$\mathcal{N} = 4$ Supersymmetric Gauge Theory, Twistor Space, and Dualities

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Saclay Lectures

Fall Term 2004

Course Overview

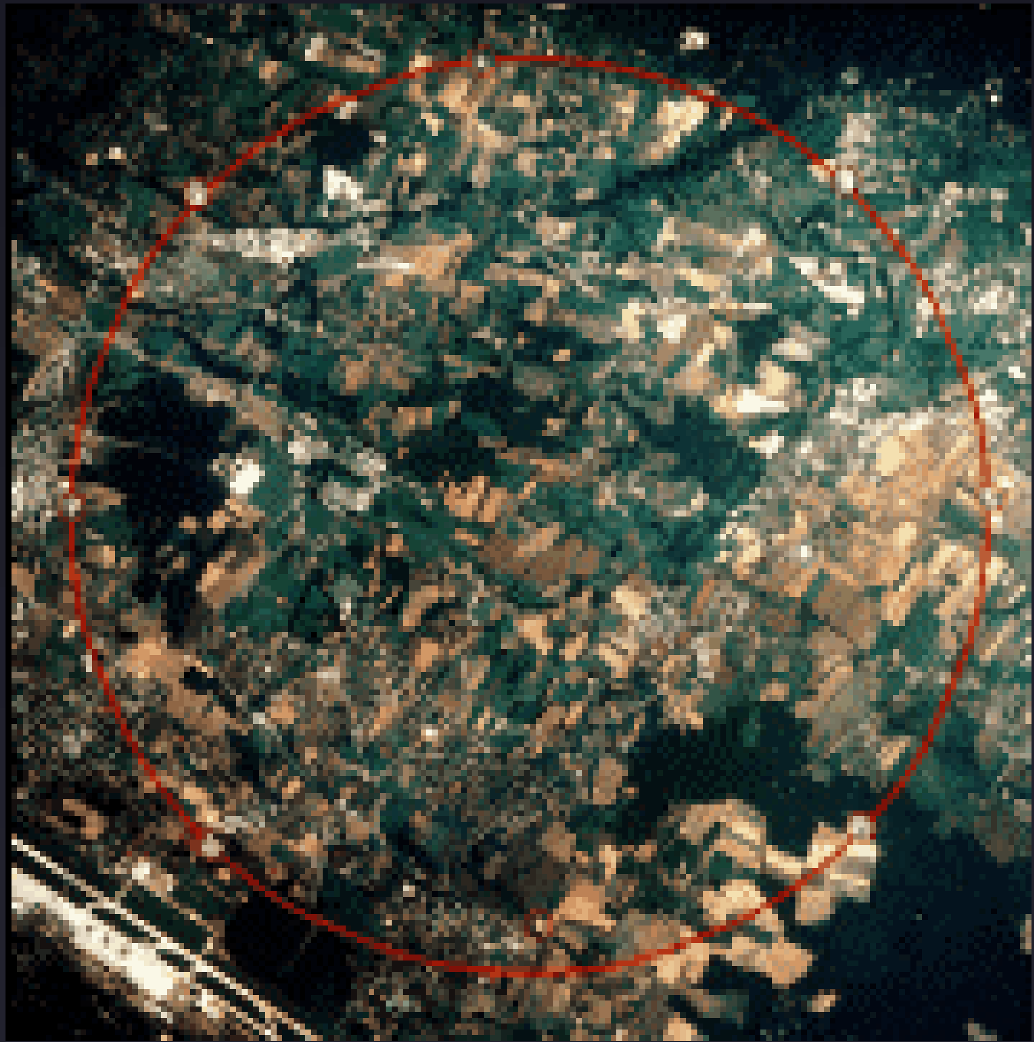
- Present advanced techniques for calculating amplitudes in gauge theories
- Motivation for hard calculations
- Review gauge theories and supersymmetry
- Color decomposition; spinor-helicity basis; recurrence relations; supersymmetry Ward identities; factorization properties of gauge-theory amplitudes
- Twistor space; Cachazo-Svrcek-Witten rules for amplitudes
- Unitarity-based method for loop calculations; loop integral reductions
- Computation of anomalous dimensions

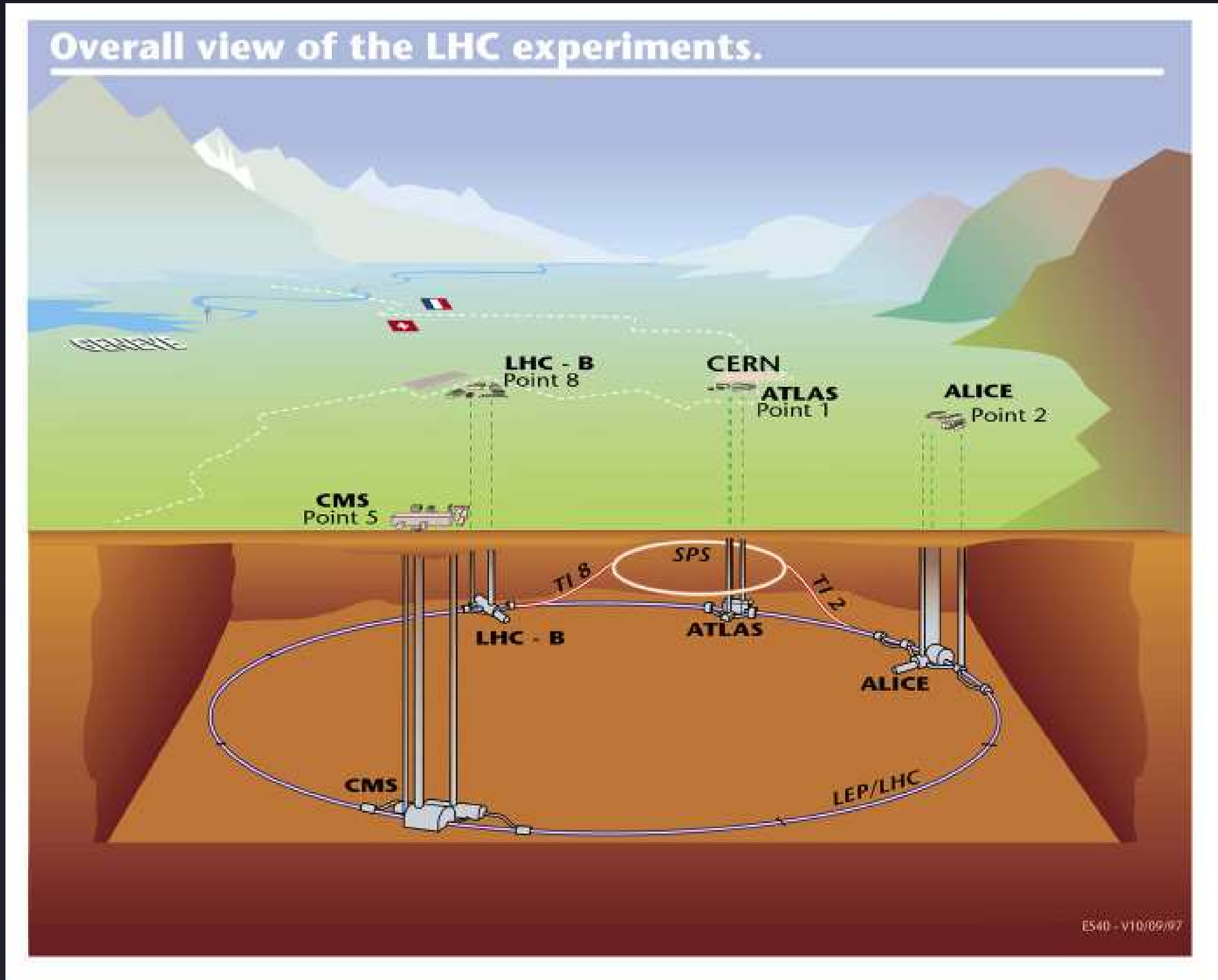
Why Calculate Amplitudes?

