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Fully Packed $O(n = 1)$ Model on Random Eulerian Triangulations

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We introduce a matrix model describing the fully packed $O(n)$ model on random Eulerian triangulations (i.e. triangulations with all vertices of even valency). For $n = 1$ the model is mapped onto a particular gravitational 6-vertex model with central charge $c = 1$, hence displaying the expected shift $c \to c + 1$ when going from ordinary random triangulations to Eulerian ones. The case of arbitrary n is also discussed.

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1. Introduction

Loop models are a general class of statistical problems where the elementary statistical objects are loops drawn on a two-dimensional lattice (for a review see for instance [1]). Loop models arise naturally in the high temperature expansion of lattice statistical models but also as the description of one-dimensional lattice objects, like self-avoiding polygons. An interesting limiting case in that of *fully packed* loop models where the set of loops is required to cover the entire lattice, without vacancies. By assigning an activity n per loop, such models can be thought of as the zero temperature limit of $O(n)$ models, more simply referred to as fully packed $O(n)$ models. In the following, we shall restrict our discussion to triangular lattices, be it the flat regular triangular lattice or random triangulations. The loops are understood here as being self-avoiding and non-intersecting. Furthermore, all the *triangles* of the lattice are assumed to be visited by a loop. For $n = 1$, the fully packed $O(n)$ model describes for instance configurations of dimers drawn on the dual of the (regular or random) lattice, the dimers occupying the edges dual to those not traversed by a loop (each triangle has exactly one such edge). The model describes equivalently the ground states of an anti-ferromagnetic Ising model. In the limit $n \to 0$, the model describes Hamiltonian cycles on the lattice, which are the compact conformations of a polymer ring. At $n = 2$, the model defined on the regular triangular lattice is equivalent to a three-coloring problem, namely the problem of coloring the edges of the lattice with three colors in such a way that the three edges adjacent to any triangle are of different color [2]. Alternatively, it describes the possible folded states of the regular triangular lattice onto itself [3]. On random triangulations, this equivalence with three-coloring and folding problems remains valid only for a restricted class of triangulations [4,5]. We shall return to this below.

A remarkable prediction for the fully packed $O(n)$ model on the *regular* two-dimensional triangular lattice is that it is not in the same universality class as the usual low temperature dense phase of the $O(n)$ model in which vacancies are allowed. If we denote by $c_d(n)$ the central charge of the dense phase fixed point, that of the fully packed phase is expected to present a shift by one, namely $c_f(n) = c_d(n) + 1$. This remarkable fact was first conjectured in [6] on the basis of transfer matrix studies, and then confirmed in [7] on the grounds of a nested Bethe Ansatz solution. As explained in [6], the shift by one in the central charge can be given a nice heuristic interpretation. Indeed, by marking those edges of the triangular lattice which are traversed by the loops, any set of fully packed loops can be viewed as a two-dimensional picture of a three-dimensional piling of cubes, whose surface defines a one-dimensional SOS height variable on the triangular lattice. This SOS degree of freedom, which emerges only if the loops are fully packed, is responsible for the shift in the central charge.

Unfortunately, the above geometrical picture breaks down when going to ordinary *random* triangulations. Indeed, the local rules which would define a height variable out of the loops in general lead to frustrations. In this case, the SOS variable cannot be properly defined anymore and the shift in the central charge does not occur. In other words, for ordinary random triangulations, the fully packed $O(n)$ model is again described by the dense phase fixed point.

Recently, in [8], it was conjectured that the shift by one in the central charge can be reinstated in the random case if the triangulations are restricted to the class of so-called *Eulerian* triangulations. An Eulerian triangulation is a triangulation where an *even* number of triangles meet at any given vertex. Eulerian triangulations arise naturally in the context of folding problems involving random lattices. Indeed, in genus zero, Eulerian triangulations are the vertex-tricolorable triangulations, namely those for which each vertex can be assigned one of three colors in such a way that any two neighbors have distinct colors [9]. This latter condition ensures the possibility of folding the triangulation in two dimensions, as explained in [8]. For Eulerian triangulations, the different possible folded states can then be mapped onto edge-three-colored states, or equivalently onto configurations of fully packed loops with a weight $n = 2$ per loop. The fully packed $O(n = 2)$ model on random Eulerian triangulations thus provides a natural description of the folding of fluid membranes. As explained in [8], for Eulerian triangulations, the construction of the SOS variable again becomes possible without frustrations, and a shift by one in the central charge should then be observed. This phenomenon was moreover confirmed numerically in [8] in the limit $n \to 0$ by a direct counting of Hamiltonian cycles on random Eulerian triangulations with up to 40 triangles. The string susceptibility exponent was found to be compatible with the value $\gamma = (-1 - \sqrt{13})/6$ expected for a central charge $c = -1$, instead of the value $\gamma = -1$ found for ordinary triangulations, corresponding to $c_d(n = 0) = -2$ [10-12].

