box of length L.*

1. *Compute the number of quantum states: calculate the integral*

$$\int_0^L \int_0^{p_{\max}} \omega\,, \tag{301}$$

with

$$[x,p] = \omega^{-1} \tag{302}$$

and $p_{\max}$ should be allowed quantum value (see a textbook on quantum mechanics). Convince yourself this integral counts the number of states.

2. *Repeat the calculation for a two-dimensional box.*

Let's take a close look at a simple example to get things going. Take the Hamiltonian of a free particle

$$H = \tfrac{1}{2}p^2\,. \tag{303}$$

Given q and p, we can always define the complex coordinates on phase space:

$$z = q + ip\,, \qquad \bar{z} = q - ip\,. \tag{304}$$

Then we have the commutation relation

$$[z, \bar{z}] = 1\,. \tag{305}$$

Note that there is no longer one of the $z, \bar{z}$ that is a unique "coordinate" or "momentum". We can take the wave function to depend on z:

$$\psi(z)\,. \tag{306}$$

Now the number of states is counted by an index

$$\mathrm{ind}\,(\bar{\partial}) = \#\ \mathrm{states}\,, \tag{307}$$

with $\bar{\partial}$ the Dolbeault operator. Note that this method needs a complex structure on phase space, which can not always be defined. For a simple manifold like $\mathbb{R}^{2N}$ it can be done. If there is a complex structure, then it turns out that the above gives a good way to quantize.

Consider now a slight extension of the free particle model. Couple it to an electromagnetic field. The Lagrangian is

$$L = \tfrac{1}{2}(\dot{q} + Aq)^2\,. \tag{308}$$

The momentum is

$$p = \dot{q} + Aq\,. \tag{309}$$

This is very different from previous examples! Even if there is no velocity, $\dot{q} = 0$, there is still a non-vanishing momentum. When there are space components of the gauge field

$$A_i \neq 0\,, \tag{310}$$

the coordinates no longer commute:

$$\omega^{ij} = [q^i, p^j] = A_j[q^i, q^j] \neq 0\,. \tag{311}$$

The non-commutatitvity of phase space becomes a non-commutativity of the physical space due to the magnetic field A_i!

8.2.2 From phase space to solution space

We prove the following claims:

1. The number of states is the symplectic volume of phase space.

2. Phase space is isomorphic to solution space (up to some caveats).

and hence:

☞ **The number of states is the symplectic volume of solution space.**

The first claim is ok from the previous arguments. The last one follows trivially from the other two. We only have to prove the second one.

Given any initial point, I can integrate to a full history. The p's act morally as velocities, and they allow to integrate$q(t_0)$ for any t_0 to a further time step. See Figure 41(a). Any point in phase space gives you a solution to the equations of motion. Conversely, a solution $q(t)$, gives you a point in phase space for any t, see Figure 41(b). This proves statement 2.

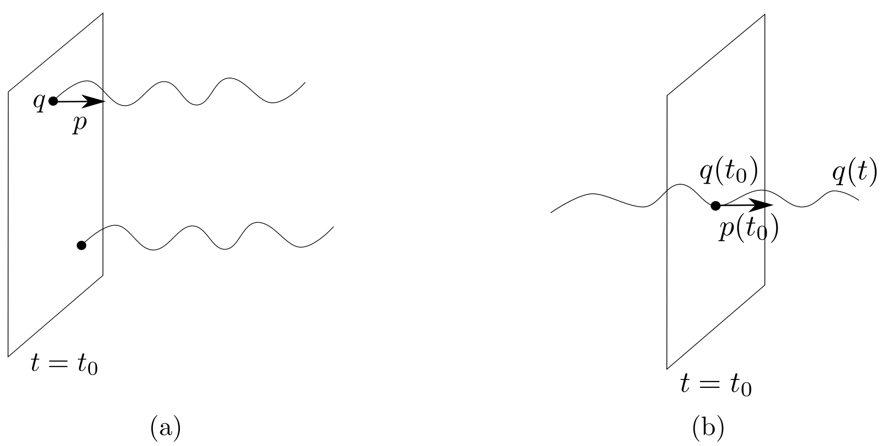


Figure 41: *Left: Given an initial configuration at $t = t_0$, we can integrate the equations of motion to obtain the full solution $q(t), p(t)$. Right: given a solution $q(t)$, we have a phase space at every t.*

There is a map to extract the symplectic form from the Lagrangian formulation. This allows to stay in solution space without going to phase space for the calculation of the number of states. Note also that we mad a statement about the full solution space and the full phase space. It is not clear if we can restrict to a subsector.