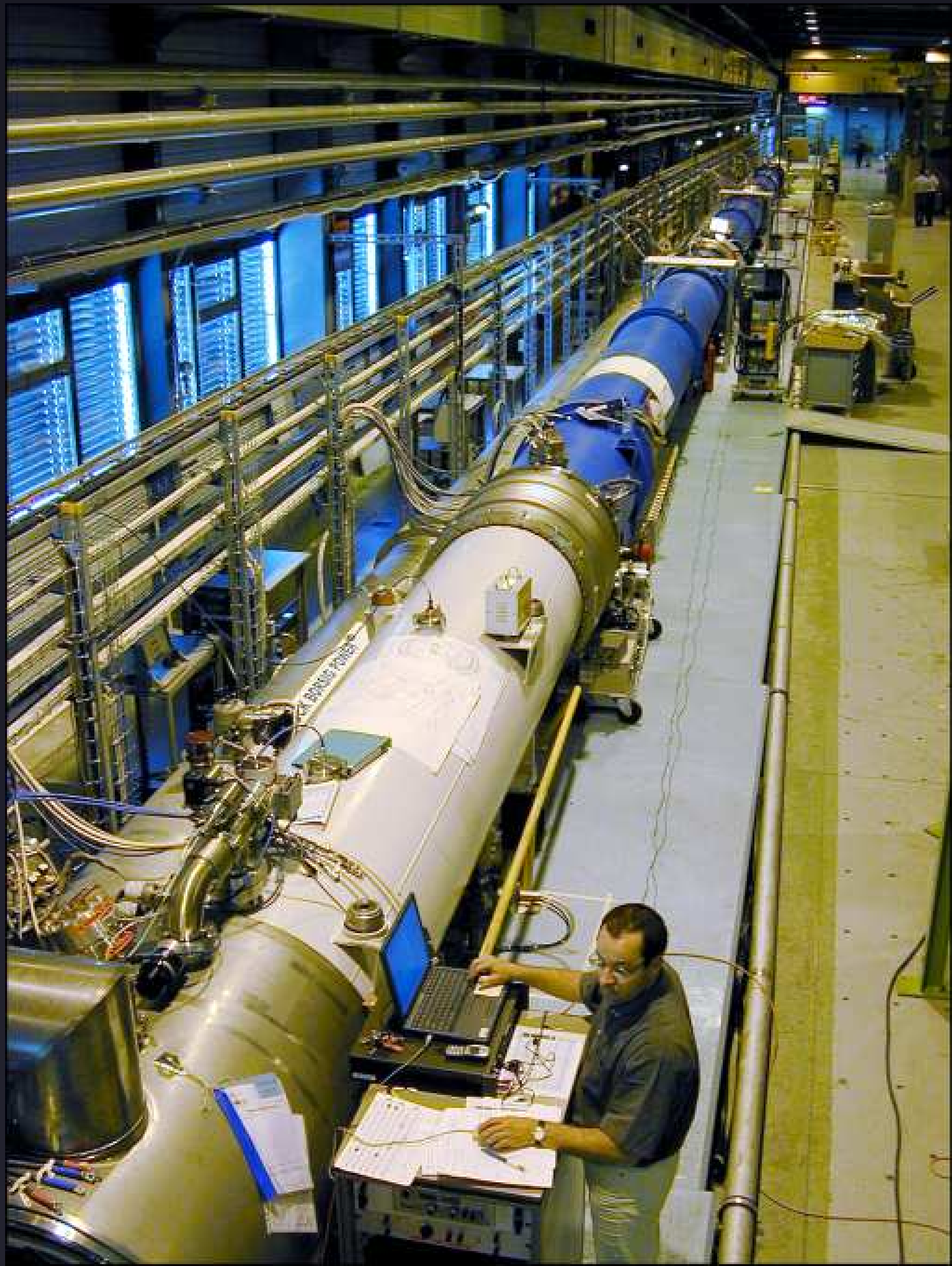
- ~~It's an easy way to while away your professional time~~
- ~~It leads to great opportunities for TV hosting slots~~

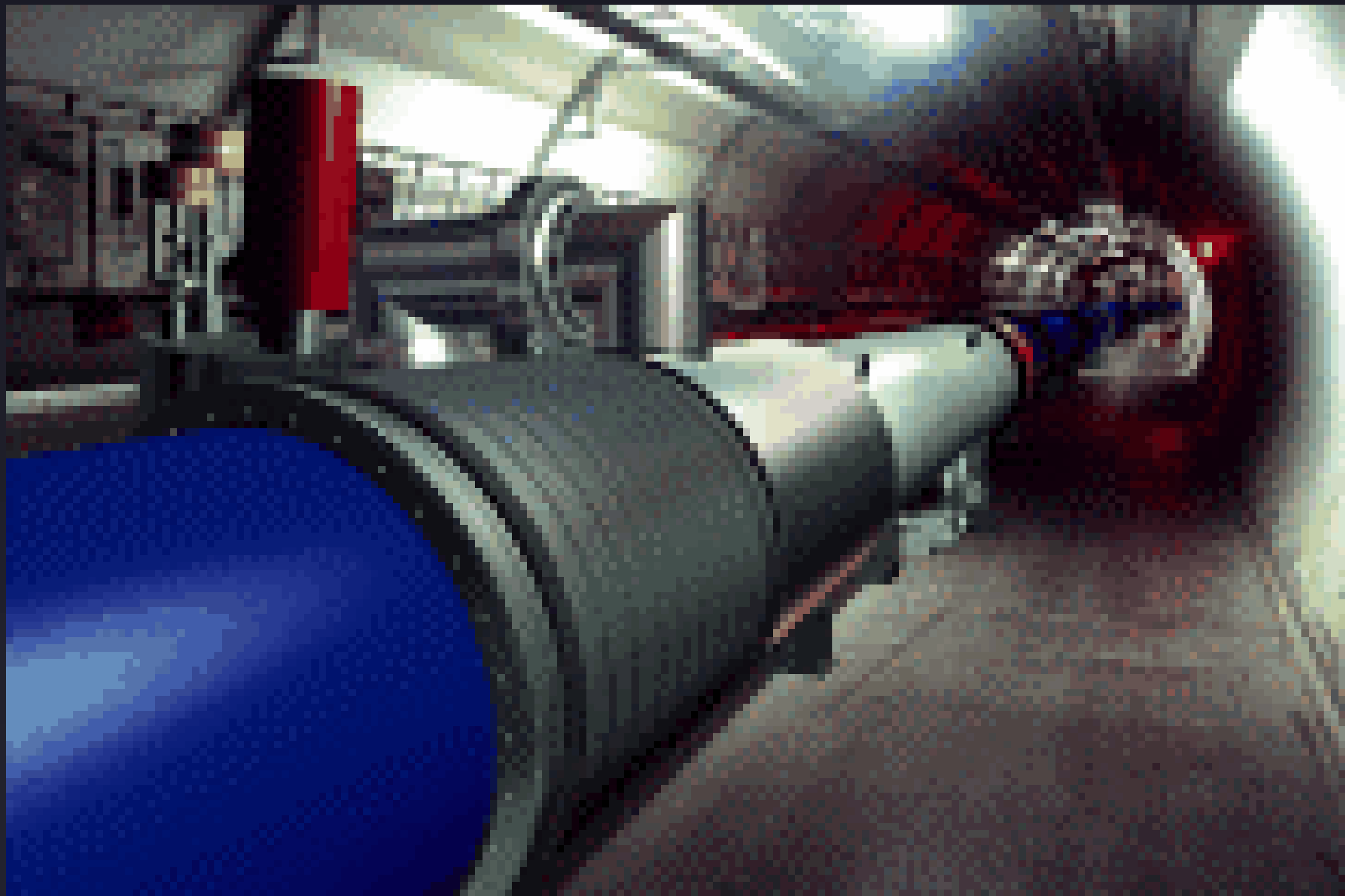
- There are strong physics motivations: LHC physics
- There are strong mathematical physics motivations: study of AdS/CFT duality