The purpose of this paper is to confirm this shift phenomenon in the case $n = 1$ by showing that the fully packed $O(n = 1)$ model has $c = 1$ when defined on random Eulerian triangulations, as opposed to the usual result $c = 0$ for the (dense or fully packed) $O(n = 1)$ model on ordinary random triangulations [13,14].

The paper is organized as follows: in Sect.2, we present a matrix model formulation for the $O(n=1)$ model on random Eulerian triangulations. This model is shown in Sect.3 to be equivalent to a particular gravitational 6-vertex model, described at criticality by a $c=1$ conformal field theory coupled to gravity. In Sect.4, we extend our matrix model formulation to arbitrary values of n. We focus in particular on the limit $n \to 0$ describing Hamiltonian cycles. A few concluding remarks are gathered in Sect.5.

2. Matrix Model for the Fully Packed $O(n=1)$ Model on Random Eulerian triangulations

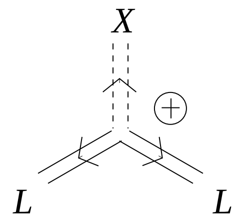
As mentioned in the introduction, an Eulerian triangulation is a closed random triangulation of arbitrary genus, for which each vertex has an *even* number of adjacent triangles. Alternatively, an Eulerian triangulation can be defined as a triangulation where one may associate a sign $+$ or $-$ to each triangle in such a way that any two adjacent triangles have opposite signs.

Here we consider random Eulerian triangulations equipped with fully packed self-avoiding loops of adjacent triangles, i.e., triangulations covered by a set of loops such that each triangle belongs to exactly one loop. In this section and in Sect.3, we address the case of an activity $n=1$ per loop; the case of general n (including the Hamiltonian cycle limit $n \to 0$) will be discussed in Sect.4. As opposed to the usual $O(n)$ model coupled to gravity [13], we insist on imposing here the two crucial restrictions: (1) the loops must be fully packed (no vacancies) and (2) the triangulations must be Eulerian. Only in this particular case is the model expected to be described by a different universality class than the usual $O(n)$ model, with the $c \to c+1$ shift in the central charge.

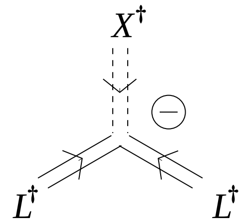
Our model is best expressed in the dual picture as that of a three-coordinate (ϕ^3) lattice with bi-colored vertices (corresponding to the above-mentioned $+$ and $-$ signs), equipped with loops visiting all *vertices*. In particular, the signs of the vertices visited by a given loop alternate along the loop. The corresponding graphs are the Feynman diagrams of the following simple matrix model. We consider a pair of complex $N \times N$ matrices (X, L), where (X, L) will correspond to the $+$ vertices of the dual graph, whereas $(X^\dagger, L^\dagger)$ will correspond to the $-$ vertices. As usual, Feynman diagrams for such objects are obtained by joining pairs of double-edges, each pair corresponding to a matrix element

M_{ij} (resp. $M^\dagger_{ij}$), where the two lines carry the matrix indices i and j, and the double-edge is oriented away from (resp. towards) a vertex. We need the following interactions

$\mathrm{Tr}(XL^2)$:



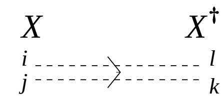
$\mathrm{Tr}(X^\dagger(L^\dagger)^2)$:



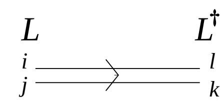
(2.1)

and propagators

$\langle (X)_{ij}(X^\dagger)_{kl}\rangle \;=\; \frac{1}{N}\delta_{il}\delta_{jk} \;=$



$\langle (L)_{ij}(L^\dagger)_{kl}\rangle \;=\; \frac{1}{N}\delta_{il}\delta_{jk} =$