8.3 Intermezzo: from QM to QFT and GR

We want to go from quantum mechanics (QM) to Quantum Field Theory (QFT). In QM, the points at time t are unconstrained, and the wave function $\psi(x)$ is a function of the unconstrained positions. In QFT, the points on each time slice are now fields ϕ that are constrained by the equations of motion, and the wave function $\psi(\phi)$ is a function of those constrained fields.

Side remark: we use the formulation of time slices and evolution of the fields from one to the other, defining wave functions on each slice. This is equivalent to the path integral forumlation

$$\langle\psi'|e^{iHt}|\psi\rangle = \int \mathcal{D}e^{-S} \,. \tag{312}$$

In field theory, the coordinates and momenta are replaced by fields:

$$\begin{aligned} q &\rightarrow \phi(x) \\ p &\rightarrow \Pi(x) = \frac{\partial L}{\partial\dot{\phi}} \,. \end{aligned} \tag{313}$$

As before for mechanics, in field theory we consider the fields on a spatial slice as in Figure 42

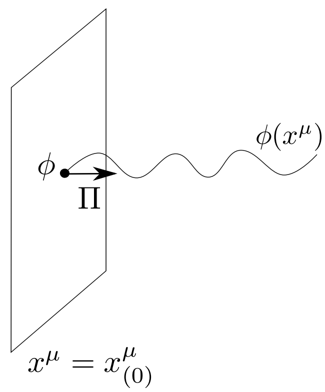


Figure 42: *Fields on a spatial slice of constant t..*

In GR, things are a little more tricky than in field theory. We define spatial slices Σ, Figure 43. We use a metric adapted to the slices

$$ds^2 = (N^2 + \beta_k\beta^k)dt^2 + 2\beta_k dx^k dt + h_{ij}dx^i dx^j \,, \tag{314}$$

in terms of the data

$$(h_{ij}, \beta_k, N). \tag{315}$$

One finds that β_k and N are non-dynamical variables, there momenta are zero:

$$\Pi^\beta = 0 \,, \qquad \Pi^N = 0 \,. \tag{316}$$

These equations are interpreted as constraints on the other fields. The only dynamical variables are the three-dimensional metric h_{ij} and its momenta Π_h:

$$\Pi_h^{ij} \equiv \frac{\delta L}{\delta\partial_t h_{ij}} \,. \tag{317}$$

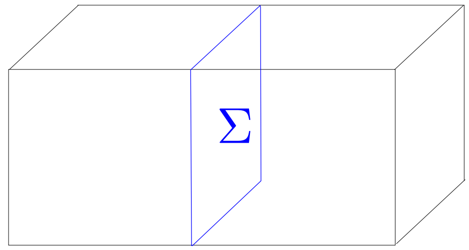


Figure 43: *GR on a spatial slices* Σ.

What terms contribute to the momentum Π_h? These are terms in the Lagrangian of the form:

$$L = \ldots + \partial_t H_{ij} \Omega^{t,ij} + \ldots \tag{318}$$

Assume first that $\beta_k = 0$. Then the metric has no mixed terms:

$$g_{\mu\nu} = g_{ii} + g_{tt}\,, \tag{319}$$

and $\partial_t h_{ij}$ can only talk to something else ($\Omega^{t,ij}$) with another time derivative and hence

$$\Pi^{ij} \sim \dot{h}^{ij}\,. \tag{320}$$

For time-independent solutions, this does not give a constribution.

If on the other hand $\beta_k \neq 0$, then $\partial_t h_{ij}$ can couple to terms like $\partial^i g^{tj}$ etc., with *spatial* derivatives. Therefore,

$$\Pi^{ij} \sim \text{time independent terms}\,, \tag{321}$$

and there are terms that survive for time-independent solutions. Remember that the multicenter metrics we were looking are of this sort, since they are stationary (time-independent with mixed terms $g_{ti} \sim k_i$ terms coming from a $(dt + k_i dx^i)^2$).

