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A recursive approach to the O(n) model on random maps via nested loops

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Abstract

We consider the $O(n)$ loop model on tetravalent maps and show how to rephrase it into a model of bipartite maps without loops. This follows from a combinatorial decomposition that consists in cutting the $O(n)$ model configurations along their loops so that each elementary piece is a map that may have arbitrary even face degrees. In the induced statistics, these maps are drawn according to a Boltzmann distribution whose parameters (the face weights) are determined by a fixed point condition. In particular, we show that the dense and dilute critical points of the $O(n)$ model correspond to bipartite maps with large faces (i.e. whose degree distribution has a fat tail). The re-expression of the fixed point condition in terms of linear integral equations allows us to explore the phase diagram of the model. In particular, we determine this phase diagram exactly for the simplest version of the model where the loops are "rigid". Several generalizations of the model are discussed.

1. Introduction and main results

1.1. General introduction

Planar maps, which are proper embeddings of graphs in the two-dimensional sphere, are fundamental mathematical objects whose combinatorics raises many beautiful enumeration problems, first addressed by Tutte in the 60's [1]. Maps are also widely used in physics as discrete models for fluctuating surfaces or interfaces in various contexts, ranging from soft matter physics (e.g. biological membranes) to high energy physics (e.g. string theory). Most problems involve "random maps", i.e. consist in the study of some particular statistical ensemble of maps, distributed according to some prescribed law. Interesting scaling limits may be reached when one considers ensembles of large random maps, giving rise to nice universal probabilistic objects. Several map enumeration techniques were developed over the years, which we may classify in three categories: Tutte's original recursive decomposition method [2], the technique of matrix integrals [3] and, more recently, the bijective method [4] where maps are coded by tree-like objects.

So far, the most advanced results were obtained for ensembles of maps with a simple control on the degree of, say the faces of the maps. Examples are ensembles of triangulations (maps with faces of degree 3 only) or of quadrangulations (degree 4) where the total number of faces in the map is either fixed or governed by some Boltzmann distribution. The statistics of these maps is now well understood and many exact enumeration results were obtained within the framework of each of the three above enumeration techniques. Of particular interest is the scaling limit of large maps with prescribed *bounded degrees*, which defines the so-called universality class of "pure gravity", and gives rise to the "Brownian map", a probabilistic object with remarkable metric properties [5]. As first recognized in physics, other universality classes of maps may be reached upon equipping the maps with statistical models, such as models of spins or particles, which present a large variety of critical phenomena that modify the statistical properties of the underlying maps. This gives rise to a large variety of universality classes for maps, whose understanding is the domain of the so-called "two-dimensional quantum gravity" [6]. It is worth mentioning that most results in this domain rely on matrix integral formulations of the models, so that the matrix integral technique appears so far as the most powerful approach to explore new universality classes.

A particular important class of models of maps equipped with statistical models are the so-called $O(n)$ loop models which consist in having maps endowed with configurations of closed self- and mutually-avoiding loops drawn on their edges, each loop receiving the weight n. A particular class of $O(n)$ loop models where loops visit only vertices of degree 3 (the degree of the other vertices being bounded) was analyzed in details by use of matrix integral techniques in [7-12]. There it was found that these models present several phases with non-trivial universal scaling limits. We will briefly recall the results of this analysis in Section 3.1 below.

It was recognized recently in [13] that another simple way to escape from the universality class of pure gravity consists in considering "maps with large faces", i.e. ensemble of maps where the degree of the faces is *unbounded* and properly controlled so that faces with arbitrarily large degree persist in the scaling limit. Using now the bijective method, it was shown that such ensembles may give rise to new probabilistic objects corresponding to maps coded by stable trees. It was then proposed that these probabilistic objects may also describe the scaling limit of $O(n)$ loop models in some of their phases.

The purpose of this paper is to show that this is indeed the case. More precisely, we show that a number of $O(n)$ loop models may be bijectively transformed into models of maps without loops and with a simple control on the degree of their faces. This control involves some degree-dependent face weights whose value is fixed by some appropriate consistency relation in the form of a fixed point condition depending on n. This implies that the possible scaling limits of the $O(n)$ loop models necessarily match the scaling limits of maps controlled simply by their face degrees. These include the pure gravity universality class as well as the new universality classes of Ref. [13] for maps with large faces, which are observed in the dense and dilute phases of the $O(n)$ loop models.

1.2. Overview of the paper

Let us start by giving a brief description of our results. Most of the paper deals with $O(n)$ loop models defined on planar *tetravalent maps* (maps with vertices of degree 4 only). By definition, the loops are self- and mutually-avoiding and each loop receives the weight n. Different models are obtained by assigning different weights to the vertices depending on whether they are visited by a loop or not. A special attention will be paid to the simplest version of the model, the *rigid loop model*, where loop turns are forbidden at the tetravalent vertices.

Our analysis is simplified by first reformulating our loop model on the dual maps, which are planar quadrangulations, and by then extending them to quadrangulations with a boundary of arbitrary (necessarily even) length. Indeed, a simple characterization of the universality class of the loop model at hand is via the large p asymptotics of its generating (or partition) function F_p^{loop} in the presence of a boundary of length $2p$. It takes the form

$$F_p^{\text{loop}} \sim \text{const.} \frac{\mathcal{A}^p}{p^a} \tag{1.1}$$

with a non-universal exponential growth factor $\mathcal{A}$ and a subleading algebraic decay term $1/p^a$ involving a *universal exponent* a in the range $3/2 \leq a \leq 5/2$. We show that, for a given n, at most four values of the exponent a may be observed, namely $a = 3/2$, corresponding to a *subcritical* model, $a = 5/2$, corresponding to a *generic critical* model, and

$$a = 2 \pm b, \quad \pi b = \arccos(n/2) \tag{1.2}$$

corresponding to a *non-generic critical* model. This last behavior requires that n lies in the range $0 < n < 2$. In the case of the rigid loop model, we give a precise description of the phase diagram which specifies the domain of physical parameters where each of the above behaviors is observed.

The expressions (1.1) and (1.2) are identical to those obtained for the the $O(n)$ loop models solved by matrix integrals in [7-12]. Here, they are obtained by different means which rely on an equivalence of our models with models of bipartite maps. More precisely, quadrangulations (with a boundary) endowed with loop configurations may be coded bijectively by bipartite maps without loops but with faces of arbitrary degree, each containing an $O(n)$ loop model configuration of its own. This bijective decomposition allows, via a simple substitution procedure, to express F_p^{loop} in terms of the well-understood generating function for bipartite maps with a boundary. As a consequence, we may identify the possible asymptotics of F_p^{loop} as those of ordinary maps with possibly large faces, leading eventually to (1.1) and (1.2).

The paper is organized as follows: Section 2 is devoted to a study of the possible critical behaviors of bipartite maps, which we classify into subcritical, generic critical and non-generic critical. The first two cases are discussed in Section 2.2 while a particular attention is paid to the non-generic critical behavior in Section 2.3. Section 2.4 discusses the so-called resolvent of bipartite maps. After recalling in Section 3.1 some known properties of the $O(n)$ loop models previously solved by matrix integrals, we then define in details in Section 3.2 our $O(n)$ loop model on tetravalent maps. We present in Section 3.3 our bijective decomposition which allows to code the configurations of our model in terms of their *gasket*, which is a bipartite map with no loops but with *holes*, which are faces of arbitrary even degree, together with a content for each hole. For consistency, the effective weights for the holes are shown to obey some crucial *fixed point condition* which determines them uniquely. This condition involves in particular some ring generating function, which accounts for the configuration in the immediate vicinity of a loop, and which we make explicit in Section 3.4. Section 4 is devoted to the consequences of the fixed point condition on the asymptotics of F_p^{loop}, leading to (1.1) and (1.2) above. We first concentrate on the rigid loop model in Section 4.1 before addressing the general case in Section 4.2. Section 5 shows how to transform the fixed point condition into a linear integral equation, both in the rigid case (Section 5.1) and in the non-rigid case (Section 5.2). We then analyze in Section 5.3 the solution of this equation as it captures all the phase diagram of the $O(n)$ loop model. Section 6 presents a detailed analysis of the rigid loop model. We first show in Section 6.1 how to transform the fixed point condition into an equation for the resolvent of the model. This equation is then solved, first along a line of non-generic critical points in Section 6.2, then in all generality in Section 6.3. The generic critical line is discussed in Section 6.4 and we summarize the resulting phase diagram in Section 6.5. Section 7 considers extensions of our results to other classes of $O(n)$ loop models. We first show how the relation (1.2) may be modified by considering loops with non-symmetric weights (Section 7.1) or with additional constraints on their length (Section 7.2). We end our study by a discussion of the wide class of $O(n)$ loop models defined on arbitrary even-valent maps (with bounded degrees) and show that our results nicely extend to this case. We gather our conclusions in Section 8.

2. Bipartite maps with arbitrarily large faces: possible critical behaviors

2.1. Reminders

Throughout this paper, we consider bipartite planar maps. Let us recall that a planar map is bipartite if and only if all its faces have even degree. To each face of degree $2k$, ($k \geq 1$), we attach a weight g_k, which is an arbitrary non-negative real number. Furthermore, we also attach a non-negative real weight u per vertex of the map, irrespectively of its degree. The unnormalized weight of a map is then the product of all its face and vertex weights. Note that, from Euler's relation, we could set $u = 1$ without loss of generality upon redefining $g_k \to g_k u^{k-1}$ but it will prove convenient to keep u as a free parameter.

It is well-known that the simplest map enumeration formulas are for *pointed rooted maps*, i.e. maps with a distinguished vertex (pointed) and a distinguished oriented edge (rooted). The generating function (i.e. the sum over all maps of the unnormalized weights above) for pointed rooted bipartite planar maps is $2u(R(u)-u)$ where $R(u)$ is the smallest non-negative (possibly infinite) solution of the equation [14]

$$R(u) = u + \sum_{k\geq 1} g_k \binom{2k-1}{k} R(u)^k \ . \tag{2.1}$$

In this paper, we choose to display explicitly the dependence in u but to hide that in the g_k's. It is easily seen that $R(u)$ is an increasing function of u with $R(0)=0$.

Other more involved generating functions can be expressed in terms of $R(u)$. For instance, we may consider the generating function for rooted maps with root degree $2k$ ($k \geq 1$), where the root degree is that of the face lying on the right of the distinguished oriented edge. This generating function reads $g_k F_k(u)$ where [15]

$$F_k(u) = \binom{2k}{k} \int_0^u R(v)^k dv \ . \tag{2.2}$$

Consistently, we set $F_0(u) = u$. Note that the weight g_k of the root face has been factored out of $F_k(u)$, thus $F_k(u)$ may as well be interpreted as the generating function for maps with a boundary of length $2k$. Furthermore, since rooted maps are obviously in bijection with maps with a boundary of length 2 and not reduced to a single edge, we may interpret $F_1(u)-u^2$ as the generating function for rooted maps. Upon integrating (2.1), we immediately deduce

$$F_1(u) - u^2 = \sum_{k\geq 1} g_k F_k(u) \tag{2.3}$$

which amounts to decomposing the generating function of rooted maps (l.h.s) according to the root degree of the maps (r.h.s).

2.2. Subcritical and generic critical ensembles

We say that the sequence $(g_k)_{k\geq 1}$ of face weights is *admissible* if $R(1)$ is finite. In this case, taking $u=1$ and dividing the unnormalized weight of a pointed rooted map by $2(R(1)-1)$, we obtain its probability in the *Boltzmann ensemble* of pointed rooted maps associated with the sequence $(g_k)_{k\geq 1}$. For example, random quadrangulations are obtained by choosing the sequence $g_k = g\,\delta_{k,2}$ ($g>0$), which is admissible for $g \leq 1/12$.

A number of properties of the Boltzmann ensemble are encoded in the expansion of $R(u)$ around $u=1$. This expansion is obtained upon inverting the expansion around $R=R(1)$ of the inverse function $u(R)$ which, from (2.1), reads explicitly:

$$u(R) = R - \varphi(R), \quad \varphi(R) = \sum_{k\geq 1} g_k \binom{2k-1}{k} R^k \ . \tag{2.4}$$

Here the function $\varphi(R)$ is defined through its series expansion at $R = 0$, whose coefficients are all non-negative. Its radius of convergence is $R_c = 1/(4 \limsup_{k\to\infty}(g_k)^{1/k})$. If the sequence is admissible, R_c is necessarily non-zero (possibly infinite), with $R(1) \le R_c$, and it is easily seen that $u'(R(1)) \ge 0$, i.e. $\varphi'(R(1)) \le 1$. The sequence is said *critical* if $\varphi'(R(1)) = 1$ (hence $u'(R(1)) = 0$ and $R'(u) \to \infty$ as $u \to 1$) and *subcritical* otherwise (hence $R'(1)$ is finite). This change of behavior is visible in the large k asymptotics of $F_k(1)$. Indeed, the integral (2.2) for $u = 1$ is asymptotically dominated by the vicinity of $v = 1$ and its behavior may be computed via Laplace's method. For a subcritical sequence, we have at large k:

$$F_k(1) \sim \frac{R(1)(1-\varphi'(R(1))}{\sqrt{\pi}} \frac{(4R(1))^k}{k^{3/2}} \ . \tag{2.5}$$

This form of the asymptotics is observed for instance when all g_k's are zero, in which case $\varphi(R) = 0$, $R(u) = u$, and $F_k(1) = \binom{2k}{k}/(k+1)$ is the k-th Catalan number.

For a critical sequence, the asymptotic behavior depends on whether the (negative) quantity $u''(R(1)) = -\varphi''(R(1))$ is finite or not. For a *generic critical sequence*, this quantity is finite and we have that

$$F_k(1) \sim \frac{R(1)^2\varphi''(R(1))}{\sqrt{\pi}} \frac{(4R(1))^k}{k^{5/2}} \ . \tag{2.6}$$

This form of the asymptotics is observed for instance for critical random quadrangulations obtained by choosing the sequence $g_k = (1/12)\delta_{k,2}$, in which case $\varphi(R) = R^2/4$, $R(1) = 2$, $\varphi''(R(1)) = 1/2$, so that (2.6) is consistent with the exact formula $F_k(1) = 2^{k+1}\frac{(2k)!}{k!(k+2)!}$.

In order to escape from this generic critical behavior, we must have $\varphi''(R(1))$ infinite, which means that $R(1)$ must coincide with the radius of convergence R_c of φ. Such non-generic critical behaviors are analyzed in the next section. Before proceeding, let us mention the following simple *monotonicity property*: if $(g_k)_{k\ge1}$ and $(\tilde{g}_k)_{k\ge1}$ are two sequences such that $g_k \le \tilde{g}_k$ for all k, the inequality being strict for at least one k, and if $(\tilde{g}_k)_{k\ge1}$ is admissible, then $(g_k)_{k\ge1}$ is subcritical. Indeed, denoting by $\tilde{R}(\cdot)$ and $\tilde{\varphi}(\cdot)$ the functions associated with the sequence $(\tilde{g}_k)_{k\ge1}$, we have $R(1) < \tilde{R}(1)$ by the intermediate value theorem ($R \mapsto R - \varphi(R)$ is continuous on $(0, \tilde{R}(1))$ with $\tilde{R}(1) - \varphi(\tilde{R}(1)) > \tilde{R}(1) - \tilde{\varphi}(\tilde{R}(1)) = 1$). Then, $\varphi'(R(1)) < \tilde{\varphi}'(\tilde{R}(1)) \le 1$ hence, by definition, $(g_k)_{k\ge1}$ is subcritical.

2.3. Non-generic critical ensembles

In order to get a non-generic critical behavior, we must ensure simultaneously the two conditions $\varphi'(R(1)) = 1$ (criticality) and $\varphi''(R(1)) = \infty$ (non-genericity). This turns out to highly constrain the possible form of the sequence $(g_k)_{k\ge1}$. As shown by Le Gall and Miermont [13], a natural general form for such a sequence is:

$$g_k = c\left(\frac{1}{4R_c}\right)^{k-1} g_k^\circ \tag{2.7}$$

where $(g_k^\circ)_{k\geq 1}$ is an arbitrary sequence of reference, such that

$$g_k^\circ \sim k^{-a} \quad \text{for } k\to\infty \quad \text{with } \frac{3}{2} < a < \frac{5}{2} \ . \tag{2.8}$$

Note that R_c is indeed the radius of convergence of $\varphi(R)$ since we have $\varphi(R) = cRf_\circ(R/(4R_c))$, where

$$f_\circ(x) = \sum_{k\geq 1} \binom{2k-1}{k} g_k^\circ \, x^{k-1} \tag{2.9}$$

is an analytic function with radius of convergence $1/4$. Since a lies in the range $]3/2, 5/2[$, we have $\varphi'(R_c) < \infty$ and $\varphi''(R_c) = \infty$. The non-genericity condition therefore reduces to demanding that

$$R(1) = R_c \ , \tag{2.10}$$

or equivalently $1 = R_c - \varphi(R_c)$. Together with the criticality condition $\varphi'(R_c) = 1$, this fixes the values of c and R_c as

$$c = \frac{4}{4f_\circ(1/4) + f_\circ'(1/4)}, \quad R_c = 1 + \frac{4f_\circ(1/4)}{f_\circ'(1/4)} \ . \tag{2.11}$$

At these values, $u(R)$ has a singular expansion around $R = R_c$ given by

$$u(R) = 1 - \frac{2c\, R_c\, \Gamma(1/2-a)}{\sqrt{\pi}} \left(1 - \frac{R}{R_c}\right)^{a-1/2} + O\left(1 - \frac{R}{R_c}\right)^2 \ . \tag{2.12}$$

Note that $\Gamma(1/2-a) > 0$ since $3/2 < a < 5/2$. Inverting this expansion, we see that the generating function $R(u)$ behaves, when we fix the face weights g_k as above and let the vertex weight u tend to 1 from below (i.e. we use the parameter u to control the approach to the critical point), as

$$R(u) = R_c(1 - \kappa\,(1-u)^{2/(2a-1)}) + O(1-u), \quad \kappa = \left(\frac{\sqrt{\pi}}{2c\,R_c\,\Gamma(1/2-a)}\right)^{2/(2a-1)} \ . \tag{2.13}$$

By standard analytic techniques, this implies that the coefficient of u^N in $R(u)$, hence the probability of having N vertices in the Boltzmann ensemble of pointed rooted maps, decays as $N^{-(2a+1)/(2a-1)}$ as $N \to \infty$.

