$N = 4$ Supersymmetric Gauge Theory, Twistor Space, and Dualities

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Saclay Lectures, III

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Course Overview

- Present advanced techniques for calculating amplitudes in gauge theories
- Motivation for hard calculations
- Review gauge theories and supersymmetry
- Color decomposition; spinor-helicity basis; recurrence relations; supersymmetry Ward identities; factorization properties of gauge-theory amplitudes
- Twistor space; Cachazo-Svrcek-Witten rules for amplitudes
- Unitarity-based method for loop calculations; loop integral reductions
- Computation of anomalous dimensions

Review of Lecture II

- Examples
- Recurrence relations
- Parke-Taylor amplitudes
- Twistor space
- Differential operators

Calculate $A_4^{\text{tree}}(1^+, 2^+, 3^+, 4^+)$

one-line calculation shows amplitude vanishes

Calculate $A_4^{\text{tree}}(1^-, 2^+, 3^+, 4^+)$

one-line calculation shows amplitude vanishes

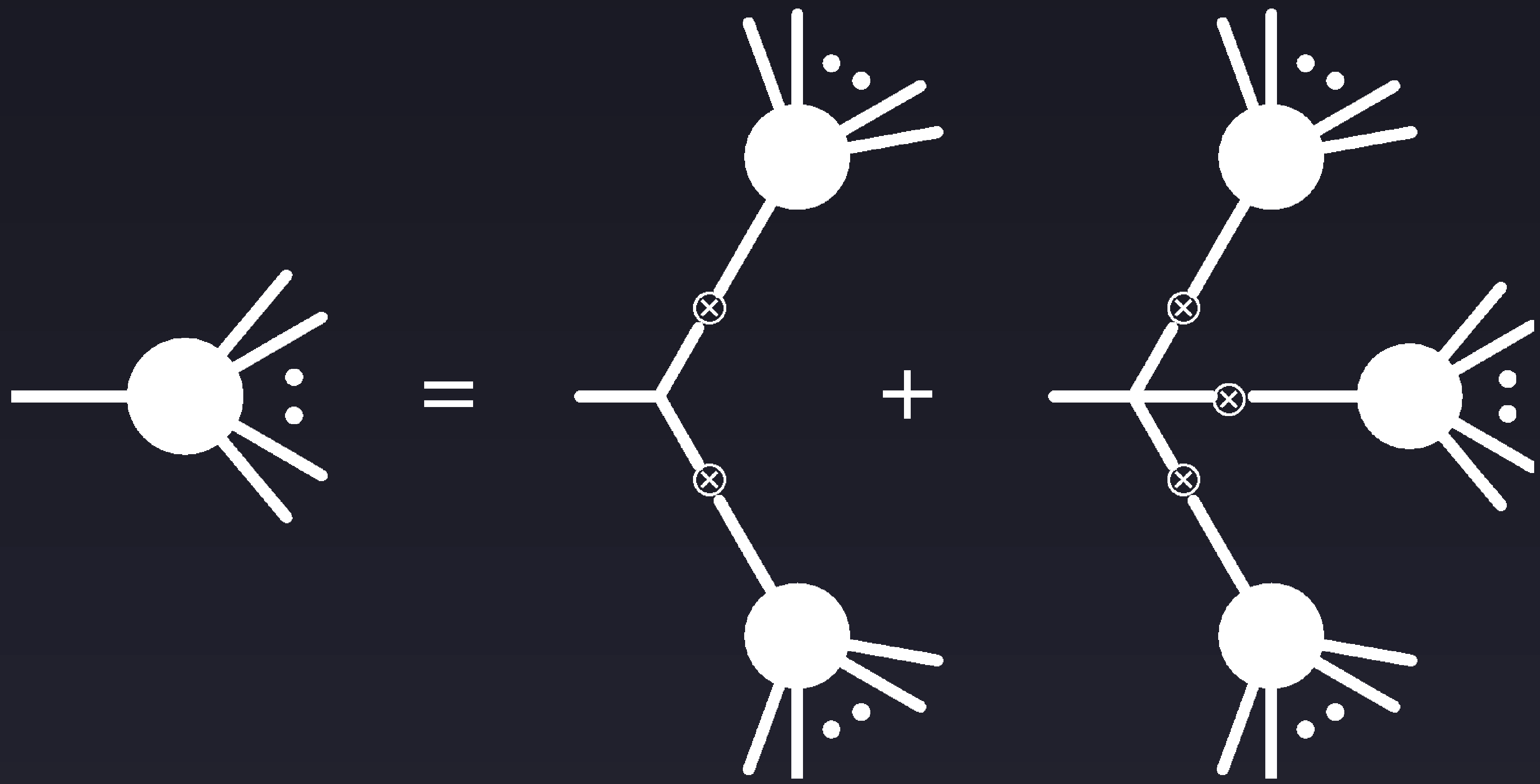
Calculate $A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$

simplified by spinor helicity

Calculate $A_4^{\text{tree}}(1^-, 2^+, 3^-, 4^+)$

using decoupling equation; no diagrammatic calculation required

Recurrence Relations



Berends & Giele (1988); DAK (1989)

⇒ Polynomial complexity per helicity

Properties of the Current

- Decoupling identity
- Reflection identity
- Conservation $K^{\mu}_{1,n} J_{\mu}(1, \ldots, n) = 0$

