$\mathcal{N}$=4 Supersymmetric Gauge Theory, Twistor Space, and Dualities

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Saclay Lectures, II

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Course Overview

- Present advanced techniques for calculating amplitudes in gauge theories
- Motivation for hard calculations
- Review gauge theories and supersymmetry
- Color decomposition; spinor-helicity basis; recurrence relations; supersymmetry Ward identities; factorization properties of gauge-theory amplitudes
- Twistor space; Cachazo-Svrcek-Witten rules for amplitudes
- Unitarity-based method for loop calculations; loop integral reductions
- Computation of anomalous dimensions

Why Calculate Amplitudes?

- There are strong physics motivations: LHC physics
- There are strong mathematical physics motivations: study of AdS/CFT duality

Color Decomposition

Standard Feynman rules ⇒ function of momenta, polarization vectors ε, and color indices

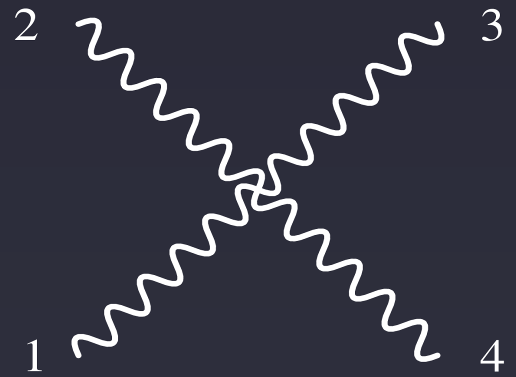
Re-organize color structure, and strip off color factors

$$\mathcal{A}_n^{\text{tree}}(\{k_i, \varepsilon_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}})$$

$$\times A_n^{\text{tree}}(k_{\sigma(1)}, \varepsilon_{\sigma(1)}; k_{\sigma(2)}, \varepsilon_{\sigma(2)}; k_{\sigma(n)}, \varepsilon_{\sigma(n)})$$

Color-Ordered Feynman Rules

$$\frac{i}{\sqrt{2}}\left[\varepsilon_1 \cdot \varepsilon_2 \,(k_1 - k_2) \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 \,(k_2 - k_3) \cdot \varepsilon_1 + \varepsilon_3 \cdot \varepsilon_1 \,(k_3 - k_1) \cdot \varepsilon_2\right]$$



$$i\varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_2 \cdot \varepsilon_4 - \frac{i}{2}\left(\varepsilon_1 \cdot \varepsilon_2 \, \varepsilon_3 \cdot \varepsilon_4 + \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_1\right)$$

Spinor Helicity

Spinor wavefunctions $|j^{\pm}\rangle \equiv u_{\pm}(k_j), \qquad \langle j^{\pm}| \equiv \overline{u_{\pm}}(k_j)\ .$

Introduce *spinor products*

$$\langle i\, j\rangle \equiv \langle i^-|j^+\rangle = \overline{u_-}(k_i)u_+(k_j)\ ,$$
$$[i\, j] \equiv \langle i^+|j^-\rangle = \overline{u_+}(k_i)u_-(k_j)$$

Explicit representation

where $u_+(k) = \begin{pmatrix} \sqrt{k_+} \\ \sqrt{k_-}e^{i\phi_k} \end{pmatrix}, \qquad u_-(k) = \begin{pmatrix} \sqrt{k_-}e^{-i\phi_k} \\ -\sqrt{k_+} \end{pmatrix}$

$$e^{\pm i\phi_k} = \frac{k^1 \pm ik^2}{\sqrt{k_+k_-}}, \qquad k_{\pm} = k^0 \pm k^3$$