(2.2)

The latter form the edges of the ϕ^3-diagram, and are of two types: those visited by the loops $\langle LL^\dagger\rangle$, and those not visited by loops $\langle XX^\dagger\rangle$. The mixed nature $(MM^\dagger)$ of the propagators guarantees the Eulerian structure. The interaction terms (2.1) describe the three-coordinate vertices with the corresponding signs and exactly two occupied (L) and one empty (X) incoming edges. The diagrams with vertices (2.1) and propagators (2.2) arise in the Feynman expansion of the following four-matrix integral

$$\begin{aligned} Z(g;N) &= \int dX dX^\dagger dL dL^\dagger e^{-N\mathrm{Tr}(V(X,L))}, \\ V(X,L) &= XX^\dagger + LL^\dagger - g(XL^2 + X^\dagger (L^\dagger)^2), \end{aligned} \tag{2.3}$$

where the standard measure over $N\times N$ complex matrices reads $dMdM^\dagger \propto \prod_{1\le i,j\le N} d\mathrm{Re}(M_{ij})\, d\mathrm{Im}(M_{ij})$, and is normalized in such a way that $Z(g=0;N)=1$.

As usual, the free energy $f(g;N) = \mathrm{Log}\, Z(g;N) = \sum_{h\ge 0} N^{2-2h} f_h(g)$ is expressed as a sum over the contributions of the connected Eulerian triangulations of genus h. The genus zero limit is therefore obtained by taking $N\to\infty$. We have represented in fig.1 an example of a connected genus zero diagram with eight triangles and two loops.

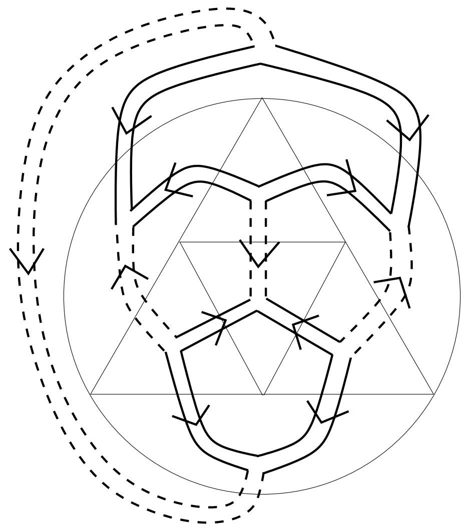


Fig. 1: A typical genus zero configuration involving two loops (solid double-lines) fully packed on an Eulerian triangulation made of 8 triangles. We have represented by dashed double-lines the un-occupied edges of the dual lattice. The orientation of the double-lines reflects the Eulerian constraint (all arrows point towards triangles with + signs, and away from those with − signs, hence the orientation alternates along each loop).

3. Mapping to a critical point of the gravitational 6-vertex model

The integral (2.3) is Gaussian in all matrices. Let us first integrate over X by setting $X = \frac{1}{\sqrt{2}}(P + iQ)$, where P and Q are two $N \times N$ Hermitian matrices, and the measure is transformed into $dXdX^\dagger \propto dPdQ$, where dP and dQ stand for the standard Haar measure for Hermitian matrices, normalized in such a way that $Z(g = 0; N) = 1$. Similarly, we set $L = \frac{1}{\sqrt{2}}(A + iB)$, with A and B Hermitian, so that the potential becomes

$$\begin{aligned}
\mathrm{Tr}(V(X,L)) &= \mathrm{Tr}\bigg(\frac{1}{2}(A^2+B^2)+\frac{1}{2}(P^2+Q^2) \\
&\qquad - \frac{g}{\sqrt{2}}\big(P(A^2-B^2)-Q(AB+BA)\big)\bigg) \\
&= \mathrm{Tr}\bigg(\frac{1}{2}(A^2+B^2) \\
&\qquad + \frac{1}{2}(P-\frac{g}{\sqrt{2}}(A^2-B^2))^2-\frac{g^2}{4}(A^2-B^2)^2 \\
&\qquad + \frac{1}{2}(Q+\frac{g}{\sqrt{2}}(AB+BA))^2-\frac{g^2}{4}(AB+BA)^2\bigg).
\end{aligned} \tag{3.1}$$