Returning to the asymptotics of $F_k(1)$, we find by Laplace's method that, for large k:

$$\int_0^1 R(v)^k dv \sim \frac{2cR_c\,(a-1/2)\Gamma(1/2-a)\Gamma(a-1/2)}{\sqrt{\pi}} \frac{R_c^k}{k^{a-1/2}} \tag{2.14}$$

and, using $\binom{2k}{k} \sim 4^k/\sqrt{\pi k}$, $(a-1/2)\Gamma(1/2-a)\Gamma(a-1/2) = \pi/\sin\pi(a-3/2)$ and $R_c = R(1)$,

$$F_k(1) \sim \frac{2c\,R(1)}{\sin\pi(a-3/2)} \frac{(4R(1))^k}{k^a} \ . \tag{2.15}$$

It is instructive to compare this expression with the "input", namely the form (2.7)-(2.8) for non-generic critical face weights, which implies the asymptotics

$$g_k \sim 4c\, R(1)\, \frac{(4R(1))^{-k}}{k^a} \ . \tag{2.16}$$

We note a striking similarity: the prefactor is simply divided by $2\sin\pi(a-3/2)$ while the exponential factor is inverted. This similarity is not merely anecdotal but will have important consequence for the $O(n)$ loop model (see Section 4).

As a concluding remark, note that the condition $R(1) = R_c$ for non-generic criticality implies that the face degree distribution has a fat tail. As seen from the derivation [14] of (2.1), the probability that the root face of a rooted pointed map has degree k is

$$\frac{g_k \binom{2k-1}{k} R(1)^k}{R(1)-1} \tag{2.17}$$

which decays exponentially as $(R(1)/R_c)^k$ when $R(1) < R_c$, but only algebraically as $k^{-a-1/2}$ at a non-generic critical point. Similarly, from (2.3), the probability that the root face of a rooted (but unpointed) map has degree k is

$$\frac{g_k F_k(1)}{F_1(1)-1} \tag{2.18}$$

which also decays exponentially as $(R(1)/R_c)^k$ when $R(1) < R_c$ and algebraically, now as k^{-2a}, at a non-generic critical point.

2.4. The resolvent

For the purposes of Section 6, it is useful to gather facts about the so-called *resolvent* (this terminology being borrowed from the matrix integral formalism), defined as:

$$W(\xi) = \sum_{k\geq 0} \frac{F_k(1)}{\xi^{2k+1}}. \tag{2.19}$$

Using (2.2) for $u = 1$ and performing the change of variable $v \to R$, we obtain

$$W(\xi) = \frac{1}{\xi} \int_0^{R(1)} \frac{u'(R)dR}{\sqrt{1-4R/\xi^2}}. \tag{2.20}$$

From this expression, it is seen that $W(\xi)$ is analytic in the complex plane minus the segment $[-\gamma, \gamma]$, with $\gamma = 2\sqrt{R(1)}$. Along this segment, it has a discontinuity encoded into the so-called *spectral density*

$$\rho(\xi) = \frac{W(\xi - \mathrm{i}0) - W(\xi + \mathrm{i}0)}{2\mathrm{i}\pi} = \frac{1}{\pi} \int_{\xi^2/4}^{R(1)} \frac{u'(R)dR}{\sqrt{4R-\xi^2}} \ , \quad \xi \in [-\gamma, \gamma]. \tag{2.21}$$

Since $u'(R) \geq 0$ for $R \leq R(1)$, $\rho(\xi)$ is nonnegative on $[-\gamma, \gamma]$. Note that, by applying the Cauchy formula around the cut, we may recover $W(\xi)$ from $\rho(\xi)$ via

$$W(\xi) = \int_{-\gamma}^{\gamma} \frac{\rho(\xi')d\xi'}{\xi - \xi'} \ , \quad \xi \notin [-\gamma, \gamma] \tag{2.22}$$

and in particular, since $W(\xi) \sim 1/\xi$ for $\xi \to \infty$, we have the normalization $\int_{-\gamma}^{\gamma} \rho(\xi) d\xi = 1$. The asymptotics (2.5), (2.6) and (2.15) translate into the respective singularities of $\rho(\xi)$ at $\xi = \pm\gamma = \pm 2\sqrt{R(1)}$:

$$\begin{aligned} \rho(\xi) &\sim \frac{1 - \varphi'(R(1))}{2\pi} \left(4R(1) - \xi^2\right)^{1/2} \quad \text{subcritical} \\ \rho(\xi) &\sim \frac{\varphi''(R(1))}{12\pi} \left(4R(1) - \xi^2\right)^{3/2} \quad \text{generic critical} \\ \rho(\xi) &\sim \frac{4^{1-a} R(1)^{3/2-a} c}{\Gamma(a) \sin \pi(a - 3/2)} (4R(1) - \xi^2)^{a-1} \quad \text{non-generic critical} \ . \end{aligned} \tag{2.23}$$

3. $O(n)$ loop model: exact recursive equations

3.1. Generalities, previous models

In this Section, we briefly describe a few of the known results about $O(n)$ models on random maps, as previously studied in [7-12] by matrix integral techniques. We will not give very precise definitions here as these models are not those that we will study in the remainder of the paper. The $O(n)$ loop model studied in [7-12] consists in having self- and mutually-avoiding loops drawn on random maps so as to visit only vertices of degree 3, the vertices not visited by loops having arbitrary (but bounded) degrees. A vertex visited by a loop receives the weight $\tilde{h}$ while a vertex not visited by a loop receives the weight $\tilde{g}_k$ if it is k-valent ($k \geq 1$). Each loop receives in addition the weight n. Allowing the dual map to have a boundary of length p (not necessarily even here), we may gather the corresponding generating functions $\tilde{F}_p^{\text{loop}}$ into the resolvent

$$\tilde{W}(\xi) = \sum_{p \geq 0} \frac{\tilde{F}_p^{\text{loop}}}{\xi^{p+1}} \ , \tag{3.1}$$

well defined for ξ large enough. In the range of weights where the model is well-defined, this formal series is in fact an analytic function on $\mathbf{C} \setminus [\gamma_-, \gamma_+]$, and has a discontinuity on some segment $[\gamma_-, \gamma_+]$. Using this information in the recursive relation for $\tilde{F}_p^{\text{loop}}$ obtained (in the spirit of the Tutte's recursive method) by removing the root edge, it can be shown that $\tilde{W}(\xi)$ is solution of a scalar non-local Riemann-Hilbert problem:

$$\forall \xi \in [\gamma_-, \gamma_+], \qquad \tilde{W}(\xi + \mathrm{i}0) + \tilde{W}(\xi - \mathrm{i}0) + n\tilde{W}(\tilde{h}^{-1} - \xi) = \sum_{k \geq 1} \tilde{g}_k \, \xi^{k-1} \ . \tag{3.2}$$

This fixes uniquely $\tilde{W}(\xi)$ as well as γ_- and γ_+, from the requirement that $\tilde{W}(\xi) \sim 1/\xi$ when $\xi \to \infty$, and that $\tilde{W}$ is holomorphic in $\mathbf{C} \setminus [\gamma_-, \gamma_+]$. A non trivial critical point is reached when $\gamma_- \to (2\tilde{h})^{-1}$, which defines a non-trivial critical surface $\tilde{h} = \tilde{h}((\tilde{g}_k)_{k\geq 1})$. The equation (3.2) was first solved on this critical surface by Kostov [8], who showed that $\tilde{W}$ may develop a singularity of the form:

$$\tilde{W}(\xi)|_{\text{sing.}} \sim \text{const.} \left(\xi - \frac{1}{2\tilde{h}}\right)^{a-1} \quad \text{for} \quad \xi \to \left(\frac{1}{2\tilde{h}}\right)^{+} , \tag{3.3}$$

with $a = 2 \pm b$, $\pi b = \arccos(n/2)$ as in (1.2). This singularity captures the large p asymptotics of $\tilde{F}_p^{\text{loop}}$ as

$$\tilde{F}_p^{\text{loop}} \sim \text{const.} \frac{(2\tilde{h})^{-p}}{p^a}. \tag{3.4}$$

The smallest value $a = 2 - b$ is observed inside the non-trivial critical surface and describes the so-called *dense phase* of the $O(n)$ model, while the largest value $a = 2 + b$ is observed only at some boundary of this critical surface and describes the so-called *dilute phase*. Other multicritical points, with $a = 2 \pm b + 2m$ ($m = 1, 2, \ldots$), can also be observed by tuning more and more coefficients $\tilde{g}_k$ [9], but this requires having some of these weights negative, which prevents interpreting these multicritical points as proper probabilistic ensembles. Eq. (3.2) was then solved in all generality (outside of the critical surface) in [11] in terms of elliptic functions and of the Jacobi theta function $\vartheta(\cdot, q)$, the non-trivial critical surface corresponding to the limit where the nome q goes to 1 (or 0 depending on the convention). In this limit, the theta function degenerates into a trigonometric function, and one can recover the critical exponents stated above. The study of the $O(n)$ model was extended recently to maps of any topology in Ref. [16].

3.2. $O(n)$ loop model on tetravalent maps: definitions

Our model of interest is the $O(n)$ loop model on planar tetravalent maps defined as follows. By *loop* we mean an undirected simple closed path on the map (visiting edges and vertices), also sometimes called an undirected cycle. A *loop configuration* is a set of disjoint loops. By construction, loops are both self- and mutually-avoiding. Alternatively, a loop configuration may be viewed as a set of *covered* edges such that each vertex is incident to either 0 or 2 covered edges.

We prefer to work with the dual map which is a planar quadrangulation. Then, the loops cross some edges of the quadrangulations. Up to a rotation, the quadrangulation has three possible types of faces, as displayed in Fig. 1: (a) empty squares not visited by a loop, (b) squares visited by a loop going straight (i.e. crossing opposite sides of the square) and (c) squares visited by a loop making a turn (i.e. crossing consecutive sides of the square). The length of a loop, i.e. the total number of faces that it visits, may be even or odd. However, the number faces of type (b) visited by a given loop is necessarily even, resulting from the fact that a planar quadrangulation is bipartite (other interesting consequences of the bipartite nature of our maps will be seen below). A loop is said *rigid* if it only visits faces of type (b).

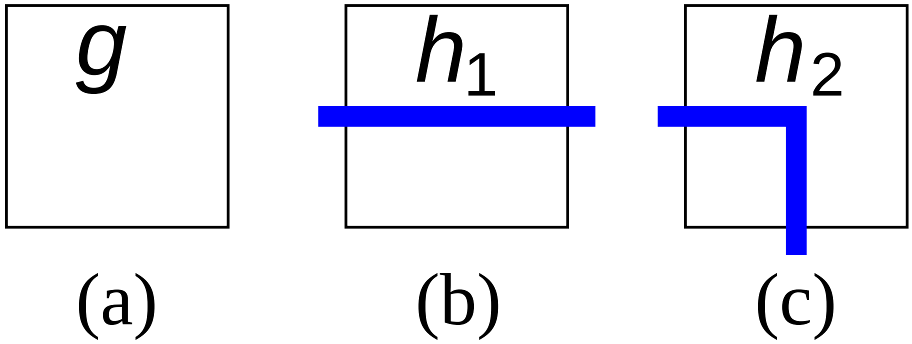

Fig. 1: A configuration of the $O(n)$ loop model on tetravalent maps, when viewed on the dual quadrangulation, is built out of three types of squares: (a) empty squares, weighted by g, (b) squares visited by a loop going straight, weighted by h_1, and (c) squares visited by a loop making a turn, weighted by h_2.

Generally speaking, the $O(n)$ loop model consists in attaching a non-local weight n to each loop, in addition to some local weights. Here, we consider an "annealed" model where the map and the loops are drawn at random altogether: a configuration of the model is the data of a planar tetravalent map endowed with a loop configuration. Local weights are attached to the vertices of the tetravalent map, or equivalently to the faces of the dual quadrangulation, and naturally these weights depend on the face type, see again Fig. 1: g per square of type (a), h_1 per square of type (b), h_2 per square of type (c). The global weight of a configuration is the product of all loop and face weights, and the generating ("partition") function of the model is the sum of the global weights over all configurations. Here n, g, h_1, h_2 are taken as non-negative real numbers. The model is said *well-defined* when the generating function is finite (this is the case for instance when $g + \max(n,1)(2h_1 + 4h_2) \leq 1/12$). It is then possible to normalize the configuration weights and define a probability distribution. The *rigid loop model* corresponds to taking $h_2 = 0$ so that loops are obliged to go straight within each visited square.

As usual it is easier to work with rooted maps. Recall that a rooted map has a distinguished oriented edge (the *root edge*). From now on, the face on the right of the root edge will be called the *external face* and we define a *quadrangulation with a boundary of length* $2p$ ($p \geq 1$) as a rooted bipartite planar map where the external face has degree $2p$ and every other face has degree 4. We shall now consider loop configurations on the dual of a quadrangulation with a boundary and, for simplicity, we assume that no loop visits the external face, to which we therefore decide to attach a local weight 1. Keeping the same weights n, g, h_1, h_2 as above, we define $F_p^{\text{loop}}(n; g, h_1, h_2)$ as the generating function for the $O(n)$ loop model on duals of quadrangulations with a boundary of length $2p$. As particular cases, for $p = 1$ we obtain (upon squeezing the bivalent external face) rooted quadrangulations where the root edge is not crossed by a loop, while for $p = 2$ we obtain (directly) rooted quadrangulations where the external face is not visited.

3.3. The gasket decomposition

The fundamental observation of this paper is that

$$F_p^{\text{loop}}(n; g, h_1, h_2) = F_p(1) \tag{3.5}$$

where $F_p(1)$ is, as in Section 2, the generating function for general bipartite maps with a boundary of length $2p$ (without loops) associated with a sequence $(g_k)_{k\geq 1}$ of face weights satisfying the fixed point condition

$$g_k = g\delta_{k,2} + n \sum_{k'\geq 0} A_{k,k'}(h_1, h_2) F_{k'}(1), \quad k \geq 1 \ . \tag{3.6}$$

Here $A_{k,k'}(h_1, h_2)$ is a polynomial in h_1 and h_2 whose expression will be discussed in the next Section. For the rigid case, it reads simply $A_{k,k'}(h_1, 0) = h_1^{2k}\delta_{k,k'}$, leading to the simpler fixed point condition

$$g_k = g\delta_{k,2} + n h_1^{2k} F_k(1) \ . \tag{3.7}$$

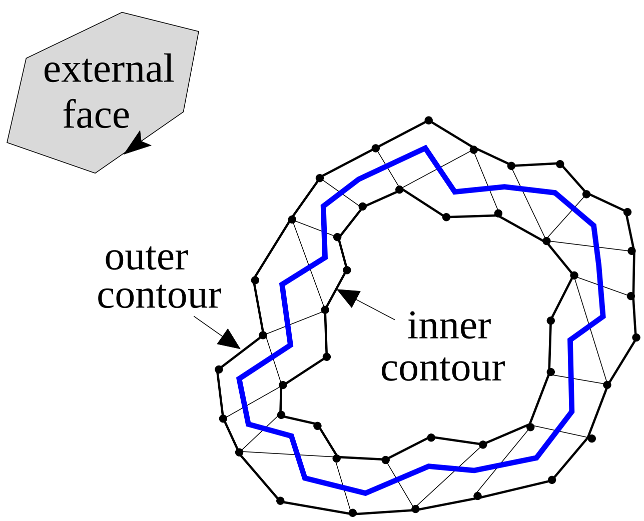

Fig. 2: The outer and inner contours of a loop.

Let us now justify our fundamental observation, by considering a quadrangulation with a boundary of length $2p$ endowed with a loop configuration. For conciseness, we will often, in this Section, call simply "quadrangulation" a quadrangulation with a boundary. Thanks to the external face, the notions of exterior and interior of a loop are well-defined. To each loop, we may associate its *outer* (resp. *inner*) *contour* formed by the edges of the quadrangulation which (i) belong to the exterior (resp. interior) of the loop and (ii) are incident to squares visited by the loop (see Fig. 2). Note that these contours are closed paths living on the edges of the quadrangulation itself and therefore

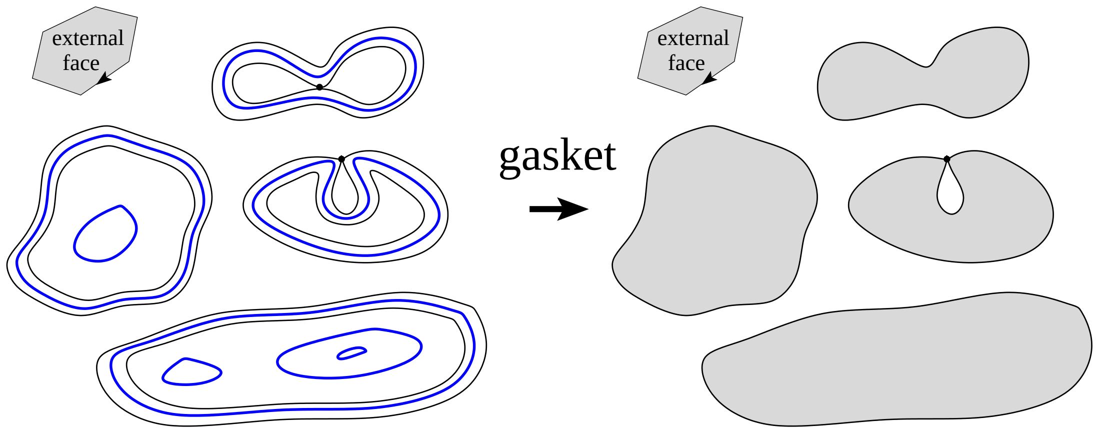

Fig. 3: The gasket is obtained by wiping out the content of the outer contours of the outermost loops, creating holes.

each of them has an even length. Note also that the inner contour may be empty, i.e. may have length 0 if the loop encircles a single vertex.