We then obtain the explicit formulæ

$$\langle i\, j\rangle = \sqrt{k_{i-}k_{j+}}e^{i\phi_{k_i}} - \sqrt{k_{i+}k_{j-}}e^{i\phi_{k_j}} \;,$$

$$[i\, j] = \langle j\, i\rangle^* = \sqrt{k_{i+}k_{j-}}e^{-i\phi_{k_j}} - \sqrt{k_{i-}k_{j+}}e^{-i\phi_{k_i}} \qquad (k^0_{i,j} > 0)$$

otherwise $[j\, i] = \mathrm{sign}(k_i^0 k_j^0)\, \langle i\, j\rangle^*$

so that the identity $\langle i\, j\rangle\, [j\, i] = 2k_i \cdot k_j$ always holds

Properties of the Spinor Product

- Antisymmetry $\langle j\, i\rangle = -\langle i\, j\rangle\,, \qquad [j\, i] = -[i\, j]$
- Gordon identity $\langle i^{\pm}|\gamma^{\mu}|i^{\pm}\rangle = 2k_i^{\mu}$
- Charge conjugation $\left\langle i^{-}\right|\gamma^{\mu}\left|j^{-}\right\rangle = \left\langle j^{+}\right|\gamma^{\mu}\left|i^{+}\right\rangle\,,$
- Fierz identity $\left\langle i^{-}\right|\gamma^{\mu}\left|j^{-}\right\rangle\left\langle p^{+}\right|\gamma^{\mu}\left|q^{+}\right\rangle = 2\,\langle i\, q\rangle\,[p\, j]$
- Projector representation $|i^{\pm}\rangle\langle i^{\pm}| = \frac{1}{2}(1 \pm \gamma_5)\not{k}_i$
- Schouten identity $\langle i\, j\rangle\,\langle p\, q\rangle = \langle i\, q\rangle\,\langle p\, j\rangle + \langle i\, p\rangle\,\langle j\, q\rangle\;.$

Spinor-Helicity Representation for Gluons

Gauge bosons also have only ± physical polarizations

Elegant — and covariant — generalization of circular polarization

$$\varepsilon_\mu^+(k,q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2}\,\langle q\, k\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2}\,[k\, q]}$$

Xu, Zhang, Chang (1984)

reference momentum q

$$q \cdot k \neq 0$$

Transverse

$$k \cdot \varepsilon^\pm(k,q) = 0$$

Normalized

$$\varepsilon^+ \cdot \varepsilon^- = -1, \qquad \varepsilon^+ \cdot \varepsilon^+ = 0$$

Properties of the Spinor-Helicity Basis

Physical-state projector

$$\sum_{\sigma=\pm} \varepsilon_\mu^\sigma(k,q)\varepsilon_\nu^{\sigma *}(k,q) = \sum_{\sigma=\pm} \varepsilon_\mu^\sigma(k,q)\varepsilon_\nu^{-\sigma}(k,q) = -g_{\mu\nu} + \frac{q_\mu k_\nu + k_\mu q_\nu}{q \cdot k}$$

Simplifications

$$q \cdot \varepsilon^{\pm}(k,q) = 0\,,$$
$$\varepsilon^+(k_1,q) \cdot \varepsilon^+(k_2,q) = \varepsilon^-(k_1,q) \cdot \varepsilon^-(k_2,q) = 0\,,$$
$$\varepsilon^+(k_1,q) \cdot \varepsilon^-(k_2,k_1) = 0$$

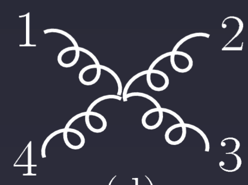
Examples

By explicit calculation (or other arguments), every term in the gluon tree-level amplitude has at least one factor of $\varepsilon_i \cdot \varepsilon_j$

Look at four-point amplitude

(a)

(b)



(d)

Recall three-point color-ordered vertex

$$\frac{i}{\sqrt{2}}\left[\varepsilon_1 \cdot \varepsilon_2 \,(k_1 - k_2) \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 \,(k_2 - k_3) \cdot \varepsilon_1 + \varepsilon_3 \cdot \varepsilon_1 \,(k_3 - k_1) \cdot \varepsilon_2\right]$$