Therefore, the commutation relations go as

$$[h_{ij}, \Pi^{kl}] \sim [h_{ij}, h^{kl}]\,, \tag{322}$$

analogous to the previous example of a particle in a magnetic field with

$$[q_i, p_j] \sim [q_i, q_j]\,. \tag{323}$$

The spatial metrics no longer commute on the phase space. This will be very important for getting the number of states.

8.3.1 Crnkovic-Witten-Zuckerman formalism

We want a covariant formalism, rather than the non-covariant GR Hamiltonian formalism.

Fix a spatial foliation:

$$\Sigma = \text{ Cauchy surface.} \tag{324}$$

Define

$$\omega := \int_\Sigma d\Sigma_\ell J^l \,, \tag{325}$$

where J^ℓ is the "syumplectic current". We have introduced the $(D-1)$-form

$$d\Sigma_\ell = \Sigma_{\mu_1 \ldots \mu_{D-1}\ell} dx^1 \wedge \ldots \wedge dx^{D-1} \,. \tag{326}$$

Then ω is a two-form on the space of fields. The symplectic current is

$$J_\ell = \delta \left[\frac{\delta L}{\delta \partial_\ell \phi_k} \right] \wedge \delta \phi^k \,, \tag{327}$$

where ϕ^k runs over the fields. If $\ell = 0$, we get $J_0 = d\Pi \wedge d\phi$, reminiscent of the symplectic form in mechanics $dp \wedge dq$. If ℓ is any other direction: we have "covariantized" that form. You don't need to take a spatial direction, can be any way you want and is covariant.

Exercise 8.3: *Play around with ω:*

1. *Show that ω is closed under a field variation*

$$\delta_\phi \omega = 0 \,. \tag{328}$$

2. *Show that the symplectic current is conserved*

$$\partial_\ell J^\ell = 0 \,. \tag{329}$$

You need to impose the equations of motion for one of these.

From the exercise, we see that ω does not vary from sllice to slice.

8.4 Back to solution space

Now we evaluate the symplectic form for the Lagrangian of M-theory. The fields are the metric and the four-form and are evaluated at the positions on solution space:

$$\phi_\ell = \{ g_{\mu\nu}[\vec{r}_p], F_{\mu\nu\rho\sigma}[\vec{r}_p] \} \,. \tag{330}$$

The symplectic form looks like

$$J^\ell = \delta \left[\frac{\delta L}{\delta \partial_\ell g[\vec{r}_p]} \right] \wedge \delta g[\vec{r}_p] + \text{ four-form term} \,. \tag{331}$$

The two-form ω will be something like

$$(\ldots) \wedge d\vec{r}_p \,, \tag{332}$$

where each $\{\vec{r}_p\}$ parametrizes a metric; these are the "coordinates" of our solution.

How to do this? Remember that the constraint equations come from the integrability condition of the defining equation for $\vec{\omega}$ (which is part of the metric $g_{\mu\nu}$):

$$\vec{\nabla} \times \vec{\omega} = V\vec{\nabla}M + \dots . \tag{333}$$

We need to find $\vec{\omega}(\vec{r}_p)$, construct $g(\vec{\omega})$ and then we can find J^ℓ. This is a very tough assignment, because inverting Equation (333) cannot be easily done.

We will follow the lazy string theorist approach and use supersymmetry to our advantage. The backreacted supergravity system is valid for $g_s N \gg 1$. As we discussed in previous sections, when $g_s N \ll 1$, we just have a quantum mechanical theory on branes at the positions of the centers on eleven-dimensional flat spacetime $\mathbb{R}^3 \times T^6 \times \mathbb{R}_t$, see Figure 44.