Using a terminology similar to [13], we call *gasket* the map formed by edges that are exterior to all the loops, see Fig. 3 for an illustration. It is a rooted bipartite planar map containing in particular all the edges incident to the external face, which therefore remains of degree $2p$. Its other faces are of two types: *regular faces* which are the squares of the original quadrangulation that lay outside of every loop, and *holes* delimited by the outer contours of the former outermost loops.

Clearly, the transformation that goes from the original quadrangulation with loops to the gasket is not reversible. To make it reversible, we must keep track of the former content of the holes. In this respect, a first remark is that the interior of a loop may itself be viewed as a quadrangulation with a boundary endowed with a loop configuration, where the boundary is nothing but the inner contour of the loop at hand (when the inner contour has length 0, we have the so-called "vertex-map" reduced to one vertex and one face, with no edge). In order not to lose any information, we must also consider the squares covered by the loop itself, which form a *ring* lying in-between the inner and outer contours. As shown in Fig. 2, this ring is a cyclic sequence of squares glued together. To summarize, the content of a hole of degree $2k$ ($k \geq 1$) is described by a pair formed by a ring with outer length $2k$ and inner length $2k'$ (for some $k' \geq 0$), and an *internal* quadrangulation with a boundary of length $2k'$ (equal to the vertex-map if $k' = 0$) endowed with a loop configuration.

Two important remarks are in order. First, to ensure bijectivity, both the ring and the internal quadrangulation must be rooted, respectively on its outer contour and on its boundary. The positions of their root edges is inherited from the rooting of the original quadrangulation via a somewhat irrelevant yet well-defined procedure: for instance we may consider the leftmost shortest path that stays within the gasket, starts with the root edge and ends with an edge of the outer contour, that edge being selected as the ring root; then, using only the data of the ring, we may easily select

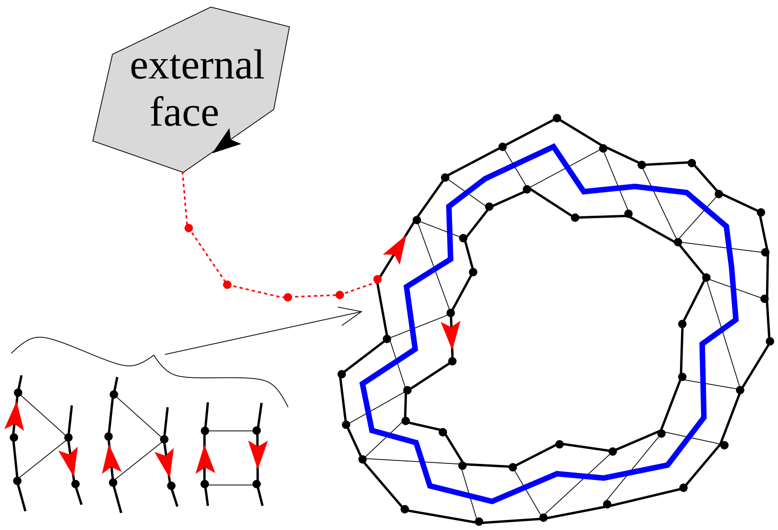

Fig. 4: Illustration of the rooting procedure. We draw the leftmost shortest path (dashed line) in the gasket starting with the map root edge and ending with an edge of the outer contour of the loop at hand, which we pick as root for the ring. We then select an edge of the inner contour by the rules displayed at the bottom left, which we pick as root for the internal quadrangulation.

an edge of the inner contour which serves as root for the internal quadrangulation (see Fig. 4). Second, an essential assumption is that the original map, the gasket and the internal quadrangulations are all possibly separable (i.e. may contain separating vertices whose removal disconnects them). In particular, a separating vertex incident to a hole in the gasket appears whenever a multiple point was present along the outer contour of the associated loop on the original quadrangulation. Similarly, a multiple point of the inner contour of a loop results into a separating vertex on the boundary of the associated internal quadrangulation. In contrast, all the vertices of a ring are considered as distinct (since the information about contacts is recorded in the gasket and internal quadrangulation). At this stage, it should be clear that the decomposition is reversible: given a ring and a quadrangulation with a boundary with compatible lengths, there is a well-defined procedure to join them and fill a hole of the gasket.

Globally, we have a one-to-one correspondence between, on the one hand, quadrangulations with a boundary endowed with a loop configuration and, on the other hand, gaskets endowed with hole contents. Translating this correspondence in the language of generating functions, we deduce that the wanted quantity $F_p^{\rm loop}(n; g, h_1, h_2)$ is equal to the generating function $F_p(1)$ for gaskets, where each regular face still receives a weight g and each hole of degree $2k$ receives a weight $n \sum A_{k,k'}(h_1, h_2) F_{k'}(1)$ (see Fig. 5). In this hole weight, n accounts for the associated loop, $F_{k'}(1)$ for the internal quadrangulation, while $A_{k,k'}(h_1, h_2)$ is the generating function for rings with sides of lengths $2k$ and $2k'$. This establishes Eq. (3.6).

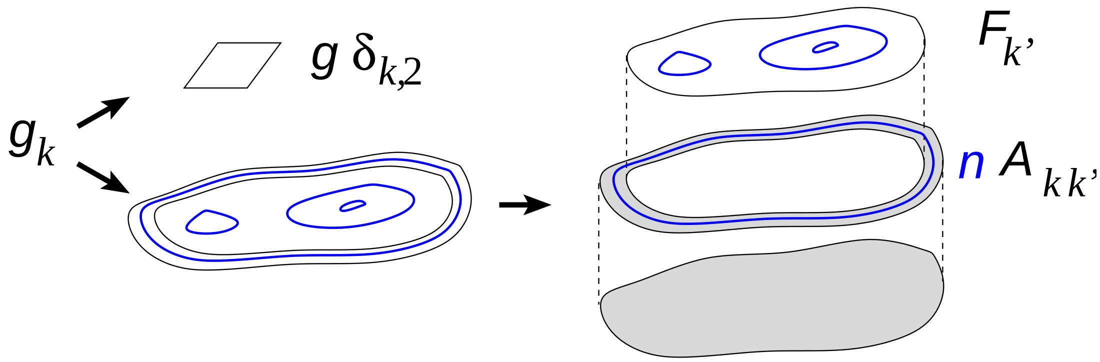

Fig. 5: A face of the gasket is either a regular square (weighted by g) or a hole of even degree, say $2k$. The content of this hole is made of a ring with outer length $2k$ and inner length $2k'$ for some $k' \geq 0$ (weighted by $nA_{k,k'}(h_1,h_2)$), and of an internal quadrangulation with a boundary of length $2k'$ (weighted by $F'_k(1)$).

3.4. The ring generating function

Let us now discuss the precise form of the ring generating function $A_{k,k'}(h_1,h_2)$. We have the explicit expression

$$A_{k,k'}(h_1,h_2) = \sum_{j=0}^{\min(k,k')} \frac{2k}{k+k'} \frac{(k+k')!}{(2j)!(k-j)!(k'-j)!} h_1^{2j} h_2^{k+k'-2j} \ , \tag{3.8}$$

which follows from a simple counting argument: the term with index j in the sum corresponds to rings with $2j$ squares of type (b), each weighted by h_1. Such rings have $k-j$ (resp. $k'-j$) squares of type (c) with their two unvisited sides along the outer (resp. inner) contour of the ring, all having a weight h_2. There are $\binom{k+k'}{2j,k-j,k'-j}$ possible sequences of such three types of squares. However, this number has to be be corrected because a ring has a distinguished edge (among $2k$) on its outer contour rather than a distinguished square (among $k+k'$) to start the sequence. This explains the corrective factor $2k/(k+k')$.

We have the alternative expression, which will prove useful in the following:

$$\sum_{k'\geq 0} A_{k,k'}(h_1,h_2) z^{-k'} = (\lambda_+(z))^k + (\lambda_-(z))^k, \quad \lambda_\pm(z) = z\left(\frac{h_1 \pm \sqrt{h_1^2 - 4h_2^2 + 4h_2 z}}{2(z-h_2)}\right)^2 \tag{3.9}$$

(note that the sum converges for $|z| > h_2$). This expression naturally follows from a transfer matrix argument. As seen in Fig. 6, we may distinguish two "states" for the edges of the outer contour depending on whether they follow clockwise a vertex adjacent to a vertex of the inner contour (state 1) or not (state 2). We may then write

$$\sum_{k'\geq 0} A_{k,k'}(h_1,h_2) z^{-k'} = \mathrm{tr} M(z)^{2k}, \qquad M(z) = \begin{pmatrix} \frac{h_1 z^{-1/2}}{1-h_2 z^{-1}} & \frac{1}{1-h_2 z^{-1}} \\ h_2 & 0 \end{pmatrix} \ , \tag{3.10}$$

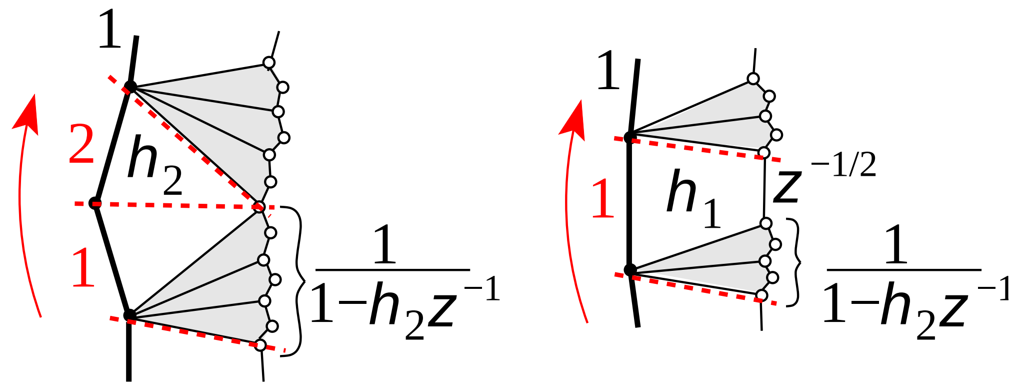

Fig. 6: The ring transfer matrix $M(z)$. An edge of the outer contour (thick edges) is in state 1 if it follows clockwise a vertex adjacent to a vertex of the inner contour, and in state 2 otherwise. The dashed lines indicate how we distribute the weights in the transfer matrix. Each edge in state 2 is followed by an edge in state 1 and receives the weight h_2 of its incident square of type (c). Each edge in state 1 first receives a weight $1/(1-h_2z^{-1})$ accounting for a sequence of squares of type (c) as shown here in grey, then a further weight $h_1 z^{-1/2}$ if and only if it is followed by another edge in state 1.

where $M(z)$ is the transfer matrix describing the transition between two successive edge states. This establishes (3.9) where $\lambda_\pm(z)$ are simply the eigenvalues of $M^2(z)$, solutions of:

$$(h_2(\lambda+z)-\lambda z)^2 - h_1^2\lambda z = 0 \; . \tag{3.11}$$

Interestingly, λ and z play a symmetric role in this equation. It follows that $\lambda(\lambda(z)) = z$ provided we pick the correct determination of λ. More precisely, it may be checked that $z \mapsto \lambda_+(z)$ is an involution on (h_2, ∞) with a unique fixed point:

$$z^* = h_1 + 2h_2 = \lambda_+(z^*) \; . \tag{3.12}$$

Finally, we observe that for rigid loops, there is a unique ring with outer length $2k$, and it has the same inner length. This yields immediately $A_{k,k'}(h_1,0) = h_1^{2k}\delta_{k,k'}$ as announced in the previous Section. This is compatible with (3.8) for $h_2 = 0$, and with (3.9) upon noting that $\lambda_+(z) = h_1^2/z$ and $\lambda_-(z) = 0$.

4. Asymptotic self-consistency

In this Section, we make a first analysis of the fixed point condition (3.6) via its asymptotic consequences. The fixed point condition is an equation for the unknown sequence $(g_k)_{k\geq1}$ that depends on the physical parameters n, g, h_1 and h_2. The model is well-defined when a finite solution exists (i.e. F_p^{loop} is finite). Since it involves the generating functions $(F_k(1))_{k\geq0}$, such a finite solution $(g_k)_{k\geq1}$ is necessarily admissible in the sense of Section 2. The purpose of this Section is to classify the possible nature (subcritical or critical, generic or not) of the solution.

4.1. Rigid case

For simplicity we first concentrate on the rigid case. In this Section, $(g_k)_{k\geq 1}$ denotes a sequence satisfying the fixed point condition (3.7). It immediately implies the asymptotic condition

$$g_k \sim n h_1^{2k} F_k(1) \qquad (k \to \infty). \tag{4.1}$$

In Section 2, we have classified the possible asymptotic behaviors of $F_k(1)$ depending on the nature of the weight sequence $(g_k)_{k\geq 1}$: the equations (2.5), (2.6) and (2.15) describe respectively the subcritical, generic critical and non-generic critical cases. Let us first remark that, for large k, we have $(F_k(1))^{1/k} \to 4R(1)$ in all cases. By comparing with the general relation $(g_k)^{1/k} \to 1/(4R_c)$, where R_c is the radius of convergence of the function $\varphi(R)$ defined in Eq.(2.4), Eq.(4.1) implies the relation

$$16 h_1^2 R_c R(1) = 1 \ . \tag{4.2}$$

For later use, we introduce the parameter

$$\tau = 4 h_1 R(1) \ . \tag{4.3}$$

Since $R(1) \leq R_c$, we have by (4.2) that $\tau \leq 1$ and the condition $\tau = 1$ amounts to $R(1) = R_c$.

We now prove that $R(1) = R_c$ ($\tau = 1$) *if and only if the sequence* $(g_k)_{k\geq 1}$ *is critical and non-generic.* The first implication ("if") was shown in Section 2 without recourse to the fixed point condition (see Eq. (2.10)). Conversely, assume that the sequence $(g_k)_{k\geq 1}$ is either subcritical or generic critical. Its asymptotic behavior is directly deduced from that of $F_k(1)$ via Eq.(4.1). In the subcritical case, due to the $k^{-3/2}$ factor in Eq.(2.5), $\varphi'(R_c)$ is infinite while, by definition of subcriticality, we have $\varphi'(R(1)) < 1$, hence $R(1) \neq R_c$. In the generic critical case, due to the $k^{-5/2}$ factor in Eq.(2.6), $\varphi''(R_c)$ is infinite while, by definition of genericity, we have $\varphi''(R(1)) < \infty$, hence $R(1) \neq R_c$. This completes the proof (note that when we do not impose the fixed point condition, we can easily construct examples of subcritical and generic critical sequences such that $R(1) = R_c$). In view of the discussion at the end of Section 2.3, we deduce that the face degree distribution of the gasket, i.e. the outer loop length distribution in the $O(n)$ model, has a fat tail iff the model is at a non-generic critical point.

If we now assume that the sequence $(g_k)_{k\geq 1}$ is critical and non-generic, and takes the general form (2.7) for a suitable sequence $(g_k^\circ)_{k\geq 1}$ satisfying (2.8), then, in addition to the condition $\tau = 1$, we see by comparing the prefactors of (2.15) and (2.16) that

$$n = 2 \sin \pi (a - 3/2) \ . \tag{4.4}$$

In particular, n must be in the range $]0, 2[$ for non-generic criticality to be possible.

To summarize, we have shown that the fixed point condition (3.7) is compatible with the model being either subcritical, generic critical, with $\tau < 1$ or, more interestingly,

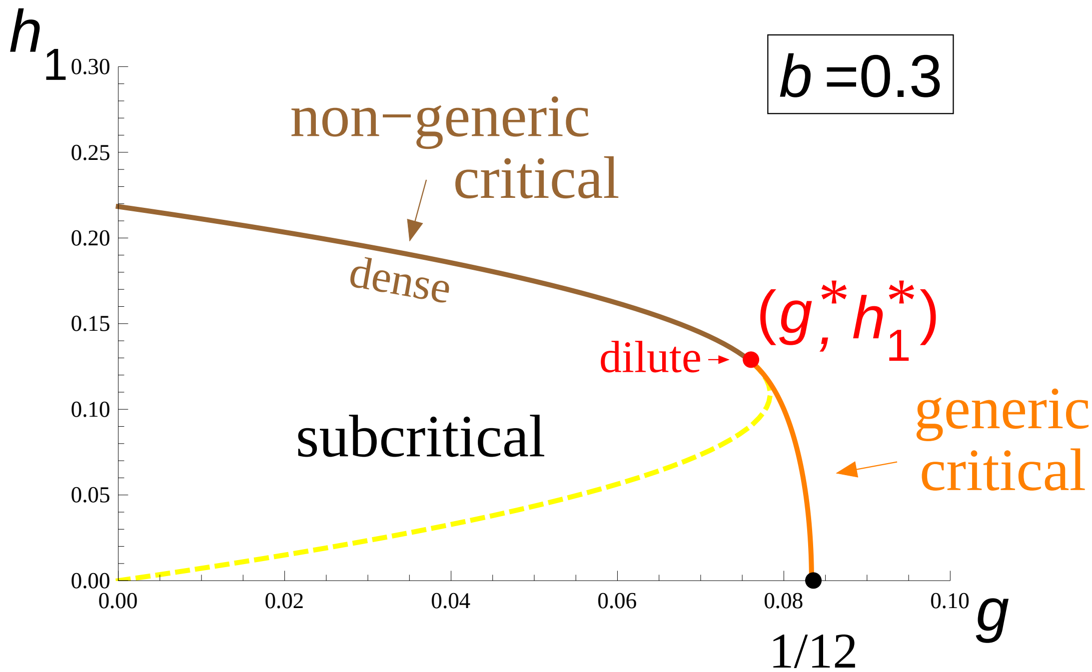

Fig. 7: The phase diagram of the $O(n)$ rigid loop model in the (g, h_1) plane, as computed exactly in Section 6 below. It is shown here for $b = 0.3$ (with $n = 2\cos\pi b$) but it is qualitatively the same for any value of b between 0 and 1/2. The critical line separates the region where the model is subcritical from the region where it is ill-defined. The type of criticality changes along the line: generic for $g > g^*$, non-generic for $g < g^*$ with an exponent $a = 2 - b$ (dense model), non-generic at $g = g^*$ with an exponent $a = 2 + b$ (dilute model). The line of non-generic critical points is an arc of parabola which we extended in dashed line for clarity.

non-generic critical with $\tau = 1$ and $a = 2 \pm b$, $\pi b = \arccos(n/2)$. As shown in Section 6, no other more exotic non-generic behavior than (2.7)-(2.8) is found.