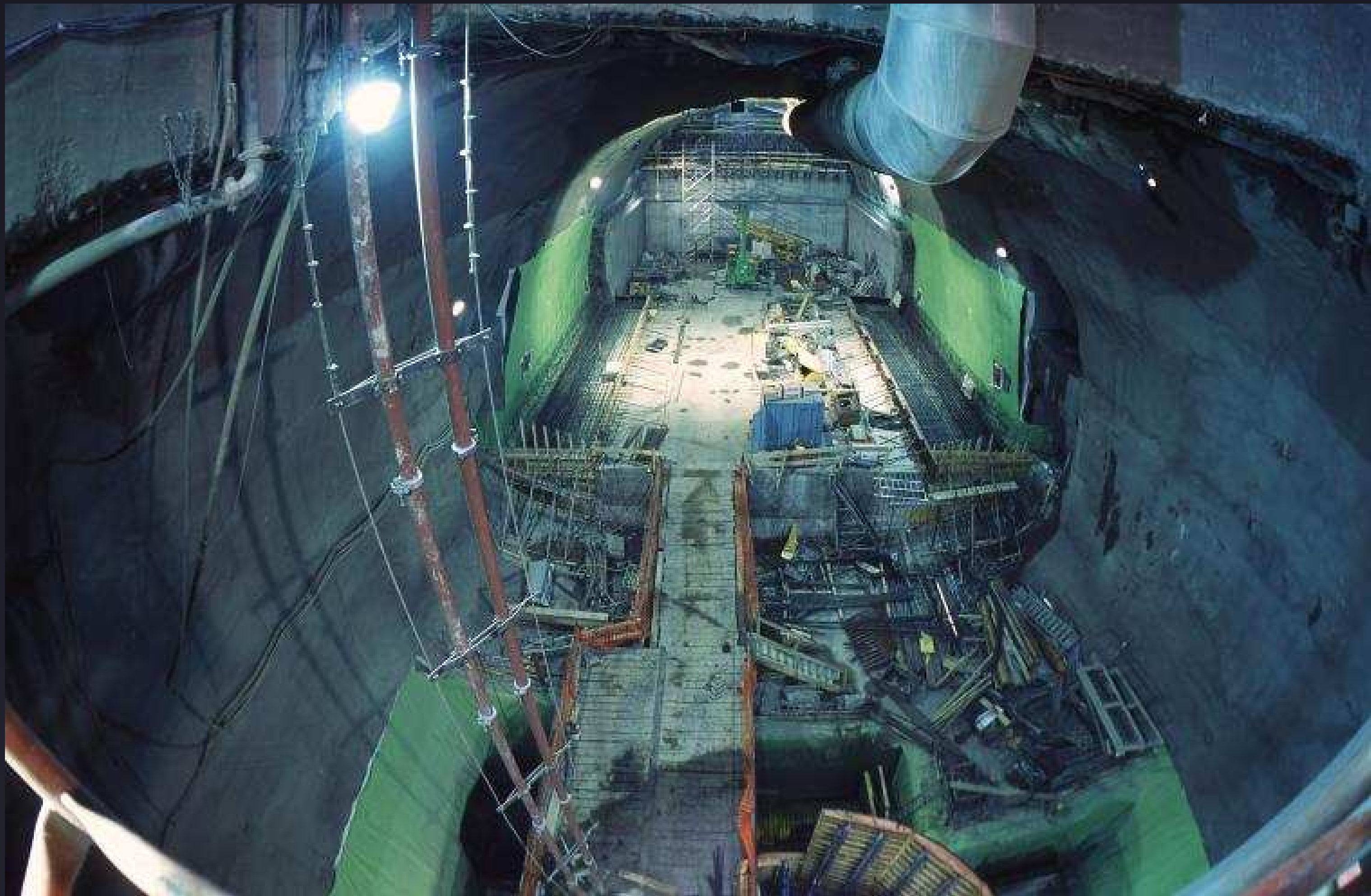
LHC Is Coming, LHC Is Coming!

