High energy hadronic interactions in QCD and applications to heavy ion collisions

IV – Saturation and the Color Glass Condensate

François Gelis

CEA / DSM / SPhT

General outline

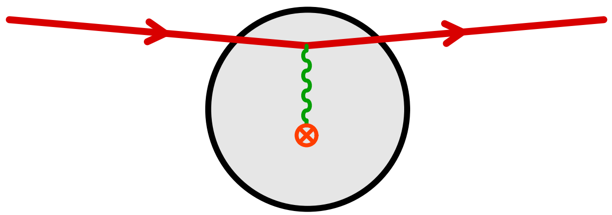
- Lecture I : Introduction and phenomenology
- Lecture II : Lessons from Deep Inelastic Scattering
- Lecture III : QCD on the light-cone
- Lecture IV : Saturation and the Color Glass Condensate
- Lecture V : Calculating observables in the CGC

Lecture IV : Saturation and CGC

- BFKL equation
- Saturation of parton distributions
- Balitsky-Kovchegov equation
- Color Glass Condensate - JIMWLK
- Analogies with reaction-diffusion processes
- Pomeron loops

High energy scattering

- Consider the following scattering process :



- Reminder : the high energy limit of the scattering amplitude $S_{\beta\alpha}$ can be written as :

$$S^{(\infty)}_{\beta\alpha} \equiv \lim_{\omega\to+\infty} \langle\beta_{\rm in}|e^{i\omega K^-}U(+\infty,-\infty)e^{-i\omega K^-}|\alpha_{\rm in}\rangle$$

$$= \langle\beta_{\rm in}|U_0(+\infty,0)\boldsymbol{F}U_0(0,-\infty)|\alpha_{\rm in}\rangle$$

with $\boldsymbol{F} \equiv \exp ig\int_{\vec{\boldsymbol{x}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)$, and :

$$\chi(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^+ \, \mathcal{A}^-(x^+,0,\vec{\boldsymbol{x}}_\perp)$$

$$\rho(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^- \, J^+(0,x^-,\vec{\boldsymbol{x}}_\perp)$$

High energy scattering

- Introduce two complete sets of intermediate states :

$$S^{(\infty)}_{\beta\alpha} = \sum_{\delta,\gamma} \int \Big[\prod_{i\in\delta} d\Phi_i \prod_{j\in\gamma} d\Phi_j \Big] \big\langle \beta_{\rm in} \big| U_0(+\infty, 0) \big| \gamma_{\rm in} \big\rangle \times \big\langle \gamma_{\rm in} \big| \boldsymbol{F} \big| \delta_{\rm in} \big\rangle \big\langle \delta_{\rm in} \big| U_0(0, -\infty) \big| \alpha_{\rm in} \big\rangle$$

- Instead of labelling the intermediate states by their variables $k^+, \vec{\boldsymbol{k}}_\perp$, use the transverse coordinate $\vec{\boldsymbol{x}}_\perp$ conjugate to $\vec{\boldsymbol{k}}_\perp$:

$$d\Phi \equiv \frac{dk^+}{4\pi k^+} d^2\vec{\boldsymbol{x}}_\perp$$

- In terms of these variables, the factor $\big\langle \delta_{\rm in} \big| U_0(0, -\infty) \big| \alpha_{\rm in} \big\rangle$ is the term of light-cone wave function of α that corresponds to δ. Let us denote :

$$\Psi_{\delta\alpha}(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\}) \equiv \big\langle \delta_{\rm in} \big| U_0(0, -\infty) \big| \alpha_{\rm in} \big\rangle$$

$$\Psi^\dagger_{\gamma\beta}(\{k_i^{+\prime}, \vec{\boldsymbol{x}}'_{i\perp}\}) \equiv \big\langle \beta_{\rm in} \big| U_0(+\infty, 0) \big| \gamma_{\rm in} \big\rangle$$

High energy scattering

- We have seen that the number and the nature of the particles is unchanged under the action of the operator $\boldsymbol{F}$. Moreover, in terms of the transverse coordinates, we simply have

$$\left\langle \gamma_{\rm in} \middle| \boldsymbol{F} \middle| \delta_{\rm in} \right\rangle = \delta_{NN'} \prod_i \left[4\pi k_i^+ \delta(k_i^+ - k_i^{+\prime}) \delta(\vec{\boldsymbol{x}}_{i\perp} - \vec{\boldsymbol{x}}'_{i\perp}) U_{R_i}(\vec{\boldsymbol{x}}_{i\perp}) \right]$$

  where $U_R(\vec{\boldsymbol{x}}_\perp)$ is a Wilson line operator, in the representation R appropriate for the particle going through the target

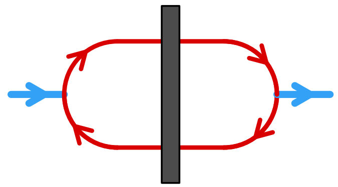
- In other words, the states δ and γ must be identical, except for the color index of the particles they contain (not written explicitly)

- Therefore, the high energy scattering amplitude can be written as :

$$S_{\beta\alpha}^{(\infty)} = \sum_\delta \int \left[\prod_{i\in\delta} d\Phi_i \right] \Psi^\dagger_{\delta\beta}(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\}) \left[\prod_{i\in\delta} U_{R_i}(\vec{\boldsymbol{x}}_{i\perp}) \right] \Psi_{\delta\alpha}(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\})$$

Scattering of a dipole

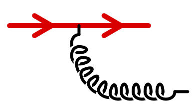
- Assume that the initial and final states α and β are a color singlet $Q\overline{Q}$ dipole. The simplest Fock state that contributes to their wave function is a $Q\overline{Q}$ pair, and the bare scattering amplitude can be written as :



$$\propto \quad \Psi^{(0)*}_{ij}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)\Psi^{(0)}_{kl}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) U_{ik}(\vec{\boldsymbol{x}}_\perp) U^\dagger_{lj}(\vec{\boldsymbol{y}}_\perp)$$

$$\propto \quad \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

- It turns out that 1-loop corrections to this contribution are enhanced by $\alpha_s \log(p^+)$, which can be large when the quark or antiquark has a large p^+
- In the gauge $A^+ = 0$, the emission of a gluon of momentum k by a quark can be written as :



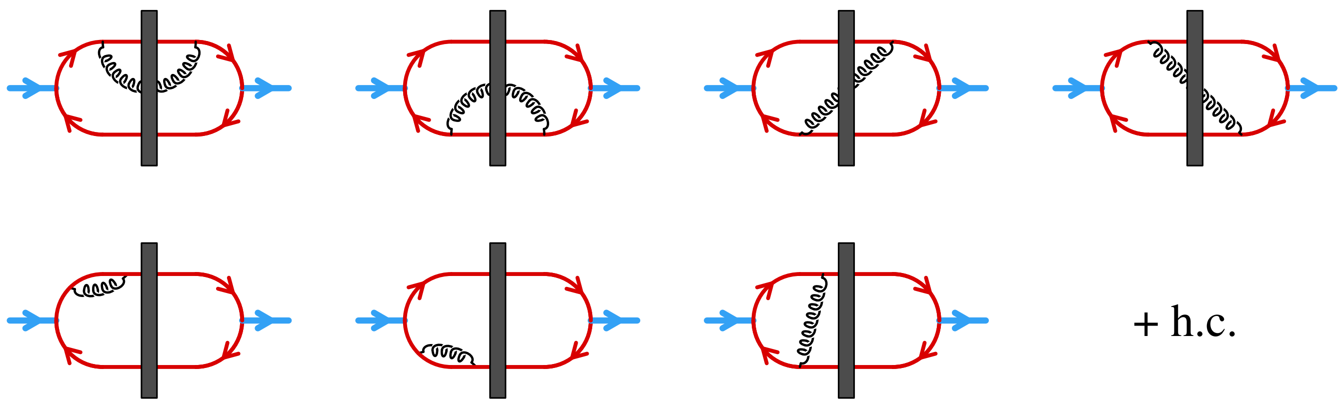
$$= \; 2gt^a \, \frac{\vec{\boldsymbol{\epsilon}}_\lambda \cdot \vec{\boldsymbol{k}}_\perp}{k_\perp^2}$$