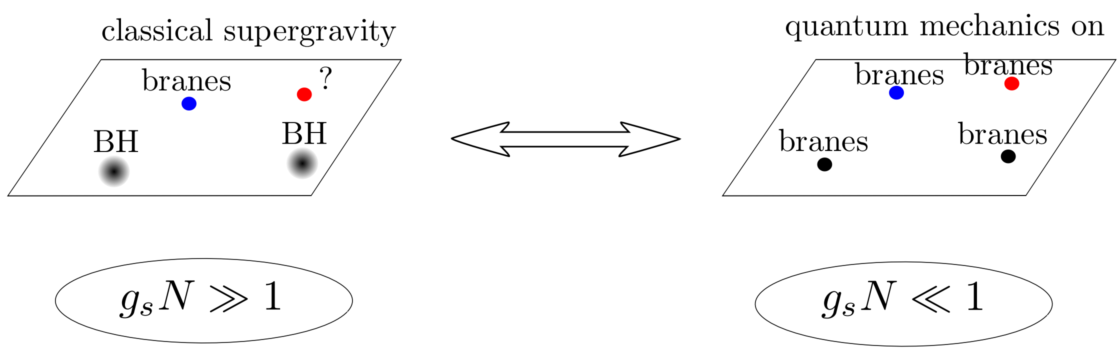


Figure 44: *At large $g_s N$, we have the supergravity multicenter solution. Each center can be either a black hole (with a horizon), or some horizonless singularity, or a smooth center etc. For small $g_s N$, we just have non-backreacting branes at several positions in flat spacetime.*

It can be shown that on each $g_s N$ side the solution space and the symplectic form are protected because of supersymmetry (the proof uses the fact that both are determined by the certain terms in the lagrangian whose form is fixed by supersymmetry and thus cannot change even as we vary of $g_s N$). Moreover one can check by explicit computation that the solution spaces at strong and weak coupling are exactly the same. For instance, for 2 centers, we still find S^2 as the solution space. Thus we are free to compute the symplectic form directly in the brane quantum mechanics which is a much easier computation.

The result we get from the $g_s N \ll 1$ quantum-mechanics-on-branes calculation is

$$\omega = \tfrac{1}{4} \sum_{p,q} \langle \Gamma_p, \Gamma_q \rangle \frac{r^i_{pq}}{|r_{pq}|^2} \epsilon_{ijk} \delta r^j_{pq} \wedge \delta r^k_{pq} \,, \tag{334}$$

and we defined

$$\vec{r}_{pq} = \vec{r}_p - \vec{r}_q \,. \tag{335}$$

The real coordinates in this calculation are the $\vec{r}_{pq}$, vectors between the centers. Morally, the $\delta r^j_{pq} \wedge \delta r^k_{pq}$ are like the $dx^i \wedge dx^j$ contributions in quantum mechanics. As before, this means that coordinates do not commute:

$$[r^i_{pq}, r^j_{pq}] = \omega^{-1} \neq 0 \,. \tag{336}$$

Note that only the r^i_{pq} talk with r^j_{pq}: the several components of a the vector between the p^{th} and q^{th} centers are non-commutative, but they commute with all the other components of all the other inter-center vectors: there is only pairwise non-commutatitivty.

The angular momentum is:

$$J = \frac{1}{2}\sum_{p,q}\langle\Gamma_p,\Gamma_q\rangle\frac{\vec{r}_{pq}}{|r_{pq}|}\,. \tag{337}$$

It is a sum of contributions from each pair of points. Each individual contribution is a vector along the line connecting two points (unit vectors $\frac{\vec{r}_{pq}}{|r_{pq}|}$) with size the angular momentum from the crossed electric and magnetic fields $\langle\Gamma_p,\Gamma_q\rangle$.

8.4.1 Two-center solutions.

Let's make things clear by an explicit example with two centers. Write $J = \langle\Gamma_1,\Gamma_2\rangle$, then the volume form on phase space is

$$\omega = J\sin\theta\, d\theta\wedge d\phi\,, \tag{338}$$

the standard symplectic form on a two-sphere. (Remember that the solutions space for two center is the S^2 of orientations of the fixed rod $\vec{r}_{12}$.) The normalization of the two-form is the angular momentum between the two centers.

The number of states is then

$$\int_{S^2}\omega = 2|J|+1\,. \tag{339}$$

We get $2|J|+1$ rather than $2|J|$ because of subtleties with fermions. This is exactly the number of states for an angular momentum multiplet.

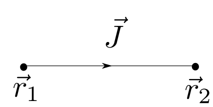


Figure 45: *Two-center solution.*

Exercise 8.4: *"Meaningless algebra" for the two-center solution space:*

- *Check that*

$$d\omega = 0 \tag{340}$$

- *Check that* ω_{S^2} *defined as* (334) *evaluates to* (338).