Calculate $A_4^{\text{tree}}(1^+, 2^+, 3^+, 4^+)$

choose identical reference momenta for all legs ⇒ all $\varepsilon \cdot \varepsilon$ vanish

⇒ amplitude vanishes

Calculate $A_4^{\text{tree}}(1^-, 2^+, 3^+, 4^+)$

choose reference momenta 4,1,1,1 ⇒ all $\varepsilon \cdot \varepsilon$ vanish

⇒ amplitude vanishes

Calculate $A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$

choose reference momenta 3,3,2,2

⇒ only nonvanishing $\varepsilon \cdot \varepsilon$ is $\varepsilon_1 \cdot \varepsilon_4$

⇒ only s_{12} channel contributes

$$\left(\frac{i}{\sqrt{2}}\right)^2 \left(-\frac{ig^{\mu\nu}}{s_{12}}\right) \left[-2k_1 \cdot \varepsilon_2^- \varepsilon_{1\mu}^-\right] \left[2k_4 \cdot \varepsilon_3^+ \varepsilon_{4\nu}^+\right]$$

$$= -\frac{2i}{s_{12}} k_1 \cdot \varepsilon_2^- k_4 \cdot \varepsilon_3^+ \varepsilon_1^- \cdot \varepsilon_4^+$$

$$= -\frac{i}{s_{12}} \left(\frac{[3\,1]\,\langle 1\,2\rangle}{[3\,2]}\right) \left(\frac{\langle 2\,4\rangle\,[4\,3]}{\langle 2\,3\rangle}\right) \left(\frac{\langle 2\,1\rangle\,[3\,4]}{\langle 2\,4\rangle\,[3\,1]}\right)$$

$$= -i \frac{\langle 1\,2\rangle^2\,[3\,4]^2}{s_{12}s_{23}}$$

$$= i \frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}$$

No diagrammatic calculation required for the last helicity amplitude,

$$A_4^{\text{tree}}(1^-, 2^+, 3^-, 4^+)$$

Obtain it from the decoupling identity

$$
\begin{aligned}
& - A_4^{\text{tree}}(3^-, 1^-, 2^+, 4^+) - A_4^{\text{tree}}(1^-, 3^-, 2^+, 4^+) \\
& = i \frac{\langle 1\,3 \rangle^3}{\langle 2\,4 \rangle} \left(- \frac{1}{\langle 1\,2 \rangle \langle 4\,3 \rangle} + \frac{1}{\langle 3\,2 \rangle \langle 4\,1 \rangle} \right) \\
& = i \frac{\langle 1\,3 \rangle^4}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,4 \rangle \langle 4\,1 \rangle}
\end{aligned}
$$

Recurrence Relations

Considered color-ordered amplitude with one leg off-shell, amputate its polarization vector