Performing the Gaussian integrals over the shifted matrices P and Q, we are left with

$$\begin{aligned} Z(g;N) &= \int dAdB e^{-N\mathrm{Tr}W(A,B)}, \\ W(A,B) &= \frac{1}{2}(A^2+B^2) - \frac{g^2}{4}(A^4+B^4) - \frac{g^2}{2}(AB)^2. \end{aligned} \tag{3.2}$$

This is nothing but the partition function of the gravitational 6-vertex model solved in the large-N limit by Kazakov and Zinn-Justin [16] (with the parameters $\alpha = \beta = g^2$), whose critical point $g = g_c = 1/(2\sqrt{\pi})$ corresponds to a compactified boson with radius $R = 1/(2\sqrt{2})$.

The crucial outcome of this equivalence is that the conformal field theory underlying our problem has central charge $c = 1$, which precisely corresponds to a shift by one of the central charge $c = 0$ of the ordinary (fully packed or not) dense phase of the $O(n = 1)$ model on arbitrary (i.e. non-necessarily Eulerian) triangulations. This proves in the particular case $n = 1$ the claim of [8] that the central charge increases by one when the fully packed model is defined on random Eulerian triangulations as opposed to ordinary random triangulations.

Two remarks are in order. From the critical value $g_c = 1/(2\sqrt{\pi})$, we deduce that the number of genus zero Eulerian triangulations with $2T$ triangles and equipped with fully packed loops behaves for large T as

$$z_{2T}^{E}(n=1) \sim (4\pi)^T, \tag{3.3}$$

to be compared with $z_{2T}^{E} \sim 8^T$ for pure Eulerian triangulations [5], to $z_{2T}^{E}(n = 0) \sim (10.10...)^T$ for Eulerian triangulations equipped with Hamiltonian cycles [8], and finally to $z_{2T}(n = 1) \sim (24)^T$ for ordinary random triangulations equipped with fully packed loops.

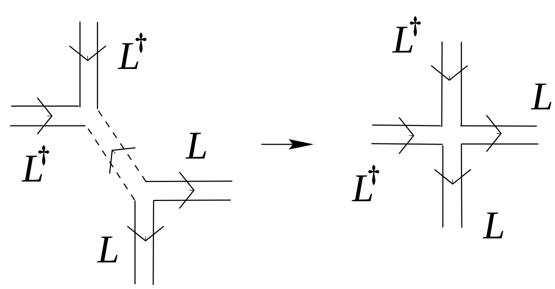


Fig. 2: Shrinking the $\langle XX^\dagger \rangle$ propagators (dashed double-edges) produces a particular 4-valent vertex of the 6-vertex model.

Secondly, let us note that the 6-vertex correspondence is best seen by shrinking the $\langle XX^\dagger\rangle$ propagators so as to form four-valent vertices with oriented edges (cf. fig.2).

4. The $O(n)$ model on random Eulerian triangulations

4.1. Matrix model for arbitrary n

The fully packed $O(n)$ model must incorporate a weight n per loop, obtained for integer n by replicating n times the $N\times N$ matrix L of (2.3). Beside the complex matrix X, we therefore introduce n complex matrices L_α, $\alpha=1,2,...,n$, with the following vertex interactions

$$\mathrm{Tr}(XL_\alpha^2) \quad \text{and} \quad \mathrm{Tr}(X^\dagger(L_\alpha^\dagger)^2), \quad \text{for } \alpha=1,2,...,n, \tag{4.1}$$

and propagators

$$\begin{aligned} \langle (X)_{ij}(X^\dagger)_{kl}\rangle &= \frac{1}{N}\delta_{il}\delta_{jk}, \\ \langle (L_\alpha)_{ij}(L_\beta^\dagger)_{kl}\rangle &= \frac{1}{N}\delta_{\alpha\beta}\delta_{il}\delta_{jk}. \end{aligned} \tag{4.2}$$

This allows only L_α-matrices of the *same* color α to form loops. The corresponding matrix model partition function reads

$$\begin{aligned} Z(n,g;N) &= \int dX dX^\dagger \prod_{\alpha=1}^{n} dL_\alpha dL_\alpha^\dagger \; e^{-N\mathrm{Tr}(V(X,L_1,...,L_n))}, \\ V(X,L_1,...,L_n) &= XX^\dagger + \sum_{\alpha=1}^{n} L_\alpha L_\alpha^\dagger \\ &\quad - g\big(X\sum_{\alpha=1}^{n}(L_\alpha)^2 + X^\dagger\sum_{\alpha=1}^{n}(L_\alpha^\dagger)^2\big). \end{aligned} \tag{4.3}$$

Here again, the integration measure is normalized in such a way that $Z(n,g=0;N)=1$, and the net result in the perturbative expansion of $Z(n,g;N)$ is to attach a weight n per loop of L-matrices.