Contract with polarization vector for last leg, amputate, and take on-shell limit

$$A_n^{\rm tree}(1^+,2^+,\ldots,n^+) = 0\,,$$
$$A_n^{\rm tree}(1^-,2^+,\ldots,n^+) = 0$$

Parke-Taylor equations

Maximally helicity-violating or 'MHV'

$$A_n^{\rm tree}(1^+,\ldots,m_1^-,(m_1{+}1)^+,\ldots,m_2^-,(m_2{+}1)^+,\ldots,n^+) = i\frac{\langle m_1\, m_2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle n\,1\rangle}$$

Proven using the Berends–Giele recurrence relations

Gauge-theory amplitude

$\downarrow$ Color decomposition & stripping

Color-ordered amplitude: function of k_i and ε_i

$\downarrow$ Spinor-helicity basis

Helicity amplitude: function of spinor products and helicities ± 1

$\downarrow$

Function of spinor variables and helicities ± 1

$\downarrow$ Half-Fourier transform

Conjectured support on simple curves in twistor space

Twistor Space

Half-Fourier transform of spinors: transform $\tilde{\lambda}_{\dot{a}}$, leave alone λ_a
$\Rightarrow$ Penrose's original twistor space, real or complex

$$\tilde{\lambda}_{\dot{a}} \to i\frac{\partial}{\partial \mu^{\dot{a}}}, \qquad -i\frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \to \mu_{\dot{a}}$$

Study amplitudes of definite helicity: introduce homogeneous coordinates $Z_I = (\lambda_a, \mu_{\dot{a}}) \equiv \tau(\lambda_a, \mu_{\dot{a}})$

$\Rightarrow$ CP^3 or RP^3 (projective) twistor space

Back to momentum space by Fourier-transforming μ

MHV amplitudes live on lines in twistor space

$$\widetilde{A}(Z) = \int d^4x \prod_j \delta^2\left(\mu_{j\dot{a}} + x_{a\dot{a}}\lambda_j^a\right) A^{\mathrm{MHV}}(\lambda_j)$$

equation for a line

Differential Operators

Equation for a line (CP^1): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K = 0$

gives us a differential ('line') operator in terms of momentum-space spinors

$$F_{123} = \langle \lambda_1 \, \lambda_2 \rangle \frac{\partial}{\partial \tilde{\lambda}_3} + \langle \lambda_2 \, \lambda_3 \rangle \frac{\partial}{\partial \tilde{\lambda}_1} + \langle \lambda_3 \, \lambda_1 \rangle \frac{\partial}{\partial \tilde{\lambda}_2}.$$

Equation for a plane (CP^2): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K Z_4^L = 0$

also gives us a differential ('plane') operator

$$K_{1234} = \langle \lambda_1 \, \lambda_2 \rangle \frac{\partial}{\partial \tilde{\lambda}_{3\dot{a}}} \frac{\partial}{\partial \tilde{\lambda}_4^{\dot{a}}} + \text{ perms}$$

Properties

$$F_{ijl}\, f(p_i + p_j + p_l) = 0$$

$$F_{ijl}\, f(\{\lambda_r\}) = 0$$

$$K_{ijlm}\, f(\{\lambda_r\}) = 0$$

Thus for example

$$F_{ijl}\, \frac{\langle m_1\, m_2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle (n-1)\,n\rangle\,\langle n\,1\rangle} = 0$$

Beyond MHV

Witten's proposal: hep-ph/0312171

- Each external particle represented by a point in twistor space
- Amplitudes non-vanishing only when points lie on a curve of degree d and genus g, where
  - d= # negative helicities – 1 + # loops
  - g ≤ # loops; g= 0 for tree amplitudes
- Integrand on curve supplied by a topological string theory
- Obtain amplitudes by integrating over all possible curves ⇒ moduli space of curves
- Can be interpreted as D_1-instantons

Simple Cases

Amplitudes with all helicities '+ ' $\Rightarrow$ degree –1 curves.
No such curves exist, so the amplitudes should vanish.
Corresponds to the first Parke–Taylor equation.

Amplitudes with one '–' helicity $\Rightarrow$ degree-0 curves: points.
Generic external momenta, all external points won't coincide (singular configuration, all collinear), $\Rightarrow$ amplitudes must vanish.
Corresponds to the second Parke–Taylor equation.

Amplitudes with two '–' helicities (MHV) $\Rightarrow$ degree-1 curves: lines.
All F operators should annihilate them, and they do.

Other Cases

Amplitudes with three negative helicities (next-to-MHV) live on conic sections (quadratic curves)

Amplitudes with four negative helicities (next-to-next-to-MHV) live on twisted cubics

Fourier transform back to spinors $\Rightarrow$ differential equations in conjugate spinors