Scattering of a dipole

- In coordinate space, this reads :

$$\int \frac{d^2\vec{\boldsymbol{k}}_\perp}{(2\pi)^2}\, e^{i\vec{\boldsymbol{k}}_\perp\cdot(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)}\; 2gt^a\,\frac{\vec{\boldsymbol{\epsilon}}_\lambda\cdot\vec{\boldsymbol{k}}_\perp}{k_\perp^2} = \frac{2ig}{2\pi}t^a\,\frac{\vec{\boldsymbol{\epsilon}}_\lambda\cdot(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$

- The following diagrams must be evaluated :

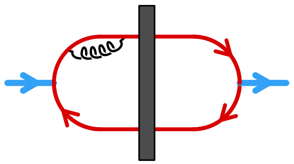


- When connecting two gluons, one must use :

$$\sum_\lambda \vec{\epsilon}^{\,i}_\lambda \vec{\epsilon}^{\,j}_\lambda = -g^{ij}$$

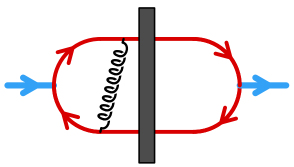
Virtual corrections

- Consider first the loop corrections inside the wavefunction of the incoming or outgoing dipole
- Examples :



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a t^a U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

$$\times -2\alpha_s\int\frac{dk^+}{k^+}\int\frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2}\frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)t^a\right]$$

$$\times 4\alpha_s\int\frac{dk^+}{k^+}\int\frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2}\frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$

- Reminder : $t^a t^a = (N_c^2-1)/2N_c \equiv C_F$

Virtual corrections

- The sum of all virtual corrections is :

$$-\frac{C_F\alpha_s}{\pi^2}\int\frac{dk^+}{k^+}\int d^2\vec{z}_\perp\,\frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2\mathrm{tr}\left[U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)\right]$$

- The integral over k^+ is divergent. It should have an upper bound at p^+ :

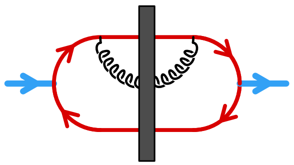
$$\int^{p^+}\frac{dk^+}{k^+}=\ln(p^+)=Y$$

  ▷ When Y is large, $\alpha_s Y$ may not be small. By differentiating with respect to Y, we will get an evolution equation in Y whose solution resums all the powers $(\alpha_s Y)^n$

- The integral over $\vec{z}_\perp$ is divergent when $\vec{z}_\perp=\vec{x}_\perp$ or $\vec{y}_\perp$

Real corrections

- There are also real corrections, for which the state that interacts with the target has an extra gluon
- Example :



$$= \left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{x}_\perp) t^b U^\dagger(\vec{y}_\perp)\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{z}_\perp}{(2\pi)^2} \widetilde{U}_{ab}(\vec{z}_\perp) \frac{(\vec{x}_\perp - \vec{z}_\perp)\cdot(\vec{x}_\perp - \vec{z}_\perp)}{(\vec{x}_\perp - \vec{z}_\perp)^2 (\vec{x}_\perp - \vec{z}_\perp)^2}$$

  - $\widetilde{U}_{ab}(\vec{z}_\perp)$ is a Wilson line in the adjoint representation

- In order to simplify the color structure, first notice that :

$$t^a \widetilde{U}_{ab}(\vec{z}_\perp) = U(\vec{z}_\perp) t^b U^\dagger(\vec{z}_\perp)$$

- Then use the $SU(N_c)$ Fierz identity :

$$t^b_{ij} t^b_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N_c}\delta_{ij}\delta kl$$

Real corrections

- The Wilson lines can be rearranged into :

$$\mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp) t^b U^\dagger(\vec{\boldsymbol{y}}_\perp)\right] \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) = \frac{1}{2}\mathrm{tr}\left[U^\dagger(\vec{\boldsymbol{z}}_\perp)U(\vec{\boldsymbol{x}}_\perp)\right]\mathrm{tr}\left[U(\vec{\boldsymbol{z}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right] - \frac{1}{2N_c}\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

  - The term in $1/2N_c$ cancels against a similar term in the virtual contribution
  - All the real terms have the same color structure

- When we sum all the real terms, we generate the same kernel as in the virtual terms :

$$\frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

- In order to simplify the notations, let us denote :

$$\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \equiv \frac{1}{N_c}\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

Evolution equation

- The 1-loop scattering amplitude reads :

$$-\frac{\alpha_s N_c^2 Y}{2\pi^2}\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2\int d^2\vec{z}_\perp\;\frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\Big\{\boldsymbol{S}(\vec{x}_\perp,\vec{y}_\perp)-\boldsymbol{S}(\vec{x}_\perp,\vec{z}_\perp)\boldsymbol{S}(\vec{z}_\perp,\vec{y}_\perp)\Big\}$$

- Reminder: the bare scattering amplitude was :

$$\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 N_c\,\boldsymbol{S}(\vec{x}_\perp,\vec{y}_\perp)$$

- Hence, we have :

$$\frac{\partial\boldsymbol{S}(\vec{x}_\perp,\vec{y}_\perp)}{\partial Y}=-\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_\perp\;\frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\Big\{\boldsymbol{S}(\vec{x}_\perp,\vec{y}_\perp)-\boldsymbol{S}(\vec{x}_\perp,\vec{z}_\perp)\boldsymbol{S}(\vec{z}_\perp,\vec{y}_\perp)\Big\}$$

  - since $\boldsymbol{S}(\vec{x}_\perp,\vec{x}_\perp)=1$, the integral over $\vec{z}_\perp$ is now regular

BFKL equation

Kuraev, Lipatov, Fadin (1977), Balitsky, Lipatov (1978)

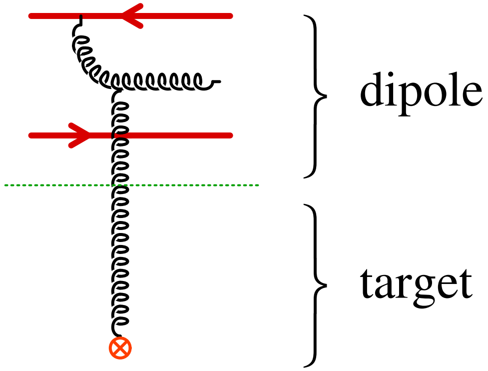
- Actually, we've got more than we need : we must simplify this equation in order to obtain the BFKL equation...

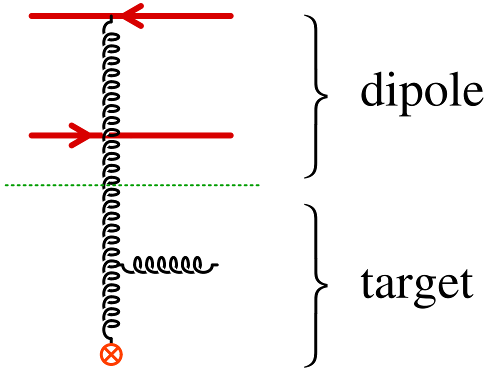
- Write $\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \equiv 1 - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)$ and assume that we are in the dilute regime, so that the scattering amplitude $\boldsymbol{T}$ is small. Drop the terms that are non-linear in $\boldsymbol{T}$:

$$\frac{\partial \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\vec{\boldsymbol{z}}_\perp \, \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$
$$\times \Big\{ \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp) + \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp) - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \Big\}$$

BFKL equation

- Note : $\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)$ is independent on the frame. In particular, it depends only on the rapidity difference between the dipole and the target
  ▷ in a frame where the dipole is held fixed, the target has to evolve in such a way as to reproduce the Y dependence of $\boldsymbol{T}$





- The corresponding evolution in the target is the radiation of a gluon

Unitarity problem

- The solution of this equation grows exponentially when $Y \to +\infty$ $\quad \triangleright$ serious unitarity problem...