Contrary to the $n=1$ case, let us first integrate over the n matrices $L_\alpha=(A_\alpha+iB_\alpha)/\sqrt{2}$, where A_α and B_α, $\alpha=1,2,...,n$ are n Hermitian matrices of size $N\times N$. To do this integration, we note that the potential V of (4.3) takes the form

$$\begin{aligned} V(X,L_1,...,L_n) &= XX^\dagger + \frac{1}{2}\sum_\alpha (A_\alpha^2 + B_\alpha^2) \\ &- \frac{g}{2} X \sum_\alpha \left(A_\alpha^2 - B_\alpha^2 + i(A_\alpha B_\alpha + B_\alpha A_\alpha)\right) \\ &- \frac{g}{2} X^\dagger \sum_\alpha \left(A_\alpha^2 - B_\alpha^2 - i(A_\alpha B_\alpha + B_\alpha A_\alpha)\right) \\ &= XX^\dagger + \frac{1}{2}\sum_{\alpha=1}^{n} (A_\alpha, B_\alpha)\mathbf{Q}\begin{pmatrix} A_\alpha \\ B_\alpha \end{pmatrix}, \end{aligned} \tag{4.4}$$

where the quadratic form $\mathbf{Q}$ reads

$$\mathbf{Q} = (I\otimes I)I_2 - \frac{g}{2}\left((X\otimes I + I\otimes X)K + (X^\dagger \otimes I + I\otimes X^\dagger)\bar{K}\right). \tag{4.5}$$

Here we have denoted by I (resp. I_2) the $N\times N$ (resp. 2×2) identity matrix, and K, $\bar{K}$ are the following 2×2 matrices

$$K = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad \bar{K} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}. \tag{4.6}$$

Performing the Gaussian integration over the A's and B's, we finally get

$$Z(n,g;N) = \int dX dX^\dagger \det(\mathbf{Q})^{-n/2} e^{-N\mathrm{Tr}(XX^\dagger)}. \tag{4.7}$$

We may further expand

$$\begin{aligned} \det(\mathbf{Q})^{-n/2} &= \exp[-\frac{n}{2}\mathrm{Tr}\ \mathrm{Log}\ \mathbf{Q}] \\ &= \exp[n\sum_{m=1}^{\infty}\frac{g^{2m}}{2m}\mathrm{Tr}\big(\big[(X\otimes I + I\otimes X)(X^\dagger\otimes I + I\otimes X^\dagger)\big]^m\big)], \end{aligned} \tag{4.8}$$

where we have used the fact that $K^2 = \bar{K}^2 = 0$, hence only the terms of the form $\mathrm{Tr}[(K\bar{K})^m] = 2^{2m}$ or $\mathrm{Tr}[(\bar{K}K)^m] = 2^{2m}$ contribute, which cancel the $1/2^{2m+1}$ pre-factor. Alternatively, we may rephrase the result (4.7)-(4.8) into

$$Z(n,g;N) = \int dXdX^\dagger e^{-N\mathrm{Tr}(XX^\dagger)} \det\left(I\otimes I - g^2(X\otimes I + I\otimes X)(X^\dagger\otimes I + I\otimes X^\dagger)\right)^{-n/2}. \tag{4.9}$$

This reduces the $O(n)$ model partition function on Eulerian triangulations to a Gaussian complex one-matrix model with some specific integrand. Note that in eqn. (4.9) the parameter n can now take any real value.