Even String Theorists Can Do Experiments

- Apply F operators to NMHV (3 –) amplitudes: products annihilate them! K annihilates them;
- Apply F operators to N^2MHV (4 –) amplitudes: longer products annihilate them! Products of K annihilate them;

$$F_{512}F_{234}F_{345}F_{451}A_5(1^-,2^-,3^-,4^+,5^+) =$$

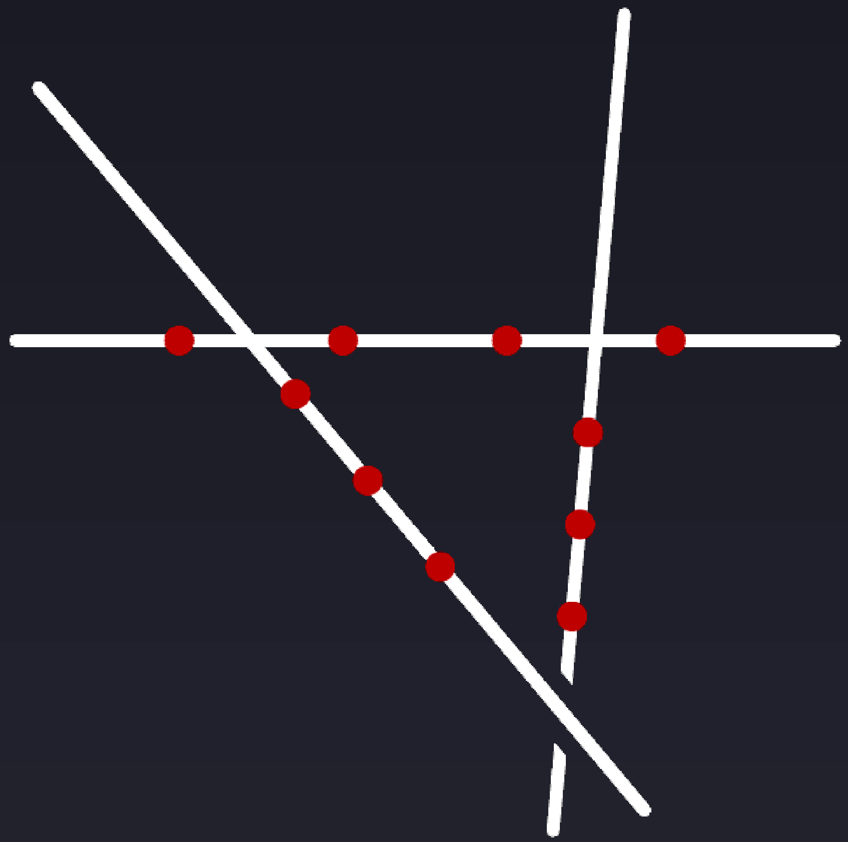
$$F_{512}F_{234}F_{345}F_{451}\frac{[4\,5]^4}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,1]} = 0$$

A more involved example

$$F_{612}F_{234}F_{345}F_{561}A_6(1^-,2^-,3^-,4^+,5^+,6^+) = 0$$

Don't try this at home!

Interpretation: twistor-string amplitudes are supported on intersecting line segments



Simpler than expected: what does this mean in field theory?

Cachazo–Svrcek–Witten Construction

hep-th/0403047

Amplitudes can be built up out of vertices ↔ line segments

Intersections ↔ propagators

Vertices are off-shell continuations of MHV amplitudes: every vertex has two '−' helicities, and one or more '+' helicities

Includes a three-point vertex

Propagators are scalar ones: i/K^2; helicity projector is in the vertices

Draw all tree diagrams with these vertices and propagator

Different sets of diagrams for different helicity configurations

Off-Shell Continuation

Can decompose a general four-vector into two massless four-vectors, one of which we choose to be the light-cone vector