- In perturbation theory, the forward scattering amplitude between a small dipole and a target made of gluons reads :

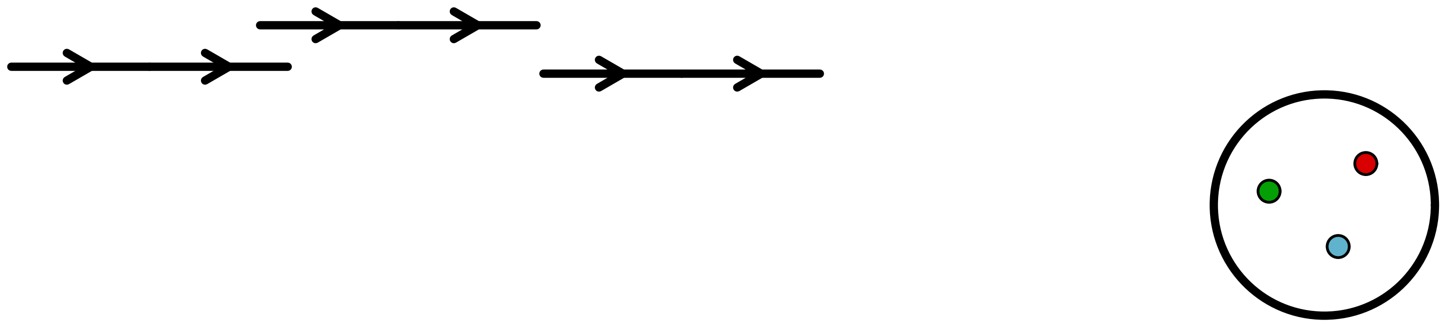
$$\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \propto |\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp|^2 \; xG(x, |\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp|^{-2})$$

where $\quad Y \equiv \ln(1/x)$

- Therefore, the exponential behavior of $\boldsymbol{T}$ implies an increase of the gluon distribution at small x

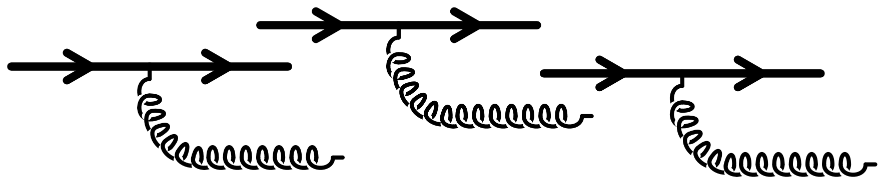
$$\boldsymbol{T} \sim e^{\omega Y} \qquad \longleftrightarrow \qquad xG(x, Q^2) \sim \frac{1}{x^\delta}$$

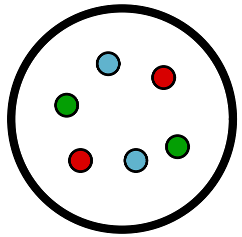
Parton evolution under boosts



▷ at low energy, only valence quarks are present in the hadron wave function

Parton evolution under boosts



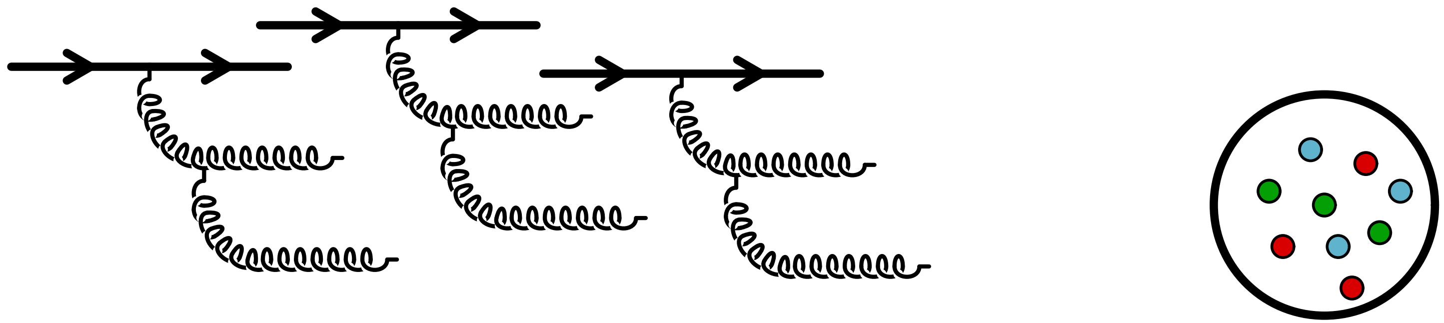


▷ when energy increases, new partons are emitted

▷ the emission probability is $\alpha_s \int \frac{dx}{x} \sim \alpha_s \ln(\frac{1}{x})$, with x the longitudinal momentum fraction of the gluon

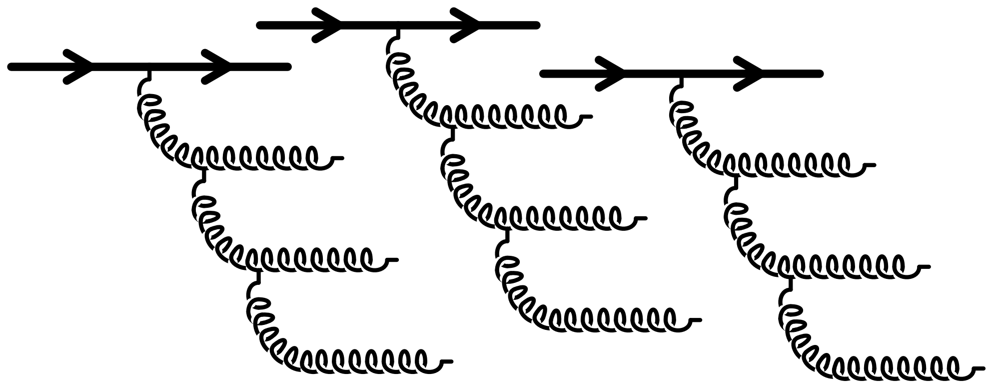
▷ at small-x (i.e. high energy), these logs need to be resummed

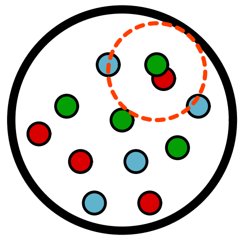
Parton evolution under boosts



▷ as long as the density of constituents remains small, the evolution is linear: the number of partons produced at a given step is proportional to the number of partons at the previous step

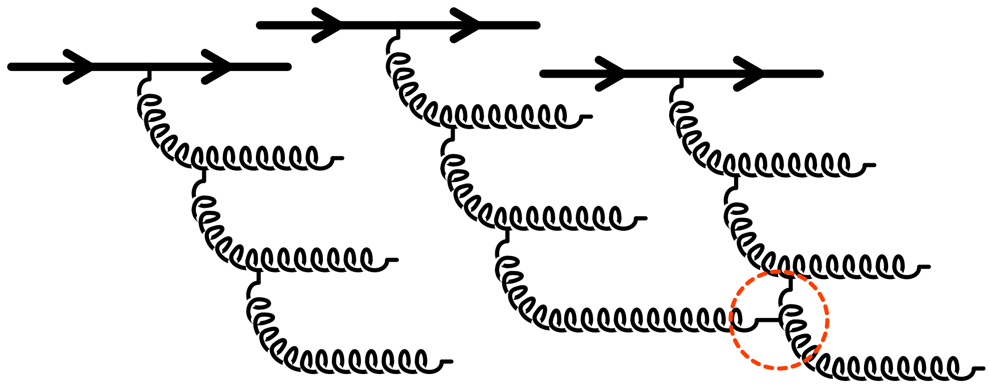
Parton recombination

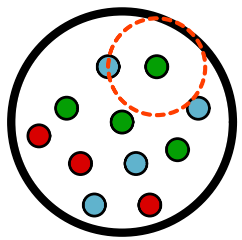




▷ eventually, the partons start overlapping in phase-space

Parton recombination





▷ parton recombination becomes favorable

▷ after this point, the evolution is non-linear:

the number of partons created at a given step depends non-linearly on the number of partons present previously

Balitsky (1996), Kovchegov (1996,2000)

Jalilian-Marian, Kovner, Leonidov, Weigert (1997,1999)

Iancu, Leonidov, McLerran (2001)

Saturation criterion

Gribov, Levin, Ryskin (1983), Mueller, Qiu (1986)