4.2. The $n \to 0$ limit: Gaussian matrix model

The result (4.7) yields in particular in the limit $n \to 0$ the partition function for Hamiltonian cycles on Eulerian triangulations, expressed as a complex one-matrix integral

$$\begin{aligned} Z_H(g;N) &= \partial_n Z(n,g;N)\big|_{n=0} \\ &= \sum_{m=1}^{\infty} \frac{g^{2m}}{2m} \langle \mathrm{Tr}\big(\big[(X\otimes I + I\otimes X)(X^\dagger \otimes I + I \otimes X^\dagger)\big]^m\big)\rangle \\ &= \sum_{m=1}^{\infty} \frac{g^{2m}}{2m} \sum_{\nu_1,\ldots,\nu_{2m}\in\{0,1\}} \langle \mathrm{Tr}(X^{\nu_1}(X^\dagger)^{\nu_2}X^{\nu_3}..)\mathrm{Tr}(X^{1-\nu_1}(X^\dagger)^{1-\nu_2}X^{1-\nu_3}..)\rangle, \end{aligned} \tag{4.10}$$

where the bracket stands for the Gaussian integration over the complex matrix X, namely $\langle f(X)\rangle = \int dX dX^\dagger f(X)\exp(-N\mathrm{Tr}(XX^\dagger))$, and is normalized in such a way that $\langle 1\rangle = 1$. Moreover, the large-N limit of (4.10), $z_H(g) = \lim_{N\to\infty} \frac{1}{N^2} Z_H(g;N)$ yields the generating function for genus zero Eulerian triangulations equipped with Hamiltonian cycles. Due to the known large-N factorization property $\langle \mathrm{Tr}(f(X)g(X))\rangle \sim \langle \mathrm{Tr}(f(X))\rangle\langle \mathrm{Tr}(g(X))\rangle$, we also have

$$z_H(g) = \sum_{m=1}^{\infty} \frac{g^{2m}}{2m} \sum_{\nu_1,\ldots,\nu_{2m}\in\{0,1\}} \langle\langle \mathrm{Tr}(X^{\nu_1}(X^\dagger)^{\nu_2}X^{\nu_3}..)\rangle\rangle\langle\langle \mathrm{Tr}(X^{1-\nu_1}(X^\dagger)^{1-\nu_2}X^{1-\nu_3}..)\rangle\rangle, \tag{4.11}$$

where the double bracket is defined as $\langle\langle \mathrm{Tr} f(X)\rangle\rangle = \lim_{N\to\infty} \frac{1}{N}\langle \mathrm{Tr} f(X)\rangle$.

The formula (4.10) can be interpreted pictorially as follows. The quantity $\big[(X\otimes I + I\otimes X)(X^\dagger \otimes I + I\otimes X^\dagger)\big]^m$ can be represented as a succession of $2m$ points along a line, from which oriented bonds originate, alternately oriented away from and towards the line to account for the alternation of X and $X^\dagger$, and going above (resp. below) the line if a term $M\otimes I$ (resp. $I\otimes M$), $M = X, X^\dagger$ is selected i.e. according to whether $\nu_i = 1$ (resp. $\nu_i = 0$) at the i-th point in (4.11). This gives rise to simple diagrams like that of fig.3, where the $2m$ points are connected pairwise by oriented arches either above or below the line, such that arrows inwards and outwards alternate along the line. This alternation can be replaced by a sequence of alternating $+$ and $-$ signs, each arch connecting a $+$ to a $-$ as in [8].

In (4.11), we see that the knowledge of all Gaussian averages of traces of words in X and $X^\dagger$ would immediately give access to $z_H(g)$. Such objects have been considered in [15], as Gaussian averages of traces of words involving Hermitian matrices, for which a complete