$$K = k^\flat + f(K) q.$$

We can solve for f using the condition $(k^\flat)^2 = 0$

$$f(K) = \frac{K^2}{2q \cdot K}$$

The rule for continuing k_j off-shell in an MHV vertices is then just

$$\langle j\, j' \rangle \to \langle j^\flat\, j' \rangle \,,$$

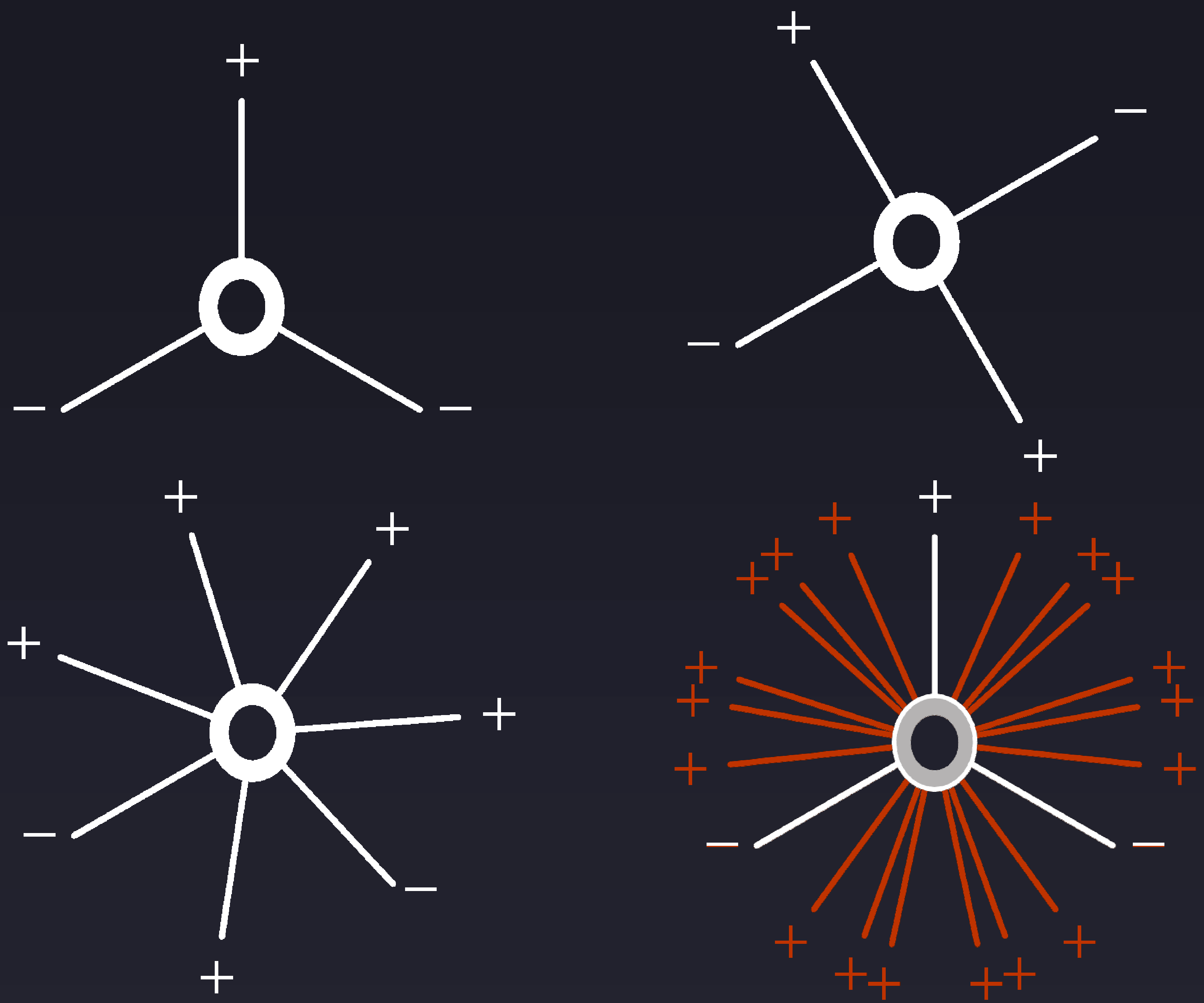
Off-Shell Vertices

General MHV vertex

$$\frac{\langle m_1\, m_2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle (j-1)\, j^\flat\rangle\,\langle j^\flat\,(j+1)\rangle \cdots \langle (n-1)\, n\rangle\,\langle n\,1\rangle}$$

Three-point vertex is just n= 3 case

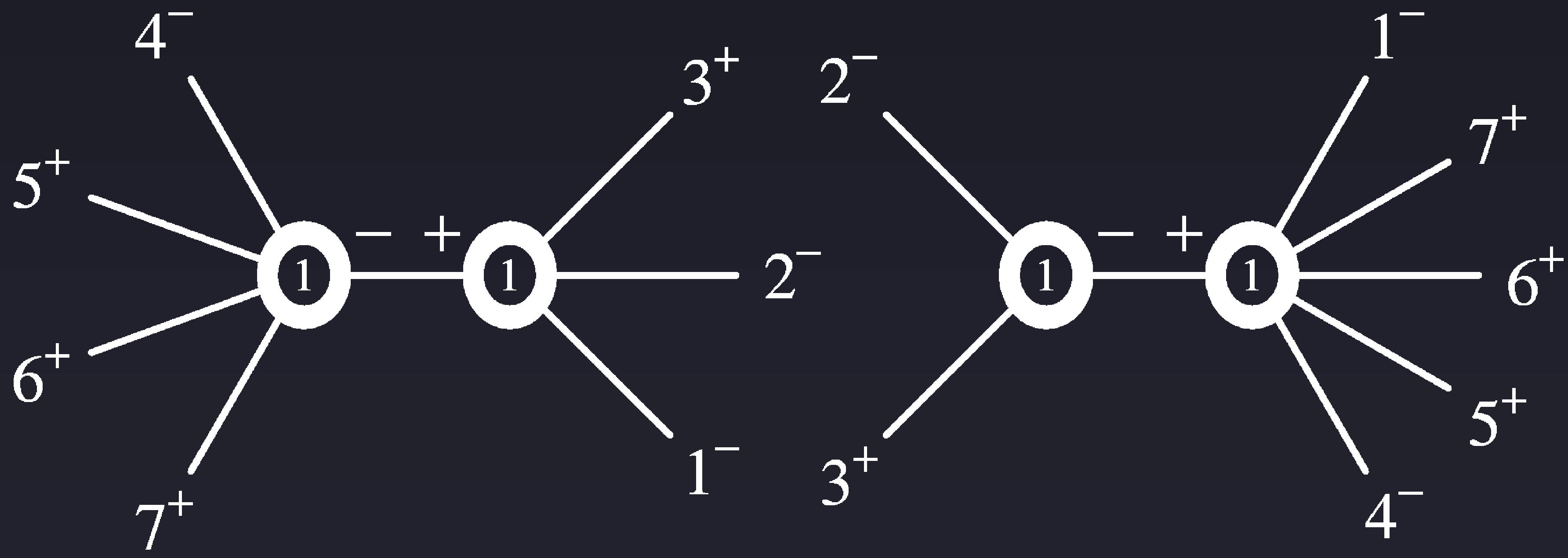
$$\frac{\langle m_1\, m_2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3^\flat\rangle\,\langle 3^\flat\,1\rangle}$$