- Number of partons per unit area:

$$\rho \sim \frac{xG(x, Q^2)}{\pi R^2}$$

- Recombination cross-section:

$$\sigma_{gg \to g} \sim \frac{\alpha_s}{Q^2}$$

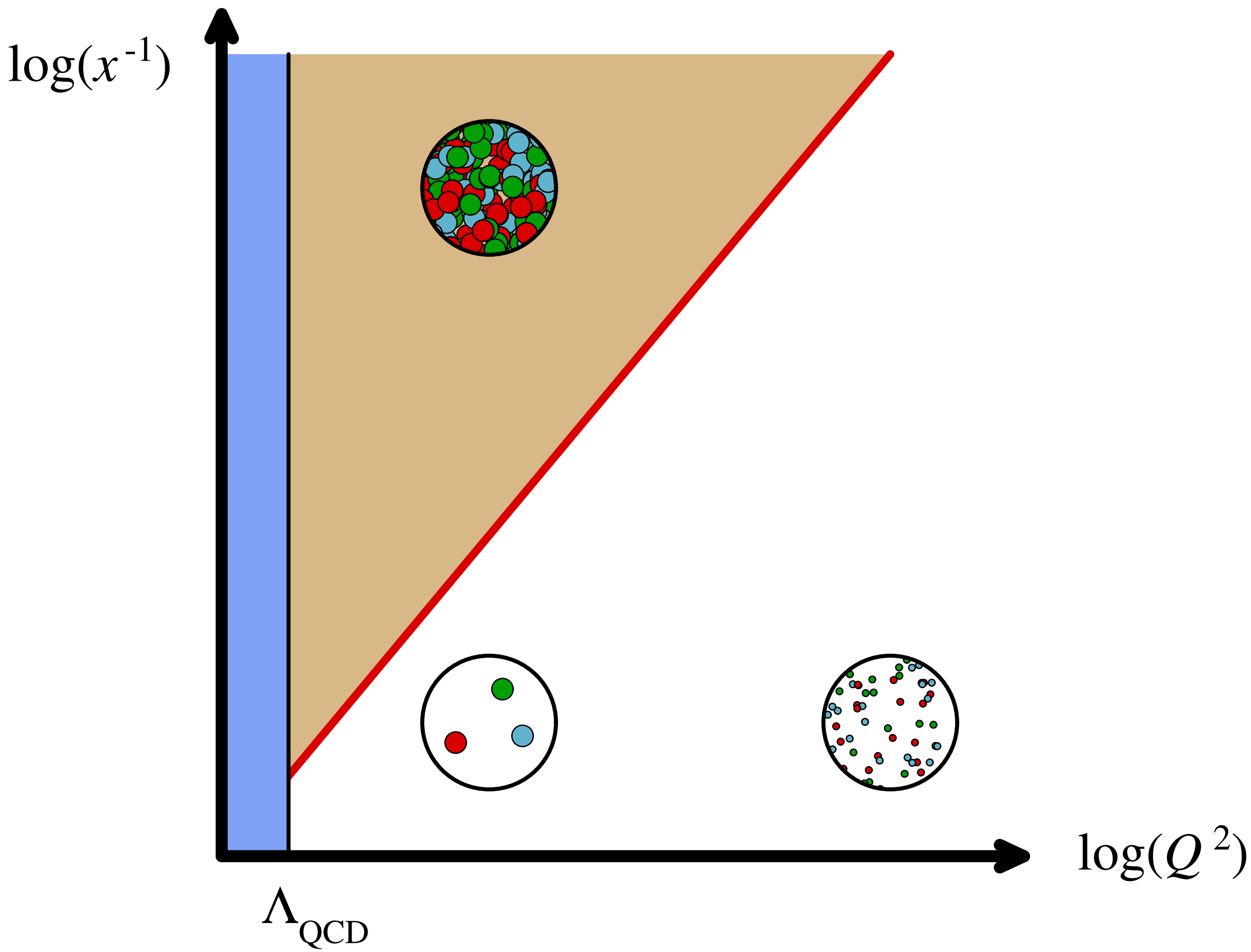
- Recombination if $\rho\sigma_{gg\to g} \gtrsim 1$, or $Q^2 \lesssim Q_s^2$, with:

$$Q_s^2 \sim \frac{\alpha_s xG(x, Q_s^2)}{\pi R^2} \qquad \sim A^{1/3} \frac{1}{x^{0.3}}$$

- At saturation, the gluon phase-space density is:

$$\frac{dN_g}{d^2\vec{\boldsymbol{x}}_\perp d^2\vec{\boldsymbol{p}}_\perp} \sim \frac{\rho}{Q^2} \sim \frac{1}{\alpha_s}$$

Saturation domain



- Boundary defined by $Q^2 = Q_s^2(x)$

Non-linear evolution equation

- In fact, the first evolution equation we derived has a bounded solution. The BFKL equation has unbounded solutions because it is an approximation in which a term quadratic in $\boldsymbol{T}$ has been neglected. The full equation reads :

$$\frac{\partial \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\vec{\boldsymbol{z}}_\perp \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$
$$\times \Big\{ \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp) + \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp) - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp)\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp) \Big\}$$

(Balitsky-Kovchegov equation)

- The r.h.s. vanishes when $\boldsymbol{T}$ reaches 1, and the growth stops. The non-linear term lets both dipoles interact after the splitting of the original dipole

- Both $\boldsymbol{T} = 0$ and $\boldsymbol{T} = 1$ are fixed points of this equation
  - $\boldsymbol{T} = \epsilon$: r.h.s.> 0 $\triangleright$ $\boldsymbol{T} = 0$ is unstable
  - $\boldsymbol{T} = 1 - \epsilon$: r.h.s.> 0 $\triangleright$ $\boldsymbol{T} = 1$ is stable

Caveats

- So far, we have studied the scattering amplitude between a color dipole and a “god given” patch of color field. This is too crude to describe any realistic situation
- One can describe Deep Inelastic Scattering as an interaction between a dipole and the proton, but for that we need to improve the treatment of the target
- At high energy, the duration of the interaction between the dipole and the proton is short. Therefore, it is legitimate to treat the proton as a frozen configuration of color fields. But an experimentally measured cross-section is an average over many collisions, and there is no reason why these fields should be the same in different collisions

Balitsky hierarchy

- Because of this average over the target configurations, the evolution equation we have derived should be written as :

$$\frac{\partial\,\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle}{\partial Y}=\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\boldsymbol{z}}_\perp\,\frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$

$$\times\Big\{\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\rangle+\langle \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle-\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle-\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle\Big\}$$

- As one can see, the equation is no longer a closed equation, since the equation for $\langle \boldsymbol{T}\rangle$ depends on a new object, $\langle \boldsymbol{T}\,\boldsymbol{T}\rangle$
- One can derive an evolution equation for $\langle \boldsymbol{T}\,\boldsymbol{T}\rangle$. Its right hand side contains objects with six Wilson lines
  - Unlike what happened previously, this combination of six Wilson lines simplifies into dipolar operators only in the large N_c limit
  - There is in fact an infinite hierarchy of nested evolution equations, whose generic structure is

$$\frac{\partial\,\langle (\boldsymbol{U}\boldsymbol{U}^\dagger)^n\rangle}{\partial Y}=\int\cdots\Big\langle(\boldsymbol{U}\boldsymbol{U}^\dagger)^n\Big\rangle\oplus\Big\langle(\boldsymbol{U}\boldsymbol{U}^\dagger)^{n+1}\Big\rangle$$

Balitsky-Kovchegov equation

- If one performs the large N_c approximation on all the equations of the Balitsky hierarchy, they can be rewritten in terms of the dipole operator $\boldsymbol{T} \equiv \mathrm{tr}(UU^\dagger)$ only. But they still contain averages like $\langle \boldsymbol{T}^n \rangle$
- In order to truncate the hierarchy of equations, one may assume that

$$\langle \boldsymbol{T}\,\boldsymbol{T} \rangle \approx \langle \boldsymbol{T} \rangle \; \langle \boldsymbol{T} \rangle$$

- This approximation gives for $\langle \boldsymbol{T} \rangle$ the same evolution equation as the one we had for a fixed configuration of the target
- Moreover, it was shown by Janik that if the initial condition is factorized :

$$\langle \boldsymbol{T}_1 \cdots \boldsymbol{T}_n \rangle_{Y_0} = \langle \boldsymbol{T}_1 \rangle_{Y_0} \cdots \langle \boldsymbol{T}_n \rangle_{Y_0}$$