8.4.2 Three-center solutions.

Solution space is $2N-2$ dimensional. For $N=3$, we get a four-dimensional solution spac M_4e. The bubble equations fix two distances, say $r_{23}(r_{12})$ and $r_{13}(r_{12})$. The four remaining parameters are

- The distance r_{12}.

- The $U(1)$ of orientations around segment r_{12}.
- The orientation of r_{12} in space (an S^2 as for the two-center solution space).

Therefore the solution space is:

$$M_4 = I \times U(1) \times S^2 \,, \tag{341}$$

where I is the line segment of r_{12}. The second product is a fibre.

Note that the size of the angular momentum is a function of the distance r_{12} as well:

$$J(r_{12}) \,. \tag{342}$$

By the bubble equations, the interval I of allowed r_{12} values is constrained

$$I = [r_{12}^{\text{in}}, r_{12}^{\text{max}}] \,. \tag{343}$$

Hence also the angular momentum is bounded between J_{min} and J_{max}, see Figure (46).

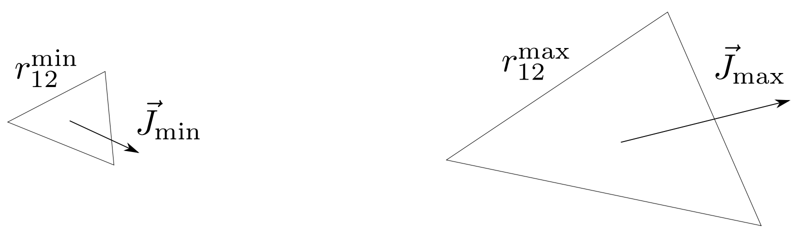


Figure 46: *The angular momentum is a function of the size of r_{12}.*

We can see the system as a whole range of angular momentum multiplets, see Figure (47) If the centers are black holes with horizons, we are not counting the horzon entropy of a single

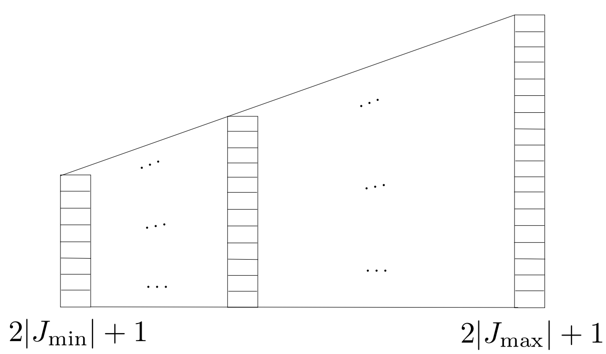


Figure 47: *The angular momentum is a function of the size of r_{12}. The states are divided into one angular momentum multiplet for each allowed value of J.*

black hole with the total charge of all the centers.

Fix the total charge

$$\Gamma = \sum_{N=1}^{\infty} \left(\sum_{q=1}^{N} \Gamma_q \right) \,. \tag{344}$$

You can have arbitrary N and arrange the centers all to be horizonless. What are all possible states corresponding to these charges? We fix Γ first, then we fix the sectors we want to divide over, and we divide the charges. All these states are in one Hilbert space, of total charge Γ. Are all these possible states reproducing the black hole entropy of a single black hole with charge Γ? Should we use smooth centers? How many can we put? Can we reproduce the entropy?

Results in the literature so far:

☞ For fully interacting centers ($\langle \Gamma_p, \Gamma_q \rangle \neq 0$), this counting has only been done for 2 and 3 centers. It has been extended to $N+1$ centers, where the first N have all charges equal $\Gamma_1 = \ldots = \Gamma_N$ and the other center has non-vanishing $\langle \Gamma_p, \Gamma_{N+1} \rangle$ with all the others.

Note that classically, there can be configurations with runaway behaviour. One of the centers can go off to infinity in the bubble equations, and this screws up the asymptotics, see Figure 48.

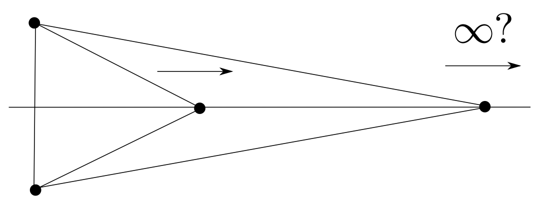


Figure 48: *Classically, one of the centers can run of to infinity.*

After quantization, there is a density on $M_4 = \mathbb{R} \times U(1) \times S^2$. This gives a finite volume. There is no more runaway, because the wave function for the positions of the centers has no support at infinity, 'the tail is vanishing'. This renders $\langle \vec{r}_p \rangle$ finite. See Figure 49.