So far, we did not tell which behavior corresponds to a given value of the "physical" parameters n, g and h_1. Since the generating function $F_p^{\text{loop}}(n; g, h_1, 0)$ is an increasing function of its parameters, there exists a decreasing function $h_c(n; g)$ such that the model is well-defined for $h_1 < h_c(n; g)$ and ill-defined for $h_1 > h_c(n; g)$. In particular, $h_c(n; g) > 0$ if $g < 1/12$. By the monotonicity property of Section 2.1, we see that the model is subcritical whenever $h_1 < h_c(n; g)$. In particular, any critical point necessarily lies on the line $h_1 = h_c(n; g)$. An exact expression for $h_c(n, g)$ will be given in Section 6 for $0 < n < 2$. There we find that the model is critical all along the line $h_1 = h_c(n; g)$. Furthermore, along this line, there exists a g^* such that (i) for $g^* < g \leq 1/12$ the model is generic critical, (ii) for $0 \leq g < g^*$ the model is non-generic critical with $a = 2 - b$ (*dense* model) and (iii) at $g = g^*$ the model is non-generic critical with $a = 2 + b$ (*dilute* model). The corresponding phase diagram is displayed in Fig. 7.

4.2. Non-rigid case

Let us now consider the general case of arbitrary non negative values of h_1 and h_2 and analyze the consequences of the fixed point condition (3.6) on the asymptotics of g_k. To treat the subcritical, generic critical and non-generic critical cases simultaneously, we write the asymptotics of $F_k(1)$ in the form

$$F_k(1) \sim \chi \frac{(z_c)^{-k}}{k^a}, \quad z_c = \frac{1}{4R(1)} , \tag{4.5}$$

with now $3/2 \leq a \leq 5/2$ and where χ may be read off Eqs.(2.5), (2.6) and (2.15) respectively. Let us introduce the generating function

$$F(z) = \sum_{k \geq 0} F_k(1)\, z^k , \tag{4.6}$$

related to the resolvent via $W(\xi) = F(1/\xi^2)/\xi$. The function $F(z)$ is singular when $z \to z_c$ with a singular part given by

$$F(z)|_{\text{sing}} \sim \chi\, \Gamma(1-a) \left(1 - \frac{z}{z_c}\right)^{a-1} \tag{4.7}$$

where we temporarily exclude the case $a = 2$.

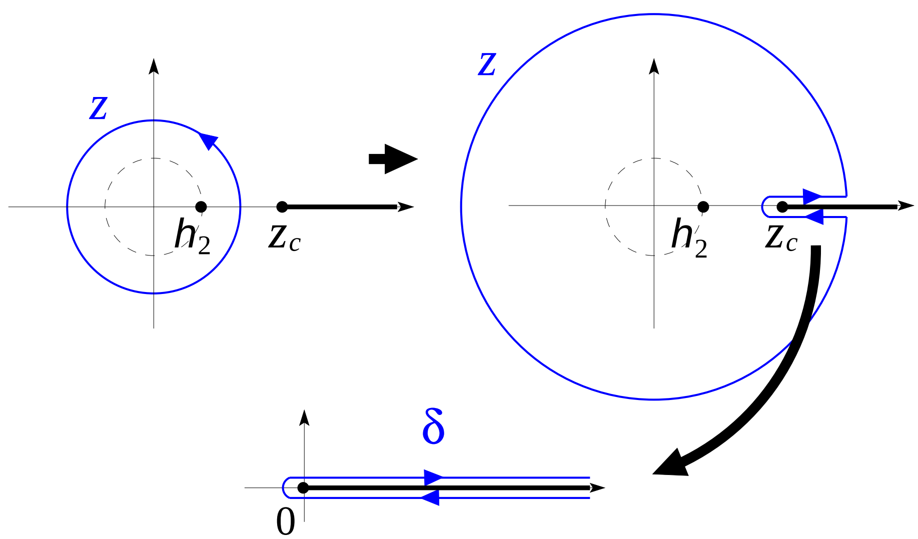

Fig. 8: Deformation of the contour of integration in Eq. (4.8): starting from a circle with radius between h_2 and z_c, we deform it by encompassing the cut of the function $F(z)$, as shown on the right, so that the dominant contribution for $k \to \infty$ is given, after the change of variable $z \to \delta$ of Eq. (4.9), by integrating the leading order of the integrand over the Hankel contour shown below.

To estimate the r.h.s of (3.6), we use Eq.(3.9) to write

$$\sum_{k'\geq 0} A_{k,k'}(h_1,h_2)F_{k'}(1) = \oint \frac{dz}{2\mathrm{i}\pi z}F(z)\ \left(\lambda_+(z)^k + \lambda_-(z)^k\right) \tag{4.8}$$

where me may choose as integration contour the circle $|z| = z_0$ for any $h_2 < z_0 < z_c$. At large k, this quantity may be evaluated via the method of Hankel contours. Indeed, the contour may be deformed as shown in Fig. 8 so that the dominant contribution to the integral comes from the vicinity of z_c. Using the scaling

$$z = z_c\left(1+\frac{\delta}{k}\right)\ , \tag{4.9}$$

the variable δ is now to be integrated back and forth from ∞ to 0 (below and above the cut of $F(z)$). Using the expansion

$$\lambda_+(z) = \lambda_+(z_c)\left(1 - \mu_+\frac{\delta}{k} + O\left(\frac{1}{k^2}\right)\right) \tag{4.10}$$

with

$$\mu_+ = -\left(z\frac{d}{dz}\log\lambda_+(z)\right)\Big|_{z=z_c}\ , \tag{4.11}$$

we get at large k

$$\begin{aligned}\sum_{k'\geq 0} A_{k,k'}(h_1,h_2)F_{k'}(1) &\sim \chi\,\Gamma(1-a)\frac{\lambda_+(z_c)^k}{k^a}\,\frac{\sin\pi(1-a)}{\pi}\int_0^\infty d\delta\,\delta^{a-1}\,\mathrm{e}^{-\mu_+\delta} \\ &= \chi\frac{\lambda_+(z_c)^k}{(\mu_+)^a\,k^a}\ .\end{aligned} \tag{4.12}$$

Here we assumed that $\lambda_+(z_c) > \lambda_-(z_c)$, which is valid as soon as $h_1 > 0$. The case $h_1 = 0$ will be discussed later. From the fixed point condition, this behavior is transferred into the asymptotics of g_k

$$g_k \sim n\,\chi\frac{\lambda_+(z_c)^k}{(\mu_+)^a\,k^a}\ . \tag{4.13}$$

By a slight modification of our reasoning, one may easily show that this asymptotics also holds in this case $a = 2$. Using again the property that $(g_k)^{1/k} \to 1/(4R_c)$ for $k \to \infty$, this yields a first consistency relation

$$\frac{1}{4R_c} = \lambda_+\left(\frac{1}{4R(1)}\right)\ . \tag{4.14}$$

This relation generalizes (4.2) which is recovered by noting that $\lambda_+(z) = h_1^2/z$ in the rigid case.

Repeating the proof of previous Section, we have here again $R(1) \leq R_c$ and $R(1) = R_c$ if and only if the sequence $(g_k)_{k\geq 1}$ is non-generic critical. Focusing on such a non-generic critical solution, the relation (4.14) allows to identify $1/(4R(1))$ as the fixed point z^* of the mapping $z \mapsto \lambda_+(z)$, namely

$$\frac{1}{4\,R(1)} = z^* = h_1 + 2h_2 \ . \tag{4.15}$$

A second consistency relation is now obtained by comparing the prefactors in (4.13) and (2.16) and using the precise value $\chi = 2cR(1)/\sin\pi(a-3/2)$. We eventually deduce the relation

$$n = 2(\mu^*)^a \sin\pi(a-3/2), \qquad \mu^* = -\left(z\frac{d}{dz}\log\lambda_+(z)\right)\Big|_{z=z^*} = -\lambda'_+(z^*) \ . \tag{4.16}$$

Now we recall that $z \mapsto \lambda_+(z)$ is an involution in the vicinity of its fixed point z^*, which implies that $\lambda'_+(z^*) = -1$ (since $\lambda_+(z)$ obviously decreases with z), a result which may trivially be checked by a direct calculation. This fixes $\mu^* = 1$ and we recover *the same relation (4.4) as in the rigid case.*

To summarize, the fixed point condition (3.6) is compatible with having the model subcritical, generic critical or non-generic critical with $a = 2 \pm b$, $\pi b = \arccos(n/2)$ as long as $h_1 > 0$. The case $h_1 = 0$ leads to a different relation since in this case $\lambda_+(z) = \lambda_-(z) = h_2 z/(z-h_2)$ hence the contribution of $\lambda_-(z)$ in (4.8) cannot be neglected. Then Eq. (4.12) must be replaced by

$$\sum_{k'\geq 0} A_{k,k'}(0,h_2)F_{k'}(1) \sim 2\chi\frac{\lambda_+(z_c)^k}{(\mu_+)^a k^a} \ . \tag{4.17}$$

At a non-generic point, we get the same relation $z_c = 1/(4R(1)) = z^*$ as before (with now $z^* = 2h_2$) and the same value $\mu^* = 1$, but comparing the prefactors as was done above now yields the new relation

$$n = \sin\pi(a-3/2) \ , \quad h_1 = 0 \tag{4.18}$$

without the factor 2 in front. We thus expect that our $O(n)$ loop model for $h_1 = 0$ and $0 < n < 1$ should have the same non-generic critical behavior as the $O(2n)$ loop model for $h_1 > 0$.

5. Linear integral equation

In this Section, we show how to rephrase the fixed point conditions (3.6) and (3.7) as linear integral equations, and then analyze their solutions. We first focus on the rigid case before addressing the general case of arbitrary h_1 and h_2.

5.1. Derivation of the equation in the rigid case

We start from the general expression (2.4) for the function $u(R)$, defining implicitly the generating function $R(u)$. Differentiating (2.4) with respect to R, we readily get

$$u'(R) = 1 - \sum_{k\geq 1} g_k \binom{2k-1}{k} k R^{k-1}. \tag{5.1}$$

Independently we observe that, upon performing the change of variable $v = u(\tilde{R})$ in (2.2), we have

$$F_k(u(R)) = \binom{2k}{k} \int_0^R \tilde{R}^k u'(\tilde{R}) d\tilde{R}. \tag{5.2}$$

So far we have only rewritten some equations of Section 2.1, related to bipartite maps. Let us now combine them with the fixed point condition (3.7) transcribing the gasket decomposition for the $O(n)$ rigid loop model: assuming that n, g and h_1 are such that the model is well-defined, we plug (5.2) at $R = R(1)$ into (3.7), then the result into (5.1), and obtain a *linear integral equation* for $u'(R)$:

$$u'(R) = 1 - 6\, g\, R - \frac{n}{2} \sum_{k\geq 1} k \binom{2k}{k}^2 h_1^{2k}\, R^{k-1} \int_0^{R(1)} \tilde{R}^k u'(\tilde{R}) d\tilde{R} \tag{5.3}$$

(note that $\binom{2k-1}{k} = \binom{2k}{k}/2$). By the change of variables

$$x = \frac{R}{R(1)}, \quad y = \frac{\tilde{R}}{R(1)}, \quad f(x) = u'(R), \quad \rho = 6g\, R(1), \quad \tau = 4h_1 R(1)\ , \tag{5.4}$$

the linear integral equation is rewritten in the more compact form

$$\begin{aligned} & f(x) + \frac{n}{2\pi} \int_0^1 K_\tau(x,y) f(y) dy = 1 - \rho x \qquad (0 \leq x \leq 1) \\ & K_\tau(x,y) = \tau^2 y\, \psi(\tau^2 xy), \qquad \psi(t) = \sum_{k\geq 1} \frac{\pi k}{16^k} \binom{2k}{k}^2 t^{k-1}\ . \end{aligned} \tag{5.5}$$

We recognize a *Fredholm integral equation of the second kind* whose unknown is the function f defined on the range $[0,1]$, and which depends on the non-negative parameters n, ρ, τ. Note the consistency relation

$$\int_0^1 f(x) = \frac{1}{R(1)} \tag{5.6}$$

which follows from $1 = u(R(1)) = \int_0^{R(1)} u'(\tilde{R}) d\tilde{R} = R(1) \int_0^1 f(x) dx$.

Let us now discuss the conditions on the function f and the parameters ρ and τ that arise from our derivation. First, the existence of the inverse function $R(u)$ for u between 0 and 1 implies that $u'(R(1)) \geq 0$ and $u'(R) > 0$ for $0 \leq R < R(1)$. This immediately translates into the positivity conditions

$$f(1) \geq 0, \qquad f(x) > 0 \text{ for } 0 \leq x < 1 \tag{5.7}$$

with $f(1) = 0$ iff the model is critical in the sense of Section 2.1. Furthermore, we also obtain that $\rho \leq 1$, since $u'(R) \leq 1 - 6\,g\,R$ by (5.3). Second, by the discussion of Section 4.1, we have $\tau \leq 1$ and the case $\tau = 1$ corresponds to a non-generic critical point. Note that this is precisely the range on which the equation (5.5) is well-defined. Indeed, since $(\pi k/16^k)\binom{2k}{k}^2 \to 1$ for $k \to \infty$, the radius of convergence of $\psi(t)$ is 1, with $\psi(t) \sim 1/(1-t)$ for $t \to 1$, hence $K_\tau(x,y)$ is a smooth function of $(x,y) \in [0,1]^2$ except for $\tau = 1$ where it has a polar singularity at $x = y = 1$ (however the integral equation still holds for $x = 1$, as $f(y) \to 0$ for $y \to 1$ sufficiently fast).

Conversely, given ρ and τ both between 0 and 1 and such that the equation (5.5) admits a solution f satisfying the positivity conditions (5.7), we may return to the original variables as follows. We first compute $R(1)$ via (5.6), then deduce $u'(R)$, g and h_1 via (5.4), and finally obtain $u(R)$ as the primitive of $u'(R)$ satisfying $u(0) = 0$. Following the steps of the above derivation backwards, we find that $u(R)$ satisfies (2.4) with the g_k's given by (3.7), so that we are indeed "solving" the $O(n)$ rigid loop model. The positivity conditions ensure that the inverse function $R(u)$, thus the model, are well-defined. In particular, the generating function $F_p^{\text{loop}}(n; g, h_1, 0) = F_p(1)$ is directly expressed from f by

$$F_p(1) = \binom{2p}{p} \frac{\int_0^1 x^p\, f(x)\, dx}{\left(\int_0^1 f(x)\, dx\right)^{p+1}} \,. \tag{5.8}$$

In conclusion, we have shown that generating functions for the $O(n)$ rigid loop model may be obtain by solving the linear integral equation (5.5). In practice, the main difficulty is that the change of parameters $(g, h_1) \to (\rho, \tau)$ is rather intricate. Nevertheless, if we were able to compute the function f for arbitrary parameters n, ρ, τ, then we could deduce $F_p^{\text{loop}}(n; g, h_1, 0)$ in a parametric form. Before further analyzing the solutions of (5.5), let us extend our formalism to the non-rigid case.