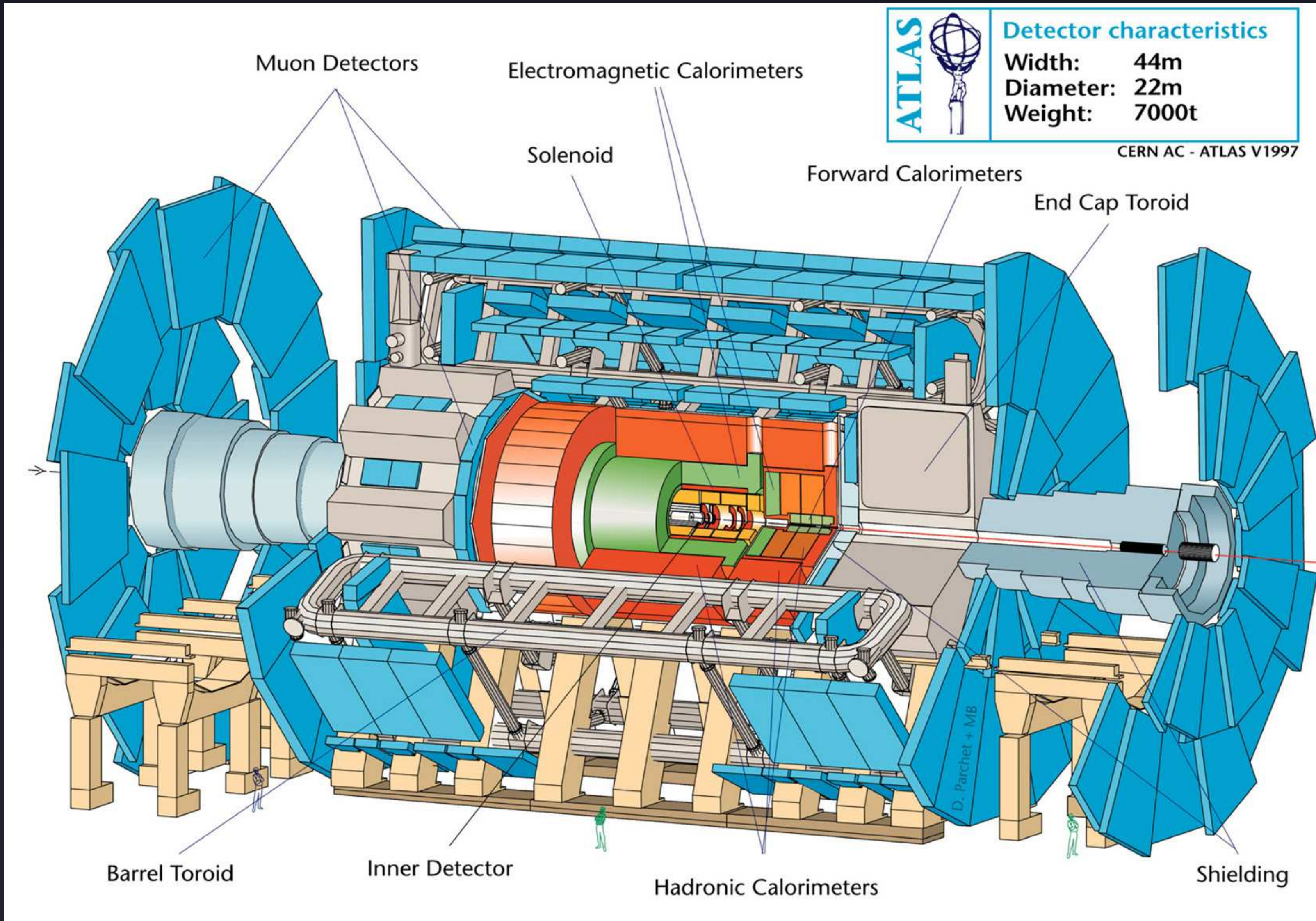


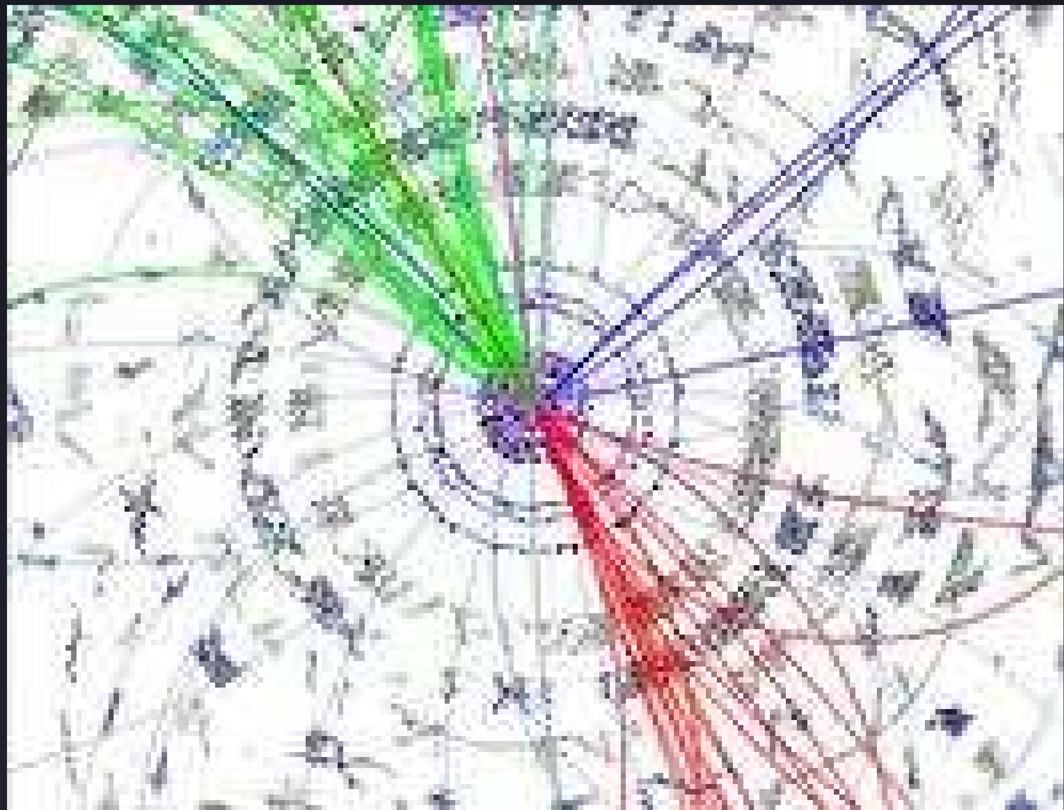

Overall view of the LHC experiments.
LHC - B
Point 8
CERN
ATLAS
Point 1
ALICE
Point 2
CMS
Point 5
SPS
TI 8
TI 2
LHC - B
ATLAS
ALICE
CMS
LEP/LHC
E540 - V10/09/97