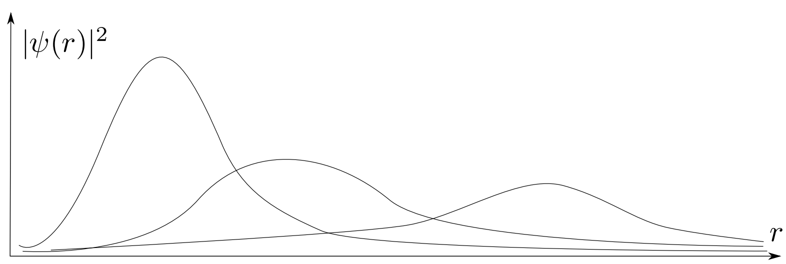


Figure 49: *In quanum mechanics, the wave function has no support at infinity.*

8.5 Scaling solutions

Let's go to solutions where the centers can come arbitrarily close. We stay in the three-center example. Remember that the bubble equations look like

$$\begin{aligned} \frac{\langle\Gamma_1,\Gamma_2\rangle}{r_{12}} + \frac{\langle\Gamma_1,\Gamma_3\rangle}{r_{13}} &= \#_1\,, \\ \frac{\langle\Gamma_2,\Gamma_1\rangle}{r_{12}} + \frac{\langle\Gamma_2,\Gamma_3\rangle}{r_{13}} &= \#_2\,, \\ \frac{\langle\Gamma_3,\Gamma_1\rangle}{r_{13}} + \frac{\langle\Gamma_3,\Gamma_2\rangle}{r_{23}} &= \#_3\,, \end{aligned} \tag{345}$$

with $\#_p$ some numbers. We look for solutions with

$$r_{pq} = \lambda\langle\Gamma_p,\Gamma_q\rangle + \mathcal{O}(\lambda^2)\,, \tag{346}$$

such that we can send $\lambda \to 0$.

As a consequence, the angular momentum is zero when $\lambda = 0$

$$\vec{J} = \sum \Gamma_{pq}\frac{\vec{r}_{pq}}{r_{pq}} = \sum \vec{r}_{pq} = 0\,, \tag{347}$$

where the last equality follows becose the $\vec{r}_{pq}$ form a closed triangle. Therefore, near $\lambda \to 0$, we have $J \to 0$. This means that we 'complete' the triangle of states in the angular momentum multiplets of Figure 47 to that of Figure 50. We can parametrize the region near $J_{\text{min}} = 0$ by the scaling parameter λ.

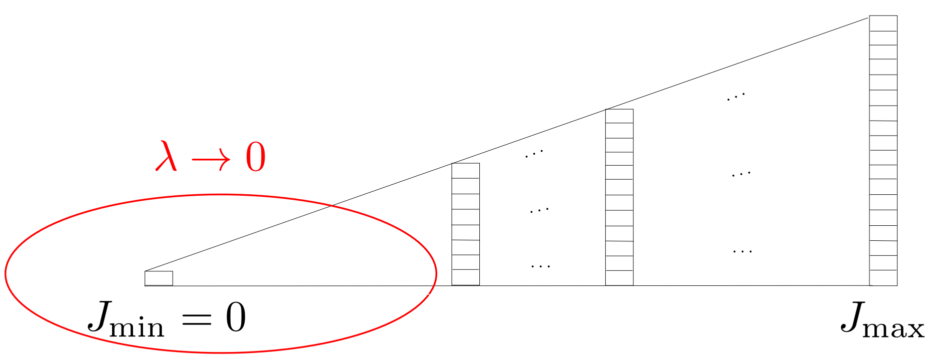


Figure 50: *The angular momentum multiplet triangle is completed for scaling solutions, since the solution space contains the limit $\lambda \to 0$, such that $J_{\text{min}} = 0$.*

When the intercenter distance $r_{pq} \sim \lambda \to 0$, the geometry develops a very deep throat of size proportional to $1/\lambda$, see Figure 51. As the centers come closer and closer, the throat becomes deeper and deeper.