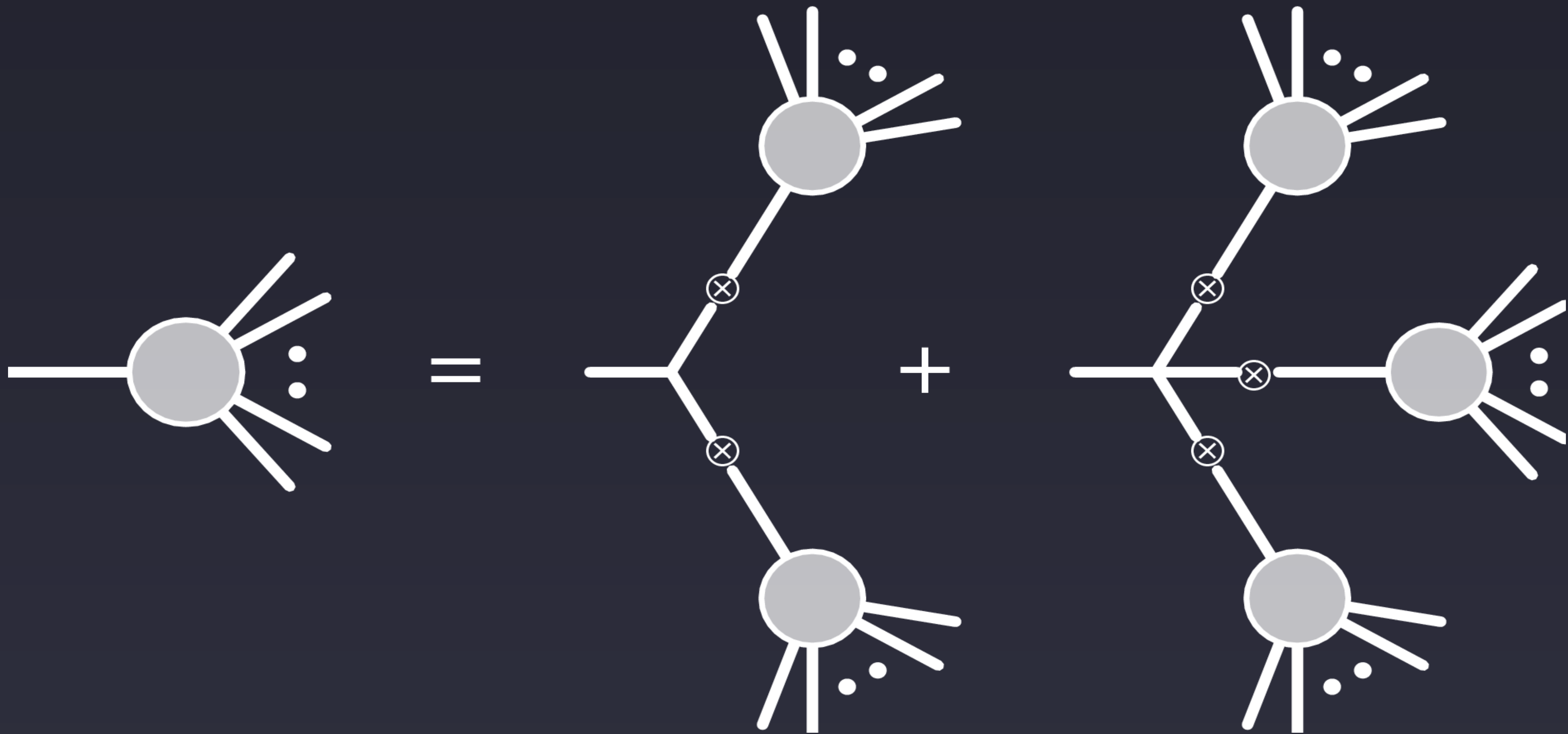
This is the Berends–Giele current $J^\mu(1,\ldots,n)$

Given by the sum of all (n+1)-point color-ordered diagrams with legs 1… n on shell

Follow the off-shell line into the sum of diagrams. It is attached to either a three- or four-point vertex.

Other lines attaching to that vertex are also sums of diagrams with one leg off-shell and other on shell, that is currents

Recurrence Relations



Berends & Giele (1988); DAK (1989)

⇒ Polynomial complexity per helicity

$$J^\mu(1,\ldots,n) = -\frac{i}{K_{1,n}^2}\left[\sum_{j=1}^{n-1} V_3^{\mu\nu\lambda}\, J_\nu(1,\ldots,j) J_\lambda(j+1,\ldots,n) + \sum_{j=1}^{n-2}\sum_{l=j+1}^{n-1} V_4^{\mu\nu\lambda\rho}\, J_\nu(1,\ldots,j) \times J_\lambda(j+1,\ldots,l) J_\rho(l+1,\ldots,n)\right]$$

Properties of the Current

- Decoupling identity
- Reflection identity
- Conservation $K^{\mu}_{1,n} J_{\mu}(1, \ldots, n) = 0$