5.2. Extension to the non-rigid case

We may repeat the above analysis in the case of arbitrary values of h_1 and h_2, replacing the fixed point condition (3.7) by the more general one (3.6). The main complication is that an extra sum over a variable k' is involved. The equation (4.8) allows to rewrite this sum as a contour integral. Furthermore by (5.2) we have

$$F(z) = \int_0^{R(1)} \frac{u'(\tilde{R}) d\tilde{R}}{\sqrt{1 - 4\,z\,\tilde{R}}}. \tag{5.9}$$

Substituting these expressions into (3.6), then into (5.1), we may explicitly evaluate the sum over the variable k and derive a linear integral equation for $u'(R)$. Again a more compact form is obtained after a suitable change of variables: we let x, y, $f(x)$, ρ be as in (5.4) while we now define

$$\tau = 4R(1)(h_1 + 2h_2). \tag{5.10}$$

After some work, we arrive at the same form for the linear integral equation

$$f(x) + \frac{n}{2\pi}\int_0^1 K_\tau(x,y)f(y)dy = 1 - \rho x \tag{5.11}$$

but with a different kernel $K_\tau(x,y)$, now given by

$$K_\tau(x,y) = \oint \frac{d\zeta}{2\mathrm{i}\zeta}\frac{1}{(1-\tau y\zeta)^{1/2}}\frac{1}{2}\left(\frac{\tau\Lambda_+(\zeta)}{(1-\tau x\Lambda_+(\zeta))^{3/2}} + \frac{\tau\Lambda_-(\zeta)}{(1-\tau x\Lambda_-(\zeta))^{3/2}}\right) \tag{5.12}$$

where we may choose as integration contour the circle $|\zeta| = \zeta_0$ for any $\Lambda_+^{-1}(1/(\tau x)) < \zeta_0 < 1/(\tau y)$, and where we introduced the rescaled functions

$$\Lambda_\pm(\zeta) = \frac{\lambda_\pm\left((h_1+2h_2)\,\zeta\right)}{h_1 + 2h_2} \ . \tag{5.13}$$

Note that Λ_+ and Λ_- are functions of the ratio h_1/h_2, and therefore $K_\tau(x,y)$ depends implicitly on this ratio. Despite the apparent complication in the expression for the kernel $K_\tau(x,y)$, much of the discussion of Section 5.1 can be generalized to the non-rigid case. Using exactly the same arguments, we find that:

- the consistency relation (5.6),
- the positivity conditions (5.7), where again $f(1) = 0$ iff the model is critical,
- the inequality $\rho \le 1$,
- the expression (5.8) for the $O(n)$ loop model generating functions $F_p^{\mathrm{loop}}(n; g, h_1, h_2) = F_p(1)$,

all still hold in the non-rigid case. The discussion of τ, in view of that of Section 4.2 and of the general expression (5.10), becomes slightly more involved. We nevertheless find that τ is still between 0 and 1, with $\tau = 1$ corresponding to a non-generic critical point. Furthermore, $K_\tau(x,y)$ is a smooth function of $(x,y) \in [0,1]^2$ except in the case $\tau = 1$ where, remarkably, we obtain the same singular behavior as in the rigid case, provided that $h_1 > 0$. More precisely, when x and y tend to 1 (keeping $(1-x)/(1-y)$ finite), we have

$$K_1(x,y) \sim \frac{1}{1-xy} \ . \tag{5.14}$$

For the record, let us briefly explain how this property results from (5.12) by a saddle-point approximation. The contour integral in the latter equation is dominated, for $x, y \to 1$ by the vicinity of $\zeta = 1$ and we set

$$\zeta = 1 + \mathrm{i}\,\epsilon\, Z\ , \quad x = 1 - \epsilon\, X\ , \quad y = 1 - \epsilon\, Y \tag{5.15}$$

with $\epsilon \to 0$. At leading order in ϵ, we have

$$\Lambda_+(\zeta) = 1 - \epsilon\,\mu^*\, Z\ + \cdots, \quad \text{with } \mu^* = -\Lambda'_+(1) = -\lambda'_+(z^*) = 1 \tag{5.16}$$

using again the "miraculous" involutivity of $\lambda_+(z)$ around z^*. We finally obtain the estimate

$$K_1(x,y) \sim \frac{1}{\epsilon}\int_{-\infty}^{\infty} \frac{dZ}{4}\frac{1}{(Y-\mathrm{i}Z)^{1/2}(X+\mathrm{i}Z)^{3/2}} = \frac{1}{\epsilon}\frac{1}{X+Y} \sim \frac{1}{1-xy}\ . \tag{5.17}$$

In the case $h_1 = 0$, this estimate must be doubled since Λ_- has then an equal, instead of negligible, contribution. We thus have $K_1(x,y) \sim 2/(1-xy)$ in this case.

In summary, the generating functions $F_p^{\rm loop}(n; g, h_1, h_2)$ for the $O(n)$ loop model on tetravalent maps may be expressed, via a change of parameters $(g, h_1, h_2) \to (\rho, \tau, h_1/h_2)$, in terms of the solution of the linear integral equation (5.11).

5.3. Discussion of the solution of (5.11) and of its singular behavior

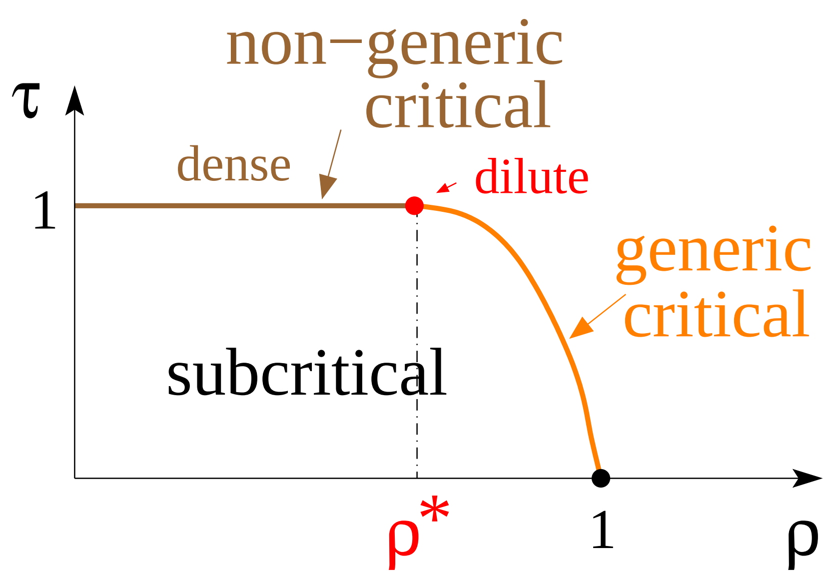

Fig. 9: Qualitative phase diagram of the $O(n)$ loop model in the (ρ,τ) plane for fixed values of n (between 0 and 2) and of h_1/h_2.

As mentioned previously, Eq. (5.11) is a Fredholm integral equation of the second kind for the function $f(x)$. This equation depends on n, ρ, τ and the ratio h_1/h_2 as follows: $-n/(2\pi)$ is the so-called *parameter of integral equation*, ρ appears only in the *right-hand side* and τ and h_1/h_2 determine the *kernel* (5.12) (in particular, for $h_1/h_2 = \infty$, we recover the rigid case of (5.5)). Here our terminology is borrowed from [17]. In this Section, we shall assume that n and h_1/h_2 are fixed quantities and will look at the dependence of $f(x)$ on τ and ρ, both varying *a priori* between 0 and 1. We shall successively consider the case $\tau < 1$ and the case $\tau = 1$, since the latter is special and corresponds, as shown before, to non-generic criticality. Here we will stay at a very general level. Still, our discussion will result in the qualitative phase diagram of Fig. 9 in the (ρ, τ) plane. This picture will be corroborated by the exact results of Section 6.

The case $\tau < 1$: here $K_\tau(x, y)$ is a smooth function of (x, y) varying in the domain $[0, 1]^2$, it is thus square-integrable. We may then apply Fredholm theory: assuming that $-(2\pi)/n$ is not a characteristic value of the integral equation (we expect this assumption to be valid when n is between 0 and 2), Eq. (5.11) has a unique solution $f(x)$ for each value of ρ. More precisely, by linearity, we have

$$f(x) = f_1(x) - \rho\, f_{\mathrm{id}}(x) \tag{5.18}$$

where $f_1(x)$ (resp. $f_{\mathrm{id}}(x)$) is the solution of the linear integral equation obtained by changing its right-hand side into 1 (resp. x). Note that $f_1(x)$ and $f_{\mathrm{id}}(x)$ do not depend on ρ but implicitly depend on τ as well as n and h_1/h_2. These functions might be expressed, for instance, via Neumann series at least for n small enough.

However, the solution $f(x)$ does not necessarily satisfy the positivity conditions (5.7) ensuring that the model is well-defined. In view of (5.18), we conjecture that these conditions amount to $\rho \leq \rho_c$, where $\rho_c = f_1(1)/f_{\mathrm{id}}(1)$. In particular, when $\rho = \rho_c$, $f(1) = 0$ and the model is critical. We identify this critical point as generic in the sense of Section 2.1, since we expect $f'(1)$ hence $u''(R(1))$ to be finite for any $\tau < 1$. As ρ_c depends on τ, we obtain a generic critical line in the (ρ, τ) plane, see Fig. 9. In the rigid case, this line corresponds precisely to the generic critical line in the (g, h_1) plane displayed on Fig. 7. Note that, for any values of n and h_1/h_2, the generic critical line starts from $(\rho = 1, \tau = 0)$, corresponding to the critical point for pure quadrangulations without loops. Indeed, since $K_0 = 0$, we have $f(x) = 1 - \rho x$ for $\tau = 0$, therefore the critical value of ρ is 1 (we also recover the known critical values $R(1) = 2$ and $g = 1/12$). Moreover, in view of the phase diagram of Fig. 7 valid for $0 < n < 2$, we expect the line of generic critical points to connect continuously with the non-generic critical line $\tau = 1$ (to be discussed below): ρ_c should be positive for all $\tau < 1$, and have a positive limit ρ^* as $\tau \to 1$.

The case $\tau = 1$: now $K_1(x, y)$ is no longer a smooth function of (x, y) in the domain $[0, 1]^2$, but diverges as (5.14) for $x, y \to 1$. We do not know whether a general theory applies to such kernels. Nevertheless, from [18], we expect that Eq.(5.11) has, for all ρ, a unique continuous solution satisfying $f(1) = 0$. Thanks to this cancellation, the integral $\int_0^1 K_1(1, y) f(y) dy$ may still be well-defined.

Furthermore, we see that the solution $f(x)$ cannot be regular for $x \to 1$ (i.e. vanish as an integer power of $x - 1$) as otherwise, the integral in (5.11) would contain singular terms (with logarithms) which are not present in the r.h.s, regular at $x = 1$. We are therefore led to look for a solution which behaves as

$$f(x) \sim C(1 - x)^\alpha \tag{5.19}$$

for some constant C and some positive non-integral exponent α (the condition $\alpha > 0$ ensures both that $f(1) = 0$ and that the integral in (5.11) converges at $x = 1$). Remarkably, α cannot take arbitrary values, but is related to n via

$$n = 2\sin\pi\alpha \tag{5.20}$$

as seen from the following argument. Subtracting to (5.11) its expression at $x = 1$, we get

$$f(x) + \frac{n}{2\pi}\int_0^1 (K_1(x,y) - K_1(1,y))f(y)\,dy = \rho(1-x) \tag{5.21}$$

with $K_1(x,y) - K_1(1,y) \sim -(1-x)y/((1-y)(1-xy))$ when x and y tend to 1. For $x \to 1$, the dominant singular term in the integral arises from y's such that $1 - y = O(1-x)$. More precisely, writing $y = 1 - s(1-x)$, we get at leading order

$$\int_0^1 (K_1(x,y) - K_1(1,y))f(y)\,dy\Big|_{\text{sing.}} \sim -C(1-x)^\alpha \int_0^\infty \frac{s^{\alpha-1}}{1+s}ds \tag{5.22}$$

with the right-hand side integral evaluated as $\pi/\sin\pi\alpha$. Combining with (5.19) back into (5.21), whose r.h.s contains no singular term, we deduce (5.20).

Note that, for a given value of n between 0 and 2, the possible $\alpha > 0$ satisfying (5.20) are of the form $\alpha = 1/2 \pm b + m$, with $\pi b = \arccos(n/2)$ and m a non-negative integer. So far we have only discussed the dominant exponent but, by linearity, all exponents appearing in the expansion of $f(x)$ at $x = 1$ should be of this form. We therefore expect $f(x)$ to be of the general form

$$f(x) = (1-x)^{1/2-b}\Phi_-(x) + (1-x)^{1/2+b}\Phi_+(x) \tag{5.23}$$

where $\Phi_-(x)$ and $\Phi_+(x)$ are functions with a regular expansion at $x = 1$ (i.e. contain only integral powers of $x - 1$). By considering the expansion of (5.11) at $x = 1$ and splitting it into singular and regular parts (i.e. separating terms with non-integral and integral exponents), we find that $\Phi_-(x)$ and $\Phi_+(x)$ should satisfy

$$(1-x)^{1/2\pm b}\Phi_\pm(x) + \frac{\cos(\pi b)}{\pi}\left\{\int_0^1 K_1(x,y)(1-y)^{1/2\pm b}\Phi_\pm(y)dy\right\}\Big|_{\text{sing.}} = 0 \tag{5.24}$$

and

$$\frac{\cos(\pi b)}{\pi}\left\{\int_0^1 K_1(x,y)\left((1-y)^{1/2-b}\Phi_-(y) + (1-y)^{1/2+b}\Phi_+(y)\right)dy\right\}\Big|_{\text{reg.}} = 1 - \rho x\ . \tag{5.25}$$

Now we expect that $\Phi_-(1)$ is not zero generically so that the leading singularity at $x = 1$ is of the form $(1-x)^{1/2-b}$. Furthermore, $\Phi_-(1)$ must be non-negative in order to satisfy the positivity conditions (5.7). This situation should hold for any $\rho < \rho^*$

and corresponds to the dense phase of the $O(n)$ loop model, see Fig. 9. At $\rho = \rho^*$, we expect $\Phi_-(1) = 0$ and $\Phi_+(1) > 0$ so that, at this special point, the leading singularity becomes of the form $(1-x)^{1/2+b}$, corresponding now to the dilute $O(n)$ loop model. For $\rho > \rho^*$, we expect $\Phi_-(1) < 0$ so that the model is ill-defined. That the critical condition $f(1) = 0$ for $\tau < 1$ coincides when $\tau \to 1$ with the condition $\Phi_-(1) = 0$ is rather natural and makes us believe that the transition from the line of generic critical points to that of non-generic ones should be continuous. Note that having an effective value of α larger than 1 for the leading singularity simply corresponds to a situation where both $\Phi_-(1)$ and $\Phi_+(1)$ would vanish. This may occur only for a particular class of right-hand sides in the integral equation and it is not expected in the present case.

Finally, recalling Section 2.2, we see that the behavior (5.19) for $f(x) = u'(R)$ corresponds precisely to the expansion (2.12) for $u(R)$. In particular, we identify

$$a = \alpha + \frac{3}{2} \tag{5.26}$$

and the relation (5.20) is nothing but (4.4).

6. Exact phase diagram of the rigid loop model

In this Section, we concentrate on the rigid loop model and explain how to derive its exact phase diagram (sketched on Fig.7).

6.1. Equations for the resolvent

The starting point is a linear integral equation, not for the function $u'(R)$ as in Section 5.1, but for the resolvent $W(\xi)$ defined as in (2.19). Those two quantities are related via the general formula (2.20). Moreover, substituting the fixed point condition (3.7) for the rigid loop model into (2.4), we have for $R < R(1)$

$$\begin{aligned} u(R) &= R - 3gR^2 - \frac{n}{2}\sum_{k\geq 1}\binom{2k}{k} h_1^{2k} R^k F_k(1) \\ &= R - 3gR^2 - \frac{n}{2}\oint \frac{d\xi W(\xi)}{2\mathrm{i}\pi}\left(\frac{1}{\sqrt{1-4h_1^2 R\,\xi^2}} - 1\right) \end{aligned} \tag{6.1}$$

where the contour of integration is, say, the circle of radius $\gamma = 2\sqrt{R(1)}$ (note that $4h_1^2 R\gamma^2$ is always smaller than 1 by the discussion of Section 4.1). Upon differentiating with respect to R, and substituting into (2.20), we obtain

$$\begin{aligned} W(\xi) &= S(\xi) - \frac{n}{\xi}\oint \frac{d\xi' W(\xi')}{2\mathrm{i}\pi}\int_0^{R(1)} dR\left(1-\frac{4R}{\xi^2}\right)^{-1/2} h_1^2\xi'^2\left(1-4h_1^2\xi'^2 R\right)^{-3/2} \\ &= S(\xi) - \frac{n}{2\xi}\oint \frac{d\xi' W(\xi')}{2\mathrm{i}\pi}\frac{h_1^2\xi'^2}{h_1^2\xi'^2 - 1/\xi^2}\left(\sqrt{\frac{1-4R(1)/\xi^2}{1-4R(1)h_1^2\xi'^2}} - 1\right) \end{aligned} \tag{6.2}$$

where

$$S(\xi) = \frac{1}{2}\left(\xi - g\xi^3 - \xi(1 - 2gR(1) - g\xi^2)\sqrt{1 - 4R(1)/\xi^2}\right) \tag{6.3}$$

corresponds to the first two terms in the r.h.s of (6.1).

Eq. (6.2) is a linear integral equation for the resolvent which rephrases that of Section 5.1 for $f(x) = u'(R)$. It implies a simpler functional equation for $W(\xi)$ as follows: for $\xi \in [-\gamma, \gamma]$, we have

$$S(\xi + \mathrm{i}0) + S(\xi - \mathrm{i}0) = \xi - g\xi^3 \tag{6.4}$$

so that, from (6.2), we may write

$$W(\xi + \mathrm{i}0) + W(\xi - \mathrm{i}0) = \xi - g\xi^3 + \frac{n}{\xi}\oint \frac{d\xi' W(\xi')}{2\mathrm{i}\pi}\frac{h_1^2\xi'^2}{h_1^2\xi'^2 - 1/\xi^2}. \tag{6.5}$$

The latter integral may be evaluated by the residue theorem: the integrand has poles at $\xi' = \pm(h_1\xi)^{-1}$, each with residue $(h_1\xi)^{-1}W((h_1\xi)^{-1})/2$, and a pole at $\xi' = \infty$ with residue $-F_0(1) = -1$. Hence the resolvent satisfies the functional equation

$$W(\xi + \mathrm{i}0) + W(\xi - \mathrm{i}0) = \xi - g\xi^3 + \frac{n}{\xi} - \frac{n}{h_1\xi^2}W\left(\frac{1}{h_1\xi}\right), \qquad \xi \in [-\gamma, \gamma]. \tag{6.6}$$

Note the similarity with Eq.(3.2) for the different model of Section 3.1. Equation (6.6) can also be obtained as a consequence of loop equations in the matrix model formulation of our $O(n)$ model [19].

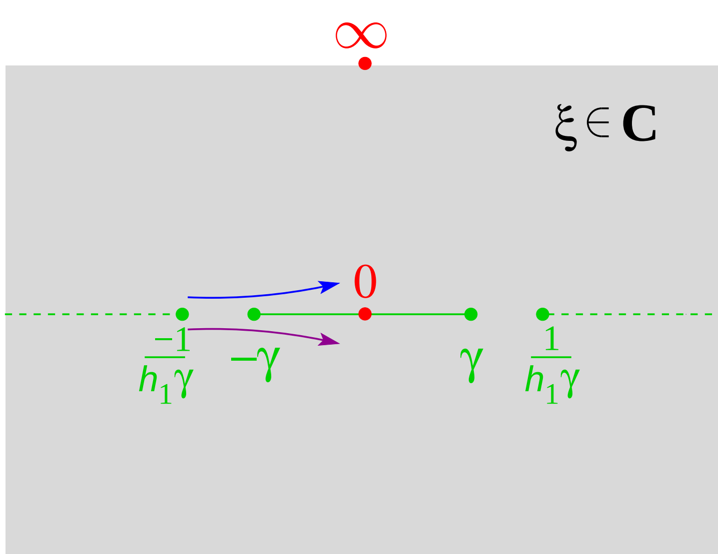

Fig. 10: The cut of $W(\xi)$, or equivalently of $W_{\mathrm{hom}}(\xi)$ (solid line) and its image (dashed line) under the involution $\xi \mapsto 1/(h_1\xi)$.