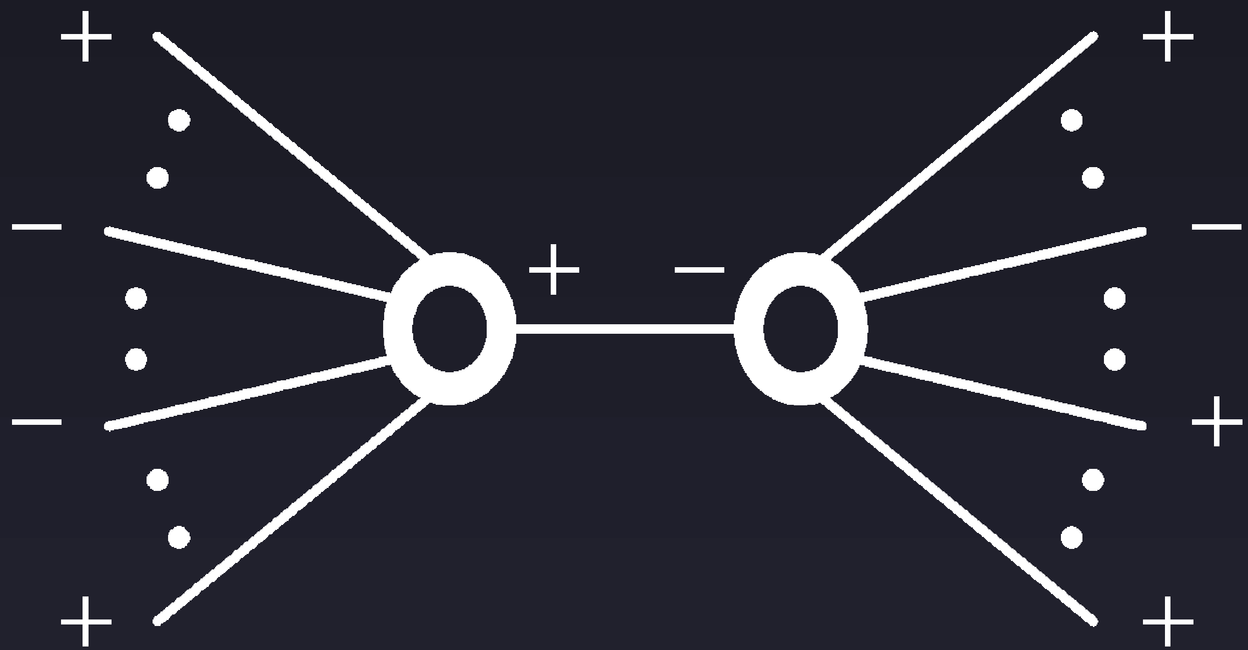
- Corresponds to all multiparticle factorizations
- Not completely local, not fully non-local
- Can write down a Lagrangian by summing over vertices

Abe, Nair, Park (2004)

- Seven-point example with three negative helicities



Next-to-MHV



$$A_n(1^-, 2^-, 3^-, 4^+, \ldots, n^+) =$$

$$\sum_{j=3}^{n-1} \frac{\langle 1\, K^\flat_{2\cdots j}\rangle^3 \langle 2\, 3\rangle^3}{K^2_{2\cdots j}\ \langle K^\flat_{2\cdots j}\, 2\rangle \langle 3\, 4\rangle \cdots \langle (j-1)\, j\rangle \langle j\, K^\flat_{2\cdots j}\rangle}$$

$$\times \frac{1}{\langle K^\flat_{2\cdots j}\, (j+1)\rangle \langle (j+1)\, (j+2)\rangle \cdots \langle (n-1)\, n\rangle \langle n\, 1\rangle}$$

$$+ \sum_{j=4}^{n} \frac{\langle 1\, 2\rangle^3 \langle K^\flat_{3\cdots j}\, 3\rangle^3}{K^2_{3\cdots j}\ \langle 2\, K^\flat_{3\cdots j}\rangle \langle 3\, 4\rangle \cdots \langle (j-1)\, j\rangle \langle j\, K^\flat_{3\cdots j}\rangle}$$

$$\times \frac{1}{\langle K^\flat_{3\cdots j}\, (j+1)\rangle \langle (j+1)\, (j+2)\rangle \cdots \langle (n-1)\, n\rangle \langle n\, 1\rangle}$$

How Do We Know It's Right?

No derivation from Lagrangian

Physicists' proof:

- Correct factorization properties (collinear, multiparticle)
- Compare numerically with conventional recurrence relations through $n = 11$ to 20 digits

A New Analytic Form

- All-n NMHV (3 – : 1, m_2, m_3) amplitude

$$\frac{i(H(1,m_2,m_3)+H(m_2,m_3,1)+H(m_3,1,m_2))}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle n\,1\rangle}.$$

$$H(m_0,m_1,m_2) =$$

$$-\langle m_1\,m_2\rangle^4 \sum_{j_1=m_2+1}^{m_0} \sum_{j_2=m_0+1}^{m_1}{}' \left(\frac{\langle m_0^-|\,\not{K}_{j_1\cdots j_2}\not{k}_{j_1}\,|m_0^+\rangle\,\langle m_0^-|\,\not{K}_{j_1\cdots j_2}\not{k}_{j_2}\,|m_0^+\rangle}{s_{j_1\cdots(j_2-1)}\,s_{(j_1+1)\cdots j_2}\,s_{j_1\cdots j_2}} + \frac{\langle m_0^-|\,\not{K}_{(j_1+1)\cdots(j_2-1)}\not{k}_{j_1}\,|m_0^+\rangle\,\langle m_0^-|\,\not{K}_{(j_1+1)\cdots(j_2-1)}\not{k}_{j_2}\,|m_0^+\rangle}{s_{(j_1+1)\cdots(j_2-1)}\,s_{j_1\cdots(j_2-1)}\,s_{(j_1+1)\cdots j_2}} \right)$$

$$+ \text{ simpler terms}$$

- Generalizes adjacent-minus result DAK (1989)