Explicit Solutions

Strategy: solve by induction

Compute explicitly

$$J^\mu(1^+, 2^+, 3^+, 4^+) = \frac{\langle q^- | \gamma^\mu \not{K}_{1,4} | q^+ \rangle}{\sqrt{2}\, \langle q\, 1 \rangle\, \langle 4\, q \rangle\, \langle\langle 1 \cdots 4 \rangle\rangle}$$

shorthand $\langle\langle 1 \cdots n \rangle\rangle \equiv \langle 1\, 2 \rangle\, \langle 2\, 3 \rangle \cdots \langle (n-1)\, n \rangle$

Compute five-point explicitly ⇒ ansatz

$$J^\mu(1^+, \ldots, n^+) = \frac{\langle q^- | \gamma^\mu \not{K}_{1,n} | q^+ \rangle}{\sqrt{2}\, \langle q\, 1 \rangle\, \langle n\, q \rangle\, \langle\langle 1 \cdots n \rangle\rangle}\,.$$

$$
\frac{1}{\sqrt{2}\,\langle q\,1\rangle\,\langle (n+1)\,q\rangle\,\langle\!\langle 1\cdots(n+1)\rangle\!\rangle\,K_{1,n+1}^2}\sum_{j=1}^{n}\frac{\langle j\,(j+1)\rangle}{\langle j\,q\rangle\,\langle q\,(j+1)\rangle}
$$

$$
\times\Big[\big\langle q^-\big|\,\gamma^\mu\,\not{K}_{j+1,n+1}\,\big|q^+\big\rangle\,K_{j+1,n+1}^\nu\,\big\langle q^-\big|\,\gamma_\nu\,\not{K}_{1,j}\,\big|q^+\big\rangle
$$

$$
-\big\langle q^-\big|\,\gamma^\mu\,\not{K}_{1,j}\,\big|q^+\big\rangle\,K_{1,j}^\nu\,\big\langle q^-\big|\,\gamma_\nu\,\not{K}_{j+1,n+1}\,\big|q^+\big\rangle\Big]
$$

$$
=\frac{\big\langle q^-\big|\,\gamma^\mu\,\not{K}_{1,n+1}\,\big|q^+\big\rangle}{\sqrt{2}\,\langle q\,1\rangle\,\langle (n+1)\,q\rangle\,\langle\!\langle 1\cdots(n+1)\rangle\!\rangle\,K_{1,n+1}^2}
$$

$$
\times\sum_{j=1}^{n}\frac{\langle j\,(j+1)\rangle}{\langle j\,q\rangle\,\langle q\,(j+1)\rangle}\,\big\langle q^-\big|\,\not{K}_{j+1,n+1}\not{K}_{1,j}\,\big|q^+\big\rangle\,.
$$

Observe that $\langle q^- | \not{K}_{1,j} \not{K}_{1,j} | q^+ \rangle = 0$

$$
- \frac{\langle q^- | \gamma^\mu \not{K}_{1,n+1} | q^+ \rangle}{\sqrt{2} \langle q\, 1 \rangle \langle (n+1)\, q \rangle \langle\langle 1 \cdots (n+1) \rangle\rangle K^2_{1,n+1}}
\times \sum_{j=1}^{n} \frac{\langle j\, (j+1) \rangle}{\langle j\, q \rangle \langle q\, (j+1) \rangle} \langle q^- | \not{K}_{1,j} \not{K}_{1,n+1} | q^+ \rangle
$$