Solving (6.2) boils down to finding a solution of (6.6) which is bounded, odd in ξ and such that $W(\xi) \sim 1/\xi$ as $\xi \to \infty$. As we shall see, these requirements fix $W(\xi)$ completely. By linearity, we may write

$$W(\xi) = W_{\rm part}(\xi) + W_{\rm hom}(\xi) \ , \tag{6.7}$$

where $W_{\rm part}(\xi)$ is the easy particular solution of (6.6)

$$W_{\rm part}(\xi) = \frac{2(\xi - g\xi^3) - n(\frac{1}{h_1^2\xi^3} - \frac{g}{h_1^4\xi^5})}{4-n^2} + \frac{n}{(2+n)\xi} \tag{6.8}$$

and where $W_{\rm hom}(\xi)$ is now an odd solution of the homogeneous equation

$$W_{\rm hom}(\xi+{\rm i}0) + W_{\rm hom}(\xi-{\rm i}0) + \frac{n}{h_1\xi^2} W_{\rm hom}\left(\frac{1}{h_1\xi}\right) = 0 \ . \tag{6.9}$$

The condition that $W(\xi) \sim 1/\xi$ for $\xi \to \infty$, and that W is bounded amounts to demanding that

$$\begin{aligned} W_{\rm hom}(\xi) &= \frac{2g}{4-n^2}\xi^3 - \frac{2}{4-n^2}\xi + \frac{2}{2+n}\xi^{-1} + O(\xi^{-2}), \quad \xi \to \infty \\ W_{\rm hom}(\xi) &= \frac{-n}{2}\left(\frac{2g}{4-n^2}\xi^{-5} - \frac{2}{4-n^2}\xi^{-3} + \frac{2}{2+n}\xi^{-1} + O(1)\right), \quad \xi \to 0 \ . \end{aligned} \tag{6.10}$$

We shall give in Section 6.3 the general expression for $W_{\rm hom}(\xi)$ when $4h_1R(1) < 1$, which corresponds to a subcritical or generic critical situation. Something special happens on the non-generic critical line $4h_1R(1) = 1$ because then, the cut $[-\gamma, \gamma]$ collides with its image under $\xi \mapsto 1/(h_1\xi)$ (see Fig. 10). As we shall now see, $W_{\rm hom}(\xi)$ has a simple expression along this line.

6.2. Non-generic critical line

As seen in Section 4.1, the non-generic critical line is characterized by $\tau = 4h_1R(1) = 1$ (or equivalently $R(1) = R_c$). This implies that the extremity of the cut $\gamma = 2\sqrt{R(1)} = 1/\sqrt{h_1}$ is a fixed point of $\xi \mapsto 1/(h_1\xi)$. It is then easy to guess the general solution of the homogeneous equation (6.9) which is odd in ξ, namely:

$$W_{\rm hom}(\xi) = \left(B(\xi) - \frac{\gamma^2}{\xi^2}\,B\left(\frac{\gamma^2}{\xi}\right)\right)\left(\frac{\xi-\gamma}{\xi+\gamma}\right)^b - \left(B(-\xi) - \frac{\gamma^2}{\xi^2}\,B\left(-\frac{\gamma^2}{\xi}\right)\right)\left(\frac{\xi+\gamma}{\xi-\gamma}\right)^b \tag{6.11}$$

where $B(\xi)$ is some arbitrary analytic function and where, again, $\pi b = \arccos(n/2)$. This in turn leads to a spectral density supported on $[-\gamma, \gamma]$:

$$\rho(\xi) = -\frac{\sin(\pi b)}{\pi}\left(\left(B(\xi) - \frac{\gamma^2}{\xi^2}\,B\left(\frac{\gamma^2}{\xi}\right)\right)\left(\frac{\gamma-\xi}{\gamma+\xi}\right)^b + \left(B(-\xi) - \frac{\gamma^2}{\xi^2}\,B\left(-\frac{\gamma^2}{\xi}\right)\right)\left(\frac{\gamma+\xi}{\gamma-\xi}\right)^b\right). \tag{6.12}$$

The requirement that $W_{\rm hom}$ is holomorphic in $\mathbf{C}\backslash[-\gamma,\gamma]$ imposes that B is a polynomial. To satisfy (6.10), $B(\xi)$ must be a polynomial of degree 3, whose four coefficients are determined from

$$B(\xi)\left(\frac{\xi-\gamma}{\xi+\gamma}\right)^b = \frac{g}{4-n^2}\xi^3 - \frac{1}{4-n^2}\xi + \frac{1}{2+n}\xi^{-1} + O(\xi^{-3}) \qquad \text{as } \xi\to\infty\ . \tag{6.13}$$

Note that we could a priori imagine terms of order ξ^2 and ξ^0 in this expansion since they would be canceled by parity in the expansion of $W_{\rm hom}(\xi)$ at large ξ. However, such terms would create poles for $\rho(\xi)$ at $\xi=0$, which are not allowed because ρ should be integrable. The condition (6.13) amounts to five equations: four of them fix the coefficients of $B(\xi)$ and the last one yields some additional relation between g and h_1 which is nothing but the equation for the non-generic critical line. We find explicitly

$$B(\xi) = \frac{g}{4-n^2}\left(\xi^3 + 2b\gamma\xi^2 + 2b^2\gamma^2\xi + \frac{2}{3}(b+2b^3)\gamma^3\right) - \frac{1}{4-n^2}(\xi+2b\gamma) \tag{6.14}$$

while the equation for the non-generic critical line reads

$$g = \frac{3}{2+b^2}\left(h_1 - \frac{2-n}{2b^2}h_1^2\right)\ . \tag{6.15}$$

In order for the expression (6.11) to be consistent, the spectral density (6.12) must be positive on $]-\gamma,\gamma[$. In particular, expanding $\rho(\xi)$ for $\xi\to\pm\gamma$, we have

$$\begin{aligned}\rho(\xi) = \frac{\sin(\pi b)}{\pi}\Bigg(&4^b\left(B(-\gamma)-\gamma B'(-\gamma)\right)\left(1-\frac{\xi^2}{\gamma^2}\right)^{1-b} \\ &+4^{-b}\left(B(\gamma)+\gamma B'(\gamma)\right)\left(1-\frac{\xi^2}{\gamma^2}\right)^{1+b}\Bigg) + O\left(1-\frac{\xi^2}{\gamma^2}\right)^{2-b}\ .\end{aligned} \tag{6.16}$$

Demanding the positivity of $\rho(\xi)$ requires $B(-\gamma)-\gamma B'(-\gamma)\geq 0$, which yields the condition

$$g \leq \frac{3h_1}{2(b^2-2b+3)}\ . \tag{6.17}$$

The non-generic critical line (6.15) therefore ends at a point (g^*,h_1^*) with

$$g^* = \frac{3b^2(2-b)^2}{2(2-n)(b^2-2b+3)^2}\ , \quad h_1^* = \frac{b^2(2-b)^2}{(2-n)(b^2-2b+3)}\ . \tag{6.18}$$

When $(g,h_1)=(g^*,h_1^*)$, we may check that the coefficient of the first subleading term is positive, i.e. $B(\gamma)+\gamma B'(\gamma)>0$. Comparing (6.16) with (2.23), we therefore read the value of the exponent a, namely $a=2-b$ for $g<g^*$ and $a=2+b$ at $g=g^*$.

6.3. General expression of the resolvent

Eq. (6.6) can be recast more elegantly in terms of the differential form $\omega(\xi) = W_{\mathrm{hom}}(\xi)\, d\xi$, namely:

$$\forall \xi \in [-\gamma, \gamma], \qquad \omega(\xi + \mathrm{i}0) + \omega(\xi - \mathrm{i}0) - n\,\omega(s(\xi)) = 0\ , \tag{6.19}$$

where s is the involution $\xi \mapsto 1/(h_1\xi)$. Note that $(s(\xi))^2 = 1/\lambda_+(1/\xi^2)$, where λ_+ is the involution of Section 3.4, specialized to the rigid case. We underline the similarity of Eq. (6.19) with that relevant in the $O(n)$ model discussed in Section 3.1, where loops visit only vertices of degree 3: in this case, we had a different involution $\xi \mapsto \tilde{h}^{-1} - \xi$. So, the techniques already developed for the $O(n)$ model where loops visit only vertices of degree 3 [11,19] can be applied to Eq. (6.19) with few modifications.

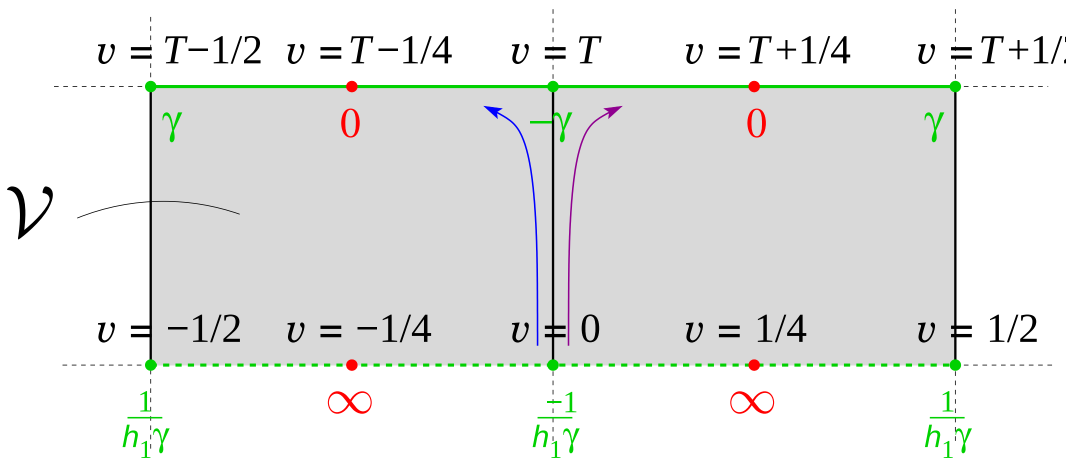

Fig. 11: Construction of the elliptic parametrization $v(\xi)$ of Eq. (6.20), which depends on the path followed from $-(h_1\gamma)^{-1}$ to ξ. The upper and lower half-planes map respectively to the left and right rectangles (whose union is denoted by $\mathcal{V}$).

The general strategy is to introduce a parametrization which opens the cut $[-\gamma, \gamma]$ as well as its image $]-\infty, -(h_1\gamma)^{-1}] \cup [(h_1\gamma)^{-1}, +\infty[$ under the involution, for instance:

$$v(\xi) = V \int_{-(h_1\gamma)^{-1}}^{\xi} \frac{d\eta}{\sqrt{-(\eta^2 - \gamma^2)(\eta^2 - (h_1\gamma)^{-2})}}\ . \tag{6.20}$$

The new variable v can be expressed in terms of Jacobi elliptic functions and is a multivalued function of ξ, depending on the path followed in the complex plane from the origin $(h_1\gamma)^{-1}$ to ξ. Conversely, one may view ξ as a function of $v \in \mathbf{C}$. We choose the constant V by demanding that $v((h_1\gamma)^{-1}) = -1/2$ when following a path with small positive imaginary part from $-(h_1\gamma)^{-1}$ to ∞ and then back to $(h_1\gamma)^{-1}$. This leads to

$$V = \frac{\mathrm{i}}{4h_1\gamma K(h_1\gamma^2)}\ , \tag{6.21}$$

where K is the complete elliptic integral. We then denote $T = v(-\gamma)$ (by the most straightforward path), and the fact that the squareroot discontinuity is included in the real axis implies that $T = \mathrm{i}\,|T|$. The function $\xi(v)$ is easily seen to have the following properties:

$$\begin{aligned} &\xi(v+1) = \xi(v)\ , \quad \xi(-v) = \xi(v)\ , \\ &\xi(v+1/2) = -\xi(v)\ , \quad \xi(T-v) = (h_1\xi(v))^{-1}\ , \end{aligned} \tag{6.22}$$

from which one can deduce also the properties of its derivative $\xi'(v)$. For bookkeeping, we mention the expansion when $v \to v_\infty = -1/4$:

$$\xi(v) = \frac{\Xi_{-1}}{v - v_\infty} + \Xi_1(v - v_\infty) + O\big((v - v_\infty)^3\big) \tag{6.23}$$

with

$$\begin{aligned} \Xi_{-1} &= \mathrm{i}V = \frac{-1}{4h_1\gamma K(h_1\gamma^2)} = \frac{\mathrm{i}\,T}{h_1\gamma K'(h_1\gamma^2)} \\ \Xi_1\Xi_{-1} &= \frac{1}{6}\left(\gamma^2 + (h_1\gamma)^{-2}\right)\ , \end{aligned} \tag{6.24}$$

where we denote $K'(\tau) = K(\sqrt{1-\tau^2})$. The image of the points of $\mathbf{C} \setminus \big([-\gamma,\gamma] \cup]-\infty, -(h_1\gamma)^{-1}] \cup [(h_1\gamma)^{-1}, +\infty[\big)$ reached by a path which does not cross the segment $[\gamma, (h_1\gamma)^{-1}]$ (for instance a straight path) is the domain $\mathcal{V} = \{v :\ \mathrm{Re}[v] \in]-1/2, 1/2[\ , \mathrm{Im}[v] \in]0, |T|[\}$, as shown in Fig. 11. Let us define

$$\varpi(v) = W_{\mathrm{hom}}(\xi(v))\xi'(v) \tag{6.25}$$

which is an analytic function on $\mathcal{V}$. Since $W_{\mathrm{hom}}(\xi)$ has no discontinuity when $\xi \in]-\infty, -(h_1\gamma)^{-1}] \cup [(h_1\gamma)^{-1}, \infty[$, $\varpi(v)$ takes opposite values when $v \leftrightarrow -v$ along the segment $[-1/2, 1/2]$. Thus, ϖ can be extended to an analytic function defined on $\mathcal{V}_2 = \mathcal{V} \cup (-\mathcal{V})$ by setting

$$\forall v \in -\mathcal{V}, \qquad \varpi(v) = -\varpi(-v)\ . \tag{6.26}$$

Likewise, the absence of discontinuity along $\xi \in [\gamma, (h_1\gamma)^{-1}]$ allows to extend ϖ as an analytic function defined on the strip $\mathcal{V}_{\mathrm{strip}} = \bigcup_{m\in\mathbf{Z}} (\mathcal{V}_2 + m)$ by setting

$$\forall v \in (\mathcal{V}_2 + m), \qquad \varpi(v) = \varpi(v - m)\ . \tag{6.27}$$

The top boundary of this strip, $\mathcal{V}_{\mathrm{cut}} = \{v :\ \mathrm{Im}[v] = |T|\}$, maps to points $\xi \in [-\gamma, \gamma]$. Eventually, looking at Eq. (6.22) and using Eq. (6.26), Eq. (6.19) turns into:

$$\forall v \in \mathcal{V}_{\mathrm{cut}}, \qquad \varpi(v) + \varpi(v - 2T) - n\varpi(v - T) = 0\ . \tag{6.28}$$

This allows to extend recursively and without ambiguity ϖ as an analytic function on $\mathbf{C} = \bigcup_{m\in\mathbf{Z}}(\mathcal{V}_{\mathrm{strip}} + 2mT)$. Since (6.26), (6.27) and (6.28) are linear relations between analytic functions, they are now valid for any value $v \in \mathbf{C}$.