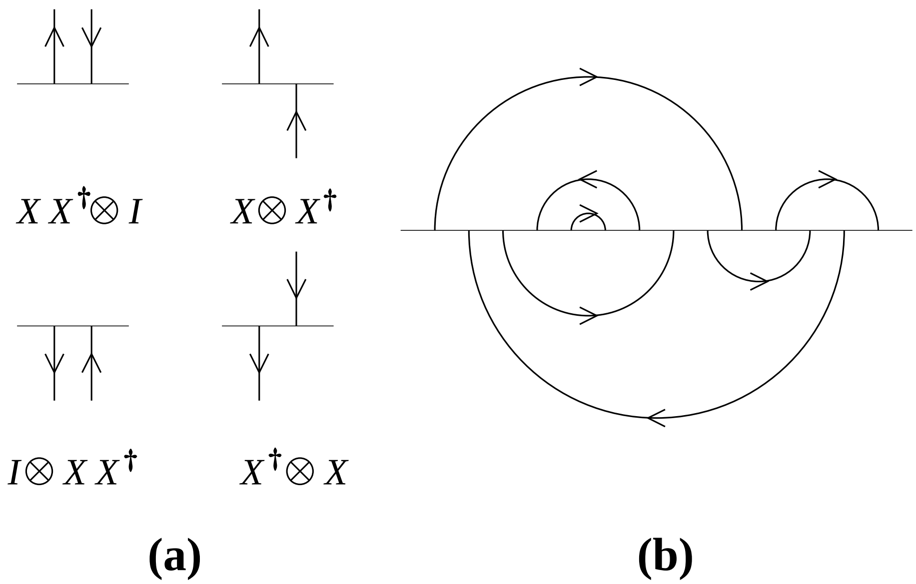


Fig. 3: (a): rules for representing a term in the expansion (4.11). The four cases correspond respectively to the values $(\nu_1,\nu_2)=(11),(10),(00),(01)$, namely to the selection of $X^{\nu_1}(X^\dagger)^{\nu_2}\otimes X^{1-\nu_1}(X^\dagger)^{1-\nu_2}$ in the product $(X\otimes I+I\otimes X)(X^\dagger\otimes I+I\otimes X^\dagger)$. It is understood that the line representing X should be drawn to the left of the line representing $X^\dagger$. In the figure (b), we have represented a typical term, with $2m=14$ points, and a choice $(\nu_1...\nu_{14})=(10011110011001)$. The value $\nu=1$ (resp. 0) corresponds to an arch going above (resp. below) the line. The Eulerian condition imposes that orientations alternate between successive points.

set of recursion relations has been found, solving in principle (but unfortunately not in practice) our problem. These were studied in the context of meander enumeration, namely the enumeration of possibly interlocking loops (roads) crossing a line (river) through $2m$ given points (bridges). The generating function for meanders with a weight 2 per connected component of the road can actually be recast in a way very similar to (4.10), namely

$$M(g^2;N)\ =\ \sum_{m=1}^{\infty}\frac{g^{2m}}{2m}\langle\mathrm{Tr}\big((X\otimes X^\dagger+X^\dagger\otimes X)^m\big)\rangle. \tag{4.12}$$

This corresponds precisely to retaining in (4.10) only the sets of ν's that satisfy $\nu_{2i}=1-\nu_{2i-1}$ for $i=1,2,...,m$ which again corresponds to retaining only the two configurations on the right hand side of fig.3-(a). These two can both be recombined into a single oriented bond crossing the line (with a weight g^2 per intersection), leading to the usual picture of a multi-component meander with two possible orientations per connected component, accounting for the factor of 2. In the planar ($N \to \infty$) limit, this yields

$$m(g^2) = \sum_{m=1}^{\infty} \frac{g^{4m}}{4m} \sum_{\nu_1,\ldots,\nu_{2m}\in\{0,1\}} \langle\langle \mathrm{Tr}(X^{\nu_1}(X^\dagger)^{1-\nu_1}X^{\nu_2}(X^\dagger)^{1-\nu_2}..)\rangle\rangle \times \langle\langle \mathrm{Tr}(X^{1-\nu_1}(X^\dagger)^{\nu_1}X^{1-\nu_2}(X^\dagger)^{\nu_2}..)\rangle\rangle. \tag{4.13}$$

5. Conclusion

In this paper, we have considered some fully packed loop models on Eulerian triangulations. In the case of the $O(n=1)$ model, we have shown that taking Eulerian triangulations rather than arbitrary ones leads to a shift $c \to c+1$ in the central charge of the conformal theory coupled to gravity describing the corresponding critical point. More generally, given any matrix model describing random triangulations, typically defined by a Hermitian multi-matrix integral with a potential of the form

$$V(A_1,...,A_p) = \frac{1}{2}\sum_{i=1}^{p} A_i^2 - \sum_{ijk} c_{ijk}A_iA_jA_k, \tag{5.1}$$

we can restrict ourselves to the class of Eulerian triangulations by replacing the Hermitian matrices A_i by complex matrices X_i, governed by the potential

$$V(X_1,...,X_p) = \sum_{i=1}^{p} X_iX_i^\dagger - \sum_{ijk}(c_{ijk}X_iX_jX_k + \bar{c}_{ijk}X_k^\dagger X_j^\dagger X_i^\dagger). \tag{5.2}$$

It would be interesting to investigate how this restriction affects the critical properties of the original model.

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