Explicit sum

Implicit sum

Exchange order of sums

$$\sum_{j=1}^{n}\sum_{l=1}^{j}\frac{\langle j\,(j+1)\rangle}{\langle j\,q\rangle\,\langle q\,(j+1)\rangle}\left\langle q^-\right|\not{k}_l\not{K}_{1,n+1}\left|q^+\right\rangle$$

$$=\sum_{l=1}^{n}\sum_{j=l}^{n}\frac{\langle j\,(j+1)\rangle}{\langle j\,q\rangle\,\langle q\,(j+1)\rangle}\left\langle q^-\right|\not{k}_l\not{K}_{1,n+1}\left|q^+\right\rangle$$

and use the "eikonal" identity

$$\sum_{j=l}^{n}\frac{\langle j\,(j+1)\rangle}{\langle j\,q\rangle\,\langle q\,(j+1)\rangle}=\frac{\langle l\,(n+1)\rangle}{\langle l\,q\rangle\,\langle q\,(n+1)\rangle}$$

$$J^\mu(1^+,\ldots,(n+1)^+) =$$

$$= -\frac{\left\langle q^- \right| \gamma^\mu \not{K}_{1,n+1} \left| q^+ \right\rangle}{\sqrt{2}\,\langle q\,1\rangle\,\langle (n+1)\,q\rangle\,\langle\langle 1\cdots(n+1)\rangle\rangle\,K^2_{1,n+1}}$$

$$\times \sum_{l=1}^{n} \frac{\left\langle (n+1)^- \right| \not{k}_l \not{K}_{1,n+1} \left| q^+ \right\rangle}{\langle q\,(n+1)\rangle}$$

$$= \frac{\left\langle q^- \right| \gamma^\mu \not{K}_{1,n+1} \left| q^+ \right\rangle}{\sqrt{2}\,\langle q\,1\rangle\,\langle (n+1)\,q\rangle\,\langle\langle 1\cdots(n+1)\rangle\rangle}$$

Contract with polarization vector for last leg, amputate, and take on-shell limit

$$A_n^{\text{tree}}(1^+, 2^+, \ldots, n^+) = 0\,,$$
$$A_n^{\text{tree}}(1^-, 2^+, \ldots, n^+) = 0$$

Parke-Taylor equations

Maximally helicity-violating or 'MHV'

$$A_n^{\text{tree}}(1^+, \ldots, m_1^-, (m_1+1)^+, \ldots, m_2^-, (m_2+1)^+, \ldots, n^+) = i\frac{\langle m_1\, m_2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle (n-1)\, n\rangle\,\langle n\,1\rangle}$$

Proven using the Berends–Giele recurrence relations

Gauge-theory amplitude

↓ Color decomposition & stripping

Color-ordered amplitude: function of k_i and ε_i

↓ Spinor-helicity basis

Helicity amplitude: function of spinor products and helicities ± 1

Spinor products → spinors

Spinor Variables

From Lorentz vectors to bi-spinors

$$p_\mu \qquad \longleftrightarrow \qquad p_{a\dot{a}} \equiv p \cdot \sigma = \begin{pmatrix} p^0 + p^3 & p^1 + ip^2 \\ p^1 - ip^2 & p^0 - p^3 \end{pmatrix}$$

$$p^2 \qquad \longleftrightarrow \qquad \det(p)$$

$$p' = \Lambda p \qquad \longleftrightarrow \qquad p' = upu^\dagger, \qquad u \in SL(2,C)$$

2×2 complex matrices
with det = 1

Null momenta $p^2 = 0 \Longrightarrow \det(p) = 0$

can write it as a bispinor $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$

phase ambiguity in $\lambda_a, \tilde{\lambda}_{\dot{a}}$ (same as seen in spinor products)

For real Minkowski *p*, take $\tilde{\lambda} = \mathrm{sign}(p^0)\overline{\lambda}$

Invariant tensor ϵ_{ab}

$$u_{aa'}u_{bb'}\epsilon_{a'b'} = \det(u) = 1$$

gives spinor products

$$\langle \lambda_1, \lambda_2 \rangle = \epsilon_{ab}\lambda_1^a \lambda_2^b$$

$$[\tilde{\lambda}_1, \tilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}}\tilde{\lambda}_1^{\dot{a}}\tilde{\lambda}_2^{\dot{b}}$$