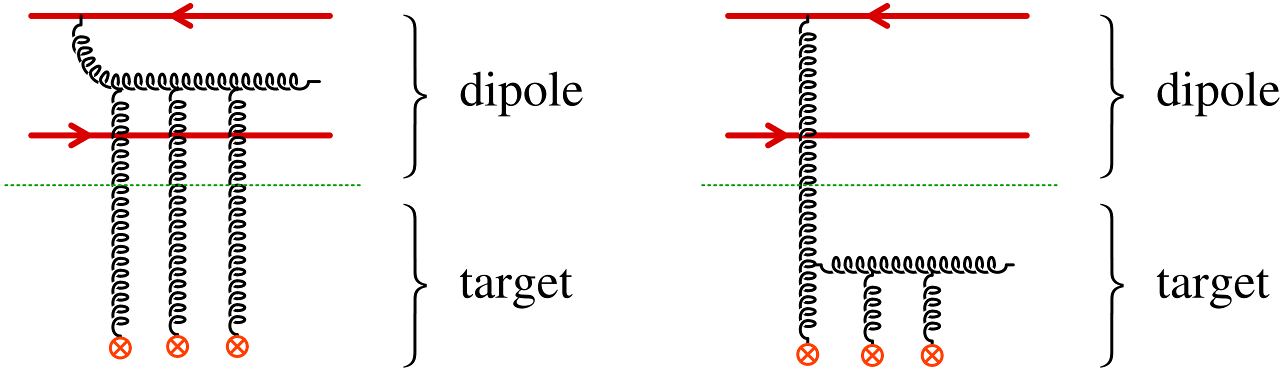
then the solution remains factorized at all $Y > Y_0$

Introduction

- One may view the Color Glass Condensate as a description centered on the target of the physics contained in Balitsky's hierarchy

- In this "target-centric" description, we need to describe how the distribution of color fields in the target changes with rapidity

- In the non-linear regime, the gluon radiation in the target must be corrected by rescatterings in the field of the target :



Degrees of freedom and their interplay

McLerran, Venugopalan (1994)
Iancu, Leonidov, McLerran (2001)

- Small-x modes have a large occupation number
  ▷ they are described by a classical color field A^μ

- The classical field obeys Yang-Mills's equation:

$$[D_\nu, F^{\nu\mu}]_a = J_a^\mu$$

- The source term J_a^μ comes from the faster partons. The large-x modes, slowed down by time dilation, are described as frozen color sources ρ_a. Hence :

$$J_a^\mu = \delta^{\mu+}\delta(x^-)\rho_a(\vec{\boldsymbol{x}}_\perp)$$

Semantics

McLerran (mid 2000)

- Color : pretty much obvious...
- Glass : the system has degrees of freedom whose timescale is much larger than the typical timescales for interaction processes. Moreover, these degrees of freedom are stochastic variables, like in “spin glasses” for instance
- Condensate : the soft degrees of freedom are as densely packed as they can (the density remains finite, of order α_s^{-1}, due to the interactions between gluons)

Target average

- The color sources ρ_a are random, and described by a distribution functional $W_Y[\rho]$, with $Y \equiv \ln(1/x_0)$, x_0 being the frontier between “small-x” and “large-x”

- The averaged dipole operator $\langle \boldsymbol{T} \rangle$ studied in the Balitsky-Kovechegov approach can be written as :

$$\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \rangle = \int [D\rho] \; W_Y[\rho] \; \left[1 - \frac{1}{N_c} \mathrm{tr}(U(\vec{\boldsymbol{x}}_\perp) U^\dagger(\vec{\boldsymbol{y}}_\perp)) \right]$$

- Since in this description, all the evolution is placed inside the target, Y must in fact be the rapidity difference between the projectile and the target

- The Y dependence of $\langle \boldsymbol{T} \rangle$ will have to come from the Y dependence of $W_Y[\rho]$

JIMWLK evolution equation

Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

- The distribution $W_Y[\rho]$ evolves with Y (more modes are included in W as x_0 decreases)
- In a high density environment, the newly created gluons can interact with all the sources already present
- The evolution is governed by a functional diffusion equation:

$$\frac{\partial W_Y[\rho]}{\partial Y} = \frac{1}{2}\int_{\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp} \frac{\delta}{\delta\rho_a(\vec{\boldsymbol{x}}_\perp)}\chi_{ab}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\frac{\delta}{\delta\rho_b(\vec{\boldsymbol{y}}_\perp)} W_Y[\rho]$$

with

$$\chi_{ab}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp) \equiv \frac{\alpha_s}{4\pi^3}\int d^2\vec{\boldsymbol{z}}_\perp \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$
$$\times\left[\left(1-\widetilde{U}^\dagger(\vec{\boldsymbol{x}}_\perp)\widetilde{U}(\vec{\boldsymbol{z}}_\perp)\right)\left(1-\widetilde{U}^\dagger(\vec{\boldsymbol{z}}_\perp)\widetilde{U}(\vec{\boldsymbol{y}}_\perp)\right)\right]_{ab}$$

  - $\widetilde{U}$ is a Wilson line in the adjoint representation, constructed from the gauge field A^+ such that $\boldsymbol{\nabla}_\perp^2 A^+ = -\rho$

JIMWLK evolution equation

- Sketch of a proof : exploit the frame independence in order to write :

$$\langle \mathcal{O} \rangle_Y = \int [D\rho]\; W_0[\rho]\; \mathcal{O}_Y[\rho] = \int [D\rho]\; W_Y[\rho]\; \mathcal{O}_0[\rho]$$

- The first formula leads to

$$\frac{\partial \langle \mathcal{O} \rangle_Y}{\partial Y} = \int [D\rho]\; W_0[\rho]\; \frac{\partial \mathcal{O}_Y[\rho]}{\partial Y}$$

  - The derivative under the integral is determined by a method similar to the derivation of the Balitsky-Kovchegov equation, by attaching one extra gluon to the operator $\mathcal{O}_Y[\rho]$ in all the possible ways
  - As pointed out by Mueller (2001), $\partial \mathcal{O}_Y[\rho]/\partial Y$ can be written as the action of an Hamiltonian on $\mathcal{O}_Y[\rho]$:

$$\frac{\partial \mathcal{O}_Y[\rho]}{\partial Y} = \mathcal{H}\left[\frac{\delta}{\delta\rho}\right]\; \mathcal{O}_Y[\rho]$$

JIMWLK evolution equation

- Then, one can write formally :

$$\mathcal{O}_Y[\rho] = \mathcal{U}(Y)\ \mathcal{O}_0[\rho]$$

with $\quad d\mathcal{U}(Y)/dY = \mathcal{H}\,\mathcal{U}(Y)\quad$ and $\quad \mathcal{U}(0) = 1$

- From there, we get :

$$\langle\mathcal{O}\rangle_Y = \int [D\rho]\ W_0[\rho]\,\mathcal{U}(Y)\ \mathcal{O}_0[\rho] = \int [D\rho]\ \left[\mathcal{U}^\dagger(Y)\ W_0[\rho]\right]\ \mathcal{O}_0[\rho]$$

and we are led to identify :

$$W_Y[\rho] = \mathcal{U}^\dagger(Y)\ W_0[\rho]$$

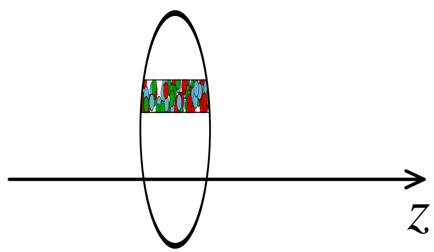
- And finally :

$$\frac{\partial W_Y[\rho]}{\partial Y} = \left[\frac{d\mathcal{U}^\dagger(Y)}{dY}\mathcal{U}(Y)\right]\ \mathcal{U}^\dagger(Y)\ W_0[\rho] = \mathcal{H}_{JIMWLK}\ W_Y[\rho]$$

with $\quad \mathcal{H}_{JIMWLK} = \left[d\mathcal{U}^\dagger(Y)/dY\right]\ \mathcal{U}(Y)$