To summarize, we reduced the problem to that of finding an analytic function $v \mapsto \varpi(v)$, which is odd, 1-periodic, and satisfies:

$$\varpi(v-2T) - n\varpi(v-T) + \varpi(v) = 0 \ . \tag{6.29}$$

Eq. (6.10) demands that $\varpi(v)$ behaves as:

$$\varpi(v) = -\frac{2g\,\Xi_{-1}^4}{4-n^2}\,\frac{1}{(v-v_\infty)^5} + \frac{2\,\Xi_{-1}^2}{4-n^2}\left[1-\frac{g}{3}\big(\gamma^2+(h_1\gamma)^{-2}\big)\right]\frac{1}{(v-v_\infty)^3} - \frac{2}{2+n}\,\frac{1}{v-v_\infty} + O(1) \tag{6.30}$$

when $v \to v_\infty = -1/4$, and:

$$\varpi(v) = \frac{n\,g\Xi_{-1}^4}{4-n^2}\,\frac{1}{(v-v_0)^5} - \frac{n\,\Xi_{-1}^2}{4-n^2}\left[1-\frac{g}{3}\big(\gamma^2+(h_1\gamma)^{-2}\big)\right]\frac{1}{(v-v_0)^3} + \frac{n}{2+n}\,\frac{1}{v-v_0} + O(1) \tag{6.31}$$

when $v \to v_0 = T - 1/4$. We now assume $n \notin \{0,2\}$, such that $e^{2\mathrm{i}\pi b} \neq 1$. The general 1-periodic solution of Eq. (6.29) is of the form $\varpi(v) = \varpi_+(v) + \varpi_-(v)$, where:

$$\varpi_\pm(v+1) = \varpi_\pm(v), \qquad \varpi_\pm(v+T) = e^{\pm\mathrm{i}\pi b}\varpi_\pm(v) \tag{6.32}$$

Such functions are generalizations of elliptic functions, and they can be constructed by taking appropriate ratios of the Jacobi theta function of nome $q = e^{\mathrm{i}\pi T}$. Let us just state the existence of a unique analytic function ζ_b, which satisfies

$$\zeta_b(v+1) = \zeta_b(v), \qquad \zeta_b(v+T) = e^{\mathrm{i}\pi b}\zeta_b(v) \tag{6.33}$$

and has a unique pole when $v = 0 \bmod \mathbf{Z} \oplus T\mathbf{Z}$, which is simple, and is such that $\zeta_b(v) \sim 1/v$ when $v \to 0$. Its construction and main properties are listed in Appendix B. One may generate functions satisfying (6.32) with poles of higher degree at $v = 0$ by considering the derivatives of ζ_b or of $\zeta_{-b}(v) = -\zeta_b(-v)$, and put this pole at any given point w by shifting the argument v to $v - w$. Since holomorphic functions satisfying (6.32) must vanish identically, one may determine ϖ by matching the divergent behavior at its poles with a linear combination of the previous functions. This leads eventually to:

$$\varpi(v) = -\frac{1}{2+n}\mathcal{D}\left\{\zeta_b(v-1/4) + \zeta_b(v+1/4) - \zeta_b(-v+1/4) - \zeta_b(-v-1/4)\right\} \tag{6.34}$$

where $\mathcal{D}$ is the differential operator:

$$\mathcal{D} = \frac{g\,\Xi_{-1}^4}{24(2-n)}\partial_v^4 + \frac{\Xi_{-1}^2}{2(2-n)}\left[-1+\frac{g}{3}\big(\gamma^2+(h_1\gamma)^{-2}\big)\right]\partial_v^2 + 1 \ . \tag{6.35}$$

Then, the spectral density is given by:

$$\begin{aligned}\rho(\xi(v))d\xi(v) &= \frac{\xi'(v)}{2\mathrm{i}\pi}\left(W_{\mathrm{hom}}(\xi(2T-v)) - W_{\mathrm{hom}}(\xi(v))\right)dv \\ &= -\frac{dw}{2\pi}\sqrt{\frac{2-n}{2+n}}\,\mathcal{D}\left\{\zeta_b\Big(w-\frac{1}{4}\Big)+\zeta_b\Big(w+\frac{1}{4}\Big)+\zeta_b\Big(-w+\frac{1}{4}\Big)+\zeta_b\Big(-w-\frac{1}{4}\Big)\right\},\end{aligned} \tag{6.36}$$

where we have set $w = T - v$ so that $w \in [0, 1/2]$ corresponds to $\xi \in [-\gamma, \gamma]$. The value of $\gamma = 2\sqrt{R(1)}$ is determined a posteriori as a function of g and h_1 by requiring that $\rho(\xi) \propto \sqrt{\xi \pm \gamma}$ when $x \to \mp\gamma$, which is equivalent to demanding that $\rho(\xi(v))\xi'(v) = O(w^2)$ when $w \to 0$.

6.4. Generic critical line

In the solution above, the generic critical line is the relation between g and h_1 obtained by demanding that $\rho(\xi) \propto (\xi \pm \gamma)^{3/2}$ when $\xi \to \mp\gamma$ (see Eq. (2.23)). In other words, in the Taylor expansion of $\rho(\xi(v))\xi'(v)$, as given by (6.36), when $w \to 0$, the generic critical line is characterized by the vanishing of the terms of order 1 and w^2. We may write these two conditions in a parametric way, with parameter $\tau = h_1\gamma^2 = 4h_1R(1)$, as

$$\begin{aligned} Z_0 + \frac{\Xi_{-1}^2}{2(2-n)}\left(-1 + \frac{g}{3h_1}(\tau + \tau^{-1})\right) Z_2 + \frac{\Xi_{-1}^4\, g}{24(2-n)} Z_4 = 0\ , \\ Z_2 + \frac{\Xi_{-1}^2}{2(2-n)}\left(-1 + \frac{g}{3h_1}(\tau + \tau^{-1})\right) Z_4 + \frac{\Xi_{-1}^4\, g}{24(2-n)} Z_6 = 0\ , \end{aligned} \tag{6.37}$$

where

$$Z_{2j} = \partial_{w=0}^{(2j)} \left(\zeta_b(w - 1/4) + \zeta_b(w + 1/4)\right)\ . \tag{6.38}$$

Note that Z_{2j} depends implicitly on τ via ζ_b which depends on T, itself related to τ via:

$$\tau = \left(\frac{\vartheta_2(0|4T)}{\vartheta_3(0|4T)}\right)^2 = \left(\frac{\vartheta_4(0|\frac{-1}{4T})}{\vartheta_3(0|\frac{-1}{4T})}\right)^2\ . \tag{6.39}$$

The solution is:

$$\begin{aligned} h_1 &= \frac{1}{2(2-n)\tau[K'(\tau)]^2}\,\frac{\Delta_8}{\Delta_6 + 4(\tau^2+1)[K'(\tau)]^2\,\Delta_4}\ , \\ g &= \frac{6}{2-n}\,\frac{\Delta_8\,\Delta_4}{(\Delta_6 + 4(\tau^2+1)[K'(\tau)]^2\,\Delta_4)^2}\ , \end{aligned} \tag{6.40}$$

where $K'(\tau) = K(\sqrt{1-\tau^2})$ and the Δ's are functions of τ defined via:

$$\Delta_4 = T^4(Z_0Z_4 - Z_2^2), \quad \Delta_6 = -T^6(Z_0Z_6 - Z_2Z_4), \quad \Delta_8 = T^8(Z_2Z_6 - Z_4^2)\ . \tag{6.41}$$

Those equations become simpler in the neighborhood of the special points $\tau \to 0$ (i.e. $h_1 \to 0$), or $\tau \to 1$ (i.e. $h_1 \to h_1^*$, the tip of the non-generic critical line). When $h_1 \to 0$, it is convenient to use the variable $q = e^{i\pi T} \to 0$ for asymptotics, and we find:

$$\begin{aligned} g &= \frac{1}{12} - \frac{n}{18}q^4 + \frac{n}{36}(7+n)\,q^8 + O(q^{12}) \\ h_1 &= \frac{q^2}{2} - (2 + \frac{n}{6})q^6 + (7 + 2n + \frac{n^2}{18})\,q^{10} + O(q^{14}) \end{aligned} \tag{6.42}$$

As expected, the generic critical line meets the critical point of quadrangulations at $(g = 1/12, h_1 = 0)$, and near this point it behaves as:

$$h_1 = \frac{3}{2}\sqrt{\frac{2}{n}}\sqrt{\frac{1}{12} - g}, \quad g \to 1/12 \tag{6.43}$$

When $\tau \to 1$, it is convenient to use the variable $q' = e^{-\mathrm{i}\pi/T} \to 0$ for asymptotics. The computation confirms that the generic critical line ends at the tip (g^*, h_1^*) of the non-generic critical line found in Eq. (6.18), and we find near this point that:

$$\begin{aligned}\left(\frac{g-g^*}{g^*}\right) &= -\frac{2(1-2b)}{2+b^2}\left(\frac{h_1-h_1^*}{h_1^*}\right) - \frac{(2-b)^2}{2+b^2}\left(\frac{h_1-h_1^*}{h_1^*}\right)^2 \\ &\quad + \frac{64(1+b)}{(1-b)(2+b^2)}\left(\frac{(2-b)^2(1-b)^2(3-2b+b^2)}{4b(1+b)^2(2+b^2)}\right)^{1/b}\left(\frac{h_1-h_1^*}{h_1^*}\right)^{1/b} \\ &\quad + o\left(\frac{h_1-h_1^*}{h_1^*}\right)^{1/b}\end{aligned} \tag{6.44}$$

The first two terms coincide exactly with Eq. (6.15): when passing from the generic to the non-generic critical line, both the slope and the curvature remain continuous. For $n = 1$ (i.e. $b = 1/3$), we find a leading discontinuity in the third derivative of g with respect to h_1, as expected for the Ising model [20].

6.5. Phase diagram

The results above are best summarized in the phase diagram of Fig. 7. We have found a line of non-generic critical points given by the arc of parabola (6.15), which links the point $(g = 0, h_1 = 2b^2/(2-n))$ to the point (g^*, h_1^*) of Eq. (6.18). Along this line, the exponent a takes the value $2 - b$. At the terminating point (g^*, h_1^*), a takes instead the value $2 + b$. We then found a line of generic critical points with a more complicated parametrization (6.40). When τ decreases from 1 to 0, this line links the point (g^*, h_1^*) to the point $(g = 1/12, h_1 = 0)$ describing pure quadrangulations. As just mentioned, the non-generic and generic critical lines connect with a continuous slope. Their concatenation forms the line $h_1 = h_c(n; g)$ of Section 4.1 as the model cannot be well-defined above this line.

Let us finally note that, in the limit $n \to 0$, then $g^* \to 1/12$ and $h_1^* \to 1/8$. The non-generic critical line tends to the arc of parabola with equation

$$g = \frac{4}{3}\left(h_1 - 4h_1^2\right), \qquad \frac{1}{8} \le h_1 \le \frac{1}{4}\ , \tag{6.45}$$

while the generic critical line becomes the vertical segment parametrized by $g = 1/12$, $0 \le h_1 \le 1/8$ (see Fig. 12). This may be understood as follows: the small n expansion of F_p^{loop} describes quadrangulations equipped with a fixed finite number of rigid loops. Using exact enumeration results for quadrangulations with multiple boundaries, it can be

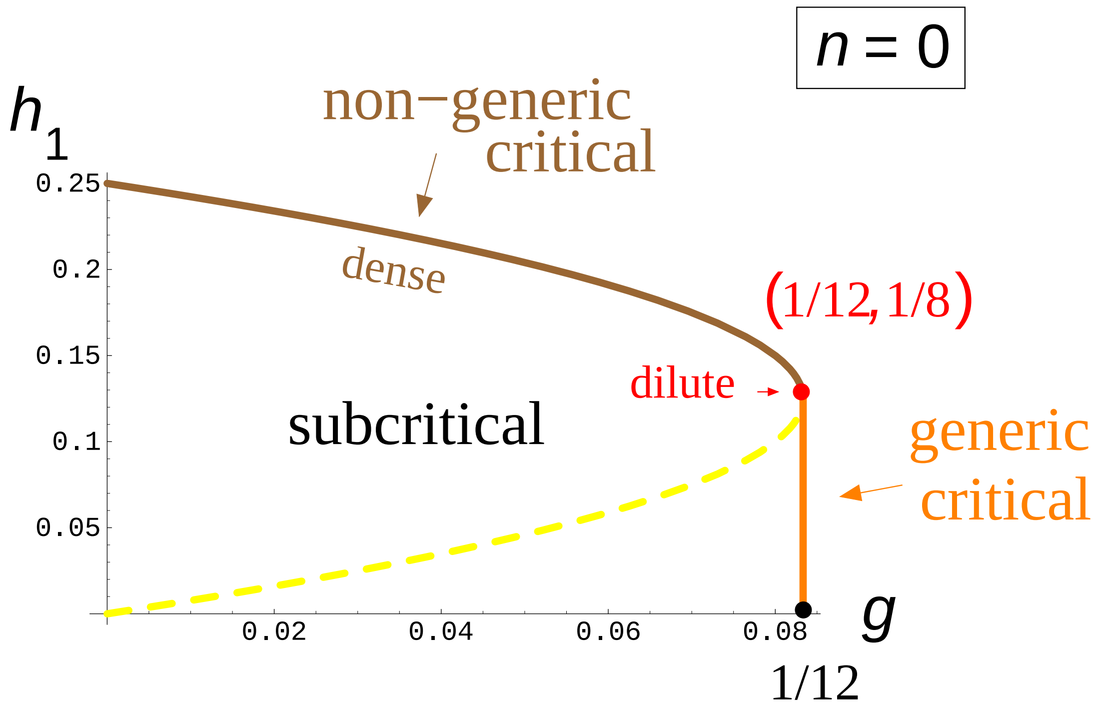

Fig. 12: The exact phase diagram of the rigid $O(n)$ model in the limit $n \to 0$.

seen that the contribution to F_p^{loop} from loops of large length $2k$ behaves as $(4h_1 R_Q)^{2k}$, where R_Q is the generating function $R(1)$ for pure quadrangulations, solution of

$$R_Q = 1 + 3g\, R_Q^2 \ . \tag{6.46}$$

Having a finite contribution from large loops requires that $4h_1 R_Q \leq 1$. Note that that R_Q ranges from 1 to 2 when g ranges from 0 to 1/12. Criticality may be obtained in two manners: either we set $4h_1 R_Q = 1$, so that the contribution from large loops decays sub-exponentially. Note then the equivalence between Eqs. (6.45) and (6.46). Or we may set $g = 1/12$, so that the pure quadrangulations are themselves critical. This last situation requires $h_1 \leq 1/8$, since $R_Q = 2$ in this case.

7. Variants of the $O(n)$ loop model

In this Section, we briefly discuss other versions of the $O(n)$ loop model on quadrangulations, including models with non-symmetric local weights and models with restricted loop lengths. More precisely, we concentrate on non-generic critical points and discuss how the relation (4.4) is modified in these cases. We finally extend our results to maps whose faces have arbitrary (but bounded) even degrees.

7.1. Non-symmetric models

At this stage, it should be clear to the reader that the relation (4.4) between the loop weight n and the exponent a only depends on a few properties of the ring transfer matrix $M(z)$. Noting $\lambda_+(z)$ the largest eigenvalue of $M^2(z)$, we used the estimate $\sum_{k'\geq 0} A_{k,k'} z^{-k'} \sim \lambda_+(z)^k$ to obtain eventually the relation

$$n = 2(\mu^*)^a \sin\pi(a-3/2), \qquad \mu^* = -\lambda'_+(z^*) \ , \tag{7.1}$$

where z^* is the fixed point of the mapping $z \mapsto \lambda_+(z)$. In the symmetric case discussed so far, this mapping is an involution in the vicinity of z^*, so that $\lambda'_+(z^*) = -1$. Let us now consider a slightly modified, non-symmetric version of our model defined as follows: the squares of type (c) in Fig.1 come in two species, those whose two edges not crossed by the loop belong to the inner contour, and those where these two edges belong to the outer contour. We may as well view these two species as corresponding to *outward*, resp. *inward turns* of the loop at hand. For instance, the loop in Fig. 2 makes 8 inward and 6 outward turns. Assigning now different weights, say $h_{2,\text{out}}$ and $h_{2,\text{in}}$ respectively to these squares, the transfer matrix $M(z)$ is replaced by

$$M(z)= \begin{pmatrix} \dfrac{h_1 z^{-1/2}}{1-h_{2,\text{out}}z^{-1}} & \dfrac{1}{1-h_{2,\text{out}}z^{-1}} \\ h_{2,\text{in}} & 0 \end{pmatrix} \ . \tag{7.2}$$

The relation (3.11) becomes

$$(h_{2,\text{out}}\lambda + h_{2,\text{in}}z - \lambda z)^2 - h_1^2 \lambda z = 0 \tag{7.3}$$

whose largest solution $\lambda_+(z)$ now has a fixed point at $z^* = h_1 + h_{2,\text{out}} + h_{2,\text{in}}$. Since λ and z do not play symmetric roles in (7.3), the mapping $z \mapsto \lambda_+(z)$ is no longer an involution and we now have a non trivial value $\mu^* = (h_1 + 2h_{2,\text{out}})/(h_1 + 2h_{2,\text{in}})$, leading to the new relation

$$n = 2\sin\pi(a-3/2) \left(\frac{h_1 + 2h_{2,\text{out}}}{h_1 + 2h_{2,\text{in}}}\right)^a \ . \tag{7.4}$$

Note in particular that the range of n allowing for non-generic criticality is modified. It is instructive to compare this result to that obtained on a regular square lattice. In this context the parametrization $n = 2\sin\pi(a-3/2) = 2\cos\pi b$ naturally appears in the Coulomb gas approach to the model [21]. Many critical exponents of the model have simple (typically polynomial) expressions in terms of a (or b). On such a regular lattice, we also have a well-defined notion of exterior and interior of a loop and we may give different weights to outward and inward turns. On a regular lattice, there are however 4 more inward than outward turns, so that the symmetry of the model may be restored at the price of a rescaling $n \to n(h_{2,\text{in}}/h_{2,\text{out}})^2$ of the weight per loop. This in turn changes the relation (4.4) into $n = 2\sin\pi(a-3/2)(h_{2,\text{out}}/h_{2,\text{in}})^2$.

As before, the case $h_1 = 0$ is special since the largest eigenvalue is degenerate in this case, leading eventually to

$$n = \sin\pi(a-3/2) \left(\frac{h_{2,\text{out}}}{h_{2,\text{in}}}\right)^a \ . \tag{7.5}$$

7.2. Loops with restricted lengths

Returning to the symmetric case, we may impose some restriction on the lengths of the loop by demanding for instance that they be multiples of a fixed integer N. Such a restriction may occur for instance when the loop model is inherited from some underlying edge coloring problem. Now the length of a loop whose outer and inner contours have length $2k$ and $2k'$ is simply $(k+k')$. The consistency relation (3.6) has to be modified to account for the new constraint, and we are naturally led to consider now the quantity

$$\sum_{\substack{k'\geq 0\\ k+k'=0 \mod N}} A_{k,k'}(h_1,h_2) z^{-k'} = \frac{1}{N}\sum_{j=0}^{N-1}\left((\omega^j \lambda_+(\omega^{-j}z))^k + (\omega^j \lambda_-(\omega^{-j}z))^k\right), \qquad \omega = \mathrm{e}^{2\mathrm{i}\pi/N}, \tag{7.6}$$

to be estimated as before in the vicinity of z^* and for large k. The above sum behaves as $(1/N)$ $(\lambda_+(z^*))^k$ with now a $1/N$ prefactor provided $|\lambda_+(z^*\omega^{-j})| < \lambda_+(z^*)$ when $j=1,\cdots,N-1$, which holds for $h_2>0$ (again we also suppose that $h_1>0$ to avoid that $\lambda_-(z^*)=\lambda_+(z^*)$). The correcting factor $(1/N)$ trivially results into a change of the relation (4.4) into

$$n = 2N\sin\pi(a-3/2) \tag{7.7}$$

(for $h_1,h_2>0$). In particular, imposing an even size for the loops takes the $O(n)$ loop model in the universality class of the $O(n/2)$ loop model without the parity constraint. This fact was already recognized in Ref. [22] in the slightly different context of loops living on triangles.