Connection to earlier spinor products

$$\langle\lambda_1,\lambda_2\rangle = \langle 1\,2\rangle$$
$$[\tilde\lambda_1,\tilde\lambda_2] = -\,[1\,2]$$

and spinor-helicity basis

$$+1: \qquad \varepsilon_{a\dot a} = \frac{\eta_a\tilde\lambda_{\dot a}}{\langle\eta,\lambda\rangle}$$
$$-1: \qquad \varepsilon_{a\dot a} = \frac{\lambda_a\tilde\eta_{\dot a}}{[\tilde\lambda,\tilde\eta]}$$

$\Rightarrow$ Amplitudes as functions of spinor variables $\lambda_a, \tilde\lambda_{\dot a}$ and helicities ± 1

Scaling of Amplitudes

Suppose we scale the spinors

also called 'phase weight'

$$\lambda_i \longmapsto \alpha_i \lambda_i \,,$$
$$\tilde{\lambda}_i \longmapsto \alpha_i^{-1} \tilde{\lambda}_i \,,$$

then by explicit computation we see that the MHV amplitude

$$A^{\mathrm{MHV}} \longmapsto i \frac{\alpha_{m_1}^2 \alpha_{m_2}^2}{\Pi_{j \neq m_1, m_2} \alpha_j^2} \frac{\langle m_1\, m_2 \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle (n-1)\, n \rangle \langle n\, 1 \rangle}$$

and that more generally

$$A \longmapsto \Pi_j \alpha_j^{-2h_j} A$$

For the non-trivial parts of the amplitude, we might as well use uniformly rescaled spinors ⇒ $\mathbb{CP}^1$ 'complex projective space'

Start with $\mathbb{C}^2$, and rescale all vectors by a common scale

$$\begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \equiv \tau \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}$$

the spinors are then 'homogeneous' coordinates on $\mathbb{CP}^1$

If we look at each factor in the MHV amplitude,

$$\frac{1}{\langle \lambda_1, \lambda_2 \rangle} = \frac{1}{\lambda_1^1 \lambda_2^1 (w_1 - w_2)} \qquad w_i = \lambda_i^2 / \lambda_i^1$$

we see that it is just a free-field correlator (Green function) on $\mathbb{CP}^1$

This is the essence of Nair's construction of MHV amplitudes as correlation functions on the 'line' = $\mathbb{CP}^1$

Let's Travel to Twistor Space!

It turns out that the natural setting for amplitudes is not exactly spinor space, but something similar. The motivation comes from studying the representation of the conformal algebra.

Half-Fourier transform of spinors: transform $\tilde{\lambda}_{\dot{a}}$, leave alone $\lambda_a \Rightarrow$ Penrose's original twistor space, real or complex

$$\tilde{\lambda}_{\dot{a}} \to i\frac{\partial}{\partial \mu^{\dot{a}}}, \qquad -i\frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \to \mu_{\dot{a}}$$

Study amplitudes of definite helicity: introduce homogeneous coordinates

$$Z_I = (\lambda_a, \mu_a)$$

$\Rightarrow$ $\mathbb{CP}^3$ or $\mathbb{RP}^3$ (projective) twistor space

Back to momentum space by Fourier-transforming μ

MHV Amplitudes in Twistor Space

Write out the half-Fourier transform including the energy-momentum conserving δ function

$$\widetilde{A}(Z) = \int \prod_j \frac{d^2\tilde{\lambda}_j}{(2\pi)^2} \; \exp[i\mu_{j\dot{a}}\tilde{\lambda}_j^{\dot{a}}] \, (2\pi)^4 \delta^4(\Sigma_r \lambda_r^a \tilde{\lambda}_r^{\dot{a}}) \, A^{\mathrm{MHV}}(\lambda_j)$$

$$\int d^4x \; \exp[\Sigma_j x_{a\dot{a}} \lambda_j^a \tilde{\lambda}_j^{\dot{a}}]$$

$$\widetilde{A}(Z) = \int d^4x \prod_j \frac{d^2\tilde{\lambda}_j}{(2\pi)^2} \; \exp[\Sigma_j i(\mu_{j\dot{a}} + x_{a\dot{a}}\lambda_j^a)\tilde{\lambda}_j^{\dot{a}}] \; A^{\mathrm{MHV}}(\lambda_j)$$