Computational Complexity

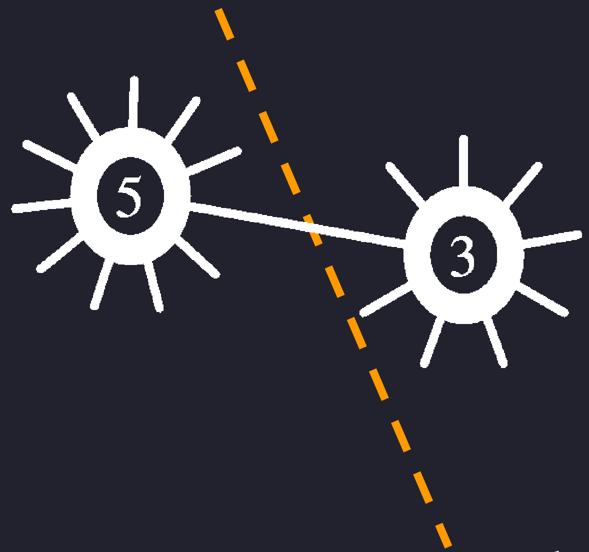
- Exponential ($2^{n-1}-1-n$) number of independent helicities
- Feynman diagrams: factorial growth ~ $n!\, n^{3/2}\, c^{-n}$, per helicity
  - Exponential number of terms per diagram
- MHV diagrams: exponential growth
  - One term per diagram
- Can one do better?

Recursive Formulation

Bena, Bern, DAK (2004)

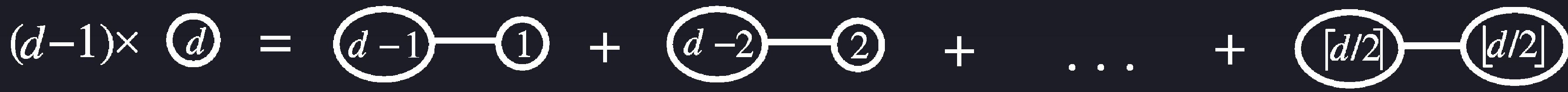
Recursive approaches have proven powerful in QCD; how can we implement one in the CSW approach?

Divide into two sets by cutting propagator



Treat as new higher-degree vertices

With higher-degree vertices, we can reformulate CSW construction in a recursive manner: uniform combinatoric factor



Compact form

$$V_n(1^+,\ldots,m_1^-,(m_1+1)^+,\ldots,m_2^-,(m_2+1)^+,\ldots,m_c^-,(m_c+1)^+,\ldots,n^+) =$$

$$\frac{1}{(c-2)}\sum_{j_1=1}^{n}\sum_{j_2=j_1+1}^{j_1-3}\frac{i}{s_{j_1\cdots j_2}}V_{j_2-j_1+2 \bmod n}(j_1,\ldots,j_2,(-K_{j_1\cdots j_2})^-)$$

$$\times V_{j_1-j_2 \bmod n}(j_2+1,\ldots,j_1-1,(-K_{(j_2+1)\cdots(j_1-1)})^+)\,,$$

Review of Supersymmetry

Equal number of bosonic and fermionic degrees of freedom

Only local extension possible of Poincaré invariance

Extended supersymmetry: only way to combine Poincaré invariance with internal symmetry

Poincaré algebra

$$[P_\mu, P_\nu] = 0$$

$$[P_\rho, M_{\mu\nu}] = \eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu$$

$$[M_{\mu\nu}, M_{\rho\lambda}] = -\eta_{\mu\rho} M_{\nu\lambda} - \eta_{\nu\lambda} M_{\mu\rho} + \eta_{\mu\lambda} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\lambda}$$

Supersymmetry algebra is graded, that is uses both commutators and anticommutators. For N=1, there is one supercharge Q, in a spin-½ representation (and its conjugate)

$$\{Q_a, \overline{Q}_{\dot{a}}\} = 2\sigma^{\mu}_{a\dot{a}} P_\mu$$
$$[P_\mu, Q_a] = [P_\mu, \overline{Q}_{\dot{a}}] = 0$$
$$[Q_a, M_{\mu\nu}] = (\sigma_{\mu\nu})_a{}^b Q_b$$
$$[\overline{Q}_{\dot{a}}, M_{\mu\nu}] = (\overline{\sigma}_{\mu\nu})_{\dot{a}}{}^{\dot{b}} \overline{Q}_{\dot{b}}$$
$$\{Q_a, Q_b\} = \{\overline{Q}_{\dot{a}}, \overline{Q}_{\dot{b}}\} = 0$$

There is also an R symmetry, a U(1) charge that distinguishes between particles and superpartners

