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Combinatorics of Hard Particles on Planar Graphs

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We revisit the problem of hard particles on planar random tetravalent graphs in view of recent combinatorial techniques relating planar diagrams to decorated trees. We show how to recover the two-matrix model solution to this problem in this purely combinatorial language.

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1. Introduction

Recent progress has been made in the enumeration of various types of planar maps, using bijective methods relating maps to decorated trees [1-4]. These techniques are different in nature from the original combinatorial approach of Tutte [5-8] and are much easier to generalize. In particular, it was shown in Ref. [9] how to extend these techniques so as to recover in a purely combinatorial way the general one-matrix model solution for the enumeration of planar maps with arbitrary valencies [10-12].

After these encouraging results, it is natural to turn to more involved problems of decorated map enumeration, and try to recover in a purely combinatorial way the results of more involved matrix models describing statistical "matter" models defined on random graphs[4] (two-dimensional quantum gravity). It is worth mentioning that there is an apparent obstruction to this type of generalization, as the local interactions on the graphs become non-local on the corresponding trees. On the other hand, the known matrix model solutions strongly suggest by their algebraic form that a simple combinatorial approach to these problems should exist.

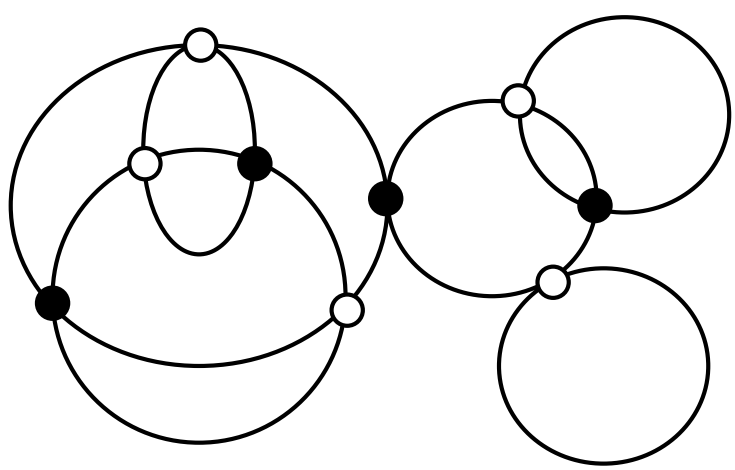

Fig. 1: A sample configuration of hard particles on a planar tetravalent graph. Empty (resp. occupied) vertices are indicated by white (resp. black) circles. The particle exclusion rule imposes that no two occupied vertices are adjacent to the same edge.

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[4] We use the denomination "graph" throughout this paper to denote maps, i.e. fatgraphs in the physics language.

The scope of this paper is to analyze a particular two-matrix model case, namely that describing *hard particles on tetravalent planar graphs.* The configurations of this model are made of arbitrary tetravalent planar maps with empty or occupied vertices, satisfying the hard-particle exclusion rule that no two occupied vertices may be adjacent to the same edge. Such a configuration is represented in Fig. 1 for illustration. This model was solved in Ref. [13] by use of a two-matrix model representation. We show here how to recover the solution of Ref. [13] for the generating function of these objects *in a purely combinatorial manner*, by establishing suitable bijections between configurations of the model and trees with particles. The techniques used here are directly borrowed from those of Ref. [9], generalizing that of Ref. [1].

The paper is organized as follows: In Sect. 2, we recall the results of Ref. [13] and introduce various types of objects for which we give closed formulas of the generating functions. Of particular interest are *two-leg* and *four-leg* diagrams as defined below, whose enumeration allows in particular to obtain the generating function for all the configurations of hard particles on tetravalent planar maps. In Sect. 3, we describe in detail the cutting procedure transforming two-leg diagrams into decorated trees which we characterize precisely. The correspondence is proved to be one-to-one by studying the inverse gluing procedure. We finally use this bijection to enumerate the two-leg diagrams at hand. Sect. 4 is devoted to the more involved case of four-leg diagrams, the enumeration of which is performed through several steps organized in several subsections. Sect. 5 discusses an interesting duality property between empty and occupied vertices of the model. We gather a few concluding remarks in Sect. 6. The combinatorial counterpart of the more general two-matrix models describing bipartite graphs is briefly discussed in Appendix A.

2. Results

Let us first recall the two-matrix model solution of Ref. [13] to the hard-particle model on random planar tetravalent graphs. The planar free energy F, i.e. the generating function for configurations of hard particles on connected planar tetravalent (fat)graphs, was derived and expressed in terms of two functions R and V determined by the following system of equations

$$\begin{aligned} R &= 3V^2 + 9zRV^2 \\ V &= \theta + R + 3zR^2 + 3zV^3 \end{aligned} \tag{2.1}$$