Result

$$\widetilde{A}(Z) = \int d^4x \prod_j \delta^2\left(\mu_{j\dot{a}} + x_{a\dot{a}}\lambda_j^a\right) A^{\mathrm{MHV}}(\lambda_j)$$

equation for a line

MHV amplitudes live on lines in twistor space

Value of the twistor-space amplitude is given by a correlation function on the line

Analyzing Amplitudes in Twistor Space

Amplitudes in twistor space turn out to be hard to compute directly. Even with computations in momentum space, the Fourier transforms are hard to compute explicitly.

We need other tools to analyze the amplitudes.

Simple 'algebraic' properties in twistor space — support on $\mathbb{CP}^1$s or $\mathbb{CP}^2$s — become differential properties in momentum space.

Construct differential operators.

Equation for a line ($\mathbb{CP}^1$): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K = 0$

gives us a differential ('line') operator in terms of momentum-space spinors

$$F_{123} = \langle\lambda_1\,\lambda_2\rangle \frac{\partial}{\partial\tilde{\lambda}_3} + \langle\lambda_2\,\lambda_3\rangle \frac{\partial}{\partial\tilde{\lambda}_1} + \langle\lambda_3\,\lambda_1\rangle \frac{\partial}{\partial\tilde{\lambda}_2}.$$

Equation for a plane ($\mathbb{CP}^2$): $\epsilon_{IJKL} Z_1^I Z_2^J Z_3^K Z_4^L = 0$

also gives us a differential ('plane') operator

$$K_{1234} = \langle\lambda_1\,\lambda_2\rangle \frac{\partial}{\partial\tilde{\lambda}_{3\dot{a}}} \frac{\partial}{\partial\tilde{\lambda}_4^{\dot{a}}} + \text{ perms}$$

Properties

$$F_{ijl}\, f(p_i + p_j + p_l) = 0$$

$$F_{ijl}\, f(\{\lambda_r\}) = 0$$

$$K_{ijlm}\, f(\{\lambda_r\}) = 0$$

Thus for example

$$F_{ijl} \frac{\langle m_1\, m_2 \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle (n-1)\, n \rangle \langle n\, 1 \rangle} = 0$$

Beyond MHV

Witten's proposal: hep-ph/0312171

- Each external particle represented by a point in twistor space
- Amplitudes non-vanishing only when points lie on a curve of degree d and genus g, where
  - d = # negative helicities – 1 + # loops
  - $g \leq$ # loops; $g = 0$ for tree amplitudes
- Integrand on curve supplied by a topological string theory
- Obtain amplitudes by integrating over all possible curves ⇒ moduli space of curves
- Can be interpreted as D_1-instantons

Strings in Twistor Space

- String theory can be defined by a two-dimensional field theory whose fields take values in target space:
  - *n*-dimensional flat space
  - 5-dimensional Anti-de Sitter × 5-sphere
  - twistor space: intrinsically four-dimensional ⇒ Topological String Theory
- Spectrum in Twistor space is $\mathcal{N} = 4$ supersymmetric multiplet (gluon, four fermions, six real scalars)
- Gluons and fermions each have two helicity states

A New Duality

- String Theory ⟷ Gauge Theory

  Topological *B*-model on $CP^{3|4}$ — $\mathcal{N}=4$ SUSY

'Twistor space'

Witten (2003); Berkovits & Motl; Neitzke & Vafa; Siegel (2004)

weak–weak