Extended Supersymmetry

We can have up to eight spinorial supercharges in four dimensions, but at most four in a gauge theory. This introduces an internal R symmetry which is U(N), and also a central extension Z^{ij}

$$\{Q^i_a, \overline{Q}^j_{\dot{a}}\} = 2\sigma^\mu_{a\dot{a}}\delta^{ij}P_\mu \qquad \{\overline{Q}_{\dot{a}}, \overline{Q}_{\dot{b}}\} = \varepsilon_{\dot{a}\dot{b}}Z^{\dagger\, ij}$$

$$[P_\mu, Q^i_a] = [P_\mu, \overline{Q}^j_{\dot{a}}] = 0 \qquad [Q^i_a, T^c] = (T_f)^{ci}{}_j Q^j_a$$

$$[Q^i_a, M_{\mu\nu}] = (\sigma_{\mu\nu})_a{}^b Q^i_b \qquad [\overline{Q}^j_{\dot{a}}, T^c] = (T^\dagger_f)^{cj}{}_j \overline{Q}^i_{\dot{a}}$$

$$[\overline{Q}^j_{\dot{a}}, M_{\mu\nu}] = (\overline{\sigma}_{\mu\nu})_{\dot{a}}{}^{\dot{b}} \overline{Q}^j_{\dot{b}} \qquad [T^c, T^d] = i f^{cd}{}_e T^e$$

$$\{Q^i_a, Q^j_b\} = \varepsilon_{ab} Z^{ij} \qquad [Z^{ij}, X] = 0$$

(Super)Conformal Symmetry

Classically all massless gauge theories have conformal (scaling) symmetry, but N=4 SUSY has it for the full quantum theory. In two dimensions, the conformal algebra is infinite-dimensional. Here it's finite, and includes a dilatation operator D, special conformal transformations K, and its superpartners.

$$[M_{\mu\nu}, K_\lambda] = \eta_{\nu\lambda} K_\mu - \eta_{\mu\lambda} K_\nu$$
$$[D, P_\mu] = -P_\mu$$
$$[D, K_\mu] = K_\mu$$
$$[P_\mu, K_\nu] = -2M_{\mu\nu} + 2\eta_{\mu\nu} D$$
$$[K_\mu, K_\nu] = 0$$

along with transformations of the new supercharges

Supersymmetric Gauge Theories

N=1: gauge bosons + Majorana fermions, all transforming under the adjoint representation

N=4: gauge bosons + 4 Majorana fermions + 6 real scalars, all transforming under the adjoint representation

Supersymmetry Ward Identities

Color-ordered amplitudes don't distinguish between quarks and gluinos $\Rightarrow$ same for QCD and N=1 SUSY

Supersymmetry should relate amplitudes for different particles in a supermultiplet, such as gluons and gluinos

Supercharge annihilates vacuum

$$\langle 0|[Q, \Phi_1\Phi_2\cdots\Phi_n]|0\rangle = 0 = \sum_{i=1}^{n}\langle 0|\Phi_1\cdots[Q,\Phi_i]\cdots\Phi_n|0\rangle$$

Grisaru, Pendleton & van Nieuwenhuizen (1977)

Use a practical representation of the action of supersymmetry on the fields. Multiply by a spinor wavefunction & Grassman parameter θ

$$[Q, G^{\pm}(k)] = \pm\Gamma^{\pm}(k,q)\Lambda^{\pm}(k)$$
$$[Q, \Lambda^{\pm}(k)] = \mp\Gamma^{\mp}(k,q)G^{\pm}(k)$$

where $\Gamma^{-} = \theta\,[q\,k]\,,\ \Gamma^{+} = \theta\,\langle q\,k\rangle$

With explicit helicity choices, we can use this to obtain equations relating different amplitudes

Typically start with Q acting on an 'amplitude' with an odd number of fermion lines (overall a bosonic object)

Supersymmetry WI in Action

All helicities positive:

$$
\begin{aligned}
0 &= \langle 0|[Q_q, \Lambda_1^+ G_2^+ \cdots G_n^+]|0\rangle \\
&= -\Gamma^-(k_1, q) A_n^{\text{tree}}(1^+, \ldots, n^+) \\
&\quad + \Gamma^+(k_2, q) A_n^{\text{tree}}(1_\Lambda^+, 2_\Lambda^+, 3^+, \ldots, n^+) \\
&\quad + \cdots + \Gamma^+(k_n, q) A_n^{\text{tree}}(1_\Lambda^+, 2^+, \ldots, (n-1)^+, n_\Lambda^+)
\end{aligned}
$$

Helicity conservation implies that the fermionic amplitudes vanish

$$
0 = -\Gamma^-(k_1, q) A_n^{\text{tree}}(1^+, \ldots, n^+)
$$

so that we obtain the first Parke–Taylor equation

With two negative helicity legs, we get a non-vanishing relation

$$
\begin{aligned}
0 &= \langle 0|[Q_q, \Lambda_1^+ G_2^- G_3^+ \cdots G_j^- G_{j+1}^+ \cdots G_n^+]|0\rangle \\
&= -\Gamma^-(k_1, q) A_n^{\text{tree}}(1^+, 2^-, \ldots, n^+) \\
&\quad - \Gamma^+(k_2, q) A_n^{\text{tree}}(1_\Lambda^+, 2_\Lambda^-, 3^+, \ldots, j^-, \ldots, n^+) \\
&\quad - \Gamma^+(k_j, q) A_n^{\text{tree}}(1_\Lambda^+, 2^-, 3^+, \ldots, j_\Lambda^-, \ldots, n^+)
\end{aligned}
$$