Putting these things together, gives a situation of the states in solution space as in Figure 52.

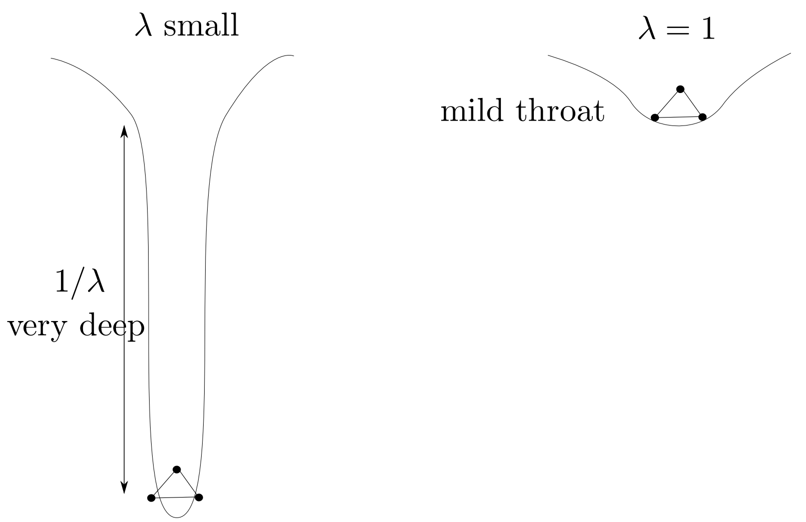


Figure 51: *By scaling down the distances between the centers as $\lambda \to 0$, the geometry develops a very deep throat whose size is inversely propotional to λ. When λ is of order 1 on the other hand, we only have a very mild throat.*

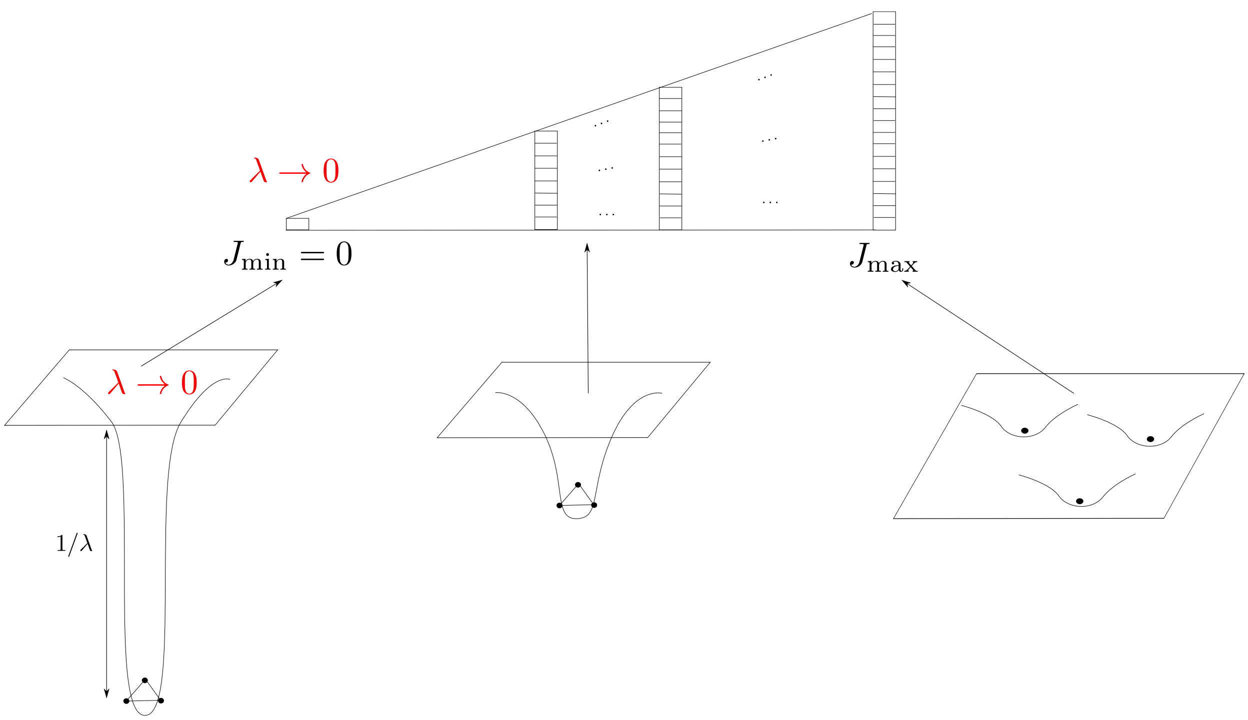


Figure 52: *The correspondence of scaling solutions of a certain size to angular momentum multiplets in the quantized solutions space.*