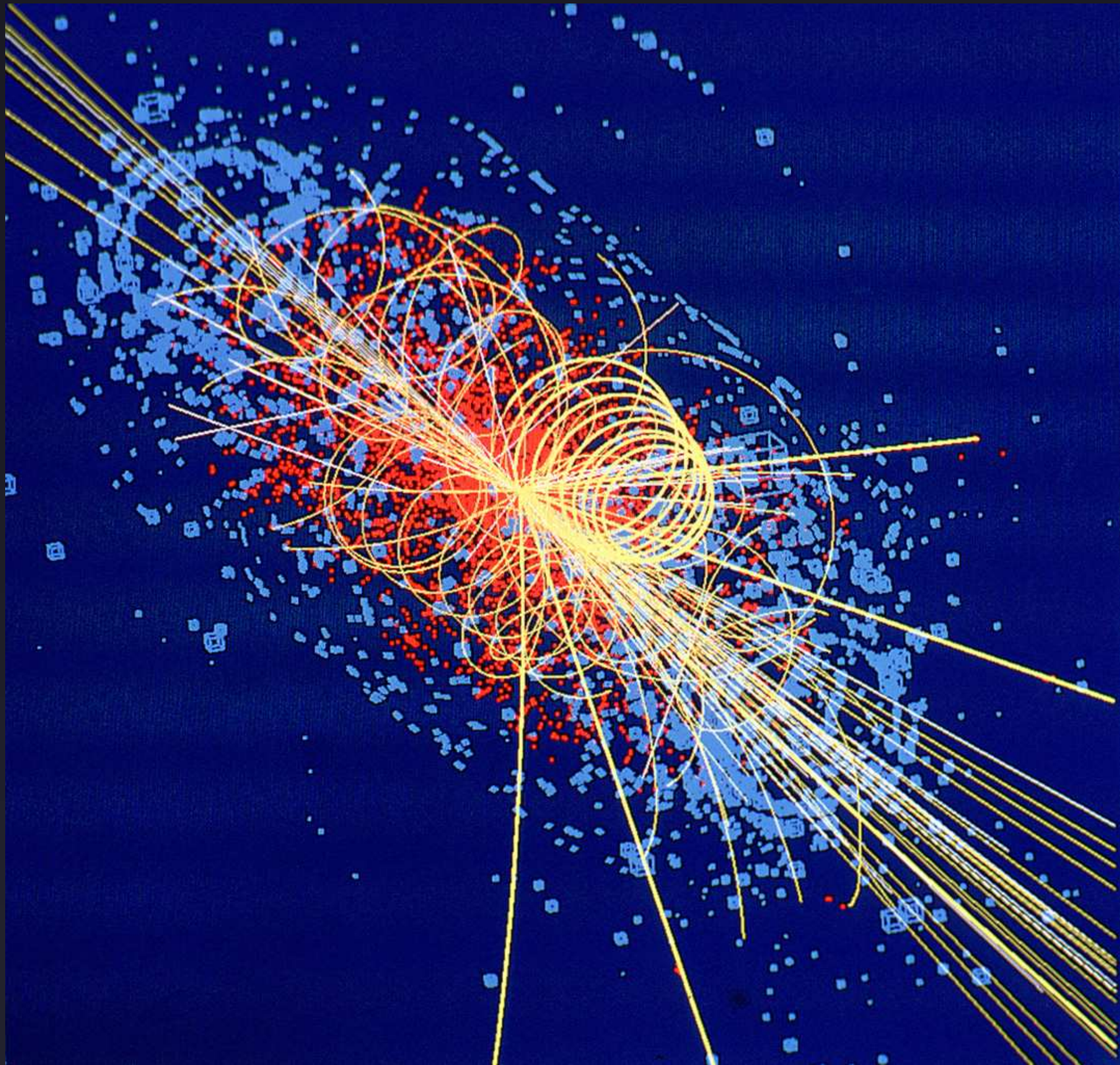




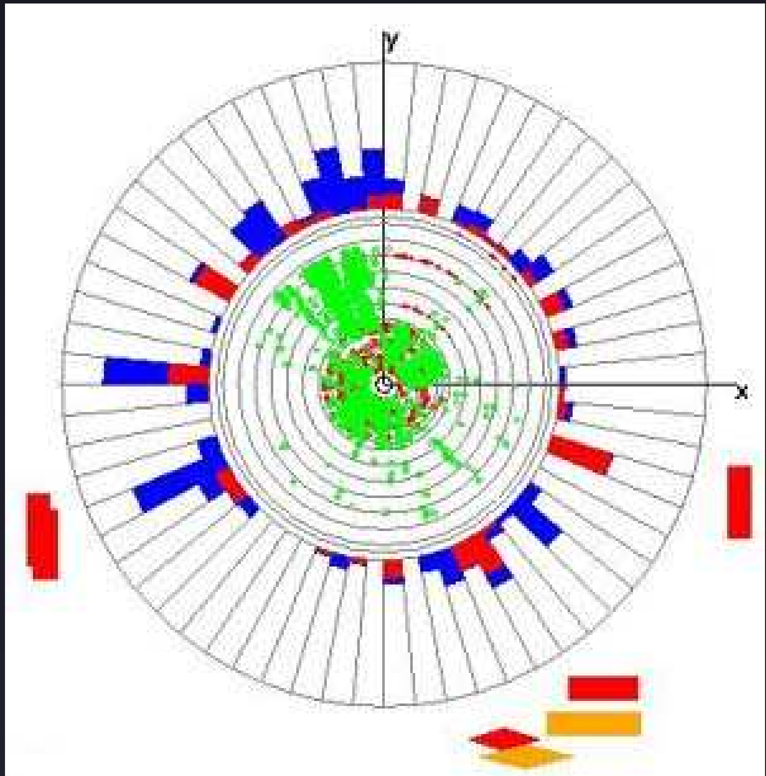

ATLAS
Detector characteristics
Width: 44m
Diameter: 22m
Weight: 7000t
CERN AC - ATLAS V1997
Muon Detectors
Electromagnetic Calorimeters
Solenoid
Forward Calorimeters
End Cap Toroid
Barrel Toroid
Inner Detector
Hadronic Calorimeters
Shielding
D. Parchet + MB

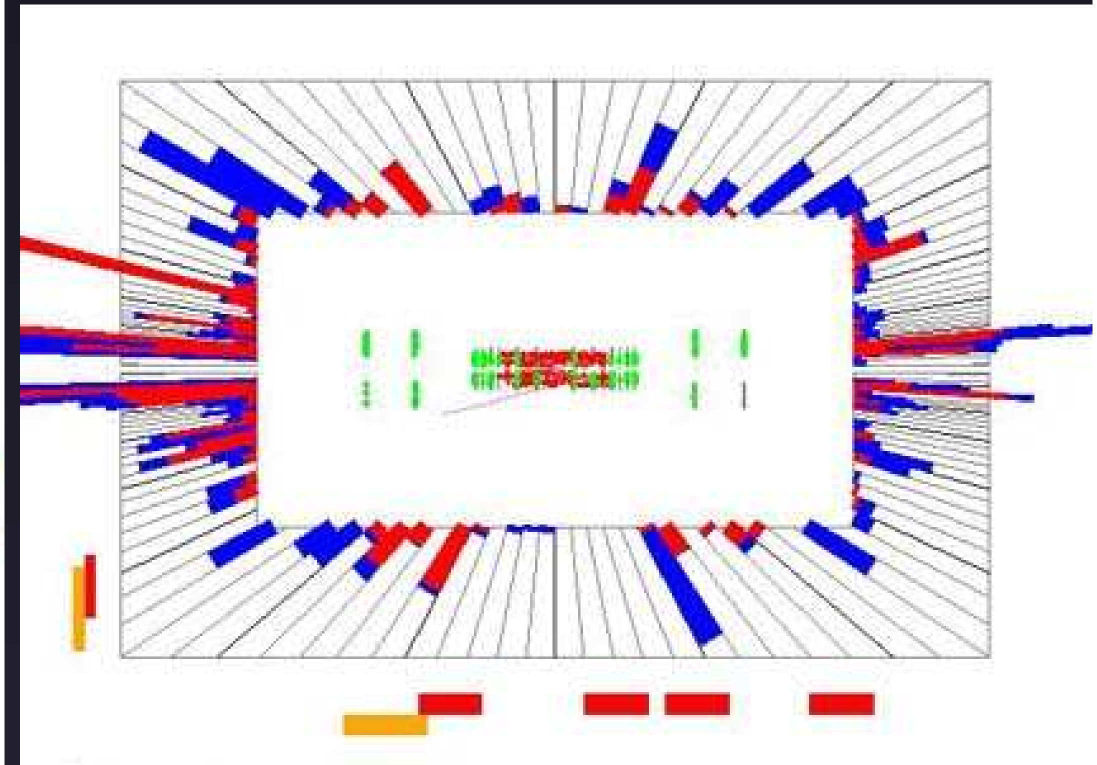


CDF event



CMS Higgs event simulation





D0 event

Event rates

Event production rates at $L=10^{33}\,cm^{-2}\,s^{-1}$ and statistics to tape

Process	Events/s	Evts on tape, 10 fb^{-1}
$W \to e\nu$	15	10^8
$Z \to ee$	1	10^7
$t\bar{t}$	1	10^6
gluinos, m=1 TeV	0.001	10^3
Higgs, m=130 GeV	0.02	10^4
Minimum bias	10^8	10^7
$b\bar{b} \to \mu X$	10^3	10^7
QCD jets p_T>150 GeV/c	10^2	10^7

assuming 1% of trigger bandwidth

⇒ statistical error negligible after few days!