Initial condition - MV model

- The JIMWLK equation must be completed by an initial condition, given at some moderate x_0
- As with DGLAP, the problem of finding the initial condition is in general non-perturbative
- The McLerran-Venugopalan model is often used as an initial condition at moderate x_0 for a large nucleus :



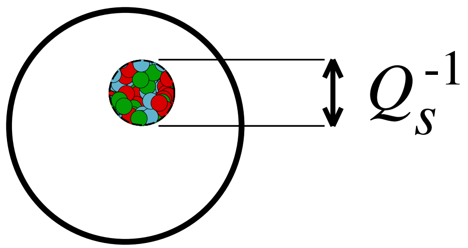
- partons distributed randomly
- many partons in a small tube
- no correlations at different $\vec{\boldsymbol{x}}_\perp$

- The MV model assumes that the density of color charges $\rho(\vec{\boldsymbol{x}}_\perp)$ has a Gaussian distribution :

$$W_{x_0}[\rho] = \exp\left[-\int d^2\vec{\boldsymbol{x}}_\perp \frac{\rho_a(\vec{\boldsymbol{x}}_\perp)\rho_a(\vec{\boldsymbol{x}}_\perp)}{2\mu^2(\vec{\boldsymbol{x}}_\perp)}\right]$$

Color correlation length

- In a nucleon at low energy, the typical correlation length among color charges is of the order of the nucleon size, i.e. $\Lambda_{QCD}^{-1} \sim 1\ \text{fm}$. This is because the typical color screening distance is Λ_{QCD}^{-1}. At low energy, color screening is due to confinement, and thus non-perturbative
- At high energy (small x), partons are much more densely packed, and it can be shown that color neutralization occurs in fact over distances of the order of $Q_s^{-1} \ll \Lambda_{QCD}^{-1}$



- This implies that all hadrons, and nuclei, behave in the same way at high energy. In this sense, the small x regime described by the CGC is universal

Analogy with reaction-diffusion

Munier, Peschanski (2003,2004)

- Assume rotation invariance, and define :

$$N(Y,k_\perp) \equiv 2\pi \int d^2\vec{\boldsymbol{x}}_\perp \; e^{i\vec{\boldsymbol{k}}_\perp\cdot\vec{\boldsymbol{x}}_\perp} \; \frac{\left\langle \boldsymbol{T}(0,\vec{\boldsymbol{x}}_\perp)\right\rangle_Y}{x_\perp^2}$$

- From the Balitsky-Kovchegov equation for $\left\langle \boldsymbol{T}\right\rangle_Y$, we obtain the following equation for N :

$$\frac{\partial N(Y,k_\perp)}{\partial Y} = \frac{\alpha_s N_c}{\pi}\Big[\chi(-\partial_L)N(Y,k_\perp) - N^2(Y,k_\perp)\Big]$$

with

$$L \equiv \ln(k^2/k_0^2)$$
$$\chi(\gamma) \equiv 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

Analogy with reaction-diffusion

- Expand the function $\chi(\gamma)$ to second order around its minimum $\gamma = 1/2$

- Introduce new variables :

$$t \sim Y$$
$$z \sim L + \frac{\alpha_s N_c}{2\pi} \chi''(1/2)\, Y$$

- The equation for N becomes :

$$\partial_t N = \partial_z^2 N + N - N^2$$

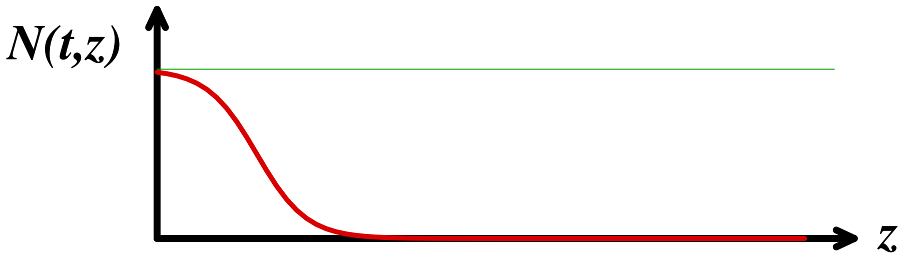
(known as the Fisher-Kolmogorov-Petrov-Piscounov equation)

Analogy with reaction-diffusion

- Interpretation : this equation is typical for all the diffusive systems in which a reaction $A \longleftrightarrow A + A$ takes place
  - $\partial_z^2 N$: diffusion term (the quantity under consideration can hop from a site to the neighboring sites)
  - $+N$: gain term corresponding to $A \to A + A$
  - $-N^2$: loss term corresponding to $A + A \to A$
- Note : this equation has two fixed points :
  - $N = 0$: unstable
  - $N = 1$: stable
- The stable fixed point at $N = 1$ exists only if one keeps the loss term. In other words, one would not have it from the BFKL equation

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



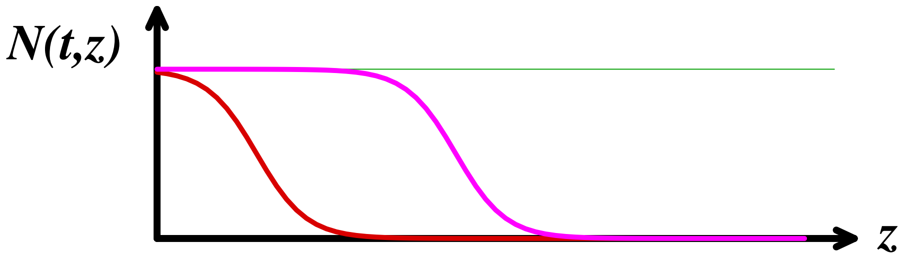
- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times (Bramson, 1983) :

$$N(t, z) \underset{t \to +\infty}{\sim} N(z - m_\beta(t))$$

with
- $m_\beta(t) = (\beta + \beta^{-1})t + \mathcal{O}(1)$ if $\beta < 1$
- $m_\beta(t) = 2t - \ln(t)/2 + \mathcal{O}(1)$ for $\beta = 1$
- $m_\beta(t) = 2t - 3\ln(t)/2 + \mathcal{O}(1)$ if $\beta > 1$

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times (Bramson, 1983) :

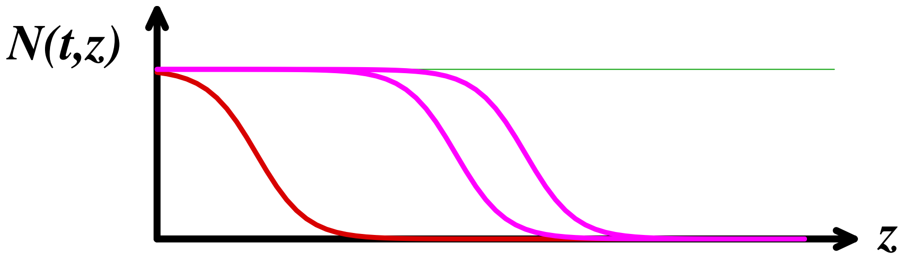
$$N(t, z) \underset{t\to+\infty}{\sim} N(z - m_\beta(t))$$

with

- $m_\beta(t) = (\beta + \beta^{-1})t + \mathcal{O}(1)$ if $\beta < 1$
- $m_\beta(t) = 2t - \ln(t)/2 + \mathcal{O}(1)$ for $\beta = 1$
- $m_\beta(t) = 2t - 3\ln(t)/2 + \mathcal{O}(1)$ if $\beta > 1$

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times (Bramson, 1983) :

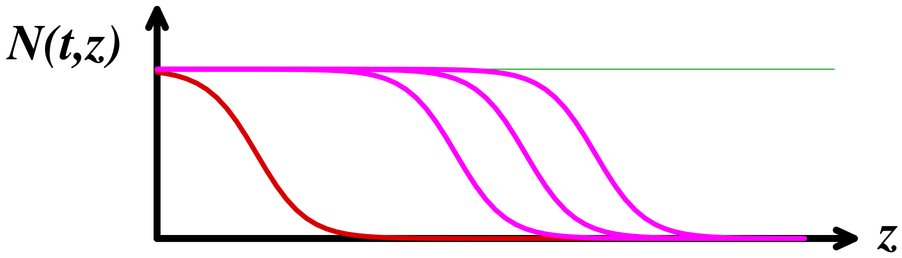
$$N(t, z) \underset{t\to+\infty}{\sim} N(z - m_\beta(t))$$

with

- $m_\beta(t) = (\beta + \beta^{-1})t + \mathcal{O}(1)$ if $\beta < 1$
- $m_\beta(t) = 2t - \ln(t)/2 + \mathcal{O}(1)$ for $\beta = 1$
- $m_\beta(t) = 2t - 3\ln(t)/2 + \mathcal{O}(1)$ if $\beta > 1$