This reveals a paradox. As $\lambda \to 0$, we get deeper and deeper microstates and we can continue like this forever. On the other hand, the number of states associated to the region of small λ of Figure 52, gives a finite number of states. Stated in a different way, in quantum mechanics, it is meaningless to put states in a cell smaller than $\hbar$-size. Remember that on solution space, we had non-commuting coordinates r^i_{pq} and r^j_{pq}. This translates to the impossibility of localizing r^i_{pq} and r^j_{pq} than $\hbar$. Therefore there is some cut-off, and all deeper and deeper microstates must correspond to one quantum state.

Hence even though we can make the throats as deep as we want classically, all these deep throats do not exists after quantization. This is related to the earlier puzzle, that due to redshift, the energy E_∞ would have a continuous spectrum for deeper and deeper throats, see Figure 53: a string stretching between two centers remains massless at ∞.

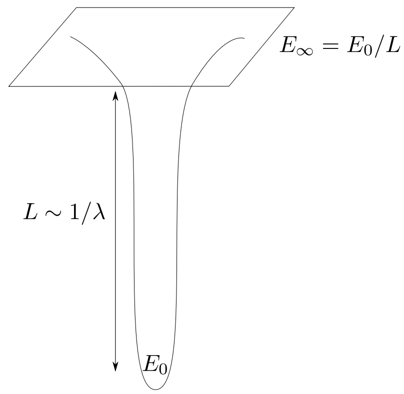


Figure 53: *The energy E_0 of an excitation down the throat is redshifted to $E_\infty \sim E_0/L$, with L the throat length.*

On the other hand, the CFT should have a discrete spectrum, otherwise the counting of microstates would not give a finite number. So the question is whether there is a cut-off in the throat, and what it is.

While the exact answer to this question depends on the state we consider and is somewhat complicated a simple order of magnitude estimate can be gleaned as follows. We consider the geometry of the throat up to the scale where λ takes its expectation value in the lowest angular momentum state (i.e. the state at $J = J_{min}$; see fig (53) above). That is, we compute $\langle\lambda\rangle$ in the state $|j = 0\rangle$ and then plug this into the harmonics to yield a solution. This gives a cuttoff on the throat and we can determine the mass gap by putting a scalar field on this background and computing the gap in its spectrum (this is analogous to a standard computation to determine the mass gap in global AdS and essentially measures the "size of the box" provided by the gravitational potential).

This computation yields a mass gap that, when measured in AdS units $1/L_{AdS}$, scales as $1/c$. Here c is a dimensionless number given by comparing the AdS length to the plank length $c \sim L_{AdS}/\ell_P$. Thus the mass gap is

$$\frac{1}{cL_{AdS}} \tag{348}$$

whereas the mass gap in global AdS is just

$$\frac{1}{L_{AdS}} \tag{349}$$

The suggestive terminology c alludes to the fact that this number is the central charge of the dual CFT so e.g. in the case where the AdS_3 is the near horizon of the D1-D5 black hole $c \sim Q_1 Q_5$.

What is the significance of this result? Recall that in our derivation of the black hole entropy in earlier sections a very important role was played by the so called "long string picture" where the entropically dominant sector of the CFT came from a string with winding $\sim Q_1 Q_5$. Consequently the momentum of this string was quantized in units $\frac{1}{Q_1 Q_5 R}$ with R the dimensionful length of the CFT circle $R \cong 2\pi L_{AdS}$.

This computation thus suggests that the quantum corrections to the deep throats microstates not only discretize the spectrum, hence resolving the issue of a continuous spectrum, but also do this by giving them a mass gap corresponding to the most entropic sector of the CFT. This suggests these states at least occupy the "typical" sector of the CFT and hence are potentially the kind of states that might account for the black hole entropy.

List of Figures

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