⇒ dominated by systematic errors (detector understanding, luminosity, theory)

Hunting for New Physics

- Compare measurements to predictions — need to calculate signals
- To extract measurements, need to understand backgrounds — and they are often huge!
- Predicting backgrounds requires precision calculations of known Standard Model physics

Precision Perturbative QCD

- Predictions of signals, signals+ jets
- Predictions of backgrounds
- Measurement of luminosity
- Measurement of fundamental parameters (α_s, m_t)
- Measurement of electroweak parameters
- Extraction of parton distributions — ingredients in any theoretical prediction

Everything at a hadron collider involves QCD

From the Formal Side

- String theory
- Understanding the nature of quantum gravity
- Gauge theory at strong coupling
- Old idea of ‘t Hooft: large-N QCD at strong coupling should be a string theory

AdS/CFT Duality

Type IIB string theory on a special background, $AdS_5 \times S^5$ is dual to N=4 supersymmetric gauge theory on the boundary at spatial infinity

Maldacena (1997)

Same theory, seen through different variables

Special example of holography in gravitational theory: can be represented by degrees of freedom on the boundary

't Hooft; Susskind; Thorne

IIB string theory on $AdS_5 \times S^5$ $\qquad$ N= 4 SUSY gauge theory

$$\begin{aligned} g_s &\leftrightarrow \frac{1}{N} \\ R &\leftrightarrow g^2_{\mathrm{YM}} N \equiv \lambda \text{ 't Hooft coupling} \\ \text{strong} &\leftrightarrow \text{weak} \\ \text{weak} &\leftrightarrow \text{strong} \end{aligned}$$

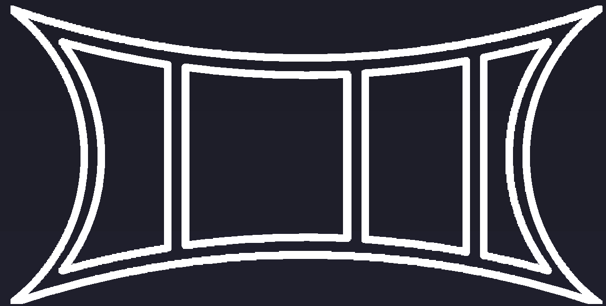
Same symmetries: SU(2,2|4) $\supset$ SO(4,2) conformal

$\supset$ SO(6) isometries of S^5= $SU(4)_R$

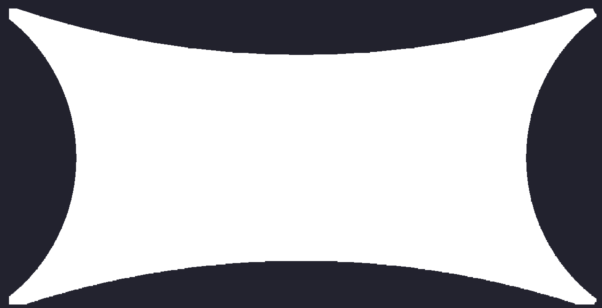
Planar Limit in Gauge Theories

't Hooft (1974)

- Consider large-N gauge theories, $g^2N \sim 1$, use double-line notation
- Planar diagrams dominate



- Sum over all diagrams $\rightarrow$ surface or string diagram



- Computations in the gauge theory good for testing duality
- Understanding string theory better
- Computing in low-energy QCD

Review of Gauge Theory

Take an SU(N) symmetry $\Psi(x) \to U(x)\Psi(x)$

and make it local. Introduce connection

$$G_\mu = \frac{1}{\sqrt{2}} T^a G^a_\mu \,, \qquad \mathrm{Tr}(T^a T^b) = \delta^{ab}$$

Structure constants $[T^a, T^b] = i\sqrt{2} f^{abc} T^c$.

Gluon transformation

$$G_\mu \to U(x) G_\mu U(x)^{-1} - \frac{i}{g}(\partial_\mu U(x)) U(x)^{-1} \,,$$

Covariant derivative

$$D_\mu \to U(x) D_\mu U^{-1}(x)$$

Construct a field-strength tensor

$$G_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]$$

$$= \frac{1}{\sqrt{2}} T^a G^a_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - ig[G_\mu, G_\nu]$$

in terms of components:

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu \,.$$

Squaring it gives us a kinetic energy term, and the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\,(G_{\mu\nu}G^{\mu\nu}) + \overline{\Psi}(i\not{D} - m)\Psi \,.$$

The $m = 0$ theory has scale invariance classically

Scattering

Scattering matrix element

$$ {}_{\text{out}}\langle p_1 p_2 \cdots \mid k_a k_b \rangle_{\text{in}} \equiv {}_{\text{asymp}}\langle p_1 p_2 \cdots \mid S \mid k_a k_b \rangle_{\text{asymp}} $$

Decompose it $S = 1 + iT$

Invariant matrix element M

$$ \langle p_1 p_2 \cdots \mid iT \mid k_a k_b \rangle = (2\pi)^4 \delta(k_a + k_b - P) i\mathcal{M}(k_a, k_b \to \{p_f\}) $$

Differential cross section

$$ d\sigma \text{ `='} \; \frac{1}{4E_a E_b |v_a - v_b|} \prod_f \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} $$

$$ \times (2\pi)^4 \delta^4(k_a + k_b - P) |\mathcal{M}(k_a, k_b \to \{p_f\})|^2 $$

Lorentz-invariant phase-space measure

$$\int \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i}\; F(p_i) = \int \frac{d^4p_i}{(2\pi)^4}\;(2\pi)\,\Theta(p_i^0)\delta(p_i^2)$$

$$\delta^{(+)}(p_i^2)$$

Compute invariant matrix element by crossing

$$\begin{aligned}\mathcal{M}(k_a,k_b \to \{p_f\}) &= \mathcal{M}(0 \to -k_a,-k_b,\{p_f\})\\ &= \mathcal{F}\,\langle \Omega \mid T\phi_a(x_a)\phi_b(x_b)\phi_1(x_1)\cdots \mid \Omega\rangle\end{aligned}$$

Functional Integration

Compute operator expectation values

$$\langle \mathcal{O} \rangle = \frac{\int [DG_\mu^a][D\cdots]\exp\{i\int \mathcal{L}\}\mathcal{O}}{\int [DG_\mu^a][D\cdots]\exp\{i\int \mathcal{L}\}} \,.$$

Gauge theories have redundant degrees of freedom

$$\frac{\int [DG_{\text{gauge}}][DG_\perp^a][D\cdots]\exp\{i\int \mathcal{L}\}\mathcal{O}}{\int [DG_{\text{gauge}}][DG_\perp^a][D\cdots]\exp\{i\int \mathcal{L}\}} = \frac{\infty}{\infty} \,.$$

Need to freeze the unphysical degrees of freedom 'gauge fixing'

Faddeev–Popov trick: functional delta function

$$1 = \int [D\alpha] \det\left(\frac{\delta\mathcal{F}(G^{[\alpha]})}{\delta\alpha}\right) \delta\left(\mathcal{F}(G^{[\alpha]})\right)$$

gauge transformation $G \to G^{[\alpha]}$

Change variables, the observable is now

$$\frac{\cancel{\left(\int [D\alpha]\right)} \int [DG][\cdots] \exp\{i \int \mathcal{L}\}\delta(\mathcal{F}(G))\,\mathcal{O}}{\cancel{\left(\int [D\alpha]\right)} \int [DG][\cdots] \exp\{i \int \mathcal{L}\}\delta(\mathcal{F}(G))}$$

Choose a covariant gauge-fixing condition

$$\mathcal{F}(G) = \partial \cdot G^a - w^a \,,$$

and use the $'$t Hooft trick to average over all ws

$$\langle f(w)\rangle_w = N(\xi) \int [Dw] \exp\left[-i \int \frac{w^2}{2\xi}\right] f(w) \,.$$