In the rigid case $h_2=0$, loops are automatically of even length by construction. We have $\omega^j\lambda_+(\omega^{-j}z^*) = \omega^{2j}\lambda_+(z^*)$ and the above sum behaves as $\lambda_+(z^*)^k$, as before, provided k is a multiple of N for N odd (respectively a multiple of $N/2$ for N even) while it vanishes otherwise. For $h_2=0$, the relation (4.4) is therefore unchanged. Finally, for $h_1=0$, we get $n=N\sin\pi(a-3/2)$ instead.

7.3. Faces with arbitrary even degrees

Our results are easily extended to the case of maps whose faces have arbitrary even degrees, provided these degrees remain bounded, say by $2M+2$ ($M\geq 1$). Faces of degree $2m$ not visited by a loop receive a non-negative weight $g^{(m)}$ ($1\leq m\leq M+1$) and those visited by a loop receive a weight $h^{(m_1,m_2)}$ ($m_1,m_2\geq 0$) if the face has m_1 (resp. m_2) incident edges belonging to the outer (resp. inner) contour of the loop at hand. Since the total degree of such a face is m_1+m_2+2, we will implicitly assume in the following that $h^{(m_1,m_2)}$ is not zero only if $m_1+m_2\leq 2M$ and m_1 and m_2 have the same parity. With these new weights, the $O(n)$ loop model is now described by the fixed point condition

$$g_k = \sum_{m=1}^{M+1} g^{(m)}\delta_{k,m} + n\sum_{k'\geq 0} A_{k,k'}F_{k'}(1) \tag{7.8}$$

where $A_{k,k'}$ is the generating function for (rooted) rings (now made of faces with arbitrary even degrees) with sides of lengths $2k$ and $2k'$. As before, this generating function is best encoded in the quantity

$$\sum_{k'\geq 0} A_{k,k'} z^{-k'} = \mathrm{tr} M^2(z) \tag{7.9}$$

involving a new transfer matrix $M(z)$ of size $2M \times 2M$ now given by:

$$M(z) = \begin{pmatrix} \frac{\sum h^{(1,m_2)} z^{-\frac{m_2}{2}}}{1-\sum h^{(0,m_2)} z^{-\frac{m_2}{2}}} & \frac{1}{1-\sum h^{(0,m_2)} z^{-\frac{m_2}{2}}} & 0 & 0 & \cdots & 0 \\ \sum h^{(2,m_2)} z^{-\frac{m_2}{2}} & 0 & 1 & 0 & \cdots & 0 \\ \sum h^{(3,m_2)} z^{-\frac{m_2}{2}} & 0 & 0 & 1 & \cdots & 0 \\ \sum h^{(4,m_2)} z^{-\frac{m_2}{2}} & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum h^{(2M,m_2)} z^{-\frac{m_2}{2}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.10}$$

Here the sum in $\sum h^{(i,m_2)} z^{-\frac{m_2}{2}}$ runs over $m_2 \geq 0$ (and in practice over values of m_2 ranging from 0 to $2M - i$ and having the same parity as i). As shown in Appendix A, the eigenvalues λ of $M^2(z)$ are the solutions of the characteristic equation

$$\Big((\lambda\, z)^M - \sum_{\substack{m_1,m_2\geq 0\\ m_1,m_2 \text{ even}}} h^{(m_1,m_2)} \lambda^{\frac{2M-m_1}{2}} z^{\frac{2M-m_2}{2}}\Big)^2 = \Big(\sum_{\substack{m_1,m_2\geq 0\\ m_1,m_2 \text{ odd}}} h^{(m_1,m_2)} \lambda^{\frac{2M-m_1}{2}} z^{\frac{2M-m_2}{2}}\Big)^2 . \tag{7.11}$$

Repeating the analysis of Section 4, we again find at a non generic critical point the consistency relation between n and the exponent a characterizing the large k asymptotics of $F_k(1)$:

$$n = 2(\mu^*)^a \sin \pi(a - 3/2), \qquad \mu^* = -\lambda_+'(z^*) \tag{7.12}$$

in terms of the largest eigenvalue $\lambda_+(z)$ of $M^2(z)$ and its fixed point z^*. Note that this relation holds when the largest eigenvalue is not degenerate. In a symmetric model, we must set $h^{(m_1,m_2)} = h^{(m_2,m_1)}$ so that λ and z play symmetric roles in (7.11). This implies as before that the mapping $z \to \lambda_+(z)$ is an involution in the vicinity of z^* and that $\mu^* = 1$ so that the simple relation (4.4) is recovered.

This generic relation (7.12) is modified whenever $h^{(m_1,m_2)} = 0$ for all odd values of m_1 and m_2. In this case, the r.h.s of Eq. (7.11) vanishes and the largest eigenvalue is degenerate, resulting in the suppression of the factor 2 in (7.12), namely $n = (\mu^*)^a \sin \pi(a - 3/2)$ with again $\mu^* = 1$ in the symmetric case.

8. Conclusion

In this paper, we have shown how to relate a number of $O(n)$ loop models to models of bipartite maps. More precisely, we have shown that the gasket of an $O(n)$ loop model configuration is distributed according to a Boltzmann ensemble of bipartite maps with appropriate degree dependent face weights $(g_k)_{k\geq 1}$. Those weights are determined by a fixed point condition inherited from a bijective decomposition of the $O(n)$ configurations along the contours of their loops. In particular, the non-generic (dense and dilute) critical points of the $O(n)$ loop models correspond to ensembles of bipartite maps with large faces belonging to the class considered in [13] with a distribution characterized by some exponent a between 3/2 and 5/2, related to n generically via

$$n = 2\sin\pi(a-3/2)\ . \tag{8.1}$$

Technically, this formula is one of a number of simple consistency relations dictated by the fixed point condition at a non-generic critical point. Their derivation involves only a few properties of a simple transfer matrix $M(z)$ describing the sequence of faces visited by a loop (the ring). For instance, the constant z^* characterizing the exponential decay of the face weights ($g_k \sim (z^*)^k$) or the exponential growth of $F_k^{\rm loop}$ ($F_k^{\rm loop} \sim (1/z^*)^k$) is simply obtained as the solution of the equation $z^* = \lambda_+(z^*)$, where $\lambda_+(z)$ denotes the largest eigenvalue of $M^2(z)$.

Noticeably, the same scheme appears to work also for $O(n)$ loop models where the loops visit only trivalent vertices or, equivalently on the dual, where the ring is made of a sequence of triangles. The ring transfer matrix reduces in this case to a scalar $\tilde{M}(z) = (\tilde{\lambda}_+(z))$, with

$$\tilde{\lambda}_+(z) = \tilde{h}/(1-\tilde{h}/z) \tag{8.2}$$

(note that we do not square $\tilde{M}$ as contours are not required to have even lengths). Here again, the exponential growth factor $(1/z^*) = (2\tilde{h})^{-1}$ for $\tilde{F}_p^{\rm loop}$ in (3.4) is the solution of $z^* = \tilde{\lambda}_+(z^*)$, while equation (8.1) still holds.

A corollary of our reformulation is that the metric properties of the gasket at a non-generic point may be obtained from those of ensembles of bipartite maps with large faces. In [13], it was shown that it these maps have a fractal dimension $2a-1$ and one may hope to be able to extract the average gasket profile from known expressions for discrete distance dependent two-point functions in bipartite maps.

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Appendix A. Eigenvalues of the transfer matrix for arbitrary even degrees

Here we consider the $O(n)$ loop model on maps with arbitrary even face degrees, as defined in Section 7.3. Let us introduce the quantity

$$S = \sum_{k\geq 1} \frac{1}{2k} w^{-k} \sum_{k'\geq 0} A_{k,k'} z^{-k'} \tag{A.1}$$

where $A_{k,k'}$ enumerates configurations of rooted rings with outer and inner sides of lengths $2k$ and $2k'$, with the face weights of Section 7.3. Due to the factor $1/(2k)$, S

enumerates *unrooted* ring configurations of arbitrary side lengths, with a weight $w^{-1/2}$ (resp. $z^{-1/2}$) per edge of the outer (resp. inner) contour. By a direct calculation, we have

$$\begin{aligned} S &= \sum_{k\geq 1} \frac{1}{2k} \mathrm{tr}(w^{-1}M^2(z))^k \\ &= -\frac{1}{2}\mathrm{tr}\log(1-w^{-1}M^2(z)) \\ &= -\frac{1}{2}\log\det(1-w^{-1}M^2(z)) \ , \end{aligned} \tag{A.2}$$

where $M(z)$ is the transfer matrix defined in (7.10). On the other hand, if we denote by ℓ the length of a ring, i.e. its number of faces or equivalently the length of the underlying loop, the configurations counted by S are simply cyclic sequences of length ℓ made of the various squares at hand. We may therefore write

$$\begin{aligned} S = \sum_{\ell\geq 1}\frac{1}{\ell}\frac{1}{2}\Big\{\Big(\sum_{m_1,m_2\geq 0} h^{(m_1,m_2)} w^{-\frac{m_1}{2}} z^{-\frac{m_2}{2}}\Big)^{\ell} + \Big(\sum_{m_1,m_2\geq 0} h^{(m_1,m_2)}(-1)^{m_1} w^{-\frac{m_1}{2}} z^{-\frac{m_2}{2}}\Big)^{\ell}\Big\} \\ - \sum_{\ell\geq 1}\frac{1}{\ell}\Big(\sum_{m_2\geq 0} h^{(0,m_2)} z^{-\frac{m_2}{2}}\Big)^{\ell} \end{aligned} . \tag{A.3}$$

Here the first two terms differ only by the factor $(-1)^{m_1}$, so that their half-sum selects configuration where the total length of the outer contour is even (and so is that of the inner contour since m_1 and m_2 have the same parity whenever $h^{(m_1,m_2)} \neq 0$). The third term is the $w^{-1} \to 0$ limit of the first two and is subtracted to account for the fact that there is no $k = 0$ term in S. Summing over ℓ, this leads to

$$S = -\frac{1}{2}\log\left(\frac{\Big(1-\sum_{m_1,m_2\geq 0} h^{(m_1,m_2)} w^{-\frac{m_1}{2}} z^{-\frac{m_2}{2}}\Big)\Big(1-\sum_{m_1,m_2\geq 0} h^{(m_1,m_2)}(-1)^{m_1} w^{-\frac{m_1}{2}} z^{-\frac{m_2}{2}}\Big)}{\Big(1-\sum_{m_2\geq 0} h^{(0,m_2)} z^{-\frac{m_2}{2}}\Big)^2}\right) \tag{A.4}$$

from which we readily extract the value of $\det(1-w^{-1}M^2(z))$ by comparison with (A.2). We end up with the final expression

$$\det(\lambda - M^2(z)) = \frac{P(\lambda, z)}{Q(z)} \tag{A.5}$$

where $P(\lambda, z)$ is the polynomial:

$$\begin{aligned} P(\lambda, z) = &\Big((\lambda\, z)^M - \sum_{m_1,m_2\geq 0} h^{(m_1,m_2)} \lambda^{\frac{2M-m_1}{2}} z^{\frac{2M-m_2}{2}}\Big) \\ &\times \Big((\lambda\, z)^M - \sum_{m_1,m_2\geq 0} h^{(m_1,m_2)} (-1)^{m_1} \lambda^{\frac{2M-m_1}{2}} z^{\frac{2M-m_2}{2}}\Big) \end{aligned} \tag{A.6}$$

and $Q(z)$ the polynomial:

$$Q(z) = \Big(z^M - \sum_{m_2\geq 0} h^{(0,m_2)} z^{\frac{2M-m_2}{2}}\Big)^2 \ . \qquad (A.7)$$

Writing $P(\lambda, z) = 0$ leads precisely to Eq. (7.11).

Appendix B. Properties of the function ζ_b

The (first) Jacobi theta function $\vartheta_1(v|T)$ is defined by [23]:

$$\vartheta_1(v|T) = \mathrm{i} \sum_{m\in\mathbf{Z}} (-1)^m \, e^{\mathrm{i}\pi(m-1/2)^2 T + \mathrm{i}\pi(2m-1)v} \qquad (B.1)$$

The series is absolutely convergent when $\mathrm{Im}[T] > 0$, and the following properties can be easily proved:

$$\vartheta_1(v+1|T) = -\vartheta_1(v|T), \quad \vartheta_1(v+T|T) = -e^{\mathrm{i}\pi T - 2\mathrm{i}\pi v}\, \vartheta_1(v|T) \ . \qquad (B.2)$$

The point $v = 0$ turns out to be the unique zero (modulo $\mathbf{Z} \oplus T\mathbf{Z}$) of $\vartheta_1(v|T)$. This function has the following modular property:

$$\vartheta_1(v|T) = \frac{\mathrm{i}}{\sqrt{-\mathrm{i}T}}\, e^{-\frac{\mathrm{i}\pi v^2}{T}}\, \vartheta_1\left(\frac{v}{T} \mid \frac{-1}{T}\right) \qquad (B.3)$$

which is useful to relate the limit $|T| \to 0$ to the limit $|T'| \to \infty$, where $T' = -1/T$. Notice that when $|T'| \to +\infty$, we have $q' = e^{\mathrm{i}\pi T'} \to 0$, and therefore the series (B.1) gives in a straightforward way the asymptotics of $\vartheta_1(v|T')$. Other Jacobi theta functions that appear in the text are:

$$\begin{aligned} \vartheta_2(v|T) &= \sum_{m\in\mathbf{Z}} e^{\mathrm{i}\pi(m-1/2)^2 T + \mathrm{i}\pi(2m-1)v} \\ \vartheta_3(v|T) &= \sum_{m\in\mathbf{Z}} e^{\mathrm{i}\pi m^2 T + 2\mathrm{i}\pi v} \\ \vartheta_4(v|T) &= \sum_{m\in\mathbf{Z}} (-1)^m \, e^{\mathrm{i}\pi m^2 T + 2\mathrm{i}\pi m v} \end{aligned} \qquad (B.4)$$

We also introduce the Weierstraß elliptic function:

$$\wp(v|T) = \frac{1}{v^2} + \sum_{(l,m)\in\mathbf{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(v+l+mT)^2} - \frac{1}{(l+mT)^2}\right) \ . \qquad (B.5)$$

The function $\wp$ is even, 1 and T periodic, and has the following properties:

$$\wp(v|T) \underset{v\to 0}{=} \frac{1}{v^2} + O(v^2), \qquad \wp(v|T) = T^{-2}\,\wp\left(\frac{v}{T} \mid \frac{-1}{T}\right) \tag{B.6}$$

Ratios of ϑ_1 may be used to construct functions having prescribed poles and zeroes, and taking a constant or linear phase when $v \to v + T$. For instance:

$$\zeta_b(v) = \frac{\vartheta_1(v - b/2|T)}{\vartheta_1(v|T)}\,\frac{\vartheta_1'(0|T)}{\vartheta_1(-b/2|T)} \tag{B.7}$$

is the unique function which is 1-periodic, takes a phase $e^{\mathrm{i}\pi b}$ when $v \to v + T$, has only a simple pole at $v = 0 \bmod \mathbf{Z} \oplus T\mathbf{Z}$, and is such that $\zeta_b(v) \sim 1/v$ when $v \to 0$. The value of the phase under translation by T implies that ζ_b has a unique zero (modulo $\mathbf{Z} \oplus T\mathbf{Z}$), located at $v = b/2$. We may also find ζ_b with another representation:

$$\zeta_b(v) = \sum_{m\in\mathbf{Z}} e^{-\mathrm{i}\pi b m}\,\pi\,\mathrm{cotan}\,\pi(v + mT)\ , \tag{B.8}$$

where $\sum_{m\in\mathbf{Z}}\cdots$ has to be understood as $\lim_{M\to\infty}\left(\sum_{m=-M}^{M}\cdots\right)$.

Let us introduce coefficients C_0 and C_1 such that

$$\zeta_b(v) = \frac{1}{v} + C_0 + C_1\,v + O(v^2), \qquad v \to 0\ . \tag{B.9}$$

In other words:

$$C_0 = (\ln\vartheta_1)'(-b/2|T), \qquad C_1 = \frac{1}{2}\,\frac{\vartheta_1''(-b/2|T)}{\vartheta_1(-b/2|T)} - \frac{1}{6}\,\frac{\vartheta_1'''(0|T)}{\vartheta_1'(0|T)}\ . \tag{B.10}$$

Several differential equations can be derived for $\zeta_b(v)$. They are all based on the fact that $\zeta_b^{(j)}(v)/\zeta_b(v)$ is 1 and T periodic, thus can be expressed just by matching the divergent behavior at the poles with help of Weierstraß function and its derivatives.

$$\begin{aligned} \zeta_b'(v) &= \frac{1}{2}\,\frac{\wp'(v) + \wp'(b/2)}{\wp(v) - \wp(b/2)}\,\zeta_b(v) \\ \zeta_b''(v) &= 2C_0\zeta_b'(v) + 2(\wp(v) - C_1)\zeta_b(v)\ . \end{aligned} \tag{B.11}$$

By the same method, one finds a "mirror relation":

$$\zeta_b(v)\zeta_b(-v) = \wp(b/2) - \wp(v) \tag{B.12}$$

which shows that $\wp(b/2) = C_0^2 - 2C_1$. As a consequence, we mention that $\tilde{\zeta}_b(v) = e^{-C_0 v}\zeta_b(v)$ satisfies the spin 1 Lamé differential equation with spectral parameter $b/2$:

$$\tilde{\zeta}_b''(v) - 2\wp(v)\tilde{\zeta}_b(v) = \wp(b/2)\tilde{\zeta}_b(v)\ . \tag{B.13}$$

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