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times (Bramson, 1983) :

$$N(t, z) \underset{t \to +\infty}{\sim} N(z - m_\beta(t))$$

with

- $m_\beta(t) = (\beta + \beta^{-1})t + \mathcal{O}(1)$ if $\beta < 1$
- $m_\beta(t) = 2t - \ln(t)/2 + \mathcal{O}(1)$ for $\beta = 1$
- $m_\beta(t) = 2t - 3\ln(t)/2 + \mathcal{O}(1)$ if $\beta > 1$

Geometrical scaling in DIS

Iancu, Itakura, McLerran (2002)
Mueller, Triantafyllopoulos (2002)
Munier, Peschanski (2003)

- In QCD, the initial condition is of the required form, with $\beta > 1$
  ▷ front velocity independent of the initial condition

- Going back to the original variables, one gets :

$$N(Y, k_\perp) = N\left(k_\perp / Q_s(Y)\right)$$

with

$$Q_s^2(Y) = k_0^2 \; Y^{-\frac{3}{2(1-\bar{\gamma})}} \; e^{\overline{\alpha}_s \chi''(\frac{1}{2})(\frac{1}{2}-\bar{\gamma}) Y}$$

- Going from $N(Y, k_\perp)$ to $\left<\boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\right>_Y$, we obtain :

$$\left<\boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\right>_Y = T(Q_s(Y) x_\perp)$$

Geometrical scaling in DIS

- The $\gamma^* p$ cross-section, measured in Deep Inelastic Scattering, can be written in terms of N:

$$\sigma_{\gamma^* p}(Y, Q^2) = 2\pi R^2 \int d^2\vec{\boldsymbol{x}}_\perp \int_0^1 dz \left|\psi(z, x_\perp, Q^2)\right|^2 \left\langle \boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\right\rangle_Y$$

  - The photon wavefunction ψ is calculable in QED :

$$\left|\psi_T(z, x_\perp, Q^2)\right|^2 = \frac{3\alpha_{\rm em}}{2\pi^2} \sum_f e_f^2 \Big\{ [z^2 + (1-z)^2]\, \overline{Q}_f^2\, K_1^2(\overline{Q}_f x_\perp) + m_f^2\, K_0^2(\overline{Q}_f x_\perp) \Big\}$$

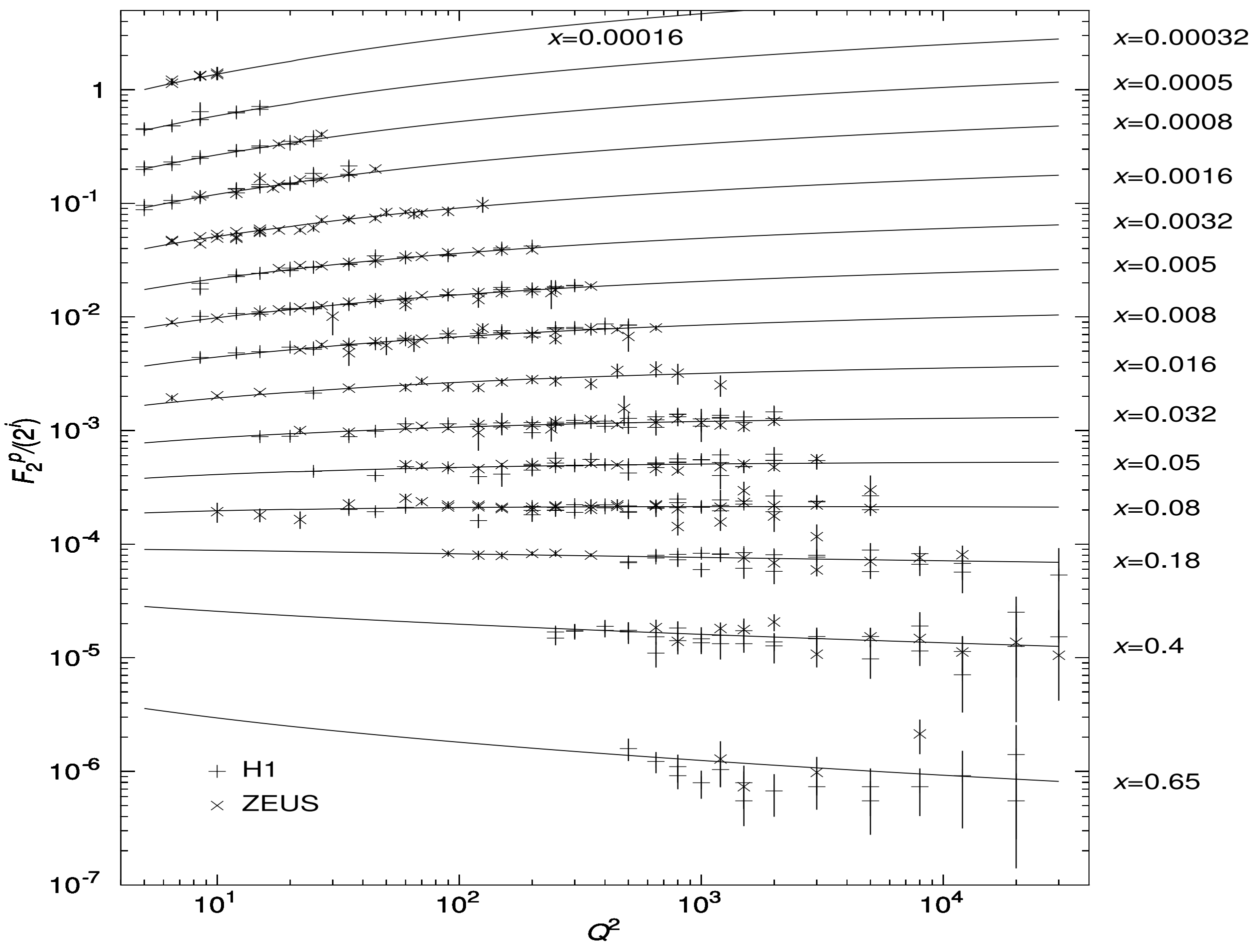
$$\left|\psi_L(z, x_\perp, Q^2)\right|^2 = \frac{3\alpha_{\rm em}}{2\pi^2} \sum_f e_f^2 \Big\{ 4\, Q^2\, z^2 (1-z)^2\, K_0^2(\overline{Q}_f x_\perp) \Big\}$$

  with $\overline{Q}_f^2 \equiv m_f^2 + Q^2 z^2 (1 - z^2)$

- If one neglects the quark masses, the scaling properties of $\langle \boldsymbol{T} \rangle_Y$ imply that $\sigma_{\gamma^* p}$ depends only on the ratio $Q^2 / Q_s^2(Y)$, rather than on Q^2 and Y separately