Our observable is now

$$\mathcal{L}_{\mathrm{gf}} = \frac{1}{2\xi} \left(\partial_\mu G^{a\mu}\right)^2 \,.$$

$$\frac{\int [DG][\cdots] \exp\{i \int \mathcal{L} - \mathcal{L}_{\mathrm{gf}}\} \det(\mathcal{F}(G))\mathcal{O}}{\int [DG][\cdots] \exp\{i \int \mathcal{L} - \mathcal{L}_{\mathrm{gf}}\} \det(\mathcal{F}(G))}$$

Lagrangian

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu\right)\left(\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu}\right)$$

$$\sum_f \overline{q}(i\partial\!\!\!/)q + \frac{g}{\sqrt{2}} G^a_\mu \sum_f \overline{q}\gamma^\mu T^a q$$

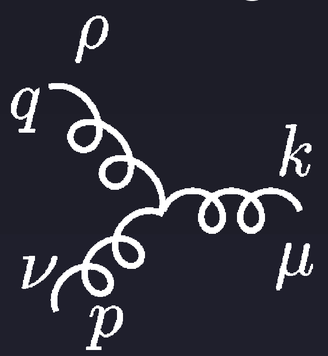
$$-\frac{g}{2} f^{abc}\left(\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu}\right) G^b_\mu G^c_\nu$$

$$-\frac{g^2}{4} f^{abe} f^{cde} G^{a\mu} G^{b\nu} G^c_\mu G^d_\nu$$

Feynman Rules

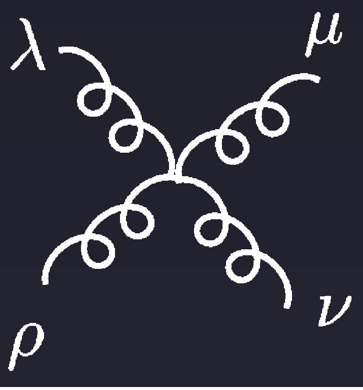
Propagator (like QED) $$-\frac{i}{k^2+i\epsilon}\delta^{ab}\left(g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

Three-gluon vertex (unlike QED)



$$g f^{abc}\left[g^{\mu\nu}(k-p)^\rho+g^{\nu\rho}(p-q)^\mu+g^{\rho\mu}(q-k)^\nu\right],$$

Four-gluon vertex (unlike QED)



$$\begin{aligned}-ig^2\Big[&f^{abe}f^{dce}(g^{\mu\rho}g^{\nu\lambda}-g^{\mu\lambda}g^{\nu\rho})\\&+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\lambda}-g^{\mu\lambda}g^{\rho\nu})\\&+f^{ace}f^{bde}(g^{\mu\nu}g^{\lambda\rho}-g^{\mu\rho}g^{\lambda\nu})\Big]\end{aligned}$$

The Functional Determinant

Delta function leads to functional determinant

Infinitesimal gauge transformation

$$\alpha\colon G^a_\mu \to G^{a[\alpha]}_\mu = G^a_\mu + \frac{1}{g} D_\mu \alpha^a$$

determinant

$$\det\left(\frac{\delta \mathcal{F}(G^{[\alpha]})}{\delta\alpha}\right) = \det \frac{1}{g}\partial \cdot D$$

In QED, this is independent of G $\Rightarrow$ over-all normalization of path integral, cancels out of Green functions

Represent the functional determinant as a path integral

$$\det \partial \cdot D = \int [Dc][D\overline{c}] \exp\left\{-i \int \overline{c}(\partial^\mu D_\mu)c\right\}$$

anticommuting scalars or ghosts

Propagator $\dfrac{i}{k^2 + i\epsilon}\delta^{ab}$

coupling to gauge bosons $g f^{abc} k^\mu$

Light-Cone Gauge

Only physical (transverse) degrees of freedom propagate

$$-\frac{i}{k^2 + i\epsilon}\delta^{ab}\left(g_{\mu\nu} - \frac{n^\mu k^\nu + n^\nu k^\mu}{n \cdot k}\right)$$

$$n^2 = 0$$

physical projector — two degrees of freedom

Color Decomposition

Standard Feynman rules ⇒ function of momenta, polarization vectors ε, and color indices

Color structure is predictable. Use representation

$$f^{abc} = -\frac{i}{\sqrt{2}}\,\mathrm{Tr}([T^a, T^b]T^c)$$

to represent each term as a product of traces,

and the Fierz identity

$$(T^a)_{i_1}{}^{\bar{\imath}_1}(T^a)_{i_2}{}^{\bar{\imath}_2} = \delta_{i_1}{}^{\bar{\imath}_2}\delta_{i_2}{}^{\bar{\imath}_1} - \frac{1}{N}\delta_{i_1}{}^{\bar{\imath}_1}\delta_{i_2}{}^{\bar{\imath}_2}$$

To unwind traces

$$f^{abc} f^{cde} = -\frac{1}{2} \operatorname{Tr}([T^a, T^b] T^c) \operatorname{Tr}(T^c [T^d, T^e])$$
$$= -\frac{1}{2} \operatorname{Tr}([T^a, T^b][T^d, T^e]) + \frac{1}{2N} \operatorname{Tr}([T^a, T^b]) \operatorname{Tr}([T^d, T^e])$$

Leads to tree-level representation in terms of single traces

$$\mathcal{A}_n^{\text{tree}}(\{k_i, \varepsilon_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \operatorname{Tr}(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}})$$
$$\times A_n^{\text{tree}}(k_{\sigma(1)}, \varepsilon_{\sigma(1)}; k_{\sigma(2)}, \varepsilon_{\sigma(2)}; k_{\sigma(n)}, \varepsilon_{\sigma(n)})$$

Color-ordered amplitude — function of momenta & polarizations alone; not Bose symmetric

Symmetry properties

- Cyclic symmetry

$$A_n^{\text{tree}}(1,\ldots,n) = A_n(2,\ldots,n,1)$$

- Reflection identity

$$A_n^{\text{tree}}(n,\ldots,1) = (-1)^n A_n(1,\ldots,n)$$

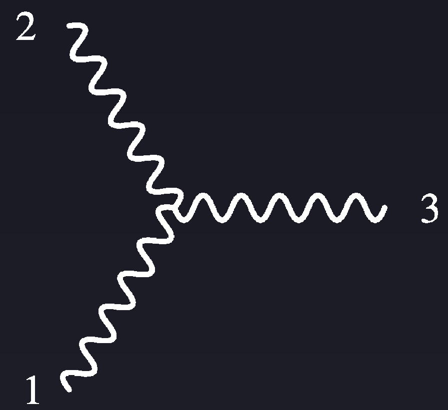
- Parity flips helicities

$$A_n^{\text{tree}}(1^{-\lambda_1},\ldots,n^{-\lambda_n}) = \left[A_n^{\text{tree}}(1^{\lambda_1},\ldots,n^{\lambda_n})\right]^\dagger$$