Choosing $q = k_2$

$$
\begin{aligned}
& A_n^{\text{tree}}(1_\Lambda^+, 2^-, 3^+, \ldots, j_\Lambda^-, \ldots, n^+) = \\
& \qquad - \frac{\langle 1\,2\rangle}{\langle 2\,j\rangle} A_n^{\text{tree}}(1^+, 2^-, 3^+, \ldots j^-, \ldots, n^+)
\end{aligned}
$$

Additional Relations in Extended SUSY

$$A_n^{\text{tree}}(1^+, \ldots, j^-, \ldots, l^-, \ldots, n^+) = \frac{\langle j\, l\rangle^4}{\langle 1\, 2\rangle^4} A_n^{\text{tree}}(1^-, 2^-, 3^+, \ldots, n^+)$$

Tree-level mplitudes with external gluons or one external fermion pair are given by supersymmetry even in QCD.

Beyond tree level, there are additional contributions, but the Ward identities are still useful.

For supersymmetric theories, they hold to all orders in perturbation theory

Factorization Properties of Amplitudes

As sums of external momenta approach poles,

$$p^2 = (k_1 + k_2)^2 \to m_X^2$$

amplitudes factorize

$$A(1+2 \to \ldots) \to A_L(1+2 \to X) \frac{i}{p^2 - m^2} A_R(X \to \ldots)$$

More generally as $p^2 = (k_1 + \cdots + k_n)^2 \to m_X^2$

$$A(1+\cdots+n \to \ldots) \to A_L(1+\cdots+n \to X) \frac{i}{p^2 - m^2} A_R(X \to \ldots)$$

In massless theories, the situation is more complicated but at tree level it can be understood as an expression of unitarity

In gauge theories, it holds (at tree level) for $n \geq 3$ but breaks down for $n = 2$: $A_3 = 0$ so we get 0/0

However A_3 only vanishes linearly, so the amplitude is not finite in this limit, but should $\sim 1/k$, that is $1/\sqrt{s_{12}}$

This is a collinear limit $(k_1 + k_2)^2 = 2k_1 \cdot k_2 \to 0$

$$\Longrightarrow k_1 \propto k_2, \ \ i.e., \ k_1 \parallel k_2$$

Universal Factorization

Amplitudes have a universal behavior in this limit

$$A_n^{\text{tree}}(\ldots, a^{h_a}, b^{h_b}, \ldots) \stackrel{k_a \| k_b}{\longrightarrow}$$

$$\sum_{h=\pm} \text{Split}_{-h}^{\text{tree}}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{tree}}(\ldots, (k_a + k_b)^h, \ldots)$$

$$+ \;\; \text{non-singular}$$

Depend on a collinear momentum fraction z

$$k_a = z(k_a + k_b), \qquad k_b = (1-z)(k_a + k_b)$$

Splitting Amplitudes

Compute it from the three-point vertex

$$\text{Split}_{-}^{\text{tree}}(a^{+},b^{+}) = -\frac{\sqrt{2}}{s_{ab}}\left[k_b\cdot\varepsilon_a\,\varepsilon_b\cdot\varepsilon_{a+b} - k_a\cdot\varepsilon_b\,\varepsilon_a\cdot\varepsilon_{a+b}\right]$$

$$= -\frac{1}{s_{ab}}\left[\frac{\langle q\,b\rangle\,[b\,a]}{\langle q\,a\rangle}\,\frac{\langle q\,(a{+}b)\rangle\,[q\,b]}{\langle q\,b\rangle\,[(a{+}b)\,q]} - \frac{\langle q\,a\rangle\,[a\,b]}{\langle q\,b\rangle}\,\frac{\langle q\,(a{+}b)\rangle\,[q\,a]}{\langle q\,a\rangle\,[(a{+}b)\,q]}\right]$$

$$= \frac{1}{\langle a\,b\rangle}\left[\sqrt{\frac{1-z}{z}}+\sqrt{\frac{z}{1-z}}\right]$$

$$= \frac{1}{\sqrt{z(1-z)}\,\langle a\,b\rangle}\,.$$

Explicit Values

$$\text{Split}_{-}^{\text{tree}}(a^{-}, b^{-}) = 0$$

$$\text{Split}_{-}^{\text{tree}}(a^{+}, b^{+}) = \frac{1}{\sqrt{z(1-z)}\ \langle a\, b\rangle}$$

$$\text{Split}_{-}^{\text{tree}}(a^{+}, b^{-}) = -\frac{z^2}{\sqrt{z(1-z)}\ [a\, b]}$$

$$\text{Split}_{-}^{\text{tree}}(a^{-}, b^{+}) = -\frac{(1-z)^2}{\sqrt{z(1-z)}\ [a\, b]}$$

Collinear Factorization at One Loop

$$A_n^{\text{1-loop; LC}}(\ldots, a^{h_a}, b^{h_b}, \ldots) \xrightarrow{k_a \| k_b}$$

$$\sum_{h=\pm} \Bigg(\text{Split}^{\text{tree}}_{-h}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{1-loop;LC}}(\ldots, (k_a + k$$

$$+ \text{Split}^{\text{1-loop}}_{-h}(a^{h_a}, b^{h_b}) A_{n-1}^{\text{tree}}(\ldots, (k_a + k$$

$$+ \ \text{non-singular}$$