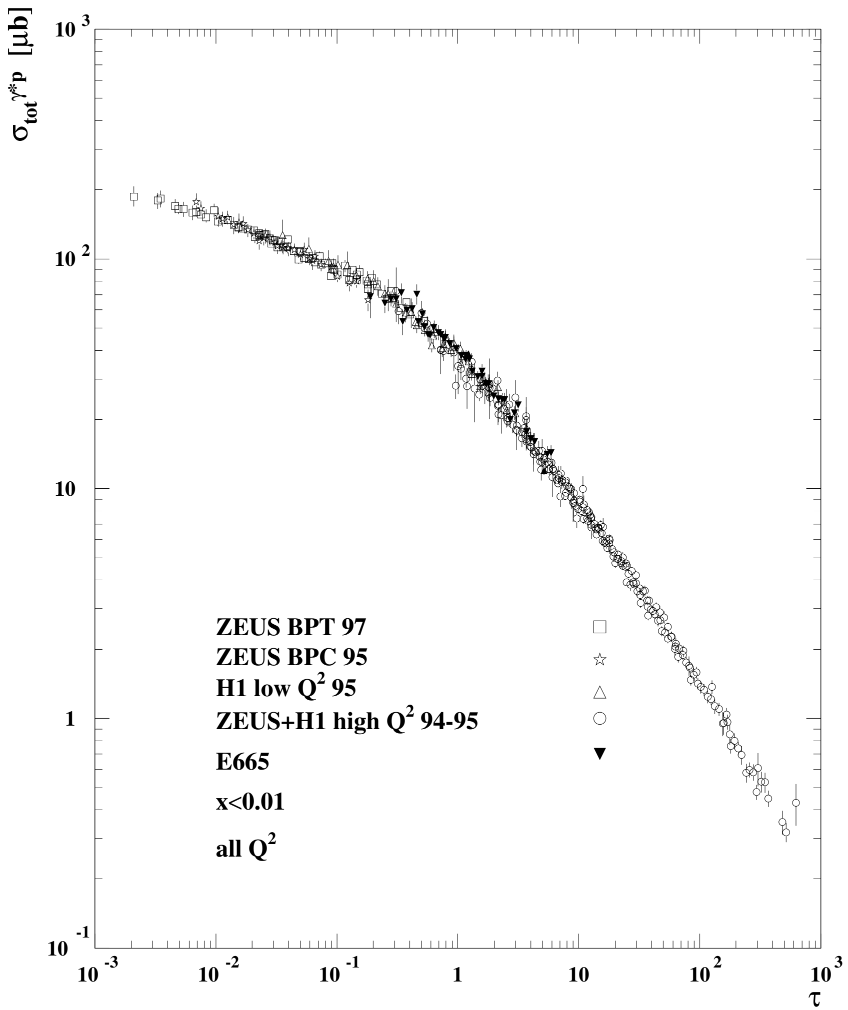
Geometrical scaling in DIS

- HERA data as a function of Q^2 and x :



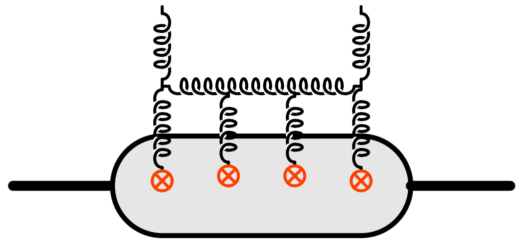
Geometrical scaling in DIS

Stasto, Golec-Biernat, Kwiecinski (2000)

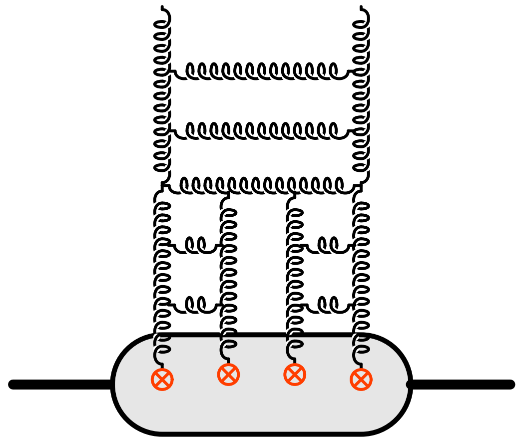


What's wrong with JIMWLK?

- In one step of evolution in Y, the JIMWLK equation allows n gluons to become 2 gluons :

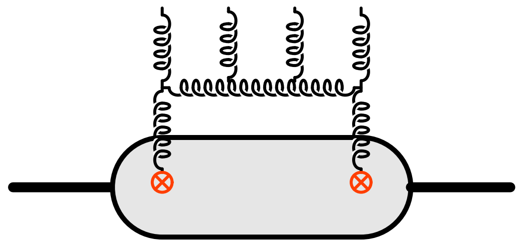


- These contributions are crucial when the color fields inside the target are large (i.e. when the parton density is large)
- When this evolution in rapidity is repeated several times, the JIMWLK equation generates the following type of diagrams :



What's wrong with JIMWLK?

- The JIMWLK equation does not include the reverse processes, where for instance 2 gluons go into n :

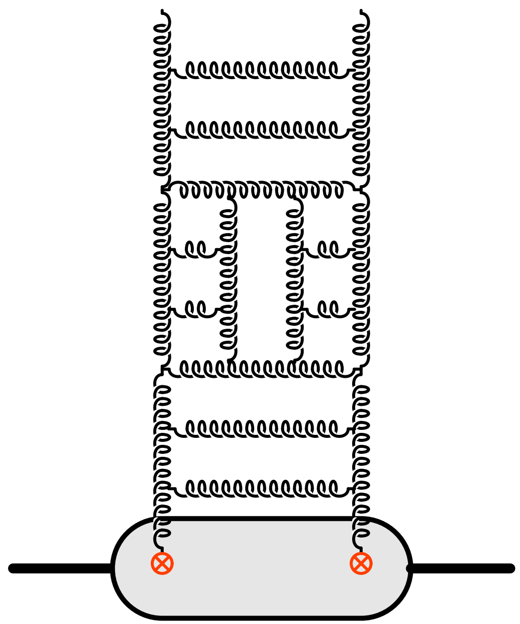


- They can be seen as a way of producing n gluons from quantum fluctuations rather than from the color field of the target
  $\triangleright$ therefore, they are important only when the field in the target is weak

- Moreover, these high multiplicity quantum fluctuations grow faster during the evolution in Y. Therefore, their effect is still felt at high Y, even if at this point these splitting processes are negligible

What's wrong with JIMWLK?

- By keeping into account both the mergings and the splittings, one gets Pomeron loops :



- Naturally, the full theory should have all the $n \to n'$ splittings

Modified Balitsky hierarchy

- Warning : in the "projectile centric" description provided by the Balitsky equations, there are splittings but no mergings...
- Loosely speaking, the first Balitsky equation reads :

$$\frac{\partial\langle \boldsymbol{T}\rangle}{\partial Y} = \int \cdots \Big\{\langle \boldsymbol{T}\rangle - \langle \boldsymbol{T}^2\rangle\Big\}$$

- The second equation of the hierarchy drives the evolution of $\langle \boldsymbol{T}^2\rangle$, and in the large N_c limit it reads :

$$\frac{\partial\langle \boldsymbol{T}^2\rangle}{\partial Y} = \int \cdots \Big\{\langle \boldsymbol{T}^2\rangle - \langle \boldsymbol{T}^3\rangle\Big\}$$

- In order to have mergings, one should add an extra term :

$$\frac{\partial\langle \boldsymbol{T}^2\rangle}{\partial Y} = \int \cdots \Big\{\langle \boldsymbol{T}^2\rangle - \langle \boldsymbol{T}^3\rangle + \alpha_s^2\,\langle \boldsymbol{T}\rangle\Big\}$$

Modified Balitsky hierarchy

- More generally, the n-th modified Balitsky equation reads :

$$\frac{\partial\langle \boldsymbol{T}^n\rangle}{\partial Y} = \int \cdots \Big\{ \langle \boldsymbol{T}^n\rangle - \langle \boldsymbol{T}^{n+1}\rangle + \alpha_s^2 \, \langle \boldsymbol{T}^{n-1}\rangle \Big\}$$

- Such a hierarchy of equations can be remapped into a Langevin equation :

$$\partial_t N = \partial_z^2 N + N - N^2 + \sqrt{N(1-N)}\xi$$

where ξ is a Gaussian white noise

Fluctuations in the F-KPP equation

Brunet, Derrida (1997,1999,2001)
Iancu, Mueller, Munier (2004)
Iancu, Triantafyllopoulos (2004)

- The properties of the front of the traveling wave are determined by the tail at $z \to +\infty$, where N is small
- This is precisely where the stochastic term is important
- N, being related to the number of partons in the target, is a quantity that should vary in discrete increments
  It cannot be arbitrary small
- When this discreteness is taken into account, one sees that the growth of N is controlled by the diffusion term $\partial_z^2 N$ rather than by the gain term $+N$
- This changes many things, in particular the velocity of the traveling front

Lecture V : Calculating observables

- Field theory coupled to time-dependent sources
- Generating function for the probabilities
- Average particle multiplicity
- Numerical methods for nucleus-nucleus collisions
  - Gluon production
  - Quark production