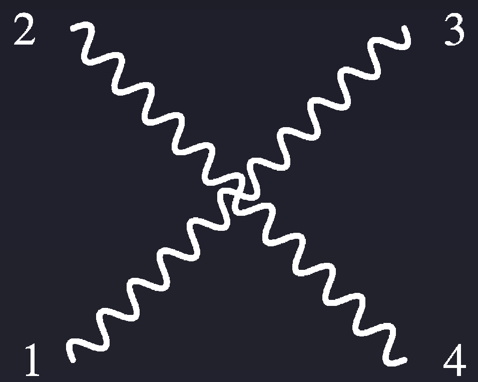
- Decoupling equation

$$A_{n+1}^{\text{tree}}(p,1,2,\ldots,n) + A_{n+1}^{\text{tree}}(1,p,2,\ldots,n) + A_{n+1}^{\text{tree}}(1,2,p,\ldots,n) + \cdots + A_{n+1}^{\text{tree}}(1,2,\ldots,p,n) = 0$$

Color-Ordered Feynman Rules



$$\frac{i}{\sqrt{2}}\left[\varepsilon_1 \cdot \varepsilon_2 \,(k_1 - k_2) \cdot \varepsilon_3 + \varepsilon_2 \cdot \varepsilon_3 \,(k_2 - k_3) \cdot \varepsilon_1 + \varepsilon_3 \cdot \varepsilon_1 \,(k_3 - k_1) \cdot \varepsilon_2\right]$$



$$i\varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_2 \cdot \varepsilon_4 - \frac{i}{2}\left(\varepsilon_1 \cdot \varepsilon_2 \, \varepsilon_3 \cdot \varepsilon_4 + \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_1\right)$$

Amplitudes

Functions of momenta k, polarization vectors ε for gluons;

momenta k, spinor wavefunctions u for fermions

Gauge invariance implies this is a redundant representation:

$\varepsilon \to k$: $A = 0$

Spinor Helicity

Spinor wavefunctions $|j^\pm\rangle \equiv u_\pm(k_j), \qquad \langle j^\pm| \equiv \overline{u_\pm}(k_j)\ .$

Introduce spinor products

$$\langle i\, j\rangle \equiv \langle i^-|j^+\rangle = \overline{u_-}(k_i)u_+(k_j)\ ,$$
$$[i\, j] \equiv \langle i^+|j^-\rangle = \overline{u_+}(k_i)u_-(k_j)$$

Explicit representation

$$u_+(k) = \begin{pmatrix} \sqrt{k_+} \\ \sqrt{k_-}e^{i\phi_k} \end{pmatrix}, \qquad u_-(k) = \begin{pmatrix} \sqrt{k_-}e^{-i\phi_k} \\ -\sqrt{k_+} \end{pmatrix}$$

where $$e^{\pm i\phi_k} = \frac{k^1 \pm ik^2}{\sqrt{k_+k_-}}, \qquad k_\pm = k^0 \pm k^3$$

We then obtain the explicit formulæ

$$\langle i\, j\rangle = \sqrt{k_{i-}k_{j+}}e^{i\phi_{k_i}} - \sqrt{k_{i+}k_{j-}}e^{i\phi_{k_j}} \;,$$

$$[i\, j] = \langle j\, i\rangle^* = \sqrt{k_{i+}k_{j-}}e^{-i\phi_{k_j}} - \sqrt{k_{i-}k_{j+}}e^{-i\phi_{k_i}} \qquad (k^0_{i,j} > 0)$$

otherwise $[j\, i] = \mathrm{sign}(k_i^0 k_j^0)\, \langle i\, j\rangle^*$

so that the identity $\langle i\, j\rangle\, [j\, i] = 2k_i \cdot k_j$ always holds

Introduce four-component representation

$$\frac{1}{\sqrt{2}}\begin{bmatrix} u_+(k) \\ u_+(k) \end{bmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} u_-(k) \\ -u_-(k) \end{bmatrix}$$

corresponding to γ matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

in order to define spinor strings

$$\langle i^\pm|\gamma^\mu|j^\pm\rangle \leftrightarrow \langle i^\pm|\sigma^\mu|j^\pm\rangle \qquad \sigma^\mu = (1, \sigma^i)$$

Properties of the Spinor Product

- Antisymmetry $\langle j\, i\rangle = -\langle i\, j\rangle\,, \qquad [j\, i] = -[i\, j]$
- Gordon identity $\langle i^{\pm}|\gamma^{\mu}|i^{\pm}\rangle = 2k_i^{\mu}$
- Charge conjugation $\left\langle i^{-}\right|\gamma^{\mu}\left|j^{-}\right\rangle = \left\langle j^{+}\right|\gamma^{\mu}\left|i^{+}\right\rangle\,,$
- Fierz identity $\left\langle i^{-}\right|\gamma^{\mu}\left|j^{-}\right\rangle\left\langle p^{+}\right|\gamma^{\mu}\left|q^{+}\right\rangle = 2\,\langle i\, q\rangle\,[p\, j]$
- Projector representation $|i^{\pm}\rangle\langle i^{\pm}| = \frac{1}{2}(1 \pm \gamma_5)\not{k}_i$
- Schouten identity $\langle i\, j\rangle\,\langle p\, q\rangle = \langle i\, q\rangle\,\langle p\, j\rangle + \langle i\, p\rangle\,\langle j\, q\rangle\ .$

Spinor-Helicity Representation for Gluons

Gauge bosons also have only ± physical polarizations

Elegant — and covariant — generalization of circular polarization

$$\varepsilon_\mu^+(k,q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2}\,\langle q\,k\rangle}, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2}\,[k\,q]}$$

$$q \cdot k \neq 0$$

Xu, Zhang, Chang (1984)

reference momentum q

$$k \cdot \varepsilon^\pm(k,q) = 0$$

Transverse

$$\varepsilon^+ \cdot \varepsilon^- = -1, \qquad \varepsilon^+ \cdot \varepsilon^+ = 0$$

Normalized

What is the significance of q?

$$\varepsilon_\mu^+(k,q') = \frac{\langle q'^- |\, \gamma_\mu \,| k^- \rangle}{\sqrt{2}\,\langle q'\,k\rangle} = \frac{\langle q'^- |\, \gamma_\mu \not{k} \,| q^+ \rangle}{\sqrt{2}\,\langle q'\,k\rangle\,\langle k\,q\rangle}$$

$$= -\frac{\langle q'^- |\, \not{k}\gamma_\mu \,| q^+ \rangle}{\sqrt{2}\,\langle q'\,k\rangle\,\langle k\,q\rangle} + \frac{\sqrt{2}\,\langle q\,q'\rangle}{\langle q'\,k\rangle\,\langle k\,q\rangle}\,k_\mu$$

$$= \varepsilon_\mu^+(k,q) + \frac{\sqrt{2}\,\langle q\,q'\rangle}{\langle q'\,k\rangle\,\langle k\,q\rangle}\,k_\mu$$

Properties of the Spinor-Helicity Basis

Physical-state projector

$$\sum_{\sigma=\pm} \varepsilon^{\sigma}_{\mu}(k,q)\varepsilon^{\sigma *}_{\nu}(k,q) = \sum_{\sigma=\pm} \varepsilon^{\sigma}_{\mu}(k,q)\varepsilon^{-\sigma}_{\nu}(k,q) = -g_{\mu\nu} + \frac{q_\mu k_\nu + k_\mu q_\nu}{q \cdot k}$$

Simplifications

$$q \cdot \varepsilon^{\pm}(k,q) = 0\,,$$

$$\varepsilon^{+}(k_1,q) \cdot \varepsilon^{+}(k_2,q) = \varepsilon^{-}(k_1,q) \cdot \varepsilon^{-}(k_2,q) = 0\,,$$

$$\varepsilon^{+}(k_1,q) \cdot \varepsilon^{-}(k_2,k_